Dynamical System on Cantor Set

Makoto MORI

Nihon University
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1. Introduction.

We will consider Cantor sets generated by piecewise $C^{1+\gamma}$ transformations ($\gamma > 0$). In this article, we only consider Markov cases. Non Markov (but piecewise linear) cases will be studied in [6]. A heuristic argument will also appear in that paper.

Let us denote $I = [0, 1]$. We assume that there exists a finite set $\mathscr{A}$ of symbols, and a subinterval $\langle a \rangle \subset I$ corresponds to a symbol $a \in \mathscr{A}$, and

1. $\bigcup_{a \in \mathscr{A}} \langle a \rangle = I$,
2. $\langle a \rangle \cap \langle b \rangle = \varnothing$ if $a \neq b$.

Take a subset $\mathscr{A}_1 \subset \mathscr{A}$, and we consider a mapping F from $\bigcup_{a \in \mathscr{A}_1} \langle a \rangle$ to I such that

1. F is monotone on each $\langle a \rangle$ and it can extend to $\overline{\langle a \rangle}$ in $C^{1+\gamma}$ ($\gamma > 0$) (piecewise $C^{1+\gamma}$),
2. if $F(\langle a \rangle) \cap \langle b \rangle \neq \varnothing$ for $a, b \in \mathscr{A}_1$, then $\overline{F(\langle a \rangle)} \supset \langle b \rangle$ (Markov),
3. $\xi = \liminf_{n \to \infty} \frac{1}{n} \operatorname{ess\,inf}_{x \in I} \log |F^{n\prime}(x)| > 0$ (expanding),
4. for each $a, b \in \mathscr{A}_1$, there exists n such that $\overline{F^n(\langle a \rangle)} \supset \langle b \rangle$ (irreducible),

where we denote the closure of a set J by $\bar{J}$. Note that from the above assumption, we get

$$\log \operatorname*{ess\,inf}_{x \in I} |F^{n\prime}(x)| > 0$$

for some $n > 0$. Here we denote by F^n the n-th iteration of F:

$$F^n(x) = \begin{cases} x & \text{if } n = 0, \\ F^{n-1}(F(x)) & \text{if } n \geq 1. \end{cases}$$

Thus, hereafter we assume without loss of generality that

$$\xi_0 = \log \operatorname*{ess\,inf}_{x \in I} |F'(x)| > 0.$$

We will consider a set

$$\mathscr{C} = \{x \in I : F^n(x) \in \bigcup_{a \in \mathscr{A}_1} \langle a \rangle, \forall n \geq 0\},$$

that is, the set of points whose orbits pass only subintervals corresponding to $\mathscr{A}_1$. One of the simplest examples of $\mathscr{C}$ is the usual Cantor set, which is defined by $F(x) = 3x$ (mod 1), $\mathscr{A} = \{0, 1, 2\}$, $\langle i \rangle = [i/3, (i+1)/3)$, and $\mathscr{A}_1 = \{0, 2\}$, and one of the most important examples is the set of points with digits 1 and 2 in its continued expansion. In the case, $F(x) = 1/x$ (mod 1), $\mathscr{A} = \{1, 2, \cdots\}$, $\langle i \rangle = (1/(i+1), 1/i]$ and $\mathscr{A}_1 = \{1, 2\}$. The continued fraction expansion has countable symbols, and does not satisfy our assumption. But the transformation on $\bigcup_{a \in \mathscr{A} \setminus \mathscr{A}_1} \langle a \rangle$ is unessential. Moreover, though $F'(1) = -1$, there exists a neighborhood of 1 such that $\{x : F(x) \notin \langle 1 \rangle \cup \langle 2 \rangle\}$. So, this case $\mathscr{A}_1 = \{1, 2\}$ essentially satisfies our assumption (cf. §6).

As usual, we express a point $x \in I$ by a sequence of symbols $a_1^x a_2^x \cdots$ called the expansion of x defined by

$$F^{i-1}(x) \in \langle a_i^x \rangle \qquad i \geq 1.$$

We call a finite sequence of symbols $w = a_1 \cdots a_m$ a word and define $|w| = m$ (the length of a word w). We define as usual a subinterval $\langle w \rangle$ corresponding to a word w, which is the set of points x such that $F^{i-1}(x) \in \langle a_i \rangle$ for any $1 \leq i \leq m$. We call a word w admissible if $\langle w \rangle \neq \emptyset$. We denote the empty word by ε, and for notational convenience we define $\langle \varepsilon \rangle = I$ (that is, ε is admissible), and $|\varepsilon| = 0$. We denote by $\mathscr{W}_m$ the set of admissible words $w = a_1 \cdots a_m$ with $a_i \in \mathscr{A}_1$. Set $\mathscr{W} = \bigcup_{m=0}^{\infty} \mathscr{W}_m$. We denote by wx ($w \in \mathscr{W}, x \in I$) a point which belongs to $\langle w \rangle$ and $F^{|w|}(wx) = x$ if it exists. Note that $\mathscr{W}$ expresses the set of all the admissible words with symbols only in $\mathscr{A}_1$. Hereafter, we only consider words with symbols in $\mathscr{A}_1$.

We also define for $x \in \langle w \rangle$ ($|w| = m$)

$$F_m^{+\prime}(x) = F_m^{+\prime}(w) = \begin{cases} +\operatorname{ess\,inf}_{y \in \langle w \rangle} |F'(y)| & \text{if } F'(x) > 0, \\ -\operatorname{ess\,inf}_{y \in \langle w \rangle} |F'(y)| & \text{if } F'(x) < 0, \end{cases}$$

$$F_m^{-\prime}(x) = F_m^{-\prime}(w) = \begin{cases} +\operatorname{ess\,sup}_{y \in \langle w \rangle} |F'(y)| & \text{if } F'(x) > 0, \\ -\operatorname{ess\,sup}_{y \in \langle w \rangle} |F'(y)| & \text{if } F'(x) < 0, \end{cases}$$

$$(F_m^{\tau})^{n\prime}(x) = \prod_{i=0}^{n-1} F_m^{\tau}(F^i(x)) \qquad (\tau \in \{+, -\}).$$

Note that a transformation F_m^{τ} is only a formal piecewise linear Markov transformation on the symbolic dynamics where F is realized, and it may not be able to express as a map from I into itself.

As in [4] and [5], we will define generating functions for $g \in L^{\infty}$ and $0 \leq \alpha \leq 1$

$$s_{g,\alpha}^w(z : F) = \sum_{n=0}^{\infty} z^n \sum_{u \in \mathscr{W}_n} \int_{ux \in \langle w \rangle} |F^{n\prime}(ux)|^{-\alpha} g(x) d\bar{x} \qquad (1)$$

$$s_{g,\alpha}^w(z : F_m^\tau) = \sum_{n=0}^\infty z^n \sum_{u \in \mathscr{W}_n} \int_{ux \in \langle w \rangle} |(F_m^\tau)^{n\prime}(ux)|^{-\alpha} g(x) d\tilde{x} \tag{2}$$

for a word $w \in \mathscr{W}$, where $d\tilde{x}$ denotes the integral by the Lebesgue measure restricted to $\bigcup_{a \in \mathscr{A}_1} \langle a \rangle$. We also define α_m^τ the maximum α for which $z = 1$ is the minimum singularity in modulus of $s_{g,\alpha}^w(z : F_m^\tau)$ for some $g \in L^\infty$ and some word w. We denote by α_0 the corresponding value for F. From the definition, if the right hand term of (1) converges, it is easy to see that

$$|s_{g,\alpha}^w(z : F)| \leq s_{|g|,\alpha}^w(|z| : F).$$

Therefore, since the coefficients of the right hand term of (1) is positive for $g \geq 0$, the minimal singularity of $s_{g,\alpha}^w(z : F)$ is nonnegative. Moreover, for a non-negative valued function g, positive z and $\alpha \geq 0$, if the right hand terms of (1) and (2) converge, we get

$$s_{g,\alpha}^w(z : F) \leq s_{g,\alpha}^\varepsilon(z : F),$$

$$s_{g,\alpha}^w(z : F_1^-) \leq s_{g,\alpha}^w(z : F_2^-) \leq \cdots \leq s_{g,\alpha}^w(z : F) \leq \cdots \leq s_{g,\alpha}^w(z : F_2^+) \leq s_{g,\alpha}^w(z : F_1^+).$$

Therefore we get

$$1 \geq \alpha_1^+ \geq \alpha_2^+ \geq \cdots \geq \alpha_0 \geq \cdots \geq \alpha_2^- \geq \alpha_1^- \geq 0.$$

REMARK.   Let v_α be a Hausdorff measure with exponent α, that is,

$$v_\alpha(J) = \liminf_{\delta \downarrow 0 \{J_i\}} (Lebes(J_i))^\alpha,$$

where infimum is taken over all coverings of $\mathscr{C}$ with countable subintervals $\{J_i\}$ with Lebesgue measure $Lebes(J_i)$ less than δ. Then as a formal expression, we can define

$$\bar{s}_{g,\alpha}^w(z : F) = \sum_{n=0}^\infty z^n \int 1_{\langle w \rangle}(x) g(F^n(x)) dv_\alpha$$

$$= \sum_{n=0}^\infty z^n \int_I \sum_{y \in \mathscr{C} : F^n(y) = x} 1_{\langle w \rangle}(y) |F^{n\prime}(y)|^{-\alpha} g(x) dv_\alpha$$

$$= \sum_{n=0}^\infty z^n \sum_{u \in \mathscr{W}_n} \int_{ux \in \langle w \rangle} |F^{n\prime}(ux)|^{-\alpha} g(x) dv_\alpha$$

$$= \int [(I - zP_\alpha)^{-1} 1_{\langle w \rangle}](x) g(x) dv_\alpha.$$

where P_α is the Perron-Frobenius operator associated with F with respect to v_α. Therefore, these formal generating functions $\bar{s}_{g,\alpha}$ will express the ergodic properties of the dynamical system. But they are only formal expression up to this point, thus we slightly modify them, and define $s_{g,\alpha}^w(z : F)$.

LEMMA 1.   *For fixed* $0 \leq \alpha \leq 1$,

$$\lim_{\substack{m \to \infty \\ x \in I}} \operatorname{ess\,sup} ||F_m^{\tau\prime}(x)|^{-\alpha} - |F'(x)|^{-\alpha}| = 0 .$$

PROOF. Note first $|X - Y|^\alpha \geq |X^\alpha - Y^\alpha|$ for X, $Y > 0$. Therefore

$$||F_m^{\tau\prime}(x)|^{-\alpha} - |F'(x)|^{-\alpha}| = ||F_m^{\tau\prime}(x)|^\alpha - |F'(x)|^\alpha|/(|F_m^{\tau\prime}(x)|^\alpha |F'(x)|^\alpha)$$

$$\leq |F_m^{\tau\prime}(x) - F'(x)|^\alpha/(|F_m^{\tau\prime}(x)|^\alpha |F'(x)|^\alpha) .$$

Because F is piecewise $C^{1+\gamma}$ and expanding, the lemma is proved.

LEMMA 2. *Set* $0 \leq \alpha \leq 1$. *For any* $\varepsilon > 0$, *we get for sufficiently large* m *independent of* n

$$|(F_m^{\tau})^{n\prime}(x)|^{-\alpha} \begin{cases} \leq (1+\varepsilon)^n |F^{n\prime}(x)|^{-\alpha} , \\ \geq (1-\varepsilon)^n |F^{n\prime}(x)|^{-\alpha} , \end{cases}$$

for any x.

PROOF. By the chain rule,

$$|(F_m^{\tau})^{n\prime}(x)|^{-\alpha} = \prod_{i=0}^{n-1} |F_m^{\tau\prime}(F^i(x))|^{-\alpha}$$

$$\leq \prod_{i=0}^{n-1} |F'(F^i(x))|^{-\alpha}$$

$$\times (1 + ||F_m^{\tau\prime}(F'(x))|^{-\alpha} - |F'(F^i(x))|^{-\alpha}||F'(F^i(x))|^\alpha) .$$

Therefore by Lemma 1, for any $\varepsilon > 0$

$$|(F_m^{\tau})^{n\prime}(x)|^{-\alpha} \leq (1+\varepsilon)^n |F^{n\prime}(x)|^{-\alpha}$$

holds for any sufficiently large m. In a similar way, we can prove the other inequality.

COROLLARY 1.   *For a fixed* $0 \leq \alpha \leq 1$, *there exists* $\beta(\alpha) > 0$ *such that for* $|z| < \beta(\alpha)$

$$\lim_{m \to \infty} s_{g,\alpha}^w(z : F_m^\tau) = s_{g,\alpha}^w(z : F)$$

for any word $w \in \mathcal{W}$.

PROOF. Note first

$$|s_{g,\alpha}^w(z : F)| \leq \sum_{n=0}^{\infty} |z|^n \sum_{u \in \mathcal{W}_n} \int_{ux \in \langle w \rangle} |F^{n\prime}(ux)|^{-\alpha} |g(x)| d\bar{x}$$

$$\leq \|g\|_\infty \sum_{n=0}^{\infty} (\#\mathcal{A}_1 |z| \operatorname*{ess\,sup}_{x \in I} |F'(x)|^{-\alpha})^n .$$

Now put

$$\beta(\alpha) = \frac{1}{2} (\#\mathscr{A}_1 \operatorname*{ess\,sup}_{x \in I} |F'(x)|^{-\alpha})^{-1} .$$

Then from Lemma 2, $s_{g,\alpha}(z : F_m^\tau)$ and $s_{g,\alpha}(z : F)$ are uniformly bounded in $|z| < \beta(\alpha)$ for sufficiently large m. Therefore the proof follows.

2. α-Fredholm matrix and α-zeta function.

Now we will construct an α-Fredholm matrix $\Phi_\alpha(z : F_m^\tau)$. For $k \geq m$, set $\mathscr{W}_k \times \mathscr{W}_k$ matrix

$$\Phi_{\alpha,k}(z : F_m^\tau)_{u,v} = \begin{cases} z |F_m^\tau(u)'|^{-\alpha} & \text{if } \overline{F(\langle u \rangle)} \supset \langle v \rangle , \\ 0 & \text{otherwise} . \end{cases}$$

We denote $\Phi_{\alpha,m}(z : F_m^\tau)$ by $\Phi_\alpha(z : F_m^\tau)$. Then we get for $u \in \mathscr{W}_k$

$$s_{g,\alpha}^u(z : F_m^\tau) = \sum_{n=0}^\infty z^n \sum_{v \in \mathscr{W}_n} \int_{vx \in \langle u \rangle} |(F_m^\tau)^{n\prime}(vx)|^{-\alpha} g(x) d\bar{x}$$

$$= \int_{\langle u \rangle} g(x) d\bar{x} + z \sum_{v \in \mathscr{W}_k : F(\langle u \rangle) \supset \langle v \rangle} |F_m^{\tau\prime}(u)|^{-\alpha} s_{g,\alpha}^v(z : F_m^\tau) .$$

Therefore we can construct a renewal equation of the form

$$(s_{g,\alpha}^u(z : F_m^\tau))_{u \in \mathscr{W}_k} = \left(\int_{\langle u \rangle} g(x) d\bar{x} \right)_{u \in \mathscr{W}_k} + \Phi_{\alpha,k}(z : F_m^\tau)(s_{g,\alpha}^v(z : F_m^\tau))_{v \in \mathscr{W}_k} .$$

Now we define α-zeta function by

$$\zeta_\alpha(z : F) = \exp\left[\sum_{n=1}^\infty \frac{z^n}{n} \sum_{y \in \mathscr{C} : y = F^n(y)} |F^{n\prime}(y)|^{-\alpha} \right]$$

$$\zeta_\alpha(z : F_m^\tau) = \exp\left[\sum_{n=1}^\infty \frac{z^n}{n} \sum_{y \in \mathscr{C} : y = F^n(y)} |(F_m^\tau)^{n\prime}(y)|^{-\alpha} \right] .$$

LEMMA 3. *for any* $k \geq m$

$$\det(I - \Phi_{\alpha,k}(z : F_m^\tau)) = \zeta_\alpha(z : F_m^\tau)^{-1} ,$$

especially

$$\det(I - \Phi_\alpha(z : F_m^\tau)) = \zeta_\alpha(z : F_m^\tau)^{-1} .$$

Note that $\Phi_{\alpha,k}(z)$ is essentially a structure matrix of the dynamical system. Namely, the trace of $\Phi_{\alpha,k}(z)^n$ corresponds to periodic orbits with period n. Thus, from the fact that, for a matrix A, $\det A = \exp[\operatorname{tr} \log A]$, we get the proof using the Taylor expansion of $\log(1-z)$, where $\operatorname{tr} A$ means the trace of a matrix A (cf. [4]).

LEMMA 4.   *For* $|z| < \beta(\alpha)$,

$$\lim_{m \to \infty} \zeta_\alpha(z : F_m^\tau) = \zeta_\alpha(z : F) .$$

The proof easily follows from Lemma 2 and the similar discussion in Corollary 1.

LEMMA 5.

$$\lim_{m \to \infty} \alpha_m^+ = \lim_{m \to \infty} \alpha_m^- = \alpha_0 .$$

PROOF.   For any $\varepsilon > 0$, take m sufficiently large such that

$$\frac{|(F_m^+)'(x)|}{|(F_m^-)'(x)|} \geq \frac{1-\varepsilon}{1+\varepsilon} .$$

From Lemma 3, for any $\alpha > \alpha_m^-$, $\zeta_\alpha(z : F_m^-)$ has no singularity in the unit disk. Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{y \in \mathscr{C} : y = F^n(y)} |(F_m^-)^{n\prime}(y)|^{-\alpha} < \infty .$$

On the other hand, for any $\alpha' > \alpha$,

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{y \in \mathscr{C} : y = F^n(y)} |(F_m^-)^{n\prime}(y)|^{-\alpha}$$

$$\geq \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1-\varepsilon}{1+\varepsilon} \right)^{n\alpha} \sum_{y \in \mathscr{C} : y = F^n(y)} |(F_m^+)^{n\prime}(y)|^{-\alpha}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1-\varepsilon}{1+\varepsilon} \right)^{n\alpha} \sum_{y \in \mathscr{C} : y = F^n(y)} |(F_m^+)^{n\prime}(y)|^{-\alpha'} \prod_{i=0}^{n-1} |(F_m^+)'(F^i(y))|^{\alpha'-\alpha} . \tag{3}$$

Now we choose any $\alpha' > \alpha$ such that

$$\inf_x |F'(x)|^{\alpha'-\alpha} \left| \frac{1-\varepsilon}{1+\varepsilon} \right|^\alpha > 1 .$$

Namely,

$$\alpha' > \alpha \frac{1 + \log[(1+\varepsilon)/(1-\varepsilon)]}{\log \inf_x |F'(x)|} . \tag{4}$$

Note here the right hand term of (4) tends to α as ε tends to 0. Thus

the right hand side of (3) $\geq \sum_{n=1}^{\infty} \frac{1}{n} \sum_{y \in \mathscr{C} : y = F^n(y)} |(F_m^+)^{n\prime}(y)|^{-\alpha'} .$

This shows that $\zeta_{\alpha'}(z : F_m^+)$ is analytic in $|z| \leq 1$. Thus $\det(I - \Phi_{\alpha'}(z : F_m^+))$ has no zero in $|z| \leq 1$, that is, $\alpha' > \alpha_m^+$. This proves the lemma. Indeed, for example, if there exists $\delta > 0$

such that $\alpha_0 > \lim_{m \to \infty} \alpha_m^- + \delta$, then we can take $\varepsilon > 0$ sufficiently small such that there exists $\alpha' < \alpha_0$ which satisfies (4). This is the contradiction, because $\alpha_m^+ < \alpha' < \alpha_0 \leq \alpha_m^+$. Therefore $\alpha_0 = \lim_{m \to \infty} \alpha_m^-$. In a similar way, we can show $\alpha_0 = \lim_{m \to \infty} \alpha_m^+$.

3. Construction of a transformation G.

First note that

LEMMA 6. *Let* $e_{m,k}^\tau = (e_{m,k}^\tau(w))_{w \in \mathscr{W}_k}$ *be an eigenvector of* $\Phi_{\alpha,k}(1 : F_m^\tau)$ *associated with maximal eigenvalue. Set* $e_m^\tau(v) = \sum_{\langle u \rangle \subset \langle v \rangle} e_{m,k}^\tau(u)$ *for* $v \in \mathscr{W}_m$. *Then* $e_m^\tau = (e_m^\tau(v))_{v \in \mathscr{W}_m}$ *is the eigenvector of* $\Phi_\alpha(1 : F_m^\tau)$ *with the eigenvalue* λ.

PROOF. Let $\lambda > 0$ be the maximal eigenvalue, and $(e_{m,k}^\tau(u))$ be an eigenvector associated with it. Since $\Phi_{\alpha_{m,k}^\tau}(1 : F_m^\tau)$ is a nonnegative irreducible matrix and 1 is its maximal eigenvalue, we can take $e_{m,k}^\tau(w) > 0$ for any $w \in \mathscr{W}_k$. For $v \in \mathscr{W}_m$, the v-component of $\Phi_\alpha(z : F_m^\tau)(e_m^\tau(v))$ $(v \in \mathscr{W}_m)$ equals

$$= \sum_{w \in \mathscr{W}_m} \Phi_\alpha(1 : F_m^\tau)_{v,w} e_m^\tau(w)$$

$$= \sum_{w \in \mathscr{W}_m : F(\langle v \rangle) \supset \langle w \rangle} |F_m^{\tau'}(v)|^{-\alpha} e_m^\tau(w)$$

$$= \sum_{w \in \mathscr{W}_k : F(\langle v \rangle) \supset \langle w \rangle} |F_m^{\tau'}(v)|^{-\alpha} e_{m,k}^\tau(w)$$

$$= \sum_{v' \in \mathscr{W}_k : \langle v' \rangle \subset \langle v \rangle} \Phi_{\alpha,k}(1 : F_m^\tau)_{v',w} e_{m,k}^\tau(w)$$

$$= \lambda \sum_{v' \in \mathscr{W}_k : \langle v' \rangle \subset \langle v \rangle} e_{m,k}^\tau(v') = \lambda e_m^\tau(v).$$

Let $e_m^\tau = (e_m^\tau(w))_{w \in \mathscr{W}_m}$ be an eigenvector associated with the eigenvalue 1 of $\Phi_{\alpha_m^\tau}(1 : F_m^\tau)$ such that $\sum_{w \in \mathscr{W}_m} e_m^\tau(w) = 1$. We can construct a piecewise linear Markov mapping $G_m^\tau : [0, 1] \to [0, 1]$ as follows:

Define a natural order on $\mathscr{A}_1$, that is, $a < b$ if and only if $x \in \langle a \rangle$ and $y \in \langle b \rangle$ satisfy $x < y$. We also introduce a natural order on $\mathscr{W}_m$. For $u = a_1 \cdots a_m$ and $v = b_1 \cdots b_m$ with $a_1 \cdots a_i = b_1 \cdots b_i$, and $a_{i+1} < b_{i+1}$. Then $u < v$ if $F^{i'}(y) > 0$ for $y \in \langle u \rangle$, and $u > v$ otherwise. Arrange all the words in $\mathscr{W}_m$ in this order

$$w_{m,1} < w_{m,2} < \cdots < w_{m, \#\mathscr{W}_m}.$$

Take

$$c_{m,i}^\tau = \begin{cases} 0 & \text{for } i = 0, \\ \sum_{j=1}^i e_m^\tau(w_{m,j}) & \text{for } 0 < i \leq \#\mathscr{W}_m. \end{cases}$$

Set

$$\langle w_{m,i}\rangle = \begin{cases} [c_{m,i-1}^{\tau}, c_{m,i}^{\tau}) & \text{for} \quad 1 \le i \le \#\mathscr{W}_m - 1, \\ [c_{m,\#\mathscr{W}_m-1}^{\tau}, 1] & \text{for} \quad i = \#\mathscr{W}_m, \end{cases}$$

$$d_{m,i}^{\tau} = \inf\left\{ \bigcup_{b \in \mathscr{A}_1 : a_2 \cdots a_m b \in \mathscr{W}_m} \langle a_2 \cdots a_m b\rangle \right\},$$

$$d_{m,i}^{\tau}{}' = \sup\left\{ \bigcup_{b \in \mathscr{A}_1 : a_2 \cdots a_m b \in \mathscr{W}_m} \langle a_2 \cdots a_m b\rangle \right\},$$

where $w_{m,i} = a_1 a_2 \cdots a_m$. Then for $x \in \langle w_{m,i}\rangle$, define

$$G_m^{\tau}(x) = \frac{d_{m,i}^{\tau}{}' - d_{m,i}^{\tau}}{c_{m,i}^{\tau} - c_{m,i-1}^{\tau}} (x - c_{m,i-1}^{\tau}) + d_{m,i}^{\tau}.$$

Note that $|G_m^{\tau}{}'(x)| = (|F_m^{\tau}{}'(y)|)^{\alpha_m^{\tau}}$ and $G_m^{\tau}{}' > 0$ if and only if $F_m^{\tau}{}'(y) > 0$, where the expansion of $y \in I$ by F equals that of $x \in [0, 1]$ by G_m^{τ}. This shows that $\Phi_{\alpha_m^r}(1 : F_m^{\tau})$ is the Fredholm matrix of G_m^{τ} (cf. [4]). Also note that G_m^{τ} is expanding and $([0, 1], G_m^{\tau})$ has the same symbolic dynamics with $(\mathscr{C}, F)$.

Set for a word $w \in \mathscr{W}$

$$s_g^w(z : G_m^{\tau}) = \sum_{n=0}^{\infty} z^n \int_0^1 1_{\langle w\rangle}(x) g((G_m^{\tau})^n(x)) dx,$$

where $\langle w\rangle \subset [0, 1]$ is the subinterval associated with a word w which is induced by G_m^{τ}. Then we can get a renewal equation of the form

$$(s_g^w(z : G_m^{\tau}))_{w \in \mathscr{W}_m} = \left(\int_{\langle w\rangle} g(x) dx \right)_{w \in \mathscr{W}_m} + \Phi_{\alpha_m^r}(z : F_m^{\tau})(s_g^w(z : G_m^{\tau}))_{w \in \mathscr{W}_m}.$$

Now we will construct a transformation $G : [0, 1] \to [0, 1]$ which corresponds to F. For $m > k$, set $c_{m,k,0}^{\tau} = 0$ and for $0 < l \le \#W_k$

$$c_{m,k,l}^{\tau} = \sum_{i=1}^{l} \sum_{u \in \mathscr{W}_m : \langle u\rangle \subset \langle w_{k,i}\rangle} e_m^{\tau}(u),$$

that is, $[c_{m,k,l-1}^{\tau}, c_{m,k,l}^{\tau})$ corresponds to a word $w_{k,l}$ with respect to the mapping G_m^{τ}. Take a subsequence which we also express by $\{m\}$ such that the sequence $c_{m,1,1}^{\tau}$ converges to a point which we express by $\bar{c}_{1,1}^{\tau}$. Next we choose again a subsequence $\{m\}$ of the above subsequence to converge $c_{m,1,2}^{\tau}$ to some point $\bar{c}_{1,2}^{\tau}$, and so on, and we can define $\bar{c}_{k,l}^{\tau}$ for all k and $0 \le l \le \#W_k$. Then $[\bar{c}_{k,l-1}^{\tau}, \bar{c}_{k,l}^{\tau})$ is a new subinterval corresponding to a word $w_{k,l} \in \mathscr{W}_K$. Using these subintervals, we can also define $\bar{d}_{k,l}^{\tau}$ and $\bar{d}_{k,l}^{\tau}{}'$. Then we can define mappings $\bar{G}_k^{\tau}$ as before. From the construction, $\bar{G}_k^{\tau}$ maps a subinterval corresponding to a word $w = a_1 \cdots a_k$ to the union of the subintervals corresponding to $a_2 \cdots a_k a$ $(a \in \mathscr{A}_1)$. Now we will fix a word $w_{k,l} = a_1 \cdots a_k \in \mathscr{W}_k$, and we assume that its image

$\langle a_2 \cdots a_k \rangle$ corresponds to a set of words $w_{k,n}$ ($l_1 \le n \le l_2$). Then we can choose a subsequence $\{m\}$ such that $c^\tau_{m,k,n}$ converges to $\bar{c}^\tau_{k,n}$ for $n = l - 1, l, l_1 - 1, l_2$. Then

$$\bar{G}_k^{\tau\prime}(w) = \frac{\bar{c}^\tau_{k,l_2} - \bar{c}^\tau_{k,l-1}}{\bar{c}^\tau_{k,l} - \bar{c}^\tau_{k,l-1}} = \lim_{m \to \infty} \frac{c^\tau_{m,k,l_2} - c^\tau_{m,k,l_1-1}}{c^\tau_{m,k,l} - c^\tau_{m,k,l-1}},$$

$$\inf_{x \in \langle w \rangle} |F'(x)|^{\alpha^\tau_m} \le \left| \frac{c^\tau_{m,k,l_2} - c^\tau_{m,k,l_1-1}}{c^\tau_{m,k,l} - c^\tau_{m,k,l-1}} \right| \le \sup_{x \in \langle w \rangle} |F'(x)|^{\alpha^\tau_m}.$$

Therefore, since $\lim_{m \to \infty} \alpha^\tau_m = \alpha_0$, this implies

$$\inf_{x \in \langle w \rangle} |F'(x)|^{\alpha_0} \le |\bar{G}_k^{\tau\prime}(w)| \le \sup_{x \in \langle w \rangle} |F'(x)|^{\alpha_0}.$$

For each word w, the endpoints of the subintervals corresponding to w converge. Therefore there exists a limit $G^{\tau\prime}(x) = \lim_{k \to \infty} \bar{G}_k^{\tau\prime}(x)$, if x does not coincide with endpoints of any word. Namely, there exists $G'(x)$ except countably many points (that is, there exists G' in L^1 sense). It is also easy to see even for an endpoint of a word whose expansion equals $s = a_1 a_2 \cdots$, there exists a limit $\lim_{m \to \infty} G_m^{\tau\prime}(s)$. However, it may not coincide with the limit corresponding to another sequence of symbols which express the same endpoint. Hereafter, we fix G^+ or G^- and denote it by G. From the construction, G and F on $\mathscr{C}$ has the same symbolic dynamics. We denote the 1 to 1, onto mapping by $\phi: [0, 1] \to \mathscr{C}$ for which $x \in [0, 1]$ and $\phi(x) \in \mathscr{C}$ has the same expansion. Then

$$G'(x) = \begin{cases} +|F'(\phi(x))|^{\alpha_0} & \text{if } F'(\phi(x)) > 0, \\ -|F'(\phi(x))|^{\alpha_0} & \text{if } F'(\phi(x)) < 0. \end{cases}$$

To emphasize the notation, we denote by $\langle w \rangle_F$ and $\langle w \rangle_G$ subintervals associated with a word w which correspond to F on I and G on $[0, 1]$, respectively.

LEMMA 7.   *Assume that $\alpha_0 > 0$. Then there exists a constant $\gamma' > 0$ such that $1/G'(x)$ is of universally bound γ'-variation, where a function f is of universally bounded p-variation if*

$$\mathrm{var}_p(f) = \sup_{0 \le x_0 < \cdots < x_n \le 1} \left(\sum_{i=1}^n |f(x_i) - f(x_{i-1})|^p \right)^{1/p} < \infty.$$

PROOF.   For $x < y$ which do not coincide with endpoints of any word,

$$|1/G'(x) - 1/G'(y)| = ||F'(\phi(x))|^{-\alpha_0} - |F'(\phi(y))|^{-\alpha_0}|$$

$$\le ||F'(\phi(x))|^{-1} - |F'(\phi(y))|^{-1}|^{\alpha_0}$$

$$\le ||F'(\phi(x))| - |F'(\phi(y))||^{\alpha_0} e^{-2\alpha_0 \xi_0}.$$

Since F' is Hölder continuous with Hölder exponent γ, it is also of universally bounded $1/\gamma$ variation (cf. [3]), therefore $1/G'$ is of universally $\gamma' = \gamma/\alpha_0$ bounded variation. This proves the lemma.

4. Hausdorff dimension.

First we will mention easy lemmas.

LEMMA 8. For $0 < \alpha < 1$, and $x_i > 0$ $(1 \le i \le k)$, we get

$$k^{\alpha-1}\left(\sum_{i=1}^{k} x_i^\alpha\right) \le \left(\sum_{i=1}^{k} x_i\right)^\alpha \le \sum_{i=1}^{k} x_i^\alpha \,.$$

LEMMA 9. There exists a constant $K > 1$ such that

$$\frac{1}{K} < \frac{Lebes(\langle wa \rangle)}{Lebes(\langle wb \rangle)} < K,$$

for any word w and any symbols a, $b \in \mathscr{A}$ such that wa, wb are admissible.

The proofs of both lemmas are trivial.

For a probability measure μ on I, we define another Hausdorff dimension $\dim_\mu$ as follows. Let for $\delta > 0$

$$\mu_\alpha(C, \delta) = \inf \sum_i \mu(\langle v_i \rangle)^\alpha \,,$$

where infimum is taken over all covering by words $\{v_i\}$ such that $\mu(\langle v_i \rangle) < \delta$. Then as usual we can define the Hausdorff dimension $\dim_\mu(\mathscr{C})$ of $\mathscr{C}$ with respect to the probability measure μ as a critical point whether $\lim_{\delta \to 0} \mu_\alpha(\mathscr{C}, \delta)$ converges or diverges.

We will use the following theorem.

THEOREM 1 (Billingsley [1]). For probability measures μ_1, μ_2 such that

$$\mathscr{C} \subset \left\{ x : \lim_{n \to \infty} \frac{\log \mu_1(\langle a^x[1, n] \rangle)}{\log \mu_2(\langle a^x[1, n] \rangle)} = \alpha \right\}$$

for some $0 \le \alpha \le \infty$,

$$\dim_{\mu_2}(\mathscr{C}) = \alpha \dim_{\mu_1}(\mathscr{C}) \,,$$

where $a^x[1, n]$ is a word with length n such that $\langle a^x[1, n] \rangle \ni x$.

We will fix $\tau \in \{+, -\}$ and express F_m^τ and G_m^τ simply by F_m and G_m.

LEMMA 10. Let $\varepsilon > 0$ be any constant. Then for sufficiently large m and any word $w \in \mathscr{W}$ $(|w| > m)$

$$Lebes(\langle w \rangle_F) \begin{cases} \le (1+\varepsilon)^{|w|} K^{2m} |F_m^{|w|'}(w)|^{-1} \,, \\ \ge (1-\varepsilon)^{|w|} K^{-2m} |F_m^{|w|'}(w)|^{-1} \,, \end{cases}$$

$$Lebes(\langle w \rangle_G) \begin{cases} \le (1+\varepsilon)^{|w|} K^{2m} |G_m^{|w|'}(w)|^{-1} \,, \\ \ge (1-\varepsilon)^{|w|} K^{-2m} |G_m^{|w|'}(w)|^{-1} \,, \end{cases}$$

where

$$K = \max\left\{\sup_{x \in I} |F'(x)|, \sup_{x \in I} |F'(x)|^{-1}\right\}.$$

PROOF. The proofs of above four inequalities are almost the same, so we will only show the first one. Note first

$$Lebes(\langle w \rangle_F) = \int_{F^{|w|}(\langle w \rangle)} |F^{|w|'}(wx)|^{-1} dx.$$

Therefore for any m and for any word w with $|w| > m$

$$Lebes(\langle w \rangle_F) = \int_{F^{|w|}(\langle w \rangle)} \prod_{n=1}^{|w|} |F'(a_n \cdots a_{|w|}x)|^{-1} dx$$

$$\leq K^m \int \prod_{n=1}^{|w|-m} |F'(a_n \cdots a_{|w|}x)|^{-1} dx,$$

where $w = a_1 \cdots a_{|w|}$. Thus by Lemma 2, we take sufficiently large m, and get

$$Lebes(\langle w \rangle_F) \leq (1 + \varepsilon)^{|w|-m} K^{2m} |F_m^{|w|'}(w)|^{-1}.$$

This proves the lemma.

Now take μ_1 as Lebesgue measure on I, and μ_2 the measure induced by ϕ from the Lebesgue measure on $[0, 1]$ where G acts. Then for $x \in \mathscr{C}$,

$$\frac{\log \mu_1(\langle a^x[1, n] \rangle)}{\log \mu_2(\langle a^x[1, n] \rangle)} = \frac{\log(Lebes(\langle a^x[1, n] \rangle_F))}{\log(Lebes(\langle a^x[1, n] \rangle_G))}$$

tends to $1/\alpha_0$ as $n \to \infty$. Because for example, from Lemma 10

$$\frac{\log(Lebes(\langle a^x[1, n] \rangle_F))}{\log(Lebes(\langle a^x[1, n] \rangle_G))} \leq \frac{n \log(1 + \varepsilon) + 2m \log K - \log |F_m^{n'}(x)|}{n \log(1 - \varepsilon) - 2m \log K - \alpha_m^\tau \log |F_m^{n'}(x)|},$$

where we take $G_m = G_m^\tau$. Taking $n \to \infty$, and as we can take ε arbitrarily small, we get the left hand term is less than or equal to $1/\alpha_0$. We can get the opposite inequality in a same way. Therefore, by Billingsley's theorem, we get

$$\dim_{\mu_1}(\mathscr{C}) = \alpha_0 \dim_{\mu_2}(\mathscr{C}) = \alpha_0 \dim_{Lebes}([0, 1]) = \alpha_0.$$

THEOREM 2. *The Hausdorff dimension of $\mathscr{C}$ equals α_0.*

PROOF. Since $\dim_{\mu_1}(\mathscr{C})$ is greater than or equal to the Hausdorff dimension of $\mathscr{C}$, we only need to show the opposite inequality. Let $\{J_i\}$ be a covering by intervals such that $\sum (Lebes(J_i))^\alpha < M < \infty$. For each J_i, let

$$n_i = \min\{n : |w| = n, \langle w \rangle \subset J_i\}.$$

If J_i intersects with $\langle u_1 \rangle$, $\cdots$, $\langle u_k \rangle$ with $|u_j| = n_i - 1$, we divide J_i into k intervals $J_i \cap u_j$ ($1 \le j \le k$). Note that $k \le \#\mathscr{A}$. We denote new covering by intervals also by $\{J_i\}$. Therefore we can assume that J_i is contained in some $\langle u \rangle$ with $|u| = n_i - 1$, and that contains at least one $\langle v \rangle$ with $|v| = n_i$. Then by Lemma 8 and the assumption,

$$\sum_i (Lebes(J_i))^\alpha \le \#\mathscr{A}^{1-\alpha} M < \#\mathscr{A} \cdot M .$$

We take all the words $w_{i,1}$, $\cdots$, w_{i,i_1} contained in J_i with length n_i. Then take all the words w_{i,i_1+1}, $\cdots$, w_{i,i_2} contained in $J_i \setminus \bigcup_{j=1}^{i_1} \langle w_{i,j} \rangle$ with length $n_i + 1$, and continue this procedure. Then we get a sequence of words $\{w_{i,j}\}$. Note that, from Lemma 9, the length of any word w with length n_i which intersect with J_i (not only words $w_{i,1}$, $\cdots$, w_{i,i_1}) is less than or equal to $KLebes(\langle w_{i_1} \rangle) \le KLebes(J_i)$. Also noticing $Lebes(\langle wa \rangle) \le e^{-\xi_0} Lebes(\langle w \rangle)$ for any word w and $a \in \mathscr{A}_1$, we get

$$\sum_i \sum_j (Lebes(\langle w_{i,j} \rangle))^\alpha \le 2\#\mathscr{A} K^\alpha \sum_i (Lebes(J_i))^\alpha / (1 - e^{-\xi_0 \alpha})$$

$$\le 2(\#\mathscr{A})^2 K^\alpha M / (1 - e^{-\xi_0 \alpha}) .$$

Now take any α which is greater than the Hausdorff dimension of $\mathscr{C}$. Then for any $\varepsilon > 0$ there exists a covering by intervals $\{J_i\}$ such that $\sum_i (Lebes(J_i))^\alpha < \varepsilon$. Then we can choose a covering by words $\{\langle w_{ij} \rangle\}$ such that

$$\sum_i \sum_j (Lebes(\langle w_{ij} \rangle))^\alpha < 2K^\alpha (\#\mathscr{A})^2 \varepsilon / (1 - e^{-\xi_0 \alpha}) .$$

This proves α_0 smaller than or equal to the Hausdorff dimension of $\mathscr{C}$. This proves the theorem.

5. Invariant measures.

We have proved in Lemma 7 that $1/G'$ is of universally bounded γ'-variation, that is, G satisfies the assumptions of Theorem 3.5 in [3]. Hence, there exists an invariant probability measure μ_G which is absolutely continuous with respect to the Lebesgue measure, and the dynamical system $([0, 1], \mu_G, G)$ is weakly mixing. We will denote by μ_F the induced measure of μ_G to $\mathscr{C}$ by $\phi: [0, 1] \to \mathscr{C}$ such that $x \in [0, 1]$ and $\phi(x) \in \mathscr{C}$ has same expansion by G and F, respectively.

LEMMA 11. *The measure μ_2, the induced measure on $\mathscr{C}$ from the Lebesgue measure on $[0, 1]$ by ϕ, is α_0-conformal measure. Here, we call a measure μ α_0-conformal if*

$$\mu(F(A)) = \int_A |F'(x)|^{\alpha_0} d\mu$$

holds for any $\mathscr{A} \subset \langle a \rangle$ ($a \in \mathscr{A}_1$).

PROOF. Since the Lebesgue measure on $[0, 1]$ is 1-conformal measure with respect to G and $|G'(x)| = |F'(y)|^{\alpha_0}$ for $y = \phi(x)$, it is easy to prove the lemma.

Thus combining the results, we get:

THEOREM 3. *The measure μ_F is an invariant probability measure absolutely continuous with respect to the α_0-conformal measure μ_2, and the dynamical system $(\mathscr{C}, \mu_F, F)$ is weakly mixing.*

Now we will study the relations between the conformal measure and the Hausdorff measure.

DEFINITION. (1) The Cantor set $\mathscr{C}$ has Darboux property if $F(\langle a \rangle \cap \mathscr{C}) = F(\langle a \rangle) \cap \mathscr{C}$.

(2) A transformation F satisfies the Misiurewicz condition if the set

$$\bigcup_{i=1}^{N} \left\{ \lim_{x \uparrow c_i} F^j(x) : j \geq 1 \right\} \cup \bigcup_{i=0}^{N-1} \left\{ \lim_{x \downarrow c_i} F^j(x) : j \geq 1 \right\}$$

has empty intersection with

$$\bigcup_{i=0}^{N-1} (c_i, c_i + \varepsilon) \cup \bigcup_{i=1}^{N} (c_i - \varepsilon, c_i)$$

for some $\varepsilon > 0$, where $\{c_i\}_{i=0}^{N}$ is the set of endpoints of $\langle a \rangle$ $(a \in \mathscr{A}_1)$.

It is easy to see that our $\mathscr{C}$ has Darboux property and F satisfies Misiurewicz condition. Then from Theorem 6 of [2], there exists a constant $c \neq 0$ such that $v_{\alpha_0} = c\mu_2$. Thus, summarizing the results, we get:
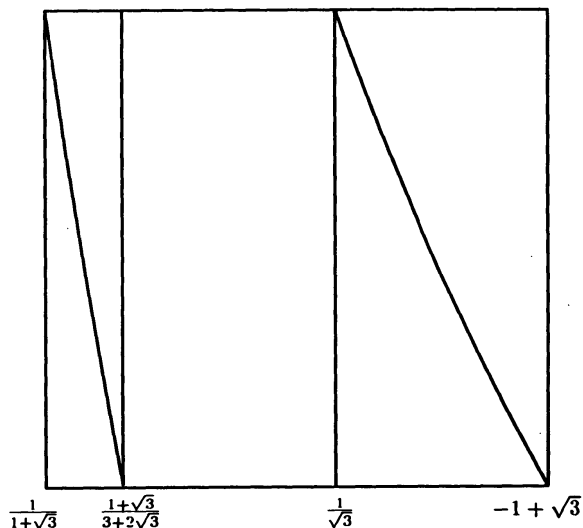
THEOREM 4. *The Hausdorff measure v_{α_0} is non-zero finite, μ_F is an invariant probability measure absolutely continuous with respect to v_{α_0}, and the dynamical system $(\mathscr{C}, \mu_F, F)$ is weakly mixing.*

6. Example.

We will calculate the Hausdorff dimension of the set which has only symbols 1 and 2 in continued fraction expansion. We can restrict this map to the interval $[1/(1 + \sqrt{3}), -1 + \sqrt{3}]$ into itself as in the figure. Namely, all the points which have symbols only 1 and 2 in continued fraction expansion are contained in this interval.

We denote the set $[1/(1 + \sqrt{3}), (1 + \sqrt{3})/(3 + 2\sqrt{3})]$, $((1 + \sqrt{3})/(3 + 2\sqrt{3}), 1/\sqrt{3})$ and $[1/\sqrt{3}, -1 + \sqrt{3}]$ by symbols $\langle 0 \rangle$, $\langle 1 \rangle$ and $\langle 2 \rangle$, respectively. Denote $\mathscr{A} = \{0, 1, 2\}$, $\mathscr{A}_1 = \{0, 2\}$ and $I = [1/(1 + \sqrt{3}), -1 + \sqrt{3})$,

$$F(x) = \begin{cases} 1/x - 2 & \text{if } x \in \langle 0 \rangle, \\ 1/x - 1 & \text{if } x \in \langle 2 \rangle. \end{cases}$$

$$\frac{1}{1+\sqrt{3}} \qquad \frac{1+\sqrt{3}}{3+2\sqrt{3}} \qquad\qquad \frac{1}{\sqrt{3}} \qquad\qquad -1+\sqrt{3}$$

Then the Cantor set which we want to calculate can be expressed by

$$\mathscr{C} = \{ x \in I : F^n(x) \notin \langle 1 \rangle \} \ .$$

We can approximate this map by formal piecewise linear transformations. First approximation is formal piecewise linear transformations F_1^+, F_1^- which are linear on each $\langle 0 \rangle$ and $\langle 2 \rangle$, and the second approximation is formal piecewise linear transformations F_2^+, F_2^- which are linear on $\langle 00 \rangle$, $\langle 02 \rangle$, $\langle 20 \rangle$ and $\langle 22 \rangle$ and so on. The Hausdorff dimension α_0 which we calculate by computer satisfies the following:

approximation	minimum	maximum
1	0.4599714039	0.6429535391
2	0.5066200906	0.5573891372
3	0.5239108226	0.5395066173
4	0.528895873377809917692	0.533552187854664990041
5	0.530600797037892992251	0.532010237643201833751

The program to get these values is very simple, but to get i-th approximation, we need to calculate determinants of 2^i dimensional matrices. Thus, it is not so easy to calculate more precise value. However, we can imagine that α_0 is not so far from the mean value of the fifth approximation $0.531305517 \cdots$.

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Present Address:
Department of Mathematics, College of Humanities and Sciences, Nihon University, Sakura-Josui, Setagaya-ku, Tokyo, 156–8550 Japan.