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On the Multiple Gamma-Functions

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In his paper [5], Shintani obtained a certain type of infinite product representations of double gamma-function Γ_2 and Stirling's modular form ρ_2 in the sense of Barnes. On the basis of them, he derived the relations of theta-function and Γ_2 and of η-function and ρ_2. This is one of the points of his new proof for classical Kronecker's limit formulas.

The main purpose of the present paper is to settle the Shintani type infinite product representations of r-ple gamma-function $\Gamma_r(w; \tilde{\omega})$ and r-ple Stirling's modular form $\rho_r(\tilde{\omega})$ in the sense of Barnes.

In §2, we prove Lemma 1, which is powerful to establish the asymptotic expansions of functions defined by some types of contour integrals. In §2.2, we introduce the function $\mathrm{LG}_m(z)$, which is a generalization of $\mathrm{LG}(z)$, first introduced and investigated by Shintani [4]. In §2.3 and §2.4, we derive the asymptotic expansions of $\mathrm{LG}_m(z+a)$ and $\log\Gamma_r(w+a; \tilde{\omega})$.

On the basis of them, we construct $\mathrm{LG}_{m+1}(z)$ in Theorem 1, §3.2, and the function $\log P_{r+1}(w; \tilde{\omega})$ in §3.3, the latter of which is a generalization of $\log P(w; z)$ of [2] and is essentially equal to $\log\Gamma_{r+1}(w; \tilde{\omega})$, up to an easier factor. Then the easier factor is determined by the difference equations (1.2.6) satisfied by $\Gamma_{r+1}(w; \tilde{\omega})$ and thus we arrive at the Shintani type infinite product representations of Γ_{r+1} and also of ρ_{r+1}.

§3.4 is a by-product and gives a 'simple proof' of inversion formulas of theta-function and of Dedekind η-function.

1. The multiple Bernoulli polynomials and the multiple gamma-functions in the sense of Barnes.

1.1. The multiple *n*-th Bernoulli polynomials. The r-ple n-th Bernoulli polynomial

$$ {}_rS_n(w; \omega_0, \omega_1, \cdots, \omega_{r-1}) $$

of w with parameters $\omega_0, \omega_1, \cdots, \omega_{r-1}$ is defined by

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(1.1.1)
$$\frac{(-1)^r te^{-wt}}{\prod_{i=0}^{r-1}(1-e^{-\omega_i t})}=\sum_{k=1}^{r}\frac{(-1)^k{}_rS_1^{(k+1)}(w;\tilde{\omega})}{t^{k-1}}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}{}_rS_n'(w;\tilde{\omega})t^n}{n!}$$

for $|t|<|2\pi/\omega_0|,\cdots,|2\pi/\omega_{r-1}|$, where ${}_rS_n(w;\tilde{\omega})$ is the abridged notation of ${}_rS_n(w;\omega_0,\omega_1,\cdots,\omega_{r-1})$ with $\tilde{\omega}=(\omega_0,\omega_1,\cdots,\omega_{r-1})$ and ${}_rS_n^{(k)}(w;\tilde{\omega})$ is the k-th derivative of ${}_rS_n(w;\tilde{\omega})$ with respect to w.

Let $B_k(w)$ be the k-th Bernoulli polynomial defined by

$$\frac{te^{wt}}{e^t-1}=\sum_{k=0}^{\infty}\frac{B_k(w)}{k!}t^k .$$

$B_k=B_k(0)$ is the k-th Bernoulli number.

Then

$$B_n(w)=(B+w)^n$$

where in the binomial expansion of the right hand side, $B=B^1$ and $B^j=B_j$ is the j-th Bernoulli number.

Further, we have

$$\frac{(-1)^r te^{-wt}}{\prod_{i=0}^{r-1}(1-e^{-\omega_i t})}=\frac{\prod_{i=0}^{r-1}(-\omega_i t)}{\prod_{i=0}^{r-1}(e^{-\omega_i t}-1)}e^{-wt}\frac{1}{t^{r-1}}\frac{(-1)^r}{\prod_{i=0}^{r-1}\omega_i}$$

$$=\exp(-({}^0B\omega_0+{}^1B\omega_1+\cdots+{}^{r-1}B\omega_{r-1}+w)t)\frac{1}{t^{r-1}}\frac{(-1)^r}{\prod_{i=0}^{r-1}\omega_i} .$$

Hence, comparing this to (1.1.1), we have

(1.1.2)
$${}_rS_n'(w;\tilde{\omega})=\frac{({}^0B\omega_0+{}^1B\omega_1+\cdots+{}^{r-1}B\omega_{r-1}+w)^{n+r-1}n!}{\prod_{i=0}^{r-1}\omega_i\cdot(n+r-1)!}$$

where in the multinomial expansion of the numerator,

$${}^iB^j=B_j \qquad i=0,\cdots,r-1$$

but

$${}^iB^j\cdot{}^{i'}B^k\neq B_{j+k} \qquad \text{for} \quad i,i'=0,\cdots,r-1 .$$

1.2. The r-ple gamma-function. Let $w,\omega_0,\cdots,\omega_{r-1}$ be complex numbers with positive real parts. We define r-ple Riemann zeta function ζ_r by

(1.2.1)
$$\zeta_r(s;w,\tilde{\omega})=\zeta_r(s;w,\omega_0,\omega_1,\cdots,\omega_{r-1})$$

$$=\sum_{\tilde{m}=\tilde{0}}^{\infty}(w+m_0\omega_0+m_1\omega_1+\cdots+m_{r-1}\omega_{r-1})^{-s} \qquad (\mathrm{Re}\,s>r)$$

where $\tilde{0}=(0,\cdots,0)$ and $\tilde{m}=(m_0,m_1,\cdots,m_{r-1})$, $m_i\in\mathbf{Z}$, $m_i\geq 0$. Here

$$w^s = \exp(s \log w)$$

and

$$\log w = \log|w| + i \arg w\,, \qquad -\pi < \arg w < \pi\,.$$

ζ_r can be continued analytically to a meromorphic function in the whole complex plane via the integral representation

$$(1.2.2) \qquad \zeta_r(s;\, w,\, \tilde{\omega}) = \frac{\Gamma(1-s)e^{-s\pi i}}{2\pi i} \int_{I(\lambda,\,\infty)} \frac{e^{-wt}t^{s-1}dt}{\prod_{i=0}^{r-1}(1-e^{-\omega_i t})}$$

where $0 < \lambda < \min_{0 \le i \le r-1}(|2\pi/\omega_i|)$ and $I(\lambda, \infty)$ is the path consisting of the infinite line from ∞ to λ, the circle of radius λ around 0 in the positive sense and the infinite line from λ to ∞ as shown below:

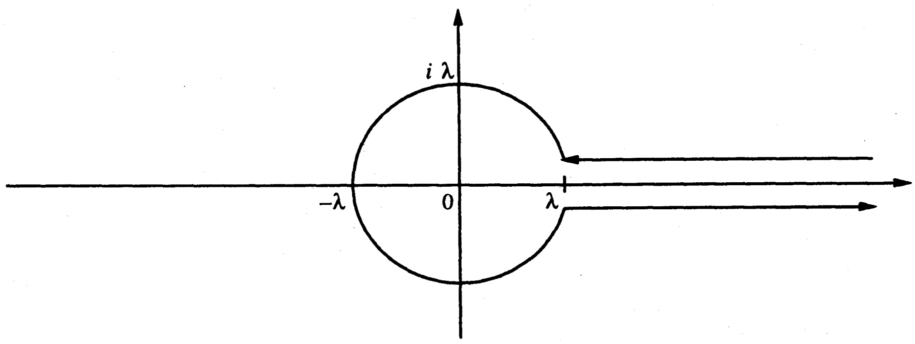

FIGURE 1. $I(\lambda, \infty)$

ζ_r is holomorphic except for simple poles at $s = 1, 2, \cdots, r$.

Put

$$-\log \rho_r(\omega_0, \omega_1, \cdots, \omega_{r-1}) = -\log \rho_r(\tilde{\omega}) = \lim_{w \to 0} \left[\left\{ \frac{\partial}{\partial s} \zeta_r(s;\, w,\, \tilde{\omega}) \right\}_{s=0} + \log w \right].$$

$\rho_r(\tilde{\omega})$ is called the Stirling modular form (in the sense of Barnes). Then the r-ple gamma-function $\Gamma_r(w;\tilde{\omega}) = \Gamma_r(w;\omega_0,\omega_1,\cdots,\omega_{r-1})$ in the sense of Barnes [1] is defined by

$$(1.2.3) \qquad \left[\frac{\partial}{\partial s} \zeta_r(s;\, w,\, \tilde{\omega}) \right]_{s=0} = \log \frac{\Gamma_r(w;\, \tilde{\omega})}{\rho_r(\tilde{\omega})}\,.$$

Then

$$(1.2.4) \qquad \lim_{w \to 0} w\Gamma_r(w;\, \tilde{\omega}) = 1\,.$$

From (1.2.3) and the integral representation of ζ_r, it follows

(1.2.5) $$\log\frac{\Gamma_r(w;\tilde\omega)}{\rho_r(\tilde\omega)}=\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\frac{e^{-wt}\log t}{\prod_{i=0}^{r-1}(1-e^{-\omega_i t})}\frac{dt}{t}+(\gamma-\pi i)\zeta_r(0;w,\tilde\omega)$$

where $\log t$ is real valued on the upper segment of $I(\lambda,\infty)$ and γ is the Euler's constant, namely

$$1+\frac{1}{2}+\cdots+\frac{1}{n}-\log(n+1)=\gamma+O(n^{-1}).$$

For $\tilde\omega=(\omega_0,\omega_1,\cdots,\omega_{r-1})$, we write $\tilde\omega^*=(\omega_0,\omega_1,\cdots,\omega_{r-1},\omega_r)$. It is known (or, easily shown from (1.2.5)) that the following difference equations hold:

(1.2.6) $$\frac{\Gamma_{r+1}(w;\tilde\omega^*)}{\Gamma_{r+1}(w+\omega_i;\tilde\omega^*)}=\frac{\Gamma_r(w;\tilde\omega_i^*)}{\rho_r(\tilde\omega_i^*)}$$

for every $i=0,1,\cdots,r$, where $\tilde\omega_i^*$ means the r-tuple obtained by omitting ω_i from $\tilde\omega^*$. In particular,

(1.2.7) $$\Gamma_{r+1}(\omega_i;\tilde\omega^*)=\rho_r(\tilde\omega_i^*).$$

It is reasonable to define $\rho_0(\ )=1$.

PROPOSITION 1.

(i) $$\zeta_r(0;w,\tilde\omega)=(-1)^r{}_rS_1'(w;\tilde\omega).$$

(ii) $$\zeta_r(-1;w,\tilde\omega)=\tfrac{1}{2}(-1)^{r+1}{}_rS_2'(w;\tilde\omega).$$

PROOF. By (1.1.1),

$$\zeta_r(0;w,\tilde\omega)=\operatorname*{Res}_{t=0}\frac{e^{-wt}t^{-1}}{\prod_{i=0}^{r-1}(1-e^{-\omega_i t})}=(-1)^r{}_rS_1'(w;\tilde\omega),$$

$$\zeta_r(-1;w,\tilde\omega)=\operatorname*{Res}_{t=0}\frac{e^{-wt}t^{-2}}{\prod_{i=0}^{r-1}(1-e^{-\omega_i t})}=\frac{1}{2}(-1)^{r+1}{}_rS_2'(w;\tilde\omega).$$ □

PROPOSITION 2. *Let t be positive.*

(i) $$\frac{\Gamma_r(tw;t\tilde\omega)}{\Gamma_r(w;\tilde\omega)}=\exp\left[\left(-(-1)^r{}_rS_1'(w;\tilde\omega)+\sum_{k=0}^{r-1}(-1)^{r-k}{}_{r-k}S_1'(\omega_k;\omega_k,\cdots,\omega_{r-1})\right)\log t\right].$$

(ii) $$\frac{\rho_r(t\tilde\omega)}{\rho_r(\tilde\omega)}=\exp\left[\left(\sum_{k=0}^{r-1}(-1)^{r-k}{}_{r-k}S_1'(\omega_k;\omega_k,\cdots,\omega_{r-1})\right)\log t\right].$$

PROOF. Differentiating $\zeta_r(s;tw,t\tilde\omega)=t^{-s}\zeta_r(s;w,\tilde\omega)$ with respect to s and putting $s=0$, we have

$$\log\frac{\Gamma_r(tw;\, t\tilde{\omega})}{\rho_r(t\tilde{\omega})} = -\zeta_r(0;\, w,\, \tilde{\omega})\log t + \log\frac{\Gamma_r(w;\, \tilde{\omega})}{\rho_r(\tilde{\omega})}\,. \tag{1.2.8}$$

Put $w=\omega_0$. Then by (1.2.7) and Proposition 1, (i),

$$\frac{\rho_r(t\tilde{\omega})}{\rho_r(\tilde{\omega})} = \frac{\rho_{r-1}(t\omega_1,\, \cdots,\, t\omega_{r-1})}{\rho_{r-1}(\omega_1,\, \cdots,\, \omega_{r-1})}\exp((-1)^r{}_rS_1'(\omega_0;\, \tilde{\omega})\log t)\,.$$

Continuing this process we have (ii) of the Proposition. Then put (ii) into (1.2.8). We have (i). □

2. Asymptotic formulas.

2.1. Key lemma. We put

$$A_m = \sum_{k=1}^{m}\frac{1}{k} \qquad \text{for an integer } m>0 \text{ and } A_0=0\,.$$

LEMMA 1. *For* $\mathrm{Re}\, z>0$ *and an integer* $m\geqq 0$,

$$\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\frac{e^{-zt}}{t^{m+1}}\log t\, dt = \frac{(-z)^m}{m!}(A_m - \log z - \gamma + \pi i)$$

where $\log t$ *is real valued on the upper segment* $(\lambda,\, +\infty)$ *of* $I(\lambda,\, \infty)$.

PROOF. We write

$$I_m = \frac{1}{2\pi i}\int_{I(\lambda,\infty)}\frac{e^{-zt}}{t^{m+1}}\log t\, dt\,,$$

$$J_m = \frac{1}{2\pi i}\int_{U(\lambda)}\frac{e^{-zt}}{t^{m+1}}\log t\, dt\,,$$

$$K_m = \int_{\lambda}^{\infty}\frac{e^{-zt}}{t^{m+1}}\, dt$$

where $U(\lambda)$ is the circle of radius λ around the origin. Then $I_m = J_m + K_m$. To compute J_m, consider the integral

$$L_m = \frac{1}{2\pi i}\int_{U(\lambda)}\frac{\log t}{t^{m+1}}\, dt$$

which is equal to

$$\frac{1}{2\pi i}\int_0^{2\pi}\frac{\log\lambda + i\theta}{\lambda^m e^{im\theta}}\, i d\theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{\log\lambda}{\lambda^m}e^{-im\theta}d\theta + \frac{i}{2\pi}\frac{1}{\lambda^m}\int_0^{2\pi}\theta e^{-im\theta}d\theta\,.$$

Then we have

$$L_0=\frac{\log\lambda}{2\pi}\int_0^{2\pi}d\theta+\frac{i}{2\pi}\int_0^{2\pi}\theta d\theta=\log\lambda+\pi i$$

and for $m>0$,

$$L_m=\frac{\log\lambda}{2\pi\lambda^m}\int_0^{2\pi}e^{-im\theta}d\theta+\frac{i}{2\pi\lambda^m}\int_0^{2\pi}\theta e^{-im\theta}d\theta=-\frac{1}{m\lambda^m}.$$

Hence

$$\begin{aligned}J_m&=\frac{1}{2\pi i}\int_{U(\lambda)}\sum_{k=0}^{\infty}\frac{1}{t^{m+1}}\frac{(-z)^k}{k!}t^k\log t\,dt\\&=\sum_{k=0}^{m-1}\frac{(-z)^k}{k!}\frac{(-1)}{(m-k)\lambda^{m-k}}+\frac{(-z)^m}{m!}(\log\lambda+\pi i)+\varepsilon(\lambda),\end{aligned}$$

where $\varepsilon(\lambda)$ denotes a function approaching 0 when λ goes to 0.

Now, for the integral K_m, $m\geqq 1$, we have

$$K_m=\left[-\frac{1}{mt^m}e^{-zt}\right]_\lambda^\infty-\frac{z}{m}\int_\lambda^\infty\frac{e^{-zt}}{t^m}dt=\frac{e^{-z\lambda}}{m\lambda^m}-\frac{z}{m}K_{m-1},$$

because of $\mathrm{Re}\,z>0$. Hence,

$$\begin{aligned}K_m=&\left\{\frac{1}{m\lambda^m}+\frac{(-z)}{m(m-1)\lambda^{m-1}}+\frac{(-z)^2}{m(m-1)(m-2)\lambda^{m-2}}+\cdots+\frac{(-z)^{m-1}}{m!\lambda}\right\}e^{-z\lambda}\\&+\frac{(-z)^m}{m!}K_0.\end{aligned}$$

Here K_0 is computed as follows:

$$\begin{aligned}K_0&=\int_\lambda^\infty\frac{e^{-zt}}{t}dt=\int_\lambda^\infty\frac{e^{-zt}-e^{-t}}{t}dt+\int_\lambda^\infty\frac{e^{-t}}{t}dt\\&=\int_\lambda^\infty\frac{e^{-zt}-e^{-t}}{t}dt-\int_\lambda^\infty\left\{\frac{1}{1-e^{-t}}-\frac{1}{t}\right\}e^{-t}dt+\int_\lambda^\infty\frac{e^{-t}}{1-e^{-t}}dt\\&=\int_0^\infty\frac{e^{-zt}-e^{-t}}{t}dt-\int_0^\infty\left\{\frac{1}{1-e^{-t}}-\frac{1}{t}\right\}e^{-t}dt+\varepsilon(\lambda)+\int_\lambda^\infty\frac{e^{-t}}{1-e^{-t}}dt\\&=-\log z-\gamma+\varepsilon(\lambda)+\int_\lambda^\infty\frac{e^{-t}}{1-e^{-t}}dt=-\log z-\log\lambda-\gamma+\varepsilon(\lambda),\end{aligned}$$

since, with $u=e^{-t}$, the last integral is equal to

$$\int_{e^{-\lambda}}^{0}\frac{-du}{1-u}=[\log|u-1|]_{e^{-\lambda}}^{0}=-\log(1-e^{-\lambda})$$

$$=-\log(\lambda(1+\cdots))=-\log\lambda+\varepsilon(\lambda)\,.$$

Summing up, we see that the integral $I_m=J_m+K_m$ is of the form

$$\frac{a_m}{\lambda^m}+\frac{a_{m-1}}{\lambda^{m-1}}+\cdots+\frac{a_1}{\lambda}+b\log\lambda+a_0+\varepsilon(\lambda)\,.$$

But the integral I_m is independent of λ. Hence I_m must be equal to a_0 which is the sum of the constant terms of J_m and K_m. The constant term of J_m is $(-z)^m\pi i/m!$ and that of K_m is

$$\frac{1}{m}\frac{(-z)^m}{m!}+\frac{(-z)}{m(m-1)}\frac{(-z)^{m-1}}{(m-1)!}+\cdots+\frac{(-z)^{m-1}}{m!}\frac{(-z)}{1!}-\frac{(-z)^m}{m!}(\log z+\gamma)$$

$$=\frac{(-z)^m}{m!}(A_m-\log z-\gamma)\,,$$

which proves our lemma. □

In general, we use the following notation: for a given integer $m\geqq 0$,

$$\left[\sum_{n=0}^{\infty}a_nx^n\right]_m=\sum_{n=0}^{m}a_nx^n\,,\qquad \left\{\sum_{n=0}^{\infty}a_nx^n\right\}_{m+1}=\sum_{n=m+1}^{\infty}a_nx^n\,.$$

For brevity, put

$$\frac{t^re^{-at}}{\prod_{i=0}^{r-1}(1-e^{-\omega_it})}=\sum_{n=0}^{\infty}\frac{{}_rT_n(a;\tilde{\omega})t^n}{n!}\,.$$

Then by (1.1.1),

$$\text{(2.1.1)}\qquad {}_rT_n(a;\tilde{\omega})=\begin{cases}(-1)^nn!\,{}_rS_1^{(r-n+1)}(a;\tilde{\omega})\,, & 0\leqq n\leqq r-1\\ \dfrac{(-1)^nn!\,{}_rS'_{n+1-r}(a;\tilde{\omega})}{(n-r+1)!}\,, & r\leqq n\,.\end{cases}$$

The following lemma is easily obtained from Lemma 1.

LEMMA 2. *For* $\mathrm{Re}\,a\geqq 0$, $\mathrm{Re}\,z>0$ *and* $\mathrm{Re}\,\omega_i>0$, $i=0,1,\cdots,r-1$,

$$\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\frac{1}{t^{m+1}}\left[\frac{t^re^{-at}}{\prod_{i=0}^{r-1}(1-e^{-\omega_it})}\right]_m e^{-zt}\log t\,dt$$

$$=\sum_{n=0}^{m}\frac{{}_rT_n(a;\tilde{\omega})}{n!}\frac{(-z)^{m-n}}{(m-n)!}(A_{m-n}-\log z-\gamma+\pi i)\,.$$

LEMMA 3.

$$\text{(1)}\qquad \sum_{k=0}^{n}\binom{n}{k}{}_rT_k(a;\tilde\omega)(-z)^{n-k}={}_rT_n(z+a;\tilde\omega)\,,$$

$$\text{(2)}\qquad \sum_{k=0}^{r}\binom{r}{k}{}_rT_k(a;\tilde\omega)(-z)^{r-k}=(-1)^r r!\,{}_rS_1'(z+a;\tilde\omega)\,.$$

PROOF. By definition,

$$\frac{t^r e^{-(z+a)t}}{\prod_{i=0}^{r-1}(1-e^{-\omega_i t})}=\sum_{n=0}^{\infty}\frac{{}_rT_n(z+a;\tilde\omega)t^n}{n!}\,.$$

On the other hand, it is equal to

$$\frac{t^r e^{-at}}{\prod_{i=0}^{r-1}(1-e^{-\omega_i t})}e^{-zt}=\sum_{n=0}^{\infty}\frac{{}_rT_n(a;\tilde\omega)t^n}{n!}\cdot\sum_{n=0}^{\infty}\frac{(-z)^n t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\sum_{k+l=n}\frac{{}_rT_k(a)}{k!}\frac{(-z)^l}{l!}t^n=\sum_{n=0}^{\infty}\left\{\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}{}_rT_k(a;\tilde\omega)(-z)^{n-k}\right\}t^n\,.$$

Comparing the right hand sides, we get (1). (2) follows easily from (1) and the formula just before Lemma 2. □

LEMMA 4. *For* $\mathrm{Re}\,z>0$, $\mathrm{Re}\,a\geqq 0$,

$$\lim_{\lambda\to 0}\int_{U(\lambda)}\left\{\frac{te^{-at}}{1-e^{-t}}\right\}_{m+1}\frac{e^{-zt}}{t^{m+1}}\log t\,dt=0\,.$$

PROOF. It is sufficient to show

$$\lim_{\lambda\to 0}\int_{U(\lambda)}t^m\log t\,dt=0\qquad\text{for}\quad m\geqq 0\,,$$

since the integrand of the lemma is the sum of the terms of type $t^m\log t$, $m\geqq 0$. We have

$$\int_{U(\lambda)}t^m\log t\,dt=\int_0^{2\pi}\lambda^{m+1}e^{(m+1)i\theta}i(\log\lambda+i\theta)d\theta$$

and get the lemma. □

LEMMA 5. *For* $\mathrm{Re}\,a\geqq 0$, $\mathrm{Re}\,z>0$ *and* $\mathrm{Re}\,\omega_i>0$, $i=0,\cdots,r-1$,

$$\int_0^{\infty}\frac{1}{t^{m+1}}\left\{\frac{t^r e^{-at}}{\prod_{i=0}^{r-1}(1-e^{-\omega_i t})}\right\}_{m+1}e^{-zt}dt$$

$$=\sum_{n=m+1}^{\infty}\frac{{}_rT_n(a;\tilde\omega)}{n(n-1)\cdots(n-m)z^{n-m}}\qquad\textit{(asymptotically for large } |z|)\,.$$

PROOF. The left hand side is asymptotically equal to

$$\sum_{n=m+1}^{\infty} \frac{{}_rT_n(a;\tilde{\omega})}{n!} \int_0^{\infty} e^{-zt} t^{n-m-1} dt .$$

Then the Lemma follows from the fact that the integral equals $\Gamma(n-m)/z^{n-m}$. □

2.2. The function $\mathrm{LG}_m(z)$. For every integer $m \geqq 0$, we define the function $\mathrm{LG}_m(z)$ by

(2.2.1)

$$\mathrm{LG}_m(z) = \frac{1}{2\pi i} \int_{I(\lambda,\infty)} \frac{t}{1-e^{-t}} \frac{e^{-zt}}{t^{m+1}} \log t\, dt + \frac{(-1)^m}{m!} B_m(z)(\gamma - \pi i) , \qquad \mathrm{Re}\, z > 0 .$$

In his paper [4], Shintani first introduced and investigated $\mathrm{LG}_2(z)$ ($=\mathrm{LG}(z)$ in his notation). The following Proposition generalizes Lemma 2 of [4].

PROPOSITION 3.

(1) $$\mathrm{LG}_m(z) = \frac{(-1)^m}{m!} \sum_{k=0}^{m} \binom{m}{k} B_k z^{m-k} (A_{m-k} - \log z) + \int_0^{\infty} \left\{ \frac{t}{1-e^{-t}} \right\}_{m+1} \frac{e^{-zt}}{t^{m+1}} dt ,$$

(2) $$\frac{d\mathrm{LG}_m(z)}{dz} = -\mathrm{LG}_{m-1}(z) , \qquad (m \geqq 1) ,$$

(3) $$\mathrm{LG}_1(z) = \log \Gamma(z) - \tfrac{1}{2} \log(2\pi) ,$$

(4) $$\frac{d\mathrm{LG}_0(z)}{dz} = -\zeta(2, z) ,$$

where $\zeta(s, z)$ is the Hurwitz zetafunction

$$\zeta(s, z) = \sum_{m=0}^{\infty} (z+m)^{-s} , \qquad \mathrm{Re}\, s > 1 ,$$

$$= \frac{\Gamma(1-s) e^{-s\pi i}}{2\pi i} \int_{I(\lambda,\infty)} \frac{e^{-zt} t^{s-1}}{1-e^{-t}} dt = \zeta_1(s; z, 1) .$$

(5) $$\mathrm{LG}_m(z+1) - \mathrm{LG}_m(z) = -\frac{(-z)^{m-1}}{(m-1)!} (A_{m-1} - \log z) \qquad (m \geqq 1) ,$$

(6) $$\mathrm{LG}_m(1) = \frac{(-1)^{m-1}}{(m-1)!} \zeta'(1-m) + \frac{(-1)^m B_m}{m!} A_{m-1} \qquad (m \geqq 2)$$

and

$$\mathrm{LG}_1(1) = \zeta'(0) = -\tfrac{1}{2} \log(2\pi) ,$$

where $\zeta(s)$ is the Riemann zetafunction defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}, \qquad \mathrm{Re}\,s>1$$

and has an expression

$$\zeta(s)=\frac{\Gamma(1-s)e^{-s\pi i}}{2\pi i}\int_{I(\lambda,\infty)}\frac{e^{-t}t^{s-1}}{1-e^{-t}}dt \tag{2.2.2}$$

valid for any s.

PROOF. (1) The integral of the defining formula (2.2.1) for $\mathrm{LG}_m(z)$ is divided into two parts:

$$\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\left\{\frac{t}{1-e^{-t}}\right\}_{m+1}\frac{e^{-zt}}{t^{m+1}}\log t\,dt \tag{2.2.3}$$
$$+\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\left[\frac{t}{1-e^{-t}}\right]_m\frac{e^{-zt}}{t^{m+1}}\log t\,dt\,.$$

The second integral is evaluated by Lemma 2 with $r=1$, $a=0$ and $\omega_0=1$. The first integral is divided to

$$\frac{1}{2\pi i}\int_{U(\lambda)}\left\{\frac{t}{1-e^{-t}}\right\}_{m+1}\frac{e^{-zt}}{t^{m+1}}\log t\,dt+\int_{\lambda}^{\infty}\left\{\frac{t}{1-e^{-t}}\right\}_{m+1}\frac{e^{-zt}}{t^{m+1}}dt\,.$$

Then (1) follows easily from Lemma 4.

(2) Note that

$$\frac{d}{dz}B_m(z)=mB_{m-1}(z)\,.$$

Then (2) is straightforward.

(3) By (1), we have

$$\mathrm{LG}_1(z)=-B_0z(1-\log z)-B_1(-\log z)+\int_0^{\infty}\left\{\frac{t}{1-e^{-t}}\right\}_2\frac{e^{-zt}}{t^2}dt\,.$$

The integral of the right hand side is equal to

$$\int_0^{\infty}\left\{\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right\}\frac{e^{-zt}}{t}dt\,.$$

Then Binet's first formula for $\log\Gamma(z)$ (Whittaker-Watson [6], p. 249) shows (3).

(4)

$$\frac{d\mathrm{LG}_0(z)}{dz}=-\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\frac{te^{-zt}}{1-e^{-t}}\log t\,dt$$
$$=-\int_0^{\infty}\frac{te^{-zt}}{1-e^{-t}}dt=-\zeta(2,z)\,.$$

(5)

$$\mathrm{LG}_m(z+1)-\mathrm{LG}_m(z)=-\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\frac{e^{-zt}}{t^m}\log t\,dt+\frac{(-1)^m z^{m-1}}{(m-1)!}(\gamma-\pi i)$$

because of $B_m(z+1)-B_m(z)=mz^{m-1}$. Then the integral is nothing but

$$I_{m-1}=-\frac{(-z)^{m-1}}{(m-1)!}(A_{m-1}-\log z-\gamma+\pi i)$$

and (5) is obtained.

(6) It is known that

$$\zeta(1-m)=-B_m/m \quad \text{for } m>1\,, \qquad \zeta(0)=-1/2\,.$$

We have, differentiating (2.2.2) and putting $s=1-m$,

$$\begin{aligned}\zeta'(1-m)&=(-1)^m(\Gamma'(m)+\pi i\Gamma(m))\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\frac{e^{-t}}{1-e^{-t}}\frac{dt}{t^m}\\&\quad+\Gamma(m)(-1)^{m-1}\mathrm{LG}_m(1)+\frac{\Gamma(m)}{m!}B_m(1)(\gamma-\pi i)\\&=\frac{-1}{(m-1)!}(\Gamma'(m)+\pi i\Gamma(m))\zeta(1-m)\\&\quad+(-1)^{m-1}(m-1)!\mathrm{LG}_m(1)+\frac{1}{m}B_m(1)(\gamma-\pi i)\,.\end{aligned}$$

Note that $B_m(1)=(-1)^m B_m$. Then $\mathrm{LG}_1(1)=\zeta'(0)=-\frac{1}{2}\log(2\pi)$ and we get

(2.2.4)
$$\begin{aligned}\mathrm{LG}_m(1)&=\frac{(-1)^{m-1}}{(m-1)!}\zeta'(1-m)+\frac{(-1)^m B_m(\Gamma'(m)+\pi i\Gamma(m))}{m!(m-1)!}\\&\quad+(-1)^m\frac{B_m(\gamma-\pi i)}{m!}\qquad \text{for } m>1\,.\end{aligned}$$

It is easily seen that $\Gamma(m)(A_{m-1}-\gamma)=\Gamma'(m)$. Then the second term plus the third term of (2.2.4) becomes $(-1)^m B_m A_{m-1}/m!$. □

2.3. Asymptotic formula for $\mathrm{LG}_m(z+a)$.

PROPOSITION 4. *For* $\mathrm{Re}\,z>0$ *and* $\mathrm{Re}\,a\geqq 0$, *we have asymptotically for large* $|z|$

$$\begin{aligned}\mathrm{LG}_m(z+a)&=\frac{(-1)^m}{m!}\sum_{k=0}^{m}\binom{m}{k}B_k(a)z^{m-k}A_{m-k}-\frac{(-1)^m}{m!}B_m(z+a)\log z\\&\quad+\sum_{n=m+1}^{\infty}\frac{(-1)^n B_n(a)}{n(n-1)\cdots(n-m)z^{n-m}}\,.\end{aligned}$$

PROOF. By Lemmas 2 and 4,

$$\begin{aligned}\mathrm{LG}_m(z+a)&=\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\frac{t}{1-e^{-t}}\frac{e^{-(z+a)t}}{t^{m+1}}\log t\,dt+\frac{(-1)^m}{m!}B_m(z+a)(\gamma-\pi i)\\&=\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\left[\frac{te^{-at}}{1-e^{-t}}\right]_m\frac{e^{-zt}}{t^{m+1}}\log t\,dt\\&\quad+\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\left\{\frac{te^{-at}}{1-e^{-t}}\right\}_{m+1}\frac{e^{-zt}}{t^{m+1}}\log t\,dt+\frac{(-1)^m}{m!}B_m(z+a)(\gamma-\pi i)\\&=\sum_{k=0}^{m}\frac{(-1)^kB_k(a)}{k!}\frac{(-z)^{m-k}}{(m-k)!}(A_{m-k}-\log z-\gamma+\pi i)\\&\quad+\int_\lambda^\infty\left\{\frac{te^{-at}}{1-e^{-t}}\right\}_{m+1}\frac{e^{-zt}}{t^{m+1}}dt+\frac{(-1)^m}{m!}B_m(z+a)(\gamma-\pi i)+\varepsilon(\lambda)\,.\end{aligned}$$

Here the terms including $\gamma-\pi i$ vanish, since the terms including $\gamma-\pi i$ in the first sum are gathered to

$$-\frac{(-1)^m}{m!}B_m(z+a)(\gamma-\pi i)\,.$$

We let λ tend to 0. Note that

$${}_1T_n(a;1)=(-1)^nB_n(a)\,,\quad n\geqq 1\,,\qquad {}_1T_0(a;1)=1\,.$$

Then Lemma 5 proves our Proposition. □

2.4. Asymptotic formula for $\log\Gamma_r(w+a;\tilde\omega)$. In [1], Barnes gave the asymptotic expansion for $\log\Gamma_r(w+a;\tilde\omega)$ in the most general setting but his proof was very much complicated. Here we present an easy proof which is based on Lemma 1.

PROPOSITION 5. *For* $\mathrm{Re}\,w>0$, $\mathrm{Re}\,a\geqq 0$ *and for* $\tilde\omega=(\omega_0,\omega_1,\cdots,\omega_{r-1})$ *with* $\mathrm{Re}\,\omega_i>0$, $i=0,\cdots,r-1$, *we have asymptotically for large* $|w|$

$$\begin{aligned}\log\frac{\Gamma_r(w+a;\tilde\omega)}{\rho_r(\tilde\omega)}&=\sum_{n=0}^{r}\frac{(-1)^n{}_rS_1^{(r-n+1)}(a;\tilde\omega)(-w)^{r-n}}{(r-n)!}(A_{r-n}-\log w)\\&\quad+\sum_{n=1}^{\infty}\frac{(-1)^{n+r}{}_rS'_{n+1}(a;\tilde\omega)}{n(n+1)w^n}\,.\end{aligned}$$

PROOF. Our proof goes on the same way as for Proposition 4. We have by (1.2.5)

$$\begin{aligned}(2.4.1)\qquad &\log\frac{\Gamma_r(w+a;\tilde\omega)}{\rho_r(\tilde\omega)}\\&=\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\frac{e^{-(a+w)t}\log t}{\prod_{i=0}^{r-1}(1-e^{-\omega_it})}\frac{dt}{t}+(\gamma-\pi i)\zeta_r(0;w+a,\tilde\omega)\end{aligned}$$

$$=\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\left[\frac{t^r e^{-at}}{\prod_{i=0}^{r-1}(1-e^{-\omega_i t})}\right]_r \frac{e^{-wt}\log t}{t^{r+1}}dt$$
$$+\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\left\{\frac{t^r e^{-at}}{\prod_{i=0}^{r-1}(1-e^{-\omega_i t})}\right\}_{r+1} \frac{e^{-wt}\log t}{t^{r+1}}dt$$
$$+(\gamma-\pi i)\zeta_r(0;\, w+a,\, \bar{\omega})\,.$$

The first integral of (2.4.1) is equal to

$$\sum_{n=0}^{r}\frac{{}_rT_n(a;\bar{\omega})}{n!}\frac{(-w)^{r-n}}{(r-n)!}(A_{r-n}-\log w-\gamma+\pi i)\,.$$

Since

$$\frac{1}{r!}\sum_{n=0}^{r}\frac{{}_rT_n(a;\bar{\omega})(-w)^{r-n}r!}{n!(r-n)!}=(-1)^r{}_rS_1'(w+a;\bar{\omega})=\zeta_r(0;\, w+a,\, \bar{\omega})$$

holds by Lemma 3 and Proposition 1, (i), $\gamma-\pi i$ disappears from (2.4.1). By Lemma 4, the second integral of (2.4.1) equals

$$\int_0^{\infty}\frac{1}{t^{r+1}}\left\{\frac{t^r e^{-at}}{\prod_{i=0}^{r-1}(1-e^{-\omega_i t})}\right\}_{r+1} e^{-wt}dt$$

which is equal to, by Lemma 5 and by (2.1.1),

$$\sum_{n=r+1}^{\infty}\frac{{}_rT_n(a;\bar{\omega})}{n(n-1)\cdots(n-r)w^{n-r}}$$
$$=\sum_{n=r+1}^{\infty}\frac{(-1)^n{}_rS_{n+1-r}'(a;\bar{\omega})}{(n-r+1)(n-r)w^{n-r}}=\sum_{n=1}^{\infty}\frac{(-1)^{n+r}{}_rS_{n+1}'(a;\bar{\omega})}{n(n+1)w^n}\,.$$

Summing up, we get our Proposition. □

3. Multiple gamma-function and Stirling's modular form.

3.1. The double gamma-function. Shintani, in his paper [5], gave the infinite product expansions of the double gamma-function $\Gamma_2(w;(1,z))$, for $w, z>0$, and the Stirling's modular form $\rho_2((1,z))$, $z>0$, as his Proposition 1. Then $\Gamma_2(w;(1,z))$ and $\rho_2((1,z))$ are continued analytically to holomorphic functions in the domains

$$\{(w,z)\ ;\ z\in\mathbf{C}-(-\infty,0],\ w\neq-(m+nz),\ m,n=0,1,\cdots\}$$

and $\{z; z\in\mathbf{C}-(-\infty,0]\}$, respectively. Also, he proved the following formulas (cf. [2], [5]), for $\operatorname{Im}z>0$,

$$\rho_2((1,-z))\rho_2((1,z))=(2\pi)^{3/2}z^{-1/2}\eta(z)\exp\left\{\pi i\left(\frac{1}{4}+\frac{1}{12z}\right)\right\}, \tag{3.1.1}$$

$$(3.1.2)\qquad \frac{\Gamma_2(w;(1,z))}{\rho_2((1,z))}\frac{\Gamma_2(1-w;(1,-z))}{\rho_2((1,-z))}\frac{\Gamma_2(1+z-w;(1,z))}{\rho_2((1,z))}\frac{\Gamma_2(w-z;(1,-z))}{\rho_2((1,-z))} = \frac{\eta(z)}{\vartheta_1(w,z)}\exp\left\{\pi i\left(-\frac{1}{6z}+\frac{w-w^2}{z}\right)\right\},$$

where

$$\eta(z)=e^{2\pi iz/24}\prod_{n=1}^{\infty}(1-e^{2\pi inz}),$$

$$\vartheta_1(w,z)=2e^{2\pi iz/12}\sin(\pi w)\eta(z)\prod_{n=1}^{\infty}(1-e^{2\pi i(w+nz)})(1-e^{2\pi i(-w+nz)}).$$

In this case, the difference equations (1.2.6) become

$$(3.1.3)\qquad \begin{cases}\Gamma_2(w+1;(1,z))=\Gamma_2(w;(1,z))(2\pi)^{1/2}\Gamma(w/z)^{-1}\exp\{(1/2-w/z)\log z\}, \\ \Gamma_2(w+z;(1,z))=\Gamma_2(w;(1,z))(2\pi)^{1/2}\Gamma(w)^{-1}.\end{cases}$$

We constructed $\log\Gamma_2$ in [2], going the converse way to Shintani's, by the Weierstrass principle from the asymptotic expansion of $\log\Gamma$. More precisely, we first defined the function $\log P(w,z)$ by

$$-\sum_{n=1}^{\infty}\left\{\log\Gamma(w+nz)-\left(w+nz-\frac{1}{2}\right)\log(nz)+nz-\frac{1}{2}\log(2\pi) -\frac{1}{12nz}-\frac{w^2-w}{2nz}\right\}, \qquad (w,z>0).$$

Here the inside of { } except for the first term gives the asymptotic expansion of $\log\Gamma(w+nz)$ up to n^{-1}-term. Thus { } is $O(n^{-2})$ and the series converges. We call such an obvious principle of construction the Weierstrass principle. $P(w,z)$ can be continued analytically to the domain of (w,z) aforementioned. Then we constructed $\Gamma_2(w;(1,z))$ by attaching easier factors to $P(w,z)$.

3.2. $\mathrm{LG}_m(w)$ by the Weierstrass principle. First, by the Weierstrass principle based on the asymptotic expansion of $\mathrm{LG}_m(w+a)$ of Proposition 4, we define

$$(3.2.1)\qquad \mathrm{PG}_m(w)=-\sum_{n=1}^{\infty}\left\{\mathrm{LG}_m(w+n+1)-\mathrm{LG}_m(w+n) -\frac{(-1)^m}{m!}\sum_{k=1}^{m}\binom{m}{k}(B_k(w+1)-B_k(w))n^{m-k}(A_{m-k}-\log n) -\frac{(-1)^{m+1}(B_{m+1}(w+1)-B_{m+1}(w))}{(m+1)!n}\right\}.$$

For $m=2$, we have

$$\mathrm{PG}_2(w)=\sum_{n=1}^{\infty}\left\{-(w+n)+(w+n)\log(w+n)+n-(w+n)\log n-\frac{w^2}{2n}\right\},$$

which was introduced and denoted by $\mathrm{PG}_2(w, 1)$ in [3]. Also, there, the formula

$$\mathrm{LG}_2(w)=w\log\sqrt{2\pi}+w\log w-w+\tfrac{1}{2}\gamma w^2+\mathrm{LG}_2(1)+\mathrm{PG}_2(w) \tag{3.2.2}$$

with $\mathrm{LG}_2(1)=1/12-\zeta'(-1)$ has been given.

The right hand side of (3.2.1) is computed as

$$\begin{aligned}
&=-\sum_{n=1}^{\infty}\Bigg\{\mathrm{LG}_m(w+n+1)-\mathrm{LG}_m(w+n)\\
&\qquad-\frac{(-1)^m}{m!}\sum_{k=1}^{m}\binom{m}{k}kw^{k-1}n^{m-k}(A_{m-k}-\log n)-\frac{(-1)^{m+1}w^m}{m!n}\Bigg\}\\
&=-\sum_{n=1}^{\infty}\Bigg\{\mathrm{LG}_m(w+n+1)-\mathrm{LG}_m(w+n)\\
&\qquad-\frac{(-1)^m}{(m-1)!}\sum_{k=1}^{m-1}\binom{m-1}{k-1}w^{k-1}n^{m-k}(A_{m-k}-\log n)\\
&\qquad\qquad+\frac{(-1)^m}{(m-1)!}w^{m-1}\log n-\frac{(-1)^{m+1}w^m}{m!n}\Bigg\}\\
&=-\sum_{n=1}^{\infty}\Bigg\{\mathrm{LG}_m(w+n+1)-\mathrm{LG}_m(w+n)\\
&\qquad-\sum_{k=0}^{m-2}\frac{(-1)^k w^k}{k!}(\mathrm{LG}_{m-k}(n+1)-\mathrm{LG}_{m-k}(n))\\
&\qquad\qquad+\frac{(-1)^m}{(m-1)!}w^{m-1}\log n-\frac{(-1)^{m+1}w^m}{m!n}\Bigg\}.
\end{aligned}$$

Here Propositon 3, (5) is used for $\mathrm{LG}_{m-k}(n+1)-\mathrm{LG}_{m-k}(n)$.

THEOREM 1.

$$\begin{aligned}
\mathrm{LG}_m(w)=&\frac{(-w)^{m-1}}{(m-1)!}(A_{m-1}-\log w)+\sum_{k=0}^{m-2}\frac{(-1)^k}{k!}w^k\mathrm{LG}_{m-k}(1)\\
&+\frac{(-1)^m}{(m-1)!}w^{m-1}\log\sqrt{2\pi}-\frac{(-1)^{m+1}}{m!}\gamma w^m+\mathrm{PG}_m(w)
\end{aligned}$$

holds for $m\geqq 1$.

PROOF. The partial sum of $-\mathrm{PG}_m(w)$,

$$(3.2.3)\qquad \sum_{n=1}^{N-1}\Big\{\mathrm{LG}_m(w+n+1)-\mathrm{LG}_m(w+n)$$
$$-\sum_{k=0}^{m-2}\frac{(-1)^k w^k}{k!}(\mathrm{LG}_{m-k}(n+1)-\mathrm{LG}_{m-k}(n))$$
$$+\frac{(-1)^m}{(m-1)!}w^{m-1}\log n-\frac{(-1)^{m+1}w^m}{m!n}\Big\},$$

is equal to

$$=\mathrm{LG}_m(w+N)-\mathrm{LG}_m(w+1)-\sum_{k=0}^{m-2}\frac{(-1)^k}{k!}w^k(\mathrm{LG}_{m-k}(N)-\mathrm{LG}_{m-k}(1))$$
$$+\frac{(-1)^m}{(m-1)!}w^{m-1}\log\Gamma(N)-\frac{(-1)^{m+1}w^m}{m!}\sum_{n=1}^{N-1}\frac{1}{n}.$$

Replace $\mathrm{LG}_m(w+N)$, $\mathrm{LG}_m(N)$, $\log\Gamma(N)$, $\sum_{n=1}^{N-1}1/n$ by

$$\mathrm{LG}_m(w+N)=\frac{(-1)^m}{m!}\sum_{k=0}^{m}\binom{m}{k}B_k(w)N^{m-k}(A_{m-k}-\log N)+O(N^{-1}),$$
$$\mathrm{LG}_m(N)=\frac{(-1)^m}{m!}\sum_{k=0}^{m}\binom{m}{k}B_kN^{m-k}(A_{m-k}-\log N)+O(N^{-1}),$$
$$\log\Gamma(N)=(N-1/2)\log N-N+\tfrac{1}{2}\log(2\pi)+O(N^{-1}),$$
$$\sum_{n=1}^{N-1}\frac{1}{n}=\gamma+\log N+O(N^{-1}).$$

Then terms going to infinity for $N\to\infty$, appearing in the formula so obtained, are of the types N^p $(p\geqq 1)$, $\log N$ and $N^q\log N$ $(q\geqq 1)$ and (3.2.3) converges when $N\to\infty$. Hence, diverging terms are canceled with each other. Finally we replace $\mathrm{LG}_m(w+1)$ by

$$\mathrm{LG}_m(w+1)=\mathrm{LG}_m(w)-\frac{(-w)^{m-1}}{(m-1)!}(A_{m-1}-\log w).\qquad\square$$

3.3. The functions $\Gamma_r(w;\tilde\omega)$ and $P_r(w;\tilde\omega)$. Recall that $\tilde\omega=(\omega_0,\omega_1,\cdots,\omega_{r-1})$ and $\tilde\omega^*=(\omega_0,\omega_1,\cdots,\omega_{r-1},\omega_r)$. Assume throughout that w, ω_i, $i=0,1,\cdots,r$, are all real positive. By the Weierstrass principle, we define $P_{r+1}(w;\tilde\omega^*)$ by

$$\log P_{r+1}(w;\tilde\omega^*)=-\sum_{n=1}^{\infty}\Big\{\log\Gamma_r(w+n\omega_r;\tilde\omega)-\log\rho_r(\tilde\omega)$$
$$-\sum_{m=0}^{r}\frac{(-1)^m{}_rS_1^{(r+1-m)}(w;\tilde\omega)(-n\omega_r)^{r-m}}{(r-m)!}(A_{r-m}-\log(n\omega_n))$$
$$-\frac{(-1)^{r+1}{}_rS_2'(w;\tilde\omega)}{2n\omega_r}\Big\}$$

based on the asymptotic expanison of $\log\Gamma_r(w;\tilde{\omega})$ given in Proposition 5 (w, a are taken as $n\omega_r$, w here, respectively). $P_2(w;(1,z))$ is denoted by $P(w;z)$ in [2].

We have, by the difference equation (1.2.6) for Γ_{r+1},

$$\log\frac{\Gamma_{r+1}(w;\tilde{\omega}^*)}{\rho_{r+1}(\tilde{\omega}^*)}-\log\frac{\Gamma_{r+1}(w+\omega_r;\tilde{\omega}^*)}{\rho_{r+1}(\tilde{\omega}^*)}=\log\frac{\Gamma_r(w;\tilde{\omega})}{\rho_r(\tilde{\omega})}.$$

Therefore, for $N\in\mathbf{Z}$, $N>0$,

$$\log\frac{\Gamma_{r+1}(w;\tilde{\omega}^*)}{\rho_{r+1}(\tilde{\omega}^*)}-\log\frac{\Gamma_{r+1}(w+N\omega_r;\tilde{\omega}^*)}{\rho_{r+1}(\tilde{\omega}^*)}=\sum_{n=0}^{N-1}\log\frac{\Gamma_r(w+n\omega_r;\tilde{\omega})}{\rho_r(\tilde{\omega})}.$$

We consider

$$\begin{aligned}(3.3.2)\qquad \log\frac{\Gamma_{r+1}(w;\tilde{\omega}^*)}{\rho_{r+1}(\tilde{\omega}^*)}&=\log\frac{\Gamma_{r+1}(w+N\omega_r;\tilde{\omega}^*)}{\rho_{r+1}(\tilde{\omega}^*)}+\log\frac{\Gamma_r(w;\tilde{\omega})}{\rho_r(\tilde{\omega})}\\
&+\sum_{n=1}^{N-1}\Bigg[\log\frac{\Gamma_r(w+n\omega_r;\tilde{\omega})}{\rho_r(\tilde{\omega})}\\
&-\sum_{m=0}^{r}\frac{(-1)^m{}_rS_1^{(r+1-m)}(w;\tilde{\omega})(-n\omega_r)^{r-m}}{(r-m)!}(A_{r-m}-\log(n\omega_r))\\
&-\frac{(-1)^{r+1}{}_rS_2'(w;\tilde{\omega})}{2n\omega_r}\Bigg]\\
&+\sum_{n=1}^{N-1}\Bigg[\sum_{m=0}^{r}\frac{(-1)^m{}_rS_1^{(r+1-m)}(w;\tilde{\omega})(-n\omega_r)^{r-m}}{(r-m)!}(A_{r-m}-\log(n\omega_r))\\
&+\frac{(-1)^{r+1}{}_rS_2'(w;\tilde{\omega})}{2n\omega_r}\Bigg].\end{aligned}$$

Note that the first sum tends to $-\log P_{r+1}(w;\tilde{\omega}^*)$ when $N\to\infty$.

The second sum is transformed to, by Proposition 3, (5),

$$\begin{aligned}&\sum_{n=1}^{N-1}\Bigg[\sum_{m=0}^{r-1}(-1)^m{}_rS_1^{(r+1-m)}(w;\tilde{\omega})\omega_r^{r-m}(\mathrm{LG}_{r-m+1}(n)-\mathrm{LG}_{r-m+1}(n+1))\\
&-(-1)^r{}_rS_1'(w;\tilde{\omega})\log n-\sum_{m=0}^{r}\frac{(-1)^m{}_rS_1^{(r+1-m)}(w;\tilde{\omega})(-n\omega_r)^{r-m}}{(r-m)!}\log\omega_r\\
&+\frac{(-1)^{r+1}{}_rS_2'(w;\tilde{\omega})}{2n\omega_r}\Bigg]\\
&=\sum_{m=0}^{r-1}(-1)^m{}_rS_1^{(r+1-m)}(w;\tilde{\omega})\omega_r^{r-m}(\mathrm{LG}_{r-m+1}(1)-\mathrm{LG}_{r-m+1}(N))\\
&-(-1)^r{}_rS_1'(w;\tilde{\omega})\log\Gamma(N)+(-1)^r{}_rS_1'(w;\tilde{\omega})\log\omega_r-N(-1)^r{}_rS_1'(w;\tilde{\omega})\log\omega_r\end{aligned}$$

$$-\sum_{n=1}^{N-1}\sum_{m=0}^{r-1}\frac{(-1)^m{}_rS_1^{(r+1-m)}(w;\tilde{\omega})(-n\omega_r)^{r-m}}{(r-m)!}\log\omega_r$$

$$+\frac{(-1)^{r+1}{}_rS_2'(w;\tilde{\omega})}{2\omega_r}\sum_{n=1}^{N-1}\frac{1}{n}.$$

Now we know that

$$\log\frac{\Gamma_{r+1}(w+N\omega_r;\tilde{\omega}^*)}{\rho_{r+1}(\tilde{\omega}^*)}$$

$$=\sum_{m=0}^{r+1}\frac{(-1)^m{}_{r+1}S_1^{(r+2-m)}(w;\tilde{\omega}^*)(-N\omega_r)^{r+1-m}}{(r+1-m)!}(A_{r+1-m}-\log(N\omega_r))+O(N^{-1})$$

$$=\sum_{m=0}^{r}(-1)^m{}_{r+1}S_1^{(r+2-m)}(w;\tilde{\omega}^*)\omega_r^{r+1-m}(\mathrm{LG}_{r-m+2}(N)-\mathrm{LG}_{r-m+2}(N+1))$$

$$-(-1)^{r+1}{}_{r+1}S_1'(w;\tilde{\omega}^*)\log N$$

$$-\sum_{m=0}^{r+1}\frac{(-1)^m{}_{r+1}S_1^{(r+2-m)}(w;\tilde{\omega}^*)(-N\omega_r)^{r+1-m}}{(r+1-m)!}\log(\omega_r)+O(N^{-1}),$$

$$\mathrm{LG}_k(N)=\frac{(-1)^k}{k!}\sum_{l=0}^{k}\binom{k}{l}B_lN^{k-l}(A_{k-l}-\log N)+O(N^{-1}),$$

$$\mathrm{LG}_k(N)-\mathrm{LG}_k(N+1)=\frac{(-1)^{k-1}}{k!}kN^{k-1}(A_{k-1}-\log N),$$

$$\sum_{n=1}^{N-1}n^{-1}=\gamma+\log N+O(N^{-1}),$$

$$\log\Gamma(N)=(N-1/2)\log N-N+\log(2\pi)^{1/2}+O(N^{-1}).$$

Put these into the right hand side of (3.3.2), which must converges when $N\to\infty$. The terms going to infinity for $N\to\infty$ are of the types N^p ($p\geqq 1$), $\log N$ and $N^q\log N$ ($q\geqq 1$). Hence diverging terms for $N\to\infty$ must be canceled out with each other. Therefore, as $N\to\infty$, we have

$$\log\frac{\Gamma_{r+1}(w;\tilde{\omega}^*)}{\rho_{r+1}(\tilde{\omega}^*)}=\log\frac{\Gamma_r(w;\tilde{\omega})}{\rho_r(\tilde{\omega})}-\log P_{r+1}(w;\tilde{\omega}^*) \tag{3.3.3}$$

$$-(-1)^{r+1}{}_{r+1}S_1'(w;\tilde{\omega}^*)\log\omega_r+(-1)^r{}_rS_1'(w;\tilde{\omega})\log\omega_r$$

$$+\sum_{m=0}^{r-1}(-1)^m{}_rS_1^{(r+1-m)}(w;\tilde{\omega})\omega_r^{r-m}\mathrm{LG}_{r-m+1}(1)$$

$$+\frac{(-1)^{r+1}{}_rS_2'(w;\tilde{\omega})\gamma}{2\omega_r}-(-1)^r{}_rS_1'(w;\tilde{\omega})\log(2\pi)^{1/2}.$$

Put

$$\Phi_{r+1}(w;\tilde{\omega}^*)=((-1)^{r+1}{}_{r+1}S_1'(w;\tilde{\omega}^*)-(-1)^r{}_rS_1'(w;\tilde{\omega}))\log\omega_r$$

$$-\sum_{m=0}^{r-1}(-1)^m{}_rS_1^{(r+1-m)}(w;\tilde{\omega})\omega_r^{r-m}\mathrm{LG}_{r-m+1}(1)-\frac{(-1)^{r+1}{}_rS_2'(w;\tilde{\omega})\gamma}{2\omega_r}$$

$$+(-1)^r{}_rS_1'(w;\tilde{\omega})\log(2\pi)^{1/2}\,,$$

$$\Psi_{r+1}(w;\tilde{\omega}^*)=\sum_{m=0}^{r}\frac{(-1)^m{}_rS_1^{(r+1-m)}(w;\tilde{\omega})(-\omega_r)^{r-m}}{(r-m)!}(A_{r-m}-\log\omega_r)$$

$$+\frac{(-1)^{r+1}{}_rS_2'(w;\tilde{\omega})}{2\omega_r}$$

for $\tilde{\omega}=(\omega_0,\omega_1,\cdots,\omega_{r-1})$, $\tilde{\omega}^*=(\tilde{\omega},\omega_r)$. Then

$$\log P_{r+1}(w;\tilde{\omega}^*)=-\sum_{n=1}^{\infty}\{\log\Gamma_r(w+n\omega_r;\tilde{\omega})-\log\rho_r(\tilde{\omega})-\Psi_{r+1}(w;\tilde{\omega},n\omega_r)\}\,,$$

$$\log\frac{\Gamma_{r+1}(w;\tilde{\omega}^*)}{\rho_{r+1}(\tilde{\omega}^*)}=\log\frac{\Gamma_r(w;\tilde{\omega})}{\rho_r(\tilde{\omega})}-\Phi_{r+1}(w;\tilde{\omega}^*)$$

$$+\sum_{n=1}^{\infty}\{\log\Gamma_r(w+n\omega_r;\tilde{\omega})-\log\rho_r(\tilde{\omega})-\Psi_{r+1}(w;\tilde{\omega},n\omega_r)\}\,.$$

Put $w=\omega_0$ in the above. Since (1.2.7) shows

$$\Gamma_{r+1}(\omega_0;\tilde{\omega}^*)=\rho_r(\omega_1,\cdots,\omega_r)\,,\qquad \Gamma_r(\omega_0;\tilde{\omega})=\rho_{r-1}(\omega_1,\cdots,\omega_{r-1})\,,$$

we have

$$\log\frac{\rho_r(\omega_1,\cdots,\omega_r)}{\rho_{r+1}(\tilde{\omega}^*)}=\log\frac{\rho_{r-1}(\omega_1,\cdots,\omega_{r-1})}{\rho_r(\tilde{\omega})}-\Phi_{r+1}(\omega_0;\tilde{\omega}^*)$$

$$+\sum_{n=1}^{\infty}\{\log\Gamma_r(\omega_0+n\omega_r;\tilde{\omega})-\log\rho_r(\tilde{\omega})-\Psi_{r+1}(\omega_0;\tilde{\omega},n\omega_r)\}\,,$$

$$\log\rho_{r+1}(\tilde{\omega}^*)=\log\frac{\rho_r(\omega_1,\cdots,\omega_r)\rho_r(\tilde{\omega})}{\rho_{r-1}(\omega_1,\cdots,\omega_{r-1})}+\Phi_{r+1}(\omega_0;\tilde{\omega}^*)$$

$$-\sum_{n=1}^{\infty}\{\log\Gamma_r(\omega_0+n\omega_r;\tilde{\omega})-\log\rho_r(\tilde{\omega})-\Psi_{r+1}(\omega_0;\tilde{\omega},n\omega_r)\}\,.$$

Then, by (3.3.3), we have

$$\log\Gamma_{r+1}(w;\tilde{\omega}^*)=\log\left[\frac{\rho_r(\omega_1,\cdots,\omega_r)}{\rho_{r-1}(\omega_1,\cdots,\omega_{r-1})}\Gamma_r(w;\tilde{\omega})\right]+\Phi_{r+1}(\omega_0;\tilde{\omega}^*)-\Phi_{r+1}(w;\tilde{\omega}^*)$$

$$+\sum_{n=1}^{\infty}\{\log\Gamma_r(w+n\omega_r;\tilde{\omega})-\log\Gamma_r(\omega_0+n\omega_r;\tilde{\omega})$$
$$+\Psi_{r+1}(\omega_0;\tilde{\omega},n\omega_r)-\Psi_{r+1}(w;\tilde{\omega},n\omega_r)\}\ .$$

Thus we get an infinite product expressions for $\Gamma_{r+1}(w;\tilde{\omega}^*)$ and $\rho_{r+1}(\tilde{\omega}^*)$ which generalize Proposition 1 of Shintani [5]:

THEOREM 2. *Let* $w>0$ *and* $\omega_i>0$, $i=0, 1, \cdots, r$.

(1) $$\Gamma_{r+1}(w;\tilde{\omega}^*)=\frac{\rho_r(\omega_1,\cdots,\omega_r)}{\rho_{r-1}(\omega_1,\cdots,\omega_{r-1})}\Gamma_r(w;\tilde{\omega})$$
$$\cdot\exp(\Phi_{r+1}(\omega_0;\tilde{\omega}^*)-\Phi_{r+1}(w:\tilde{\omega}^*))$$
$$\cdot\prod_{n=1}^{\infty}\left[\frac{\Gamma_r(w+n\omega_r;\tilde{\omega})}{\Gamma_r(\omega_0+n\omega_r;\tilde{\omega})}\exp\{\Psi_{r+1}(\omega_0;\tilde{\omega},n\omega_r)-\Psi_{r+1}(w;\tilde{\omega},n\omega_r)\}\right]$$

and this is continued holomorphically to the domain

$$\{(w,\tilde{\omega}^*)\ ;\ \omega_i\in\mathbf{C}-(-\infty,0],\ i=0,1,\cdots,r,$$
$$w\neq-\textstyle\sum_{i=0}^{r}m_i\omega_i,\ m_i\in\mathbf{Z},\ i=0,1,\cdots,r\}\ .$$

(2) $$\rho_{r+1}(\tilde{\omega}^*)=\frac{\rho_r(\omega_1,\cdots,\omega_r)\rho_r(\tilde{\omega})}{\rho_{r-1}(\omega_1,\cdots,\omega_{r-1})}\exp(\Phi_{r+1}(\omega_0;\tilde{\omega}^*))$$
$$\cdot\prod_{n=1}^{\infty}\{\Gamma_r(\omega_0+n\omega_r;\tilde{\omega})^{-1}\rho_r(\tilde{\omega})\exp(\Psi_{r+1}(\omega_0;\tilde{\omega},n\omega_r))\}$$

and this is continued holomorphically to the domain

$$\{\tilde{\omega}^*\ ;\ \omega_i\in\mathbf{C}-(-\infty,0],\ i=0,1,\cdots,r\}\ .$$

For $r=1$, take $\omega_0=1$ and put $\omega_1=z$. Then

$$_2S_1'(w;(1,z))=\frac{1}{12z}+\frac{z}{12}+\frac{w^2}{2z}+\frac{1}{4}-\frac{w}{2z}-\frac{w}{2}\,,\qquad {}_1S_1'(w;1)=-\frac{1}{2}+w\,,$$

$$_1S_1^{(2)}(w;1)=1\,,\quad {}_1S_2'(w;1)=w^2-w+\frac{1}{6}\,,\quad \mathrm{LG}_2(1)=\frac{1}{12}-\zeta'(-1)\,,$$

$$\Phi_2(w;(1,z))=\left(\frac{1}{12z}+\frac{z}{12}+\frac{w^2}{2z}-\frac{1}{4}-\frac{w}{2z}+\frac{w}{2}\right)\log z-\left(\frac{1}{12}-\zeta'(-1)\right)z$$
$$-\frac{\gamma(w^2-w+1/6)}{2z}-\left(-\frac{1}{2}+w\right)\log(2\pi)^{1/2}\,,$$

$$\Psi_2(w;(1,z))=-z(1-\log z)+\left(-\frac{1}{2}+w\right)\log z+\frac{w^2-w+1/6}{2z}\,.$$

Hence we have

$$\Gamma_2(w;(1,z))=(2\pi)^{w/2}\exp\left\{\left(\frac{w-w^2}{2z}-\frac{w}{2}\right)\log z+\frac{(w^2-w)\gamma}{2z}\right\}$$

$$\cdot\Gamma(w)\prod_{n=1}^{\infty}\frac{\Gamma(w+nz)}{\Gamma(1+nz)}\exp\left\{\frac{w-w^2}{2nz}+(1-w)\log(nz)\right\},$$

$$\rho_2((1,z))=(2\pi)^{3/4}\exp\left\{-\frac{\gamma}{12z}-\frac{z}{12}+z\zeta'(-1)+\left(\frac{z}{12}-\frac{1}{4}+\frac{1}{12z}\right)\log z\right\}$$

$$\cdot\prod_{n=1}^{\infty}(2\pi)^{1/2}\Gamma(1+nz)^{-1}\exp\left\{\frac{1}{12nz}+\left(\frac{1}{2}+nz\right)\log(nz)-nz\right\}.$$

These formulas coincide with the formulas of Proposition 1 in Shintani [5].

3.4. The inversion formula.

PROPOSITION 6. *$\Gamma_r(w;\omega_0,\omega_1,\cdots,\omega_{r-1})$ and $\rho_r(\omega_0,\omega_1,\cdots,\omega_{r-1})$ are symmetric with respect to $\omega_0,\omega_1,\cdots,\omega_{r-1}$.*

For short, we say that $\Gamma_r(w;\tilde{\omega})$ and $\rho_r(\tilde{\omega})$ are symmetric (with respect to $\tilde{\omega}$).

PROOF. (1.2.5) shows that $\log(\Gamma_r(w;\tilde{\omega})/\rho_r(\tilde{\omega}))$ is symmetric with respect to $\tilde{\omega}$. Add $\log w$ to the both hand sides of (1.2.5) and let w tend to 0. Then by (1.2.4), $\rho_r(\tilde{\omega})$ is symmetric and the symmetricity of $\Gamma_r(w;\tilde{\omega})$ follows. □

PROPOSITION 7. *Assume that $z_0=1$, $z_l\in\mathbf{C}-(-\infty,0]$ for $l=1,2,\cdots,r-1$, and w is such that $w\neq-\sum_{i=0}^{r-1}m_iz_i$, $m_i\in\mathbf{Z}$, $i=0,1,\cdots,r-1$. Then for every $l=1,2,\cdots,r-1$,*

(i)
$$\Gamma_r\left(\frac{w}{z_l};1,\frac{z_1}{z_l},\cdots,\overset{l}{\frac{1}{z_l}},\cdots,\frac{z_{r-1}}{z_l}\right)=\Gamma_r(w;1,z_1,\cdots,z_{r-1})$$
$$\cdot\exp\left\{\left((-1)^r{}_rS_1'(w;1,z_1,\cdots,z_{r-1})-\sum_{k=0}^{r-1}(-1)^{r-k}{}_{r-k}S_1'(z_k;z_k,\cdots,z_{r-1})\right)\log z_l\right\}.$$

(ii)
$$\rho_r\left(1,\frac{z_1}{z_l},\cdots,\overset{l}{\frac{1}{z_l}},\cdots,\frac{z_{r-1}}{z_l}\right)=\rho_r(1,z_1,\cdots,z_{r-1})$$
$$\cdot\exp\left\{-\left(\sum_{k=0}^{r-1}(-1)^{r-k}{}_{r-k}S_1'(z_k;z_k,\cdots,z_{r-1})\right)\log z_l\right\}.$$

In particular, for $r=2$ and $z_1=z$,

(iii)
$$\Gamma_2\left(\frac{w}{z};\left(1,\frac{1}{z}\right)\right)=\Gamma_2(w;(1,z))\exp\left\{\left(1+\frac{w^2}{2z}-\frac{w}{2}\left(1+\frac{1}{z}\right)\right)\log z\right\},$$

(iv) $$\rho_2\left(\left(1, \frac{1}{z}\right)\right)=\rho_2((1, z))\exp\left\{\left(\frac{3}{4}-\frac{1}{12z}-\frac{z}{12}\right)\log z\right\}.$$

PROOF. In Proposition 2, (i), (ii), put $\omega_0=z_0=1$, $\omega_k=z_k$, $k=1, 2, \cdots, r-1$, and $t=1/z_l$. (First, apply them to $t>0$. Then make analytic continuation.) Then (i), (ii) in the present Proposition easily follow from the homogeneity (Proposition 2) and the symmetricity (Proposition 6). □

COROLLARY. *For* $\operatorname{Im} z>0$,

(i) $$\vartheta_1\left(\frac{w}{z}, -\frac{1}{z}\right)=e^{\pi i w^2/z}\frac{1}{i}\sqrt{\frac{z}{i}}\,\vartheta_1(w, z),$$

(ii) $$\eta\left(-\frac{1}{z}\right)=\sqrt{\frac{z}{i}}\,\eta(z).$$

PROOF. We first derive (ii). Since $\operatorname{Im}(-1/z)>0$, we have, by (3.1.1),

$$\rho_2\left(\left(1, -\frac{1}{z}\right)\right)\rho_2\left(\left(1, \frac{1}{z}\right)\right)=(2\pi)^{3/2}\left(-\frac{1}{z}\right)^{-1/2}\eta\left(-\frac{1}{z}\right)\exp\left\{\pi i\left(\frac{1}{4}-\frac{z}{12}\right)\right\}.$$

Then our Proposition shows

$$\rho_2\left(\left(1, -\frac{1}{z}\right)\right)=\rho_2((1, -z))\exp\left\{\left(\frac{3}{4}+\frac{1}{12z}+\frac{z}{12}\right)\log(-z)\right\},$$

$$\rho_2\left(\left(1, \frac{1}{z}\right)\right)=\rho_2((1, z))\exp\left\{\left(\frac{3}{4}-\frac{1}{12z}-\frac{z}{12}\right)\log z\right\}.$$

Then (ii) is obtained by a straight forward calculation. Here note that $\log(-z)=\log z-\pi i$.

Proof of (i). In (3.1.2), replace w, z by w/z, $-1/z$, respectively. Then it becomes

(3.4.1)
$$\frac{\Gamma_2(\frac{w}{z};(1, -\frac{1}{z}))\Gamma_2(1-\frac{w}{z}; (1, \frac{1}{z}))\Gamma_2(1-\frac{1}{z}-\frac{w}{z}; (1, -\frac{1}{z}))\Gamma_2(\frac{w}{z}+\frac{1}{z}; (1, \frac{1}{z}))}{\rho_2((1, -\frac{1}{z}))\rho_2((1, \frac{1}{z}))\rho_2((1, -\frac{1}{z}))\rho_2((1, \frac{1}{z}))}$$
$$=\frac{\eta(-1/z)}{\vartheta_1(w/z, -1/z)}\exp\pi i\left\{\frac{z}{6}-z\left(\frac{w}{z}-\frac{w^2}{z^2}\right)\right\}.$$

Now, by (3.1.3) (=the difference equations),

$$\Gamma_2\left(\frac{w}{z}; \left(1, -\frac{1}{z}\right)\right)=\Gamma_2\left(\frac{w}{z}-\frac{1}{z}; \left(1-\frac{1}{z}\right)\right)(2\pi)^{1/2}\Gamma\left(\frac{w}{z}\right).$$

Then by Proposition 7, (iii) and the symetricity (Proposition 6), we have

$$\Gamma_2\left(\frac{w}{z}-\frac{1}{z};\left(1,\ -\frac{1}{z}\right)\right)$$

$$=\Gamma_2(1-w;(1,\ -z))\exp\left\{\left(1-\frac{(1-w)^2}{2z}-\frac{1}{2}\left(1-\frac{1}{z}\right)(1-w)\right)\log(-z)\right\}.$$

Thus

$$\Gamma_2\left(\frac{w}{z};\left(1,\ -\frac{1}{z}\right)\right)=\Gamma_2(1-w;(1,\ -z))(2\pi)^{-1/2}\Gamma\left(\frac{w}{z}\right)$$

$$\cdot\exp\left\{\left(1-\frac{(1-w)^2}{2z}-\frac{1}{2}\left(1-\frac{1}{z}\right)(1-w)\right)\log(-z)\right\}.$$

In the same way, we have

$$\Gamma_2\left(1-\frac{w}{z};\left(1,\frac{1}{z}\right)\right)=\Gamma_2\left(1+\frac{1}{z}-\frac{w}{z};\left(1,\frac{1}{z}\right)\right)(2\pi)^{-1/2}\Gamma\left(1-\frac{w}{z}\right)$$

$$=\Gamma_2(1+z-w;(1,z))(2\pi)^{-1/2}\Gamma\left(1-\frac{w}{z}\right)$$

$$\cdot\exp\left\{\left(1+\frac{(1+z-w)^2}{2z}-\frac{1}{2}\left(1+\frac{1}{z}\right)(1+z-w)\right)\log z\right\},$$

$$\Gamma_2\left(1-\frac{1}{z}-\frac{w}{z};\left(1,\ -\frac{1}{z}\right)\right)=\Gamma_2\left(1-\frac{w}{z};\left(1,\ -\frac{1}{z}\right)\right)(2\pi)^{1/2}\Gamma\left(1-\frac{w}{z}\right)^{-1}$$

$$=\Gamma_2(w-z;(1,\ -z))(2\pi)^{1/2}\Gamma\left(1-\frac{w}{z}\right)^{-1}$$

$$\cdot\exp\left\{\left(1-\frac{(w-z)^2}{2z}-\frac{1}{2}\left(1-\frac{1}{z}\right)(w-z)\right)\log(-z)\right\},$$

$$\Gamma_2\left(\frac{1}{z}+\frac{w}{z};\left(1,\frac{1}{z}\right)\right)=\Gamma_2\left(\frac{w}{z};\left(1,\frac{1}{z}\right)\right)(2\pi)^{1/2}\Gamma\left(\frac{w}{z}\right)^{-1}$$

$$=\Gamma_2(w;(1,z))(2\pi)^{1/2}\Gamma\left(\frac{w}{z}\right)^{-1}$$

$$\cdot\exp\left\{\left(1+\frac{w^2}{2z}-\frac{1}{2}\left(1+\frac{1}{z}\right)w\right)\log z\right\}.$$

Put these into (3.4.1). Then a straight forward calculation shows (i). □

It will be worth to note that the inversion formulas for $\vartheta_1(w, z)$ and $\eta(z)$ are derived from the homogeneity and the symmetricity of $\rho_2(1, z)$ and $\Gamma_2(w; (1, z))$ (only on the basis of Shintani [5]).

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