TOKYO J. MATH.
VOL. 21, NO. 1, 1998

# On the First Eigenvalue of the $p$-Laplacian in a Riemannian Manifold

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(Communicated by Y. Maeda)

## 1. Introduction and results.

Let $\Omega$ be a bounded domain in a Riemannian manifold $(M, g)$ of dimension $m$. We consider the following Dirichlet problem:

$$\tag{1} \begin{aligned} \Delta_p u+\lambda|u|^{p-2}u=0 \quad &\text{in } \Omega\,, \\ u=0 \quad &\text{on } \partial\Omega\,, \end{aligned}$$

where $\Delta_p u=\mathrm{div}(|\nabla u|_g^{p-2}\nabla u)$ is the $p$-Laplacian with $1<p<\infty$. In local coordinates,

$$\Delta_p u=\frac{1}{\sqrt{\det(g_{ij})}}\sum_{i,j=1}^{m}\frac{\partial}{\partial x^i}\left(\sqrt{\det(g_{ij})}g^{ij}|\nabla u|^{p-2}\frac{\partial u}{\partial x^j}\right),$$

where $|\nabla u|^2=|\nabla u|_g^2=\sum_{ij}g^{ij}(\partial u/\partial x^i)(\partial u/\partial x^j)$, and $(g^{ij})=(g_{ij})^{-1}$. The *first eigenvalue* $\lambda_{1,p}(\Omega)$ *of the p-Laplacian* is defined as the least real number $\lambda$ for which the Dirichlet problem (1) has a nontrivial solution $u\in W_0^{1,p}(\Omega)$. Here the Sobolev space $W_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the Sobolev norm $\|u\|_{1,p}=\{\int_\Omega(|u|^p+|\nabla u|^p)dv_g\}^{1/p}$. It can be also characterized by

$$\tag{2} \lambda_{1,p}(\Omega)=\inf_{u\neq 0}\frac{\int_\Omega|\nabla u|^p dv_g}{\int_\Omega|u|^p dv_g}\,,$$

where $u$ runs over $W_0^{1,p}(\Omega)$ and $dv_g$ denotes the volume element of $M$. We would like to estimate the $\lambda_{1,p}(\Omega)$. For the case $p=2$, there have been several results, such as the Faber-Krahn inequality [1], the Cheeger inequality [2], and the Cheng inequality [3]. The purpose of this paper is to give inequalities for their $p$-Laplacian analogue. More precisely we show the following theorems.

THEOREM 1 (the Faber-Krahn type inequality). *Let $M_k$ be a complete simply connected Riemannian manifold of constant sectional curvature $\kappa$. Let $B$ be the geodesic*

Received November 8, 1996

*ball in $M_\kappa$, whose volume is equal to that of the domain $\Omega$ in $M_\kappa$. Then the following inequality holds*:

$$\lambda_{1,p}(\Omega) \geq \lambda_{1,p}(B) . \tag{3}$$

*The equality holds only for the case the domain $\Omega$ is the ball $B$ in $M_\kappa$.*

Next we define the Cheeger constant $h(\Omega)$ of $\Omega$ to be

$$h(\Omega) = \inf_{\Omega'} \frac{\mathrm{Vol}(\partial\Omega')}{\mathrm{Vol}(\Omega')} ,$$

where $\Omega'$ ranges over all open submanifold of $\Omega$ with compact closure in $\Omega$ and smooth boundary $\partial\Omega'$. $\mathrm{Vol}(\Omega')$ and $\mathrm{Vol}(\partial\Omega')$ denote the volumes of $\Omega'$ and $\partial\Omega'$ respectively.

THEOREM 2 (the Cheeger type inequality). *For any bounded domain $\Omega$ with piecewise smooth boundary in a complete Riemannian manifold, we have the following inequality*:

$$\lambda_{1,p}(\Omega) \geq \left( \frac{h(\Omega)}{p} \right)^p . \tag{4}$$

THEOREM 3 (the Cheng type inequality). *Let $M$ be an $m$-dimensional complete Riemannian manifold with Ricci curvature satisfying* $\mathrm{Ric}(v) \geq k(m-1)$ *for any unit vector $v \in TM$. Let $B(x_0, r)$ be the geodesic ball in $M$ of radius $r$ with center $x_0$, and $V(k, r)$ be a ball of radius $r$ with center $\tilde{x}_0$ in the space form of curvature $k$. Then we have*

$$\lambda_{1,p}(B(x_0, r)) \leq \lambda_{1,p}(V(k, r)) , \tag{5}$$

*with equality if and only if $B(x_0, r)$ is isometric to $V(k, r)$.*

The author would like to express his gratitude to Professors T. Sakai and A. Katsuda for helpful discussions and advice.

## 2. Proof of Theorem 1.

Let $f$ be a nonnegative eigenfunction of $p$-Laplacian in $\Omega$ associated with $\lambda_{1,p}(\Omega)$. Consider the set $\Omega_t = \{x \in \Omega ; f(x) > t\}$ and $\Gamma_t = \{x \in \Omega ; f(x) = t\}$. Using a symmetrization procedure, we construct the geodesic ball $B_t$ in $M_k$ such that $\mathrm{Vol}(B_t) = \mathrm{Vol}(\Omega_t)$ for each $t$, and $B_0 = B$. We define a function $F : B \to \mathbf{R}^+$ such that $F$ is a radially decreasing function and $\partial B_t = \{x \in B ; F(x) = t\}$.

Then it suffices to prove

$$\int_\Omega f^p dv_g = \int_B F^p dv_g , \tag{6}$$

(7) $$\int_{\Omega} |\nabla f|^p dv_g \geq \int_B |\nabla F|^p dv_g .$$

Indeed for (6), using coarea formula [4],

$$\begin{aligned}\int_{\Omega} f^p dv_g &= \int_0^{\infty} \int_{\Gamma_t} \frac{f^p}{|\nabla f|} dA_t dt = \int_0^{\infty} t^p \left( \int_{\Gamma_t} \frac{dA_t}{|\nabla f|} \right) dt \\ &= -\int_0^{\infty} t^p \frac{d}{dt} \mathrm{Vol}(\Omega_t) dt = -\int_0^{\infty} t^p \frac{d}{dt} \mathrm{Vol}(B_t) dt = \int_B F^p dv_g ,\end{aligned}$$

where $dA_t$ is the $(m-1)$-dimensional volume element on $\Gamma_t$. Here we have used the identity

$$\frac{d}{dt} \mathrm{Vol}(\Omega_t) = -\int_{\Gamma_t} |\nabla f|^{-1} dA_t .$$

Next we shall prove (7). Using the Hölder inequality, we have

$$\begin{aligned}\int_{\Gamma_t} dA_t &= \int_{\Gamma_t} |\nabla f|^{1-1/p} \cdot |\nabla f|^{-1+1/p} dA_t \\ &\leq \left( \int_{\Gamma_t} |\nabla f|^{p-1} dA_t \right)^{1/p} \left( \int_{\Gamma_t} |\nabla f|^{-1} dA_t \right)^{(p-1)/p} \\ &= \left( \int_{\Gamma_t} |\nabla f|^{p-1} dA_t \right)^{1/p} \left( -\frac{d}{dt} \mathrm{Vol}(\Omega_t) \right)^{(p-1)/p} .\end{aligned}$$

Thus we have, using isoperimetric inequality,

$$\begin{aligned}\int_{\Gamma_t} |\nabla f|^{p-1} dA_t &\geq \frac{\mathrm{Vol}(\Gamma_t)^p}{\left( -\frac{d}{dt} \mathrm{Vol}(\Omega_t) \right)^{p-1}} \\ &\geq \frac{\mathrm{Vol}(\Gamma_t^*)^p}{(\int_{\Gamma_t^*} |\nabla F|^{-1} dA_t^*)^{p-1}} = \int_{\Gamma_t^*} |\nabla F|^{p-1} dA_t^* ,\end{aligned}$$

where $\Gamma_t^* = \{x \in B\,;\, F(x) = t\}$, and $dA_t^*$ is the $(m-1)$-dimensional volume element on $\Gamma_t^*$. Integrating in $t$, we get (7).

## 3. Proof of Theorem 2.

Let $u$ be a nonnegative eigenfunction of the $p$-Laplacian in $\Omega$ associated with $\lambda_{1,p}(\Omega)$. Then we may assume $u(x) > 0$ for $x \in \Omega$. Integrating the identity

$$-u\Delta_p u = \lambda_{1,p}(\Omega) |u|^{p-2} u^2$$

by parts, we have

$$\lambda_{1,p}(\Omega)=\frac{\int_\Omega|\nabla u|^p dv_g}{\int_\Omega|u|^p dv_g}.$$

Hence we have used Green's formula:

$$\int_\Omega -u\Delta_p u dv_g=-\int_\Omega u\,\mathrm{div}(|\nabla u|^{p-2}\nabla u)dv_g=\int_\Omega|\nabla u|^p dv_g.$$

By $\nabla u^p=pu^{p-1}\nabla u$ and the Hölder inequality,

$$\lambda_{1,p}(\Omega)\geq\left(\frac{\int_\Omega|\nabla u^p|dv_g}{p\int_\Omega|u|^p dv_g}\right)^p. \tag{8}$$

Now by the coarea formula,

$$\begin{aligned}\int_\Omega|\nabla u^p|dv_g&=\int_{-\infty}^\infty \mathrm{Vol}(A(t))dt\\ &\geq\inf_t\left(\frac{\mathrm{Vol}(A(t))}{\mathrm{Vol}(V(t))}\right)\int_{-\infty}^\infty \mathrm{Vol}(V(t))dt\\ &\geq h(\Omega)\int_\Omega|u|^p dv_g,\end{aligned} \tag{9}$$

where $A(t)=\{x\,;|u(x)|^p=t\}$ and $V(t)=\{x\,;|u(x)|^p>t\}$. Combining (8) and (9), we get (4) in Theorem 2.

## 4. Proof of Theorem 3.

Let $\tilde{f}$ be a nonnegative first eigenfunction of $p$-Laplacian on $\overline{V(k,r)}$. Let $d_{\tilde{x}_0}$ be the distance function with respect to the center $\tilde{x}_0$ of $\overline{V(k,r)}$. Since $\tilde{f}$ depends only on the distance $d_{\tilde{x}_0}$, we may write $\tilde{f}=\varphi\circ d_{\tilde{x}_0}$, where $\varphi$ is a positive function on $(0,r)$. We define a $C^\infty$ map $\Theta:(0,r)\times S^{m-1}\to M$ by

$$\Theta(t,v)=\exp_x tv,$$

where $S^{m-1}$ is the unit sphere in $T_xM$ and $\exp_x$ is a local diffeomorphism from a neighbourhood of 0 in $T_xM$ onto a neighbourhood of $x$ in $M$. We set $\theta(t,v)=t^{m-1}\sqrt{\det g_{ij}(\Theta(t,v))}$, which is a $C^\infty$ function on $(0,r)\times S^{m-1}$. Then we have

$$\Theta^*dv_g=\theta(t,v)dtdv,$$

where $dtdv$ denotes the canonical product measure on $(0,r)\times S^{m-1}$. When we define $\theta(t,v)$ on $\overline{V(k,r)}$ in the same manner, $\theta(t,v)$ does not depend on $v\in S^{m-1}$. We denote it simply by $\tilde{\theta}(s)$. We have for $0\leq s\leq r$,

$$(10)\quad \begin{gathered}(p-1)|\varphi'(s)|^{p-2}\varphi''(s)+\frac{\tilde{\theta}'(s)}{\tilde{\theta}(s)}|\varphi'(s)|^{p-2}\varphi'(s)+\lambda_{1,p}(V(k,r))|\varphi(s)|^{p-2}\varphi(s)=0\,,\\ \varphi(r)=0\,,\qquad \varphi'(0)=0\,.\end{gathered}$$

We take $f(x)=\varphi\circ d_{x_0}(x)$ as a test function on a ball $B(x_0, r)$, which satisfies the boundary condition $f|_{\partial B(x_0,r)}=\varphi(r)=0$. Then we get

$$(11)\qquad \lambda_{1,p}(B(x_0, r))\leq\frac{\int_{B(x_0,r)}|\nabla f|^p dv_g}{\int_{B(x_0,r)}|f|^p dv_g}\,.$$

From $|\nabla f|^p=|\varphi'|^p$ we have

$$(12)\qquad \int_{B(x_0,r)}|\nabla f|^p dv_g=\int_{S^{m-1}}dS^{m-1}\int_0^{a(v)}|\varphi'(s)|^p\theta(s, v)ds\,,$$

$$(13)\qquad \int_{B(x_0,r)}|f|^p dv_g=\int_{S^{m-1}}dS^{m-1}\int_0^{a(v)}|\varphi(s)|^p\theta(s, v)ds\,,$$

where $a(v)\leq r$ such that $\exp_{x_0}(a(v)\cdot v)$ is the cut point of $x_0$ along the geodesic $t\to\exp_{x_0}(tv)$. By

$$\{\tilde{\theta}(s)|\varphi'(s)|^{p-2}\varphi'(s)\}'=-\lambda_{1,p}(V(k, r))|\varphi(s)|^{p-2}\varphi(s)\tilde{\theta}(s)\leq 0$$

and $\varphi'(0)=0$, we can see that $\varphi'(s)\leq 0$. Integrating the above equation (12) by parts, we have

$$\begin{aligned}\int_{B(x_0,r)}|\nabla f|^p dv_g&=-\int_{S^{m-1}}dS^{m-1}\int_0^{a(v)}\left[\{\varphi|\varphi'|^{p-1}\theta(s, v)\}'-\varphi(|\varphi'|^{p-1}\theta)'\right]ds\\ &=\int_{S^{m-1}}dS^{m-1}\int_0^{a(v)}\varphi(s)|\varphi'(s)|^{p-2}\left\{-(p-2)\varphi''(s)-\frac{\theta'(s, v)}{\theta(s, v)}\cdot\varphi'(s)\right\}\theta(s, v)ds\,,\end{aligned}$$

where $\theta'(s, v)$ denotes the partial derivative with respect to $s$. By the Bishop comparison theorem we have $\{\theta(s, v)/\tilde{\theta}(s)\}'\leq 0$. Recalling $\varphi'\leq 0$, we get

$$\varphi'(s)\cdot\theta'(s, v)/\theta(s, v)\geq\varphi'(s)\cdot\tilde{\theta}'(s)/\tilde{\theta}(s)\,.$$

Thus we have

$$\begin{aligned}&\int_{B(x_0,r)}|\nabla f|^p dv_g\\ &\leq\int_{S^{m-1}}dS^{m-1}\int_0^{a(v)}\varphi(s)|\varphi'(s)|^{p-1}\left\{-(p-1)\varphi''(s)-\tilde{\theta}'(s)/\tilde{\theta}(s)\cdot\varphi'(s)\right\}\theta(s, v)ds\\ &=\int_{S^{m-1}}dS^{m-1}\int_0^{a(v)}\lambda_{1,p}(V(k, r))\varphi^p(s)\theta(s, v)ds=\lambda_{1,p}(V(k, r))\int_{B(x_0,r))}\varphi^p dv_g\,.\end{aligned}$$

This implies that $\lambda_{1,p}(B(x_0, r))\leq\lambda_{1,p}(V(k, r))$. If the equality holds, then $\{\theta(s, v)/\tilde{\theta}\}'=0$.

Since the equality holds in the Bishop comparison theorem, $B(x_0, r)$ is of constant curvature $k$. It follows that $B(x_0, r)$ is isometric to $V(k, r)$.

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