TOKYO J. MATH.
VOL. 21, NO. 1, 1998

On a Variety of Algebraic Minimal Surfaces in Euclidean 4-Space

Katsuhiro MORIYA

Tokyo Metropolitan University

(Communicated by N. Sasakura)

Abstract. In this paper, we show that the moduli space of the Weierstrass data for algebraic minimal surfaces in Euclidean 4-space with fixed topological type, orders of branched points and ends, and total curvature, has the structure of a real analytic variety. We provide the lower bounds of its dimension. We also show that the moduli space of the Weierstrass data for stable algebraic minimal surfaces in Euclidean 4-space has the structure of a complex analytic variety.

1. Introduction.

Let M be a Riemann surface and $f: M \to \mathbf{R}^n$ a branched conformal minimal immersion whose induced degenerate Riemannian metric ds^2 is complete in the sense that any locally rectifiable divergent path has infinite length. Then, by modifying the Chern-Osserman theorem [ChOs, Theorem 1], we can prove that the total curvature is finite if and only if the Gauss map Φ_f is algebraic, i.e. M is biholomorphic to a compact Riemann surface M_g punctured at a finite set of points and Φ_f extends to a holomorphic map from M_g to $Q_{n-2}(\mathbf{C})$. We call a branched immersed minimal surface with finite total curvature an *algebraic minimal surface*.

X. Mo constructed the moduli space of pairs of certain meromorphic functions on a compact Riemann surface which give algebraic minimal surfaces in $\mathbf{R}^3$ by the Weierstrass formula. He proved that the moduli space has the structure of a real analytic variety and that it contains a subset having the structure of a complex analytic variety. We can see this work in the book written by Yang [Ya2, Chapter 3]. His idea of the proof is to clarify the conditions satisfied by the divisors of meromorphic functions.

In this paper, by following this idea, we will construct the moduli space of the triples of certain meromorphic functions on a compact Riemann surface which give algebraic minimal surfaces in $\mathbf{R}^4$, and give a lower bound of the dimension of the moduli space.

Received August 16, 1996
Revised July 31, 1997

We fix a compact Riemann surface M_g of genus g, a holomorphic (if $g=0$, then a meromorphic) 1-form Ω on M_g, integers k, r ($k\geq 0, r\geq 1$), and an integer vector $B_{k,r}=(J_j; I_i)\in \mathbf{Z}^k\times\mathbf{Z}^r$ ($J_j\geq 1$, $I_i\geq 2$).

We denote by $AM=AM(M_g, B_{k,r})$ the set of algebraic minimal surfaces $f=(f^1, f^2, f^3, f^4)\colon M\to(\mathbf{R}^4, ds^2)$ in $\mathbf{R}^4$ satisfying the following conditions:

The Riemann surface M is biholomorphic to $M_g-\{\text{puncture points}\}$;
It is branched at k points with order J_j ($j=1, \cdots, k$) and
punctured at r points with order I_i ($i=1, \cdots, r$);
$f^1+\sqrt{-1}f^2$ is not holomorphic.

We denote by $FD=FD(M_g, \Omega, B_{k,r}, \alpha, \beta)$ the set of the triples $(F, \varphi_1, \varphi_2)$ of meromorphic functions on M_g satisfying the following conditions:

$$F\not\equiv 0\,,\quad \deg(\varphi_1)_\infty=\alpha\,,\quad \deg(\varphi_2)_\infty=\beta\,,$$

$$\text{(D)}\qquad -(\varphi_1)_\infty-(\varphi_2)_\infty+(F)+(\Omega)=\sum_{j=1}^{k}J_jb_j-\sum_{i=1}^{r}I_ip_i\,,$$

$$\text{(P)}\qquad \frac{1}{2}\Re\left\{\int_\gamma E_aF\Omega\right\}=0$$

for any $\gamma\in H_1(M_g-\{p_1, \cdots, p_r\})$ ($a=1, \cdots, 4$); where $\{b_j; p_i\}$ are distinct points. We denote by $(\varphi)=(\varphi)_0-(\varphi)_\infty$ the divisor of a meromorphic function on M_g where $(\varphi)_0$ is the zero divisor and $(\varphi)_\infty$ is the polar divisor. In particular, $\deg(c)_\infty=0$ for $c\in\mathbf{C}^*$ and we define $(0)=\mathbf{0}$ and $\deg(0)_\infty=\infty$ for later use. Similarly, (Ω) is the divisor of a meromorphic 1-form on M_g. We define

$$E_1=1+\varphi_1\varphi_2\,,\qquad E_2=\sqrt{-1}(1-\varphi_1\varphi_2)\,,$$
$$E_3=\varphi_1-\varphi_2\,,\qquad E_4=-\sqrt{-1}(\varphi_1+\varphi_2)\,.$$

We call the condition (D) *divisor condition* and the condition (P) *period condition*.

By using the result of Osserman [Os], we will show the following lemma (§3):

LEMMA 1.1. *There is a bijective correspondence between $AM(M_g, B_{k,r})/\sim$ and $\coprod_{\alpha,\beta}FD(M_g, \Omega, B_{k,r}, \alpha, \beta)$, where for G and $H\in AM(M_g, B_{k,r})$, $G\sim H$ means G and H are congruent by a parallel transformation in $\mathbf{R}^4$.*

The above lemma provides a correspondence between the space $AM/\sim$ of algebraic minimal surfaces in $\mathbf{R}^4$ and the moduli space FD of triples of meromorphic functions on a compact Riemann surface.

We fix $\alpha, \beta\in\{0, 1, 2, \cdots\}\cup\{\infty\}$ and let l be the number of 0 or ∞ in $\{\alpha, \beta\}$. When α, β are finite, i.e. $(F, \varphi_1, \varphi_2)\in FD(M_g, \Omega, B_{k,r}, \alpha, \beta)$ are functions not identically zero, our results are

THEOREM 1.2. *If $FD(M_g, \Omega, B_{k,r}, \alpha, \beta)$ is nonempty, then it has the structure of a real analytic variety of real dimension at least* $2[(k+2\alpha+2\beta+5)-\{(7-l)g+r\}]$.

THEOREM 1.3. *If $FD(M_g, \Omega, B_{k,r}, \alpha, \beta)$ is nonempty, then it contains a subset which has the structure of a complex analytic variety of complex dimension at least* $(k+2\alpha+2\beta+7)-\{(11-l)g+3r\}$.

When α or $\beta=\infty$, i.e. φ_1 or $\varphi_2\equiv 0$, our minimal surfaces are considered as branched holomorphic curves in $\mathbf{C}^2$ which is identified with $\mathbf{R}^4$ in a certain manner (§4). In this case, we can construct the moduli space in a similar fashion as above. Let m be the number of $\infty\in\{\alpha, \beta\}$. Then, $1\leq m\leq l\leq 2$. For $\alpha\in\mathbf{Z}\cup\{\infty\}$, we define α' by $\alpha'=0$ if $\alpha=\infty$ and by $\alpha'=\alpha$ otherwise. Using the results of Micallef ([Mi1, Corollary 5.2] and [Mi2, Theorem]), we can prove

THEOREM 1.4. *If α or $\beta=\infty$, then the element of $FD(M_g, \Omega, B_{k,r}, \alpha, \beta)$ corresponds to a branched complete stable minimal surface in $\mathbf{R}^4$ of finite total curvature. If FD is nonempty, then it has the structure of a complex analytic variety of complex dimension at least* $\{k+2\alpha'+2\beta'+2(3-m)\}-\{(2-m)r+(9-l-2m)g\}$.

The author would like to express his gratitude to Professor Y. Ohnita for his encouragement and for his advice on Theorem 1.4. He also would like to thank Doctor M. Kokubu for his useful suggestion in the proof of Lemma 1.1.

2. A modified Chern-Osserman theorem.

In the theory of immersed algebraic minimal surfaces, the Chern-Osserman theorem [ChOs, Theorem 1] plays an important role. In this section, we shall modify it to apply to the theory of branched immersed algebraic minimal surfaces.

First, we shall define a singular Hermitian metric on a Riemann surface (cf. [Ya1], p. 141). Let M be a Riemann surface and U a coordinate neighborhood of M. We define a $(1, 0)$-form η of meromoprhic type on U as a form $\eta=z^{J_p}hdz$ for each $p\in U$, where z is a holomorphic coordinate with $z(p)=0$, h is a complex-valued smooth function with $h(p)\neq 0$, and J_p is an integer. We call the integer J_p the order of η at p and denote it by $\mathrm{ord}_p\eta$. If $J_p>0$ for each $p\in U$, we call η a $(1, 0)$-form of holomorphic type. We write $(\eta)=\sum_{p\in U}(\mathrm{ord}_p\eta)p$ and call it a divisor of η. We say that ds^2 is a singular Hermitian metric on M if it is given locally as $ds^2=\eta\cdot\bar{\eta}$, where $\eta\not\equiv 0$ is a $(1, 0)$-form of meromorphic type. We call ds^2 degenerate at p if $\mathrm{ord}_p\eta>0$, regular at p if $\mathrm{ord}_p\eta=0$, and divergent at p if $\mathrm{ord}_p\eta<0$. We note that ds^2 is a Hermitian metric on M if ds^2 is regular for any $p\in M$. We call $p\in M$ a singular point of ds^2 if ds^2 is degenerate or divergent at p. We define the singular divisor S of a singular Hermitian metric ds^2 as the divisor of η, i.e., $S=\sum_{p\in M}(\mathrm{ord}_p\eta)p$.

Next, we generalize the Gauss-Bonnet theorem. Let M be a Riemann surface, $ds^2=\eta\cdot\bar{\eta}$ a singular Hermitian metric on M with finitely many singular points, dA the area element of ds^2, and K the Gaussian curvature of ds^2. We denote by U an open subset of M such that its closure $\bar{U}$ is compact and that the boundary of $\bar{U}$ consists of finitely many smooth Jordan curves β_i $(i=1, \cdots, m)$ whose orientation is chosen as U

lies on the left-hand side. We define $k_{g,i}$ to be the geodesic curvature of β_i. We assume that there is no singular point on each β_i. Then we state a generalized local Gauss-Bonnet theorem as follows:

LEMMA 2.1. *Under the above situation, we have*

$$\int_U KdA = 2\pi(\chi(U) + \deg(S|_U)) - \sum_{i=1}^{m} \int_{\beta_i} k_{g,i} ds ,$$

where $\chi(U)$ is the Euler number of U.

PROOF. Let $\{q_1, \cdots, q_e\}$ be all the singular points of ds^2 contained in U. Since ds^2 is a singular Hermitian metric, we can write $\eta = z^{J_a} h_a(z) dz$ on a neighborhood of q_a $(a=1, \cdots, e)$, where z is a holomorphic coordinate around q_a with $z(q_a)=0$, $h_a(z)$ is a complex valued smooth function with $h(0) \neq 0$, and J_a is the order of η at q_a. We denote by $D(q_a, R)$ the set $\{|z| \leq R\}$. We choose a sufficiently small $R>0$ such that $D(q_i, R) \cap D(q_j, R) = \varnothing$ for $i \neq j$. Let $\mu_{q_a,R} = \partial D(q_a, R)$ where its orientation is chosen as $D(q_a, R)$ lies on the left-hand side. We denote by $k_{g,q_a,R}$ the geodesic curvature along $\mu_{q_a,R}$, and $U_R = U \setminus \bigcup_{a=1}^{e} D(q_a, R)$. Then, by the local Gauss-Bonnet theorem, we have

$$\int_{U_R} KdA = 2\pi\chi(U_R) - \sum_{j=1}^{m_i} \int_{\beta_i} k_{g,i} ds + \sum_{a=1}^{e} \int_{\mu_{q_a,R}} k_{g,q_a,R} ds .$$

We express $k_{g,q_a,R} ds$ explicitly. Let $ds^2 = \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2$ be the singular Hermitian metric, where $\{\theta^1, \theta^2\}$ is a oriented orthonormal frame. We define e_i to be the dual of θ^i $(i=1, 2)$. We denote by ω^i_j the Levi-Civita connection form satisfying $d\theta^i = -\omega^i_j \wedge \theta^j$, $\omega^i_j = -\omega^j_i$ $(i, j = 1, 2)$. Let γ be the curve in M such that $d\gamma/ds = \xi^1 e_1 + \xi^2 e_2$ where s is the arc-length parameter, and ν be the vector field normal to $d\gamma/ds$ expressed by $\nu = -\xi^2 e_1 + \xi^1 e_2$. We denote by k_g the geodesic curvature along γ. Then we have

$$k_g ds = [(d\xi^1 + \xi^2 \omega^1_2) e_1 + (d\xi^2 + \xi^1 \omega^2_1) e_2]|_\gamma \cdot \nu .$$

Introducing polar coordinates (r, t) to a neighborhood around q_a, we can express the singular Hermitian metric ds^2 in the form $ds^2 = r^{2J_a} |h_a|^2 (dr \otimes dr + r^2 dt \otimes dt)$. We assume $\theta^1 = r^{J_a} |h_a| dr$ and $\theta^2 = r^{J_a+1} |h_a| dt$. Then we have

$$e_1 = \frac{r^{-J_a}}{|h_a|} \frac{\partial}{\partial r} , \qquad e_2 = \frac{r^{-(J_a+1)}}{|h_a|} \frac{\partial}{\partial t} .$$

In terms of polar coordinates, we can express the curve μ in the form $\mu_{q_a,R} = (R, t)$, $t \in [0, 2\pi]$. Hence, $d\mu_{q_a,R}/ds = e_2$. Since

$$d\theta^1 = -\frac{1}{r} \frac{\partial \log|h_a|}{\partial t} dr \wedge \theta^2 , \qquad d\theta^2 = -\left((J_a + 1) + r \frac{\partial \log|h|}{\partial r} \right) dt \wedge \theta^1 ,$$

we obtain

$$\omega_2^1 = \frac{1}{r} \frac{\partial \log|h_a|}{\partial t} dr - \left((J_a+1) + r \frac{\partial \log|h_a|}{\partial r}\right) dt .$$

Thus,

$$k_{g,q_a,R} ds = \left((J_a+1) + R \frac{\partial \log|h_a|}{\partial r}\right) dt .$$

Hence,

$$\begin{aligned}\int_{U_R} K dA &= 2\pi\chi(U_R) - \sum_{i=1}^{m} \int_{\beta_i} k_{g,i} ds + \sum_{a=1}^{e} \int_0^{2\pi} \left((J_a+1) + R \frac{\partial \log|h_a|}{\partial r}\right) dt \\ &= 2\pi(\chi(U) + \deg(S|_U)) - \sum_{j=1}^{m} \int_{\beta_i} k_{g,i} ds + \sum_{a=1}^{e} R \int_0^{2\pi} \frac{\partial \log|h_a|}{\partial r} dt .\end{aligned}$$

Since $\partial \log|h_a|/\partial r$ is bounded on $D(q_a, R)$, we have

$$\lim_{R\to 0} R \int_0^{2\pi} \frac{\partial \log|h_a|}{\partial r} dt = 0 .$$

Thus, as R tends to 0, we obtain

$$\int_U K dA = 2\pi(\chi(U) + \deg(S|_U)) - \sum_{i=1}^{m} \int_{\beta_i} k_{g,i} ds . \qquad \square$$

Immediately, we also obtain

COROLLARY 2.2. *If M is a compact Riemann surface with a singular Hermitian metric ds^2, then we have*

$$\int_M K dA = 2\pi(\chi(M) + \deg(S)) .$$

The following lemma is an analogue of the theorem of Huber [Hu, Theorem 13] in the case where a singular Hermitian metric with finitely many degenerate point and no divergent point is equipped on a Riemann surface.

LEMMA 2.3. *Let M be an infinitely connected Riemann surface, $ds^2 = \eta \cdot \bar{\eta}$ a singular Hermitian metric on M with finitely many degenerate points and no divergent point. If ds^2 is complete, then*

$$\int_M K^- dA = +\infty ,$$

where $K^- = \max\{0, -K\}$.

PROOF. We prove that if $\int_M K^- < +\infty$, then ds^2 is not complete.

We denote by $\{U_i\}$ an exhaustion of M, i.e., a sequence of open subsets of M such that $U_i \subset U_j$ for $i<j$, that the closure $\bar{U}_i$ of each U_i is compact, that the boundary of $\bar{U}_i$ consists of finitely many smooth Jordan curves $\beta_{i,j}$ ($j=1, \cdots, m_i$), and that $\bigcup_{i=1}^{\infty} U_i = M$. We choose the orientation of each β_{ij} as U_i lies on the left-hand side. Let $M \setminus U_i = \coprod_j^{m_i} \Omega_{ij}$, where $\partial \Omega_{ij} = \beta_{ij}$. We assume that all the singular points $\{b_a\}$ ($a=1, \cdots, e$) of ds^2 on M are contained in the U_1. By Lemma 2.1, we have

$$\int_{U_i} KdA = 2\pi(\chi(U_i) + \deg(S)) - \sum_{j=1}^{m_i} \int_{\beta_{ij}} k_{g,ij} ds .$$

Hence,

$$-\int_{U_i} KdA + 2\pi(\chi(U_i) + \deg(S)) = \sum_{j=1}^{m_i} \int_{\beta_{ij}} k_{g,ij} ds . \tag{2.1}$$

As i tends to ∞, the left-hand side of (2.1) tends to $-\infty$. Therefore, for sufficiently large I,

$$\sum_{j=1}^{m_I} \int_{\beta_{Ij}} k_{g,Ij} ds < -2 \int_M K^- dA .$$

Hence, there exists J, $\varepsilon > 0$ such that

$$\int_{\beta_{IJ}} k_{g,IJ} ds = -2 \left\{ \int_{\Omega_{IJ}} K^- dA + \varepsilon \right\} .$$

We can choose a Jordan curve δ in Ω_{IJ} homotopic to β_{IJ} and satisfying $\int_{(\beta_{IJ}, \delta)} K^+ dA < \varepsilon$, where $K^+ = \max\{0, K\}$, (β_{IJ}, δ) is a domain surrounded by β_{IJ} and δ. The following two lemmas are proved in [Hu, p. 62, Lemma 6] and [Hu, p. 23, Lemma 2]:

LEMMA 2.4. *Under the above situation, there exists a number $C>0$ which satisfies the following property*:

> *For any integer i, there exists a rectifiable curve α_i: $[0, 1) \to M$ such that $\alpha_i(0) \in \delta$, $\lim_{t\to 1} \alpha_i(t) \in \partial U_i$, $\int_{\alpha_i} ds < C$.*

LEMMA 2.5. *We denote by Ω a doubly connected region in S^2. Let Γ, γ denote the two boundaries of Ω, and Ω_0 the simply connected open set containing γ and surrounded by Γ. Assume that there exist a sequence of rectifiable curves $\{\sigma_n\}$, σ_n: $[0, 1) \to \Omega$, a compact subset $K \subset \Omega_0$ and a number $C>0$ such that they satisfy the following conditions*:

> *For each σ_n, $\mathrm{Im}\{\sigma_n\} \cap K \neq \emptyset$;*
> *For any compact subset $L \subset \Omega_0$, $\bigcup_n \mathrm{Im}\{\sigma_n\}$ is not contained in L;*
> *$\int_{\sigma_n} ds < C$ for all n.*

Then there exists a locally rectifiable divergent path σ in Ω such that $\int_\sigma ds < +\infty$ and that $\lim_{t\to 1} \sigma(t) \in \Gamma$.

By Lemma 2.4, we obtain a sequence of curves in Ω_{IJ}, a compact set δ and a number $C>0$ satisfying the assumption of Lemma 2.5. Hence, there is a locally rectifiable divergent path $\sigma\colon [0, 1) \to M$ such that $\int_\sigma ds < +\infty$. Therefore, ds^2 is not complete. $\square$

Now, we can modify the Chern-Osserman theorem as follows:

PROPOSITION 2.6. *Let $f\colon M \to \mathbf{R}^n$ be a branched conformal minimal immersion such that the singular Riemannian metric ds^2 induced by f is complete. Then, the total curvature is finite if and only if the Gauss map Φ_f is algebraic.*

PROOF. First, we observe that we can extend Φ_f over all branch points. Indeed, a branch point b is locally a common zero point of holomorphic functions $p \mapsto (\partial f^a/\partial z)(p)$ $(a=1, \cdots, n)$. Hence there exists the minimum of orders of their functions at a branch point b, which we denote by k. We define

$$\Phi_f(b) = \left[\frac{1}{z^k}\frac{\partial f^1}{\partial z}(b), \frac{1}{z^k}\frac{\partial f^2}{\partial z}(b), \cdots, \frac{1}{z^k}\frac{\partial f^n}{\partial z}(b)\right].$$

Then Φ_f becomes holomorphic at b. We also observe that ds^2 is a singular Hermitian metric on M with no divergent point in this case. Indeed, we have locally

$$ds^2 = 2\sum_{a=1}^{n}\left|\frac{\partial f^a}{\partial z}\right|^2 dz \cdot d\bar{z}.$$

Since $\partial f^a/\partial z$ $(a=1, \cdots, n)$ is holomorphic, we have $\partial f^a/\partial z = z^{u_a}h_a(z)$ $(a=1, \cdots, n)$, where u_a is a nonnegative integer and h_a is a holomorphic function not equal to 0 at 0. Thus, $ds^2 = |z|^{2u}h(z)dz \cdot d\bar{z}$, where $u = \min\{u_a \mid a=1, \cdots, n\}$ is a nonnegative integer and h is a local real-valued positive smooth function. When we set $\eta = z^u\sqrt{h(z)}dz$, we see that $ds^2 = \eta \cdot \bar{\eta}$ is a singular Hermitian metric with no divergent point.

We assume that the total curvature is finite. Then M is finitely connected by Lemma 2.3. Then, in the same way as the proof of the Chern-Osserman theorem ([ChOs], Theorem 1), we can prove that M is biholomorphic to a compact Riemann surface M_g punctured at finite points and that Φ_f is extended to be holomorphic at all puncture points. Thus Φ_f is algebraic.

Conversely, we assume that Φ_f is algebraic. Let M_g be the compact Riemann surface on which Φ_f is extended to a holomorphic map, $\{b_1, \cdots, b_k\}$ the branch points, $\{p_1, \cdots, p_r\}$ the puncture points. Then ds^2 is a singular Hermitian metric on M_g degenerate at b_j $(j=1, \cdots, k)$ and divergent at p_i $(i=1, \cdots, r)$. By Corollary 2.2, we have

$$\int_{M_g} KdA = 2\pi(\chi(M_g) + \deg(S)).$$

Since both $\chi(M_g)$ and $\deg(S)$ are finite, the total curvature is finite. $\square$

3. Representation formula.

We shall prove Lemma 1.1. Let $CD = CD(M_g, B_{k,r}, \alpha, \beta)$ be the set of all the quadruplets $(\zeta^1, \zeta^2, \zeta^3, \zeta^4)$ of meromorphic 1-forms on M_g satisfying the following conditions:

$$\zeta^1 - \sqrt{-1}\zeta^2 \not\equiv 0; \qquad \sum_{a=1}^{4} \zeta^a \otimes \zeta^a = 0;$$

$$\deg\left(\frac{\zeta^3 + \sqrt{-1}\zeta^4}{\zeta^1 - \sqrt{-1}\zeta^2}\right)_\infty = \alpha, \qquad \deg\left(\frac{-\zeta^3 + \sqrt{-1}\zeta^4}{\zeta^1 - \sqrt{-1}\zeta^2}\right)_\infty = \beta;$$

$$(\zeta) = \sum_{j=1}^{k} J_j b_j - \sum_{i=1}^{r} I_i p_i;$$

$$\Re\left\{\int_\gamma \zeta^a\right\} = 0,$$

for each $\gamma \in H_1(M_g - \{p_1, \cdots, p_r\})$ and each a $(a = 1, \cdots, 4)$, where for $\zeta^a \not\equiv 0$, let $(\zeta) = \sum_{p \in M_g} \min_a (\mathrm{ord}_p \zeta^a) p$. Then, by the relations

$$\zeta^a(f^1, f^2, f^3, f^4) = \frac{\partial f^a}{\partial z} dz; \qquad f^a(\zeta^1, \zeta^2, \zeta^3, \zeta^4)(z) = \Re\left\{\int^z \zeta^a dz\right\}$$

$(a = 1, \cdots, 4)$, we can define a bijective correspondence between $AM(M_g, B_{k,r})/\sim$ and $\coprod_{\alpha,\beta} CD(M_g, B_{k,r}, \alpha, \beta)$. Indeed, it is clear that an element of $AM/\sim$ corresponds to an element of some CD, and an element of CD corresponds to a minimal surface branched at k points with orders J_j and punctured at r points with orders I_i. Let (f^1, f^2, f^3, f^4) be a minimal surface corresponding to an element of CD. For a puncture point p of order I, we take a local holomorphic coordinate z such that $z(p) = 0$. Then the singular Hermitian metric ds^2 induced by f becomes as follows:

$$ds^2 = \frac{h(z)}{|z|^{2I}} dz \cdot d\bar{z},$$

where $h(z)$ is a positive smooth function. Let $\sigma(t) = x(t) + \sqrt{-1}y(t)$ be a smooth locally rectifiable curve tending to p as t tends to ∞. Then, we have

$$\left\|\frac{d\sigma}{dt}\right\|^2 = h(\sigma(t)) \cdot \frac{(dx/dt)^2 + (dy/dt)^2}{(x(t)^2 + y(t)^2)^I}.$$

Hence, $\|d\sigma/dt\|$ tends to ∞ as t tends to ∞. Thus σ has infinite length, and we see that the induced metric is complete. Therefore, (f^1, f^2, f^3, f^4) gives an element of AM. Hence, $AM/\sim$ is in one-to-one correspondence with $\coprod CD$ through the relation above.

On the other hand, there is the relations defined by

$$
(3.1)\quad
\begin{aligned}
&F(\zeta^1, \zeta^2, \zeta^3, \zeta^4) = \frac{\zeta^1 - \sqrt{-1}\zeta^2}{\Omega}, \\
&\varphi_1(\zeta^1, \zeta^2, \zeta^3, \zeta^4) = \frac{\zeta^3 + \sqrt{-1}\zeta^4}{\zeta^1 - \sqrt{-1}\zeta^2}, \\
&\varphi_2(\zeta^1, \zeta^2, \zeta^3, \zeta^4) = \frac{-\zeta^3 + \sqrt{-1}\zeta^4}{\zeta^1 - \sqrt{-1}\zeta^2}; \\
&\zeta^a(F, \varphi_1, \varphi_2) = \frac{1}{2} E_a F\Omega \qquad (a = 1, \cdots, 4).
\end{aligned}
$$

Using (3.1), we can define a bijective correspondnece between $CD(M_g, B_{k,r}, \alpha, \beta)$ and $FD(M_g, \Omega, B_{k,r}, \alpha, \beta)$ (cf. [Os, Section 4] or [HoOs, §3, Remark 4]). Then, for each $(\zeta^1, \zeta^2, \zeta^3, \zeta^4) \in CD$ and its corresponding $(F, \varphi_1, \varphi_2) \in FD$, we have

$$
(3.2)\qquad (\zeta) = -(\varphi_1)_\infty - (\varphi_2)_\infty + (F) + (\Omega).
$$

We finished proving Lemma 1.1.

4. Proof of the theorems.

We shall prove Theorem 1.2, Theorem 1.3, and Theorem 1.4. We fix M_g, Ω, k, r, $B_{k,r}$, α, and β as above. We denote by $\mathrm{Div}_+^d(M_g)$ the space of effective divisors of degree d on M_g. We observe that $\mathscr{D} = \mathscr{D}(M_g, B_{k,r}, \alpha, \beta) = \mathrm{Div}_+^J(M_g) \times \mathrm{Div}_+^I(M_g) \times \mathrm{Div}_+^{\alpha'}(M_g) \times \mathrm{Div}_+^{\alpha'}(M_g) \times \mathrm{Div}_+^{\beta'}(M_g) \times \mathrm{Div}_+^{\beta'}(M_g)$, where $J = \sum_{j=1}^k J_j$ and $I = \sum_{i=1}^r I_i$, has the structure of a compact complex manifold of dimension $J + I + 2\alpha' + 2\beta'$ (cf. [GrHa, p. 236]). Let $L = L(M_g, B_{k,r}, \alpha, \beta)$ be the open subset of $M_g \times \cdots \times M_g$ ($k + r + 2\alpha' + 2\beta'$ times) consisting of the elements $(b_j; p_i; s_\delta; t_\delta; x_\varepsilon; y_\varepsilon)$ such that $\{b_j; p_i\}$ are distinct points and that $\{s_\delta\} \cap \{t_\delta\} = \{x_\varepsilon\} \cap \{y_\varepsilon\} = \varnothing$. We will see that $\{s_\delta; t_\delta\}(\{x_\varepsilon; y_\varepsilon\}$, respectively) corresponds to the support of the divisor of φ_1 (φ_2, respectively). Let $DAD'(M_g, \Omega, B_{k,r}, \alpha, \beta)$ be the set of $\tilde{D}$'s defined by

$$
\tilde{D} = \begin{cases}
(D_1, \mathbf{0}, \mathbf{0}) & \text{if } \alpha' = \beta' = 0 \\
(D_1, D_2, \mathbf{0}), & \text{if } \alpha' \neq 0 \text{ and } \beta' = 0 \\
(D_1, \mathbf{0}, D_3), & \text{if } \alpha' = 0 \text{ and } \beta' \neq 0 \\
(D_1, D_2, D_3), & \text{otherwise},
\end{cases}
$$

where D_1, D_2, and D_3 are divisors on M_g satisfying the following conditions:

$$
D_1 = \sum_{j=1}^k J_j b_j - \sum_{i=1}^r I_i p_i + \sum_{\delta=1}^{\alpha'} t_\delta + \sum_{\varepsilon=1}^{\beta'} y_\varepsilon - (\Omega),
$$

$$D_2=\sum_{\delta=1}^{\alpha'} s_\delta-\sum_{\delta=1}^{\alpha'} t_\delta\,,\qquad D_3=\sum_{\varepsilon=1}^{\beta'} x_\varepsilon-\sum_{\varepsilon=1}^{\beta'} y_\varepsilon\,,$$

for $(b_j; p_i; s_\delta; t_\delta; x_\varepsilon; y_\varepsilon)\in L$. When $(\zeta^1, \cdots, \zeta^4)\in CD$ and $(F, \varphi_1, \varphi_2)\in FD$ are the elements corresponding to each other such that $(\varphi_1)=D_2$ and that $(\varphi_2)=D_3$, we have

$$(F)=D_1\,, \tag{4.1}$$

by (3.1) and (3.2). We will prove the following lemma:

LEMMA 4.1. *The set DAD' has the structure of a complex analytic subvariety of $\mathscr{D}$ with the complex dimension $k+r+2\alpha'+2\beta'$.*

PROOF. Let $\mathscr{C}=\mathscr{C}(M_g, B_{k,r}, \alpha, \beta)$ be the subset of $\mathscr{D}$ consisting of the elements

$$\left(\sum_{j=1}^{k} J_j b_j,\ \sum_{i=1}^{r} I_i p_i,\ \sum_{\delta=1}^{\alpha'} s_\delta,\ \sum_{\delta=1}^{\alpha'} t_\delta,\ \sum_{\varepsilon=1}^{\beta'} x_\varepsilon,\ \sum_{\varepsilon=1}^{\beta'} y_\varepsilon\right)$$

such that $(b_j; p_i; s_\delta; t_\delta; x_\varepsilon; y_\varepsilon)\in L$. Then $\mathscr{C}$ is an analytic subvariety of $\mathscr{D}$ and

$$\dim_{\mathbf{C}}\mathscr{C}=k+r+2\alpha'+2\beta'\,.$$

Clearly, we can define a bijective correspondence between $\mathscr{C}$ and DAD'. We have thus proved Lemma 4.1. □

Let $DAD(M_g, \Omega, B_{k,r}, \alpha, \beta)$ be the subset of DAD' such that each element consists of principal divisors on M_g.

LEMMA 4.2. *The set DAD is a complex analytic subvariety of DAD'. If DAD is nonempty, then*

$$\dim_{\mathbf{C}} DAD\geq k+r+2\alpha'+2\beta'-(3-l)g\,.$$

PROOF. Let $J(M_g)$ be the Jacobian variety of M_g and $u\colon \mathrm{Div}(M_g)\to J(M_g)$ the Jacobi map. We define $\tilde{u}\colon DAD\to J(M_g)^3$ by

$$\tilde{u}(\tilde{D})=\begin{cases}(u(D_1), 0, 0)\,, & \text{if}\ \ \alpha'=\beta'=0\\ (u(D_1), u(D_2), 0)\,, & \text{if}\ \ \alpha'\neq 0 \text{ and } \beta'=0\\ (u(D_1), 0, u(D_3))\,, & \text{if}\ \ \alpha'=0 \text{ and } \beta'\neq 0\\ (u(D_1), u(D_2), u(D_3))\,, & \text{otherwise}\,.\end{cases}$$

We note that $\deg D_1=\deg D_2=\deg D_3=0$. Indeed, by (4.1), $\deg D_1=0$. $\deg D_2=\deg D_3=0$ is clear. By Abel's theorem (see [GrHa], p. 225), $D\in \mathrm{Div}^0(M_g)$ is a principal divisor if and only if $u(D)=0$. Thus, $DAD=\tilde{u}^{-1}(0, 0, 0)$. Since $\tilde{u}$ is holomorphic with respect to the complex structure induced as above, DAD is a complex analytic subvariety of DAD'. By the definition of l, we also have

$$\begin{aligned}\dim_{\mathbf{C}} DAD &\geq \dim_{\mathbf{C}} DAD' - \dim_{\mathbf{C}} (J(M_g))^{3-l} \\ &= k + r + 2\alpha' + 2\beta' - (3-l)g\,. \qquad \square\end{aligned}$$

We assume $\alpha' \neq 0$ and $\beta' \neq 0$. Other cases are similar. Let $AD(M_g, \Omega, B_{k,r}, \alpha, \beta)$ be the set of triples $(F, \varphi_1, \varphi_2)$ of meromorphic functions on M_g such that $((F), (\varphi_1), (\varphi_2)) \in DAD$, $\deg(\varphi_1)_\infty = \alpha$, and $\deg(\varphi_2)_\infty = \beta$. We define $\eta: AD \to DAD$ by the projection $\eta(F, \varphi_1, \varphi_2) = ((F), (\varphi_1), (\varphi_2))$ and we set $V = DAD - \{\text{all singular points}\}$.

LEMMA 4.3. *The set AD has the structure of a complex analytic variety, and then $\eta: \eta^{-1}(V) \to V$ becomes a holomorphic principal $(\mathbf{C}^*)^{3-m}$ bundle. If AD is nonempty, then*

$$\dim_{\mathbf{C}} AD \geq k + r + 2\alpha' + 2\beta' - (3-l)g + (3-m)\,.$$

PROOF. Assume $(F, \varphi_1, \varphi_2) \in AD$. Then $((F), (\varphi_1), (\varphi_2)) = ((w_1 \cdot F), (w_2 \cdot \varphi_1), (w_3 \cdot \varphi_2))$ for any $(w_1, w_2, w_3) \in (\mathbf{C}^*)^3$. Hence $(\mathbf{C}^*)^3$ acts on AD. Moreover, we can easily see that $(\mathbf{C}^*)^3$ acts on $\eta^{-1}(V)$.

To simplify the proof, we prove the claim for only one of the factors corresponding to the functions not vanishing identically. We locally induce a complex structure from DAD and prove that this complex structure is globally defined on AD.

First, we assume that $g \geq 1$. Let ϑ be the Riemann theta function, and $D = \sum_{i=1}^d b_i - \sum_{i=1}^d p_i$ a divisor of M_g with $u(D) = 0$. The following lemma is proved in [Mu, Chapter 2, §3].

LEMMA 4.4. *There exists a constant Δ in $\mathbf{C}^g$ depending only on the choice of the normalized basis for the space of holomorphic 1-forms on M_g and satisfying the following conditions:*

For a point $v = (v_1, \cdots, v_{g-1})$ in $(M_g)^{g-1}$ with $\{b_1, \cdots, b_d, p_1, \cdots, p_d\} \cap \{v_1, \cdots, v_{g-1}\} = \varnothing$, the mapping $h_v: V \times M_g \to \mathbf{C} \cup \{\infty\} \cong \mathbf{C}P^1$ defined by

$$h_v(D)(z) = \frac{\prod_{i=1}^d \vartheta(\Delta - \sum_{j=1}^{g-1} u(v_j) + u(z) - u(b_i))}{\prod_{i=1}^d \vartheta(\Delta - \sum_{j=1}^{g-1} u(v_j) + u(z) - u(p_i))}$$

is a meromorphic function on M_g such that $(h_v(D)) = D$.

We define $h_v(\mathbf{0}) = 1$. We fix such a v for each divisor B in V and denote it by v_B. Then $h_{v_B}(D)(z)$ is locally a holomorphic function with respect to D. Assume that U_B is a sufficiently small neighborhood of B in V. Then $(h_{v_B}(D)) = D$ for D in U_B.

We define

$$\tau_{U_B}: \eta^{-1}(U_B) \to U_B \times \mathbf{C}^*\,, \qquad f \mapsto \left((f), \frac{f}{h_{v_B}((f))}\right).$$

Then this is a bijective map between $\eta^{-1}(U_B)$ and $U_B \times \mathbf{C}^*$. Hence we can give $\eta^{-1}(U_B)$ a complex structure $c(v_B)$. If H is another divisor and $U_B \cap U_H \neq \varnothing$, then $U_B \cap U_H$ has two complex structures $c(v_B)$ and $c(v_H)$. But

$$\tau_{U_B} \circ \tau_{U_H}^{-1}(D, w) = (D, (g_{U_B,U_H}(D)) \cdot w), \qquad g_{U_B,U_H} = h_{v_H}/h_{v_B}$$

for each $D \in U_B \cap U_H$ and $w \in \mathbf{C}^*$, and g_{U_B,U_H} is holomorphic with respect to D. Hence the two complex structures are compatible. In the same fashion as above, this complex structure is independent of the choice of $\{v_B\}$. Therefore, we can induce the complex structure c to $\eta^{-1}(V)$, where $\eta: (\eta^{-1}(V), c) \to V$ and $\tau_{U_B}: (\eta^{-1}(U_B), c) \to U_B \times \mathbf{C}^*$ are holomorphic and the following becomes a commutative diagram.

$$\begin{tikzcd}
\eta^{-1}(U_B) \arrow[r, "\tau_{U_B}"] \arrow[d, "\eta"'] & U_B \times \mathbf{C}^* \arrow[d, "\text{projection}"] \\
U_B \arrow[r, "\text{id}"'] & U_B
\end{tikzcd}$$

We also have

$$\tau_{U_B}^{-1}(D, w_1 w_2) = w_1 \cdot w_2 \cdot h_{v_B} = \tau_{U_B}^{-1}(D, w_1) \cdot w_2 .$$

Hence, we can give $\eta: (\eta^{-1}(V), c) \to V$ a structure of a holomorphic principal $\mathbf{C}^*$ bundle. If B' is a singular point of DAD, then $B' \times \mathbf{C}^*$ is a singular locus of $U_{B'} \times \mathbf{C}^*$. Thus we can give AD the structure of a complex analytic variety. Since the number of components AD is $3-m$, we have

$$\dim_{\mathbf{C}} AD \geq k + r + 2\alpha' + 2\beta' - (3-l)g + (3-m) .$$

In the case where $g=0$, we can prove the lemma in a similar fashion as above only by taking $\prod_{i=1}^{d}(z-b_i)/\prod_{i=1}^{d}(z-p_i)$ instead of h_{v_B}. □

Now, we shall prove our theorems. We note that $FD(M_g, \Omega, B_{k,r}, \alpha, \beta)$ consists of all the triples of meromorphic functions in AD satisfying the period condition.

PROOF OF THEOREM 1.2. We note that $m=0, \alpha'=\alpha, \beta'=\beta$ in this case. We fix $(F_0, \varphi_{10}, \varphi_{20}) \in FD$ and denote $-(\varphi_{10})_\infty - (\varphi_{20})_\infty + (F_0) + (\Omega) = \sum J_j b_{j0} - \sum I_i p_{i0}$. Let $\Gamma := \{\gamma_1, \cdots, \gamma_{2g}; \gamma_{2g+1}, \cdots, \gamma_{2g+r-1}\}$ be a basis for $H_1(M_g - \{p_{10}, \cdots, p_{r0}\})$ such that $\{\gamma_1, \cdots, \gamma_{2g}\}$ is a basis for $H_1(M_g)$ and that γ_{2g+i} is a simple closed curve around p_{i0} $(i=1, \cdots, r-1)$. We denote by W_0 a neighborhood of $(F_0, \varphi_{10}, \varphi_{20})$ in AD such that for $(F, \varphi_1, \varphi_2) \in W_0$, Γ is still a basis for $H_1(M_g - \{p_1, \cdots, p_r\})$ where p_i are puncture points of $(F, \varphi_1, \varphi_2)$. We define holomorphic functions $\lambda_i^a: W_0 \to \mathbf{C}$ $(i=1, \cdots, 2g+r-1, a=1, \cdots, 4)$ as

$$\lambda_i^a(F, \varphi_1, \varphi_2) = \int_{\gamma_i} \zeta^a(F, \varphi_1, \varphi_2) .$$

Then,

$$FD \cap W_0 = \bigcap_{i,a} (\Re\{\lambda_i^a\})^{-1}(0) .$$

Hence, FD is a real analytic subvariety of AD and

$$\begin{aligned}\dim_{\mathbf{R}} FD &\geq 2\{k+r+2\alpha'+2\beta'-(3-l)g+3\}-4(r+2g-1)\\ &=2[(k+2\alpha+2\beta+5)-\{(7-l)g+r\}]\,.\end{aligned}$$ □

PROOF OF THEOREM 1.3. We pay attention to the elements of W_0 whose periods are equal to the $(F_0, \varphi_{10}, \varphi_{20})$'s. Since

$$FD \cap W_0 \supset \bigcap_{i,a} (\lambda_i^a)^{-1}\left(\int_{\gamma_i} \zeta^a(F_0, \varphi_{10}, \varphi_{20})\right) \neq \varnothing\,,$$

we have that $FD \cap W_0$ contains a complex analytic subvariety of AD and

$$\begin{aligned}\dim_{\mathbf{C}} FD &\geq \dim_{\mathbf{C}} FD \cap W_0\\ &\geq \{k+r+2\alpha'+2\beta'-(3-l)g+3\}-4(r+2g-1)\\ &=(k+2\alpha+2\beta+7)-\{(11-l)g+3r\}\end{aligned}$$ □

PROOF OF THEOREM 1.4. We may assume that $\varphi_2 \equiv 0$. Then, $E_1=1$, $E_2=\sqrt{-1}$, $E_3=\varphi_1$, and $E_4=-\sqrt{-1}\varphi_1$. Hence, the period condition becomes as follows:

$$\int_\gamma F\Omega=0\,, \quad \int_\gamma \varphi_1 F\Omega=0 \qquad \text{for any } \gamma \in H_1(M_g-\{\text{puncture points}\})\,.$$

Since

$$FD \cap W_0 = \bigcap_{i,a} (\lambda_i^a)^{-1}(0)\,,$$

we know that FD is a complex analytic subvariety of AD and since the number of λ_i^a's not vanishing identically is at least $(3-m)(r+2g-1)$, we obtain

$$\begin{aligned}\dim_{\mathbf{C}} FD &\geq k+r+2\alpha'+2\beta'-(3-l)g+(3-m)-(3-m)(r+2g-1)\\ &=\{k+2\alpha'+2\beta'+2(3-m)\}-\{(2-m)r+(9-l-2m)g\}\,.\end{aligned}$$

For the corresponding $(f_1, f_2, f_3, f_4) \in AM$, we see that $f^1-\sqrt{-1}f^2$ and $f^3+\sqrt{-1}f^4$ are holomorphic functions on $M_g-\{\text{puncture points}\}$. Hence $(F, \varphi_1, \varphi_2)$ corresponds to a branched complete holomorphic surface in $\mathbf{R}^4$ of finite total curvature via identification $\mathbf{R}^4$ and $\mathbf{C}^2$ by $(x_1, x_2, x_3, x_4) \sim (x_1-\sqrt{-1}x_2, x_3+\sqrt{-1}x_4)$. It is known that such a surface is a stable minimal surface (cf. [La, Chapter I, §7, Corollary 28]). Micallef showed that any branched complete stable minimal surface of finite total curvature in $\mathbf{R}^4$ is congruent to such a surface by an isometry of $\mathbf{R}^4$ (see [Mi1, Corollary 5.2] and [Mi2, Theorem]). Hence, we obtain Theorem 1.4. □

References

[ChOs] S. Chern and R. Osserman, Complete minimal surfaces in Euclidean n-space, J. Analyse Math. **19** (1967), 15–34.

[GrHa] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley (1978).

[HoOs] D. Hoffman and R. Osserman, *The Geometry of the Generalized Gauss Map*, Mem. Amer. Math. Soc. **236** (1980).

[Hu] A. Huber, On subharmonic functions and differential geometry in the large, Comment. Math. Helv. **32** (1957), 181–206.

[La] H. B. Lawson, *Lectures on Minimal Submanifolds Vol. I*, Math. Lecture Ser. **9**, Publish or Perish (1980).

[Mi1] M. J. Micallef, Stable minimal surfaces in Euclidean space, J. Differential Geom. **19** (1984), 57–84.

[Mi2] M. J. Micallef, A note on branched stable two-dimensional minimal surfaces, Proc. Centre Math. Anal. Austral. Nat. Univ. (L. Simon and N. S. Trudinger, ed.), *Miniconference on Geometry and Partial Differential Equations* (*Canberra, August 1–3, 1985*), *vol. 10*, Austral. Nat. Univ. (1985), 157–162.

[Mu] D. Mumford, *Tata Lectures on Theta. I*, Progr. Math. **28**, Birkhäuser (1982).

[Os] R. Osserman, *Global properties of minimal surfaces in E^3 and E^n*, Ann. of Math. **80** (1964), 340–364.

[Ya1] K. Yang, *Compact Riemann Surfaces and Algebraic Curves*, Ser. Pure. Math. **10**, World Scientific (1988).

[Ya2] K. Yang, *Complete Minimal Surfaces of Finite Total Curvature*, Math. Appl. **294** (1994), Kluwer Academic Publishers.

Present Address:
Department of Mathematics, Tokyo Metropolitan University,
Minami-Ohsawa, Hachioji-shi, Tokyo, 192–0397, Japan.
e-mail: kmoriya@math.metro-u.ac.jp