

## Isometric Shift Operators on the Disc Algebra

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### Introduction.

The purpose of this note is to study linear isometries on function algebras, especially isometric shift operators on the disc algebra. For a compact Hausdorff space  $X$ , we denote by  $C(X)$  the Banach space of all complex-valued continuous functions on  $X$ . Recently, A. Gutek, D. Hart, J. Jamison and M. Rajagopalan [5] and F. O. Farid and K. Varadarajan [3] have obtained many significant results concerning isometric shift operators on Banach spaces, especially on  $C(X)$ . Here we investigate linear isometries on function algebras and isometric shift operators on the disc algebra.

In section 1, we give a representation of a codimension 1 linear isometry on a function algebra and in section 2, on the disc algebra  $A$ , we establish the form of a codimension 1 linear isometry  $\varphi$  and give equivalent conditions under which  $\varphi$  is a shift operator.

### 1. Codimension 1 linear isometries on function algebras.

Let  $E$  be a Banach space and  $\varphi$  a linear isometry from  $E$  into  $E$ . Then we call  $\varphi$  a *codimension 1 linear isometry* on  $E$  if the range of  $\varphi$  has codimension 1. A bounded linear operator  $\varphi$  on  $E$  is called a *shift operator* on  $E$  if the following conditions are satisfied: (i)  $\varphi$  is injective; (ii) the range of  $\varphi$  has codimension 1; and (iii)  $\bigcap_{n=1}^{\infty} \varphi^n(E) = \{0\}$ . A linear isometry on  $E$  which is a shift operator is an *isometric shift operator* on  $E$ .

Let  $X$  be a compact Hausdorff space. We say that  $A$  is a *function algebra* on  $X$  if it is a closed subalgebra of  $C(X)$ , the Banach algebra of all complex-valued continuous functions on  $X$  with the supremum norm, which separates points in  $X$  and contains the constants. After now, we consider codimension 1 linear isometries on function algebras and isometric shift operators on the disc algebra.

The following extends a theorem of Gutek, Hart, Jamison and Rajagopalan [5, Theorem 2.1] to the case of the function algebras (cf. [9]).

**THEOREM 1.1.** *Let  $A$  be a function algebra on a compact Hausdorff space  $X$ . Suppose*

the Choquet boundary  $\text{Ch}(A)$  for  $A$  is  $X$ . Let  $\varphi$  be a codimension 1 linear isometry on  $A$ . Then there is a closed subset  $F$  of  $X$ , where either,

- (i)  $F = X \setminus \{p\}$ , where  $p$  is an isolated point of  $X$  or
- (ii)  $F = X$

such that there is a continuous map  $\tau$  from  $F$  onto  $X$  and a unimodular function  $u \in C(F)$  such that  $(\varphi f)(x) = u(x)f(\tau(x))$  for any  $f \in A$  and  $x \in F$ .

**PROOF.** Let  $B = \varphi(A)$ . Then by Novinger [7], there are a continuous map  $\tau$  from  $\text{Ch}(B)$  onto  $\text{Ch}(A)$  and a unimodular continuous function  $u$  on  $\text{Ch}(B)$  such that  $(\varphi f)(x) = u(x)f(\tau(x))$  for  $f \in A$  and  $x \in \text{Ch}(B)$ . Since  $\text{Ch}(A)$  is closed in  $X$  by the hypothesis,  $\text{Ch}(B)$  is also closed in  $X$  [7, Corollary 2]. We here assert that  $X \setminus F$  consists of at most one point if we put  $F = \text{Ch}(B)$ . Otherwise,  $X \setminus F$  contains two distinct points  $p, q$ . Since  $p, q \in X = \text{Ch}(A)$  and  $A$  is a function algebra, there are  $f, g \in A$  such that  $\|f\| = \|g\| = 1$ ,  $f(p) = 1$ ,  $|f(x)| \leq 1/4$  ( $x \in F \cup \{q\}$ ) and  $g(q) = 1$ ,  $|g(x)| \leq 1/4$  ( $x \in F \cup \{p\}$ ) (see [1]). We here show that if  $af + bg \in B = \varphi(A)$  ( $a, b \in \mathbb{C}$ ) then  $a = b = 0$ . Since  $h = af + bg \in B$ , there is a  $k \in A$  such that  $h = \varphi k$ . So  $af(x) + bg(x) = (\varphi k)(x) = u(x)k(\tau(x))$  ( $x \in F$ ). Hence

$$|k(\tau(x))| \leq |a||f(x)| + |b||g(x)| \leq \frac{1}{4}(|a| + |b|) \quad x \in F.$$

Since  $\tau$  is surjective, we have

$$(1) \quad \|h\| = \|\varphi k\| = \|k\| \leq \frac{1}{4}(|a| + |b|).$$

It implies the following since  $h(p) = a + bg(p)$  and  $h(q) = af(q) + b$ .

$$(2) \quad |a + bg(p)| \leq \frac{1}{4}(|a| + |b|), \quad |af(q) + b| \leq \frac{1}{4}(|a| + |b|).$$

Since  $|g(p)| \leq 1/4$ , by the first part of (2) we have  $3|a| \leq 2|b|$ . Similarly, by the latter of (2) we have  $3|b| \leq 2|a|$  and  $a = b = 0$ . Thus, the codimension of  $\varphi$  is at least two. This contradiction tells us that  $X \setminus F$  has at most one point.

The following lemma was shown in the case of  $C(X)$  in [5, Lemma 2.2], but we observe that this holds true in the case of function algebras.

**LEMMA 1.2.** *Let  $A$  be a function algebra on a compact Hausdorff space  $X$  and suppose that  $\text{Ch}(A) = X$ . Let  $\varphi$  be a codimension 1 linear isometry on  $A$  and let  $F, \tau$  and  $u$  be as in Theorem 1.1. Then  $\tau^{-1}(x)$  has at most two elements for any  $x \in X$ . Furthermore, if  $\tau^{-1}(x_0)$  has two elements for some  $x_0 \in X$ , then  $\tau^{-1}(x)$  is a singleton for any  $x \in X \setminus \{x_0\}$ .*

Let  $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$ . If  $A$  is a function algebra on  $\Gamma$ , we can have the following

**THEOREM 1.3.** *Let  $A$  be a function algebra on  $\Gamma$  with  $\text{Ch}(A) = \Gamma$  and  $\varphi$  a codimension 1 linear isometry on  $A$ . Then  $\tau$  is a homeomorphism from  $\Gamma$  onto  $\Gamma$ .*

PROOF. Since  $\Gamma$  is connected,  $F$  in Theorem 1.1 is equal to  $\Gamma$ . From Lemma 1.2, it suffices to show that there is no element  $x_0 \in \Gamma$  such that  $\tau^{-1}(x_0)$  has two elements. Suppose  $\tau(a) = \tau(b) = x_0$  for some distinct points  $a, b \in \Gamma$ . Let  $L_1$  be a closed arc on  $\Gamma$  having  $a, b$  as the end points. If  $t_1 \in L_1, t_1 \neq a, t_1 \neq b$ , then  $\tau(t_1) \neq x_0$ . Since  $\tau$  is a continuous map from  $\Gamma$  onto  $\Gamma$ ,  $\tau(L_1)$  is a closed arc on  $\Gamma$  containing  $x_0$  and  $\tau(t_1)$ . The fact that  $\tau^{-1}(x)$  is a singleton for any  $x \in \Gamma \setminus \{x_0\}$  follows that  $\tau(L_1) = \Gamma$ . If  $L_2$  is another closed arc on  $\Gamma$  having  $a, b$  as the end points, a similar argument as above implies that  $\tau(L_2) = \Gamma$ . This is a contradiction since  $\tau^{-1}(x)$  is a singleton for any  $x \in \Gamma \setminus \{x_0\}$ .

## 2. Isometric shift operators on the disc algebra.

Let  $\Gamma = \{z \in \mathbf{C} : |z| = 1\}$ ,  $D = \{z \in \mathbf{C} : |z| < 1\}$  and  $\bar{D} = \Gamma \cup D$ . We put  $A_0 = \{f \in C(\bar{D}) : f \text{ is analytic on } D\}$  and  $A = A_0|_{\Gamma}$ .  $A$  is called the *disc algebra*. We here consider isometric shift operators on the disc algebra  $A$ . We set  $M_a(z) = (z - a)/(1 - \bar{a}z)$  for  $a \in D$ .

We begin with the following theorem.

**THEOREM 2.1.** *Let  $A$  be the disc algebra and  $\varphi$  a codimension 1 linear isometry on  $A$ . Then there are  $\alpha, \beta \in \mathbf{C}$  ( $|\alpha| = |\beta| = 1$ ) and  $a, b \in D$  such that  $(\varphi f)(z) = \alpha M_a(z) f(\beta M_b(z))$  ( $f \in A, z \in \Gamma$ ).*

PROOF. Since  $A$  is a function algebra on  $\Gamma$  with  $\text{Ch}(A) = \Gamma$ , by Theorem 1.1 and Theorem 1.3, there are a continuous map  $u$  from  $\Gamma$  into  $\Gamma$  and a homeomorphism  $\tau$  from  $\Gamma$  onto  $\Gamma$  such that  $(\varphi f)(z) = u(z) f(\tau(z))$  ( $f \in A, z \in \Gamma$ ). By putting  $f = 1$ , we see that  $u \in A$ . For every  $f \in A$ , there is a unique function  $f_0 \in A_0$  such that  $f_0|_{\Gamma} = f$ . In the rest we also write  $f$  instead of  $f_0$ .

(i) We first assume that  $u$  does not have zeros in  $D$ . Since  $|u| = 1$ ,  $u$  is a constant function  $\alpha$ . By putting  $f = z$  (the coordinate function), we have  $u\tau \in A$  and so  $\tau \in A$ . If  $\tau$  does not have zeros in  $D$ ,  $\tau$  is constant since  $|\tau| = 1$ . Hence  $\tau$  has (not necessarily distinct) zeros  $b_1, b_2, \dots, b_n$  in  $D$  and so  $\tau(z)$  is of form  $\beta \prod_{i=1}^n M_{b_i}(z)$  ( $|\beta| = 1$ ). We here assert that  $n = 1$ . Suppose  $n \geq 2$ . When  $z$  turns around on  $\Gamma$  one time, it is not hard to see that  $\tau(z)$  rotates on  $\Gamma$   $n$ -times. This contradicts that  $\tau$  is a homeomorphism of  $\Gamma$  onto  $\Gamma$ . Thus,  $\varphi f$  is of form  $\alpha f(\beta M_b)$ , and so  $\varphi$  does not have 1 codimension.

(ii) We next assume that  $u$  has zeros  $a_1, a_2, \dots, a_m$  in  $D$ . Then  $u(z)$  is of form  $\alpha \prod_{j=1}^m M_{a_j}(z)$  ( $|\alpha| = 1$ ). By putting  $f = z^k, u\tau^k \in A$  ( $k = 1, 2, 3, \dots$ ). Since  $u(u\tau^2) = (u\tau)^2$ ,  $u\tau \in A$  and  $u\tau^2 \in A$ ,  $u\tau$  has zeros in  $D$ . From that  $|u\tau| = 1$ , it follows that  $(u\tau)(z)$  is of form  $\beta \prod_{i=1}^n M_{b_i}(z)$  ( $|\beta| = 1, b_i \in D$ ). We first consider  $a_1$ . Since  $u(u\tau^2) = (u\tau)^2$  again, there is some  $b_j$  with  $b_j = a_1$ . Let  $m_1$  be the number of  $a_i$  such that  $a_i = a_1$  and  $a_i \in \{a_j\}_{j=1}^m$ . From that  $u^{k-1}(u\tau^k) = (u\tau)^k$  ( $k = 1, 2, 3, \dots$ ), if  $n_1$  is the number of  $b_j$  such that  $b_j = a_1$  and  $b_j \in \{b_i\}_{i=1}^n$ , we get  $(k-1)m_1 \leq kn_1$ . By tending  $k$  to  $\infty$ ,  $m_1 \leq n_1$ . A similar argument for any  $a_j$  implies that  $\tau \in A$ . Since  $\tau \in A$ , as we saw in (i),  $\tau$  is of form  $\beta M_b$ . Finally we show that  $m = 1$ . For otherwise,  $m \geq 2$ . Suppose first that  $\{a_j\}_{j=1}^m$  contains two distinct elements; call them  $a_1$  and  $a_2$ . If  $pM_{a_1} + qM_{a_2} \in \varphi(A)$  ( $p, q \in \mathbf{C}$ ), there is an  $f \in A$  such

that  $pM_{a_1}(z) + qM_{a_2}(z) = \alpha \prod_{j=1}^m M_{a_j}(z) f(\beta M_b(z))$ . By putting  $z = a_1$  and  $z = a_2$ ,  $qM_{a_2}(a_1) = 0$ ,  $pM_{a_1}(a_2) = 0$ , and so  $p = q = 0$ . Suppose next that  $a_1 = a_2 = \cdots = a_m = a$ . If  $p + qM_a(z) \in \varphi(A)$ ,  $p + qM_a(z) = \alpha (M_a(z))^m f(\beta M_b(z))$  for an  $f \in A$ . By setting  $z = a$ , we have  $p = 0$ . So  $q = \alpha (M_a(z))^{m-1} f(\beta M_b(z))$ . By putting  $z = a$  again,  $q = 0$ . This means  $\varphi$  has at least 2 codimension either way. This contradiction shows  $m = 1$  and  $\varphi f$  is of form  $\alpha M_a f(\beta M_b)$ .

We next discuss when a codimension 1 linear isometry on the disc algebra  $A$  becomes an isometric shift operator. To do this, we describe the form of  $\varphi^n$  as follows:

Let  $\varphi$  be a codimension 1 linear isometry on the disc algebra  $A$ . By Theorem 2.1, there are  $\alpha, \beta \in \mathbb{C}$  ( $|\alpha| = |\beta| = 1$ ) and  $a, b \in D$  such that

$$(\varphi f)(z) = \alpha M_a(z) f(\beta M_b(z)) \quad (f \in A, z \in \Gamma).$$

Hence,

$$(*) \quad (\varphi^n f)(z) = \alpha^n M_a(z) M_a(\beta M_b(z)) \cdots M_a[(\beta M_b)^{n-1}(z)] f[(\beta M_b)^n(z)]$$

for every positive integer  $n$ ,  $f \in A$  and  $z \in \Gamma$ , where  $(\beta M_b)^k$  denotes the  $k$ -times composition of  $\beta M_b$ .

Now, for  $n = 0, 1, 2, \dots$ , we take  $d_n \in D$  such that  $a = (\beta M_b)^n(d_n)$ . We call  $\{d_n\}$  the *backward orbit* of  $a$  by  $\beta M_b$ .

Our final aim is to give equivalent conditions under which a codimension 1 linear isometry  $\varphi$  on the disc algebra  $A$  is a shift operator.

We start with the following lemmas.

LEMMA 2.2 (cf. [8], [2]). *Let  $D = \{z \in \mathbb{C}; |z| < 1\}$  and  $\Gamma = \{z \in \mathbb{C}; |z| = 1\}$  and let  $m$  be an analytic automorphism of  $D$ . Then it occurs either of the following four cases.*

- (i)  *$m$  is the identity, that is,  $m(z) = z$  ( $z \in D$ ).*
- (ii)  *$m$  has only one fixed point in  $D$ . Then  $m$  is said to be elliptic.*
- (iii)  *$m$  has distinct two fixed points on  $\Gamma$ . Then  $m$  is said to be hyperbolic.*
- (iv)  *$m$  has only one fixed point on  $\Gamma$ . Then  $m$  is called parabolic.*

We fix a point  $z_0 \in D$  and set  $z_n = m^n(z_0)$ , where  $m^n$  denotes the  $n$ -times composition of  $m$ . Then we obtain the following.

LEMMA 2.3. (a) *If  $m$  satisfies (i) or (ii) of Lemma 2.2, then  $\sum_{n=0}^{\infty} (1 - |z_n|) = \infty$ .*  
 (b) *If  $m$  satisfies (iii) or (iv) of Lemma 2.2, then  $\sum_{n=0}^{\infty} (1 - |z_n|) < \infty$ .*

PROOF. (a) It is clear if  $m$  has (i). Suppose that  $m$  satisfies (ii). Let  $p$  be the fixed point of  $m$  in  $D$ . By putting  $k(z) = (z - p)/(1 - \bar{p}z)$ ,  $h = k \circ m \circ k^{-1}$  is an analytic automorphism of  $D$  and  $h(0) = 0$ . Hence  $h(z) = \lambda z$  for a  $\lambda \in \mathbb{C}$  ( $|\lambda| = 1$ ). If we set  $w_n = k(z_n)$  ( $n = 0, 1, 2, \dots$ ), then  $w_n = k \circ m^n \circ k^{-1}(w_0) = \lambda^n w_0$ . Hence  $\{w_n\}$  is a relatively compact subset in  $D$ , and so is  $\{z_n\}$  since  $k^{-1}$  is an analytic automorphism on  $D$  and  $z_n = k^{-1}(w_n)$  ( $n = 0, 1, 2, \dots$ ). It follows that  $\sum_{n=0}^{\infty} (1 - |z_n|) = \infty$ .

(b) Suppose that  $m$  satisfies (iii). Let  $p$  be the Denjoy-Wolff point of  $m$  and  $q$  be another fixed point of  $m$  on  $\Gamma$  (cf. [2, p. 59]). Let  $l$  be a bi-holomorphic map of  $D$  onto the upper half plane  $H$  of  $\mathbb{C}$  such that  $l(p)=\infty$  and  $l(q)=0$ . Then  $l \circ m \circ l^{-1}(w)=\alpha w$  ( $w \in H$ ) for some  $\alpha > 0$  ( $\alpha \neq 1$ ) since it is an analytic automorphism on  $H$  which fixes 0 and  $\infty$  only [2, p. 59]. Since  $p$  is the Denjoy-Wolff point of  $m$ ,  $z_n$  converges to  $p$  and so  $l(z_n)$  converges to  $l(p)=\infty$ . If we set  $w_n=l(z_n)$  ( $n=0, 1, 2, \dots$ ), then  $w_n=\alpha^n w_0$ . Since  $w_n=l(z_n)$  converges to  $\infty$ , it follows  $\alpha > 1$ . Therefore,

$$\sum_{n=0}^{\infty} \frac{\operatorname{Im} w_n}{1+|w_n|^2} \leq \sum_{n=0}^{\infty} \frac{1}{|w_n|} = \frac{1}{|w_0|} \sum_{n=0}^{\infty} \frac{1}{\alpha^n} < \infty.$$

It follows that  $\{w_n\}$  is the zeros of a Blaschke product defined on  $H$  [4, p. 55]. Since  $z_n=l^{-1}(w_n)$  ( $n=0, 1, 2, \dots$ ) and  $l$  is a bi-holomorphic map of  $D$  onto  $H$ , it guarantees that  $\{z_n\}$  is the zeros of a non-zero bounded analytic function on  $D$ , and so  $\sum_{n=0}^{\infty} (1-|z_n|) < \infty$  (cf. [6]).

Next suppose that  $m$  has (iv). Let  $p$  be the unique fixed point of  $m$  on  $\Gamma$ . Let  $l$  be the bi-holomorphic map of  $D$  onto  $H$  such that  $l(p)=\infty$  and  $l(-p)=0$ . Then  $l \circ m \circ l^{-1}(w)=w+\gamma$  ( $w \in H$ ) for some non-zero real number  $\gamma$ , since it is an analytic automorphism on  $H$  which fixes  $\infty$  only [2, p. 59]. If we set  $w_n=l(z_n)$ , then  $w_n=w_0+n\gamma$  ( $n=0, 1, 2, \dots$ ). Since  $\sum_{n=0}^{\infty} \operatorname{Im} w_n/(1+|w_n|^2) \leq \alpha \sum_{n=1}^{\infty} 1/n^2 < \infty$  for some  $\alpha > 0$ ,  $\{w_n\}$  is the zeros of a Blaschke product defined on  $H$  and so  $\{z_n\}$  is the zeros of a non-zero bounded analytic function on  $D$ . It follows that  $\sum_{n=0}^{\infty} (1-|z_n|) < \infty$ .

Let  $\beta$  be a complex number with  $|\beta|=1$ . Then  $(1+\beta)/\sqrt{\beta}$  is real. Since the trace of  $\beta M_b$  is  $(1+\beta)/\sqrt{\beta(1-|b|^2)}$ , we have the following by [8, Theorem, p. 5].

**LEMMA 2.4.** *Let  $\beta$  be a complex number with  $|\beta|=1$  and  $b \in D$ . Then  $\beta M_b$  is elliptic if and only if  $(1+\beta)/\sqrt{\beta(1-|b|^2)} < 2$ , where a branch of  $\sqrt{\beta(1-|b|^2)}$  is chosen so that  $(1+\beta)/\sqrt{\beta(1-|b|^2)}$  is non-negative.*

We are now in a position to discuss conditions under which a codimension 1 linear isometry on the disc algebra is a shift operator.

**THEOREM 2.5.** *Let  $\varphi$  be a codimension 1 linear isometry on the disc algebra  $A$  and  $\varphi f = \alpha M_a f(\beta M_b)$  for  $f \in A$ . Let  $\{d_n\}$  be the backward orbit of  $a$  by  $\beta M_b$ . Then the following four conditions are mutually equivalent.*

- (a)  $\varphi$  is a shift operator.
- (b)  $\beta M_b$  is the identity or elliptic.
- (c)  $\sum_{n=0}^{\infty} (1-|d_n|) = \infty$ .
- (d)  $\beta=1$  and  $b=0$ , or  $(1+\beta)/\sqrt{\beta(1-|b|^2)} < 2$ , where a branch of  $\sqrt{\beta(1-|b|^2)}$  is chosen so that  $(1+\beta)/\sqrt{\beta(1-|b|^2)}$  is non-negative.

**PROOF.** The equivalence of (b) and (d) follows from Lemma 2.4. By Lemma 2.2 and 2.3, (b) and (c) are equivalent.

(c)→(a). If  $f \in \bigcap_{n=1}^{\infty} \varphi^n(A)$ , by (\*), we get  $f(d_n)=0$  ( $n=0, 1, 2, \dots$ ). Since  $\sum_{n=0}^{\infty} (1-|d_n|) = \infty$  and  $f$  is bounded and analytic on  $D$ , it follows  $f=0$ .

To prove the theorem, it remains only to show that (a)→(b). Suppose that  $\beta M_b$  is hyperbolic or parabolic. Then  $m=(\beta M_b)^{-1}$  is also hyperbolic or parabolic and  $d_n=m^n(d_0)$  ( $n=0, 1, 2, \dots$ ). Hence  $d_n$  converges to the Denjoy-Wolff point  $d$  of  $m$  and  $\sum_{n=0}^{\infty} (1-|d_n|) < \infty$  by (b) of Lemma 2.3.

Let  $B$  be the Blaschke product having  $\{d_n\}$  as its zeros. If we put  $f(z)=(z-d)B(z)$ , then  $f \in A$ ,  $f \neq 0$  and  $f(d_n)=0$  ( $n=0, 1, 2, \dots$ ). Hence by (\*), for any positive integer  $n$  we can find a  $g \in A$  such that  $f=\varphi^n g$ . Thus  $\bigcap_{n=1}^{\infty} \varphi^n(A) \neq \{0\}$  and  $\varphi$  is not a shift operator. The proof is completed.

EXAMPLES. Let  $\varphi$  be a codimension 1 linear isometry on the disc algebra  $A$  and  $\varphi f = \alpha M_a f(\beta M_b)$  for  $f \in A$ . From Theorem 2.5, the following are immediate.

- (a) If  $\beta=1$ , then  $\varphi$  is a shift operator on  $A$  if and only if  $b=0$ .
- (b) If  $\beta=-1$ , then  $\varphi$  is always a shift operator on  $A$ .
- (c) If  $\beta=\pm i$ , then  $\varphi$  is a shift operator on  $A$  if and only if  $|b|^2 < 1/2$ .

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