TOKYO J. MATH.
VOL. 21, NO. 1, 1998

Nonlinear Ergodic Theorems in a Banach Space Satisfying Opials's Condition

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(Communicated by Ma. Kato)

1. Introduction.

Let C be a nonempty closed convex subset of a real Banach space E. Then a mapping $T: C \to C$ is called nonexpansive, if $\|Tx-Ty\| \le \|x-y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. Let G be a commutative semigroup with identity and let $\mathscr{S}=\{T(s): s \in G\}$ be a family of nonexpansive mappings of C into itself satisfying $T(s+t)=T(s)T(t)$ for all $s, t \in G$, which is called a nonexpansive semigroup on C. Then, $u: G \to C$ is called an almost-orbit of $\mathscr{S}=\{T(s): s \in G\}$ if

$$\lim_s \sup_t \|u(t+s)-T(t)u(s)\|=0\,,$$

where the binary relation $\le$ on G is defined by $a \le b$ if and only if there exists $c \in G$ such that $a+c=b$. The notion of such an almost-orbit was introduced by Takahashi and Park [24]; see Bruck [4] in the case of $G=\{1, 2, 3, \cdots\}$ and Miyadera and Kobayasi [15] in the case of $G=\{t: 0 \le t < \infty\}$.

The first nonlinear ergodic theorem for nonexpansive mappings in a Hilbert space was established by Baillon [1]: Let C be a nonempty closed convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself. If the set $F(T)$ is nonempty, then for each $x \in C$, the Cesàro means

$$S_n(x)=\frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converge weakly to some $y \in F(T)$. In Baillon's theorem, putting $y=Px$ for each $x \in C$, P is a nonexpansive retraction of C onto $F(T)$ such that $PT^n=T^nP=P$ for all positive integers n and $Px \in \overline{co}\{T^n x: n=1, 2, \cdots\}$ for each $x \in C$, where $\overline{co}A$ is the closure of the convex hull of A. Takahashi [20, 22] proved the existence of such retractions, "ergodic retractions", for noncommutative semigroups of nonexpansive mappings in a

Received June 3, 1996

Hilbert space. Rodé [19] found a sequence of means on the semigroups, generalizing the Cesàro means on the positive integers, such that the corresponding sequence of mappings converges to an ergodic retraction onto the set of common fixed points: see also [24]. On the other hand, Miyadera and Kobayasi [15] proved nonlinear ergodic theorems for almost-orbits in the case when $G=\{t: 0\leq t<\infty\}$ and a Banach space E satisfies Opial's condition or has a Fréchet differentiable norm. Hirano, Kido and Takahashi [9, 10] proved nonlinear ergodic theorems for commutative semigroups of nonexpansive mappings in a uniformly convex Banach space with a Fréchet differentiable norm.

The purpose of this paper is, among other things, to prove a nonlinear ergodic theorem for almost-orbits of commutative semigroups of nonexpansive mappings in a uniformly convex Banach space which satisfies Opial's condition. Further, we consider some applications of this result.

2. Preliminaries.

Throughout this paper, we assume that a Banach space E is real and G is a commutative semigroup with identity unless others specified. In this case, $(G, \leq)$ is a directed system when the binary relation $\leq$ on G is defined by $a\leq b$ if and only if there is $c\in G$ with $a+c=b$.

We denote by E^* the dual space of E and by $\mathbf{R}$ the set of real numbers. In addition, $\mathbf{R}^+$ denotes the set $[0, +\infty)$ of nonnegative real numbers. We also denote by $\langle y, x^*\rangle$ the value of $x^*\in E^*$ at $y\in E$. For a subset A of E, coA (resp. $\overline{co}A$) means the convex hull of A (resp. the closure of convex hull of A). We say that E satisfies Opial's condition [17] if for any sequence $\{x_n\}\subset E$ with $x_n \rightharpoonup x\in E$, the inequality

$$\liminf_n \|x_n - x\| < \liminf_n \|x_n - y\| \tag{1}$$

holds for every $y\in E$ with $y\neq x$, where $\rightharpoonup$ means weak convergence. In a reflexive Banach space, this condition is equivalent to the analogous condition for a bounded net which has been introduced in [13].

PROPOSITION 2.1. *If a Banach space E is reflexive, the following conditions are equivalent*:

(i) *For any sequence $\{x_n\}\subset E$ with $x_n\rightharpoonup z_0\in E$, the inequality*

$$\liminf_n \|x_n - z_0\| < \liminf_n \|x_n - y\|$$

holds for every $y\in E$ with $y\neq z_0$;

(ii) *For any bounded net $\{x_\alpha\}\subset E$ with $x_\alpha\rightharpoonup y_0\in E$, the inequality*

$$\liminf_\alpha \|x_\alpha - y_0\| < \liminf_\alpha \|x_\alpha - y\|$$

holds for every $y \in E$ *with* $y \neq y_0$.

PROOF. We know from Browder [3] that if C is a bounded subset of a reflexive Banach space E and x is a point in the weak closure of C, then there exists an infinite sequence $\{x_n\} \subset C$ converging weakly to x. Since (ii) $\Rightarrow$ (i) is obvious, we show (i) $\Rightarrow$ (ii). Assume that there exists a bounded net $\{x_\alpha\} \subset E$ satisfying $x_\alpha \rightharpoonup x_0$ and

$$\liminf_\alpha \|x_\alpha - y_0\| \leq \liminf_\alpha \|x_\alpha - x_0\|$$

for some $y_0 \in E$ with $y_0 \neq x_0$. There exists a subnet $\{x_{\alpha_\beta}\}$ of $\{x_\alpha\} \subset E$ such that

$$\liminf_\alpha \|x_\alpha - y_0\| = \lim_\beta \|x_{\alpha_\beta} - y_0\| .$$

Then, we have

$$\begin{aligned} \lim_\beta \|x_{\alpha_\beta} - y_0\| &= \liminf_\alpha \|x_\alpha - y_0\| \\ &\leq \liminf_\alpha \|x_\alpha - x_0\| \leq \liminf_\beta \|x_{\alpha_\beta} - x_0\| . \end{aligned} \tag{2}$$

On the other hand, there exists a subnet $\{x_{\alpha_{\beta_\gamma}}\}$ of $\{x_{\alpha_\beta}\}$ such that

$$\liminf_\beta \|x_{\alpha_\beta} - x_0\| = \lim_\gamma \|x_{\alpha_{\beta_\gamma}} - x_0\| .$$

Then, from (2), we have

$$\lim_\gamma \|x_{\alpha_{\beta_\gamma}} - y_0\| = \lim_\beta \|x_{\alpha_\beta} - y_0\| \leq \liminf_\beta \|x_{\alpha_\beta} - x_0\| = \lim_\gamma \|x_{\alpha_{\beta_\gamma}} - x_0\| .$$

Put $K_0 = \lim_\gamma \|x_{\alpha_{\beta_\gamma}} - y_0\|$ and $L_0 = \lim_\gamma \|x_{\alpha_{\beta_\gamma}} - x_0\|$. We have that $K_0 \leq L_0$. Let $X = E \times \mathbf{R} \times \mathbf{R}$, $C = \{(x_{\alpha_{\beta_\gamma}}, \|x_{\alpha_{\beta_\gamma}} - y_0\|, \|x_{\alpha_{\beta_\gamma}} - x_0\|)\}$ and $x = (x_0, K_0, L_0)$. Since $x_\alpha \rightharpoonup x_0$, $\|x_{\alpha_{\beta_\gamma}} - y_0\| \to K_0$ and $\|x_{\alpha_{\beta_\gamma}} - x_0\| \to L_0$, we see that x is a point in the weak closure of C. So, from Browder [3], there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup x = (x_0, K_0, L_0)$. Put $\{x_n\} = \{(z_n, \|z_n - y_0\|, \|z_n - x_0\|)\}$. Then, we have

$$\lim_n \|z_n - y_0\| = K_0 \quad \text{and} \quad \lim_n \|z_n - x_0\| = L_0 .$$

We also have that $z_n \rightharpoonup x_0$. Then, it follows from (i) that

$$L_0 = \lim_n \|z_n - x_0\| < \lim_n \|z_n - y_0\| = K_0 ,$$

which contradicts $K_0 \leq L_0$. □

It is known that all Hilbert spaces and ℓ^p $(1 < p < \infty)$ satisfy Opials's condition. It is also known that every separable Banach space can be equivalently renormed so that it satisfies Opial's condition (see [6]). We also know that if a Banach space E has a duality mapping which is weakly sequentially continuous at 0, then E satisfies Opial's condition (see [7]). However, the spaces L^p with $1<p<\infty$ and $p\neq 2$ do not satisfy Opial's condition (see also [17]).

Let $m(G)$ be the Banach space of all bounded real-valued functions on G with the supremum norm. Then, for each $s\in G$ and $f\in m(G)$, we can define $r_s f\in m(G)$ by $(r_s f)(t)=f(t+s)$ for all $t\in G$. We also denote by r_s^* the conjugate operator of r_s. Let D be a subspace of $m(G)$ and let μ be an element of D^*. Then, we denote by $\mu(f)$ the value of μ at $f\in D$. Sometimes, $\mu(f)$ will be also denoted by $\mu_t(f(t))$ or $\int f(t)d\mu(t)$. When D contains constants, a linear functional μ on D is called a mean on D if $\|\mu\|=\mu(1)=1$. Further, let D be invariant under every r_s, $s\in G$. Then, a mean μ on D is invariant if $\mu(r_s f)=\mu(f)$ for all $s\in G$ and $f\in D$. For $s\in G$, we can define a point evaluation δ_s by $\delta_s(f)=f(s)$ for every $f\in m(G)$. A convex combination of point evaluations is called a finite mean on G. A finite mean μ on G is also a mean on any subspace D of $m(G)$ containing constants. The following definition which was introduced by Takahashi [20] (see also [10]) is crucial in the nonlinear ergodic theory for abstract semigroups. Let f be a function of G into E such that the weak closure of $\{f(t): t\in G\}$ is weakly compact. Let D be a subspace of $m(G)$ containing constants and invariant under every r_s, $s\in G$. Assume that for each $x^*\in E^*$, the function $t\to\langle f(t), x^*\rangle$ is in D. Then, for any $\mu\in D^*$ there exists a unique element $f_\mu\in E$ such that

$$\langle f_\mu, x^*\rangle=\int\langle f(t), x^*\rangle d\mu(t)$$

for all $x^*\in E^*$. If μ is a mean on D, then f_μ is contained in $\overline{co}\{f(t): t\in G\}$ (for example, see [11, 12, 20]). Sometimes, f_μ will be denoted by $\int f(t)d\mu(t)$.

Let C be a subset of a Banach space E. Then, a family $\mathcal{S}=\{T(s): s\in G\}$ of mappings of C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

(i) $T(s+t)=T(s)T(t)$ for all $s, t\in G$;

(ii) $\|T(s)x-T(s)y\|\leq\|x-y\|$ for all $x, y\in C$ and $s\in G$.

We denote by $F(\mathcal{S})$ the set of common fixed points of $T(t)$, $t\in G$, that is, $F(\mathcal{S})=\bigcap_{t\in G}F(T(t))$. If C is a bounded closed convex subset of a uniformly convex Banach space E and G is commutative, then we know that $F(\mathcal{S})$ is nonempty (for example, see [2]). A function $u: G\to C$ is called an almost-orbit of $\mathcal{S}=\{T(t): t\in G\}$ if

$$\lim_s \sup_t \|u(t+s)-T(t)u(s)\|=0$$

(see [15, 24]). We denote by $AO(\mathcal{S})$ the set of almost-orbits of $\mathcal{S}=\{T(t): t\in G\}$.

3. Lemmas.

In this section, we give some lemmas which are used to prove the main theorem in Section 4. The following lemma was proved in [9, 16].

LEMMA 3.1. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let $\mathscr{S}=\{T(t): t\in G\}$ be a nonexpansive semigroup on C. Let u be an almost-orbit of $\mathscr{S}=\{T(t): t\in G\}$. Then, for each $\varepsilon>0$ and a finite mean λ on G, there exists $w'=w'(\varepsilon, \lambda)$ such that*

$$\left\| T(h)\left(\int u(w+t)d\lambda(t)\right)-\int T(h)u(w+t)d\lambda(t)\right\| <\varepsilon$$

for every $h\in G$ and $w\geq w'$.

By using Lemma 3.1, we can now show the following lemma which plays an important role in the proof of Lemma 3.5.

LEMMA 3.2. *Let E, C, $\mathscr{S}=\{T(t): t\in G\}$ and u be as in Lemma 3.1. Let $\{\mu_\alpha : \alpha\in I\}$ and $\{\lambda_\beta : \beta\in J\}$ be nets of finite means on G such that*

$$\lim_\alpha \|\mu_\alpha - r_t^*\mu_\alpha\| = 0 \quad \textit{and} \quad \lim_\beta \|\lambda_\beta - r_t^*\lambda_\beta\| = 0 \qquad \textit{for every} \quad t\in G. \tag{$*$}$$

Then, there exist $\{p_\alpha\}$, $\{q_\beta\}\subset G$ such that for any $z\in F(\mathscr{S})$,

$$\lim_\alpha \left\| \int u(p_\alpha+t)d\mu_\alpha(t)-z\right\| = \lim_\beta \left\| \int u(q_\beta+t)d\lambda_\beta(t)-z\right\| . \tag{3}$$

PROOF. Define $\phi: G\to \mathbf{R}^+$ by $\phi(s)=\sup_t \|u(t+s)-T(t)u(s)\|$ for every $s\in G$ and let $\varepsilon>0$. Then, for $\alpha\in I$ and $\beta\in J$, from Lemma 3.1, there exist p_α, $q_\beta\in G$ such that

$$\phi(w+p_\alpha)<\varepsilon\,, \qquad \phi(w+q_\beta)<\varepsilon\,,$$

$$\sup_{h\in G} \left\| \int T(h)u(w+p_\alpha+t)d\mu_\alpha(t) - T(h)\left(\int u(w+p_\alpha+t)d\mu_\alpha(t)\right)\right\| <\varepsilon$$

and

$$\sup_{h\in G} \left\| \int T(h)u(w+q_\beta+s)d\lambda_\beta(s) - T(h)\left(\int u(w+q_\beta+s)d\lambda_\beta(s)\right)\right\| <\varepsilon$$

for every $w\in G$. Fix $z\in F(\mathscr{S})$ and consider

$$L=\left\| \int u(p_\alpha+t)d\mu_\alpha(t)-z\right\| ,$$

$$I_1=\left\| \int u(p_\alpha+t)d\mu_\alpha(t)-\int\int u(p_\alpha+t+q_\beta+s)d\lambda_\beta(s)d\mu_\alpha(t)\right\| ,$$

$$I_2=\left\| \int\int u(p_\alpha+t+q_\beta+s)d\lambda_\beta(s)d\mu_\alpha(t)-z\right\| ,$$

$$J_1^{(2)}=\left\| \int\int u(p_\alpha+t+q_\beta+s)d\lambda_\beta(s)d\mu_\alpha(t)-\int\int T(p_\alpha+t)u(q_\beta+s)d\lambda_\beta(s)d\mu_\alpha(t)\right\| ,$$

$$J_2^{(2)}=\left\| \int\int T(p_\alpha+t)u(q_\beta+s)d\lambda_\beta(s)d\mu_\alpha(t)-\int T(p_\alpha+t)\left(\int u(q_\beta+s)d\lambda_\beta(s)\right)d\mu_\alpha(t)\right\|$$

and

$$J_3^{(2)}=\left\| \int T(p_\alpha+t)\left(\int u(q_\beta+s)d\lambda_\beta(s)\right)d\mu_\alpha(t)-z\right\| .$$

Then, we have $L\le I_1+I_2$ and $I_2\le J_1^{(2)}+J_2^{(2)}+J_3^{(2)}$. Suppose

$$\mu_\alpha=\sum_{i=1}^n a_i\delta_{t_i}\ \left(a_i\ge 0,\ \sum_{i=1}^n a_i=1\right)\quad\text{and}\quad \lambda_\beta=\sum_{j=1}^m b_j\delta_{s_j}\ \left(b_j\ge 0,\ \sum_{j=1}^m b_j=1\right). \tag{4}$$

Then, we have

$$\begin{aligned}J_1^{(2)}&\le\sum_{i=1}^n\sum_{j=1}^m a_ib_j\|u(p_\alpha+t_i+q_\beta+s_j)-T(p_\alpha+t_i)u(q_\beta+s_j)\|\\&\le\sum_{i=1}^n\sum_{j=1}^m (a_ib_j)\sup_h\|u(h+q_\beta+s_j)-T(h)u(q_\beta+s_j)\|=\sum_{j=1}^m b_j\phi(q_\beta+s_j)\end{aligned}$$

and

$$\begin{aligned}J_2^{(2)}&\le\sum_{i=1}^n a_i\left\| \int T(p_\alpha+t_i)u(q_\beta+s)d\lambda_\beta(s)-T(p_\alpha+t_i)\left(\int u(q_\beta+s)d\lambda_\beta(s)\right)\right\|\\&\le\sup_{h\in G}\left\| \int T(h)u(q_\beta+s)d\lambda_\beta(s)-T(h)\left(\int u(q_\beta+s)d\lambda_\beta(s)\right)\right\| .\end{aligned}$$

Since $z\in F(\mathscr{S})$, we obtain

$$J_3^{(2)}\le\sum_{i=1}^n a_i\left\| T(p_\alpha+t_i)\left(\int u(q_\beta+s)d\lambda_\beta(s)\right)-z\right\|\le\left\| \int u(q_\beta+s)d\lambda_\beta(s)-z\right\| .$$

Then, we have

$$I_2\le J_1^{(2)}+J_2^{(2)}+J_3^{(2)}<\varepsilon+\varepsilon+\left\| \int u(q_\beta+s)d\lambda_\beta(s)-z\right\| .$$

On the other hand, from (4), we obtain

$$\begin{aligned}I_1&=\left\| \int u(p_\alpha+t)d\mu_\alpha(t)-\sum_{j=1}^m b_j\int u(p_\alpha+t+q_\beta+s_j)d\mu_\alpha(t)\right\|\\&\le\sum_{j=1}^m b_j\left\| \int u(p_\alpha+t)d\mu_\alpha(t)-\int u(p_\alpha+t)d(r_{q_\beta+s_j}^*\mu_\alpha)(t)\right\|\\&\le\sum_{j=1}^m b_j\sup_{g\in G}\|u(g)\|\ \|\mu_\alpha-r_{q_\beta+s_j}^*\mu_\alpha\| .\end{aligned}$$

Therefore, from $\lim_\alpha I_1 = 0$, we have

$$\limsup_\alpha \left\| \int u(p_\alpha + t) d\mu_\alpha(t) - z \right\| = \limsup_\alpha L$$

$$\leq \limsup_\alpha (I_1 + I_2) \leq 2\varepsilon + \left\| \int u(q_\beta + s) d\lambda_\beta(s) - z \right\| .$$

Then, we have

$$\limsup_\alpha \left\| \int u(p_\alpha + t) d\mu_\alpha(t) - z \right\| \leq 2\varepsilon + \liminf_\beta \left\| \int u(q_\beta + s) d\lambda_\beta(s) - z \right\| .$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\limsup_\alpha \left\| \int u(p_\alpha + t) d\mu_\alpha(t) - z \right\| \leq \liminf_\beta \left\| \int u(q_\beta + s) d\lambda_\beta(s) - z \right\| .$$

Similarly, we have

$$\limsup_\beta \left\| \int u(q_\beta + s) d\lambda_\beta(s) - z \right\| \leq \liminf_\alpha \left\| \int u(p_\alpha + t) d\mu_\alpha(t) - z \right\| .$$

Therefore, we have

$$\lim_\alpha \left\| \int u(p_\alpha + t) d\mu_\alpha(t) - z \right\| = \lim_\beta \left\| \int u(q_\beta + t) d\lambda_\beta(t) - z \right\| . \qquad \square$$

REMARK 3.3. In Lemma 3.2, take $\{p'_\alpha\}$, $\{q'_\beta\} \subset G$ such that $p'_\alpha \geq p_\alpha$ and $q'_\beta \geq q_\beta$. Then, we can see that

$$\lim_\alpha \left\| \int u(p'_\alpha + t) d\mu_\alpha(t) - z \right\| = \lim_\beta \left\| \int u(q'_\beta + t) d\lambda_\beta(t) - z \right\|$$

for every $z \in F(\mathscr{S})$.

For each $\varepsilon > 0$ and $h \in G$, set

$$F_\varepsilon(T(h)) = \{x \in C : \|T(h)x - x\| \leq \varepsilon\} .$$

Then, as in the proofs of [10, 16], we have the following lemma.

LEMMA 3.4. *Let E, C, $\mathscr{S} = \{T(t) : t \in G\}$ and u be as in Lemma 3.1. Let $\{\mu_\alpha : \alpha \in I\}$ be a net of finite means on G such that*

$$\lim_\alpha \|\mu_\alpha - r_t^* \mu_\alpha\| = 0 \qquad \text{for every} \quad t \in G . \tag{$*$}$$

Then, for any $\varepsilon > 0$ and $h \in G$, there exists $\alpha_0(\varepsilon, h) \in I$ such that

$$\int u(p + t) d\mu_\alpha(t) \in F_\varepsilon(T(h))$$

for all $\alpha \geq a_0(\varepsilon, h)$ *and* $p \in G$.

By using Lemma 3.2 and Lemma 3.4, we can show the following lemma which is crucial to prove the main theorem (Theorem 4.1).

LEMMA 3.5. *Let* $E, C, \mathcal{S}=\{T(t): t \in G\}$ *and* u *be as in Lemma* 3.1. *Additionally, assume that* E *satisfies Opial's condition. Let* $\{\mu_\alpha: \alpha \in I\}$ *be a net of finite means on* G *such that*

$$\lim_\alpha \|\mu_\alpha - r_t^* \mu_\alpha\| = 0 \qquad \text{for every} \quad t \in G\,. \tag{$*$}$$

Then, $\int u(h+t)d\mu_\alpha(t)$ *converges weakly to some* $y \in F(\mathcal{S})$ *uniformly in* $h \in G$. *Furthermore, such an element* y *of* $F(\mathcal{S})$ *is independent of* $\{\mu_\alpha\}$ *and for any invariant mean* μ *on* D, $y = u_\mu = \int u(t)d\mu(t)$.

PROOF. Let $\{\mu_\alpha: \alpha \in I\}$ and $\{\lambda_\beta: \beta \in J\}$ be nets of finite means on G such that

$$\lim_\alpha \|\mu_\alpha - r_t^* \mu_\alpha\| = 0 \quad \text{and} \quad \lim_\beta \|\lambda_\beta - r_t^* \lambda_\beta\| = 0 \tag{$*$}$$

for every $t \in G$. From Lemma 3.4, for each $h \in G$, we have

$$\lim_\alpha \sup_p \left\| \int u(p+t)d\mu_\alpha(t) - T(h)\left(\int u(p+t)d\mu_\alpha(t)\right)\right\| = 0\,. \tag{5}$$

Further, we can take $\{p_\alpha\} \subset G$ such that for any $z \in F(\mathcal{S})$, $\lim_\alpha \|\int u(p_\alpha + t)d\mu_\alpha(t) - z\|$ exists. Let $\{\Phi_\alpha\} = \{\int u(p_\alpha + t)d\mu_\alpha(t): \alpha \in I\}$. Then, we first prove that $\{\Phi_\alpha\}$ converges weakly to some $y \in F(\mathcal{S})$. Since E is uniformly convex and C is a bounded closed convex subset of E, $\{\Phi_\alpha\}$ must contain a subnet which converges weakly to a point in C. So, let $\{\Phi_{\alpha_\gamma}\}$ and $\{\Phi_{\alpha_\delta}\}$ be two subnets of $\{\Phi_\alpha\}$ such that

$$w\text{-}\lim_\gamma \Phi_{\alpha_\gamma} = v \quad \text{and} \quad w\text{-}\lim_\delta \Phi_{\alpha_\delta} = v'\,,$$

where $w\text{-}\lim_\alpha x_\alpha = x$ means $x_\alpha \rightharpoonup x$. Then, from (5) and demiclosedness principle (see [3]), we have that v and v' are common fixed points of $T(t)$, $t \in G$. Suppose $v \neq v'$. From Lemma 3.2 and Opial's conditions, we obtain

$$\lim_\alpha \|\Phi_\alpha - v\| = \lim_\gamma \|\Phi_{\alpha_\gamma} - v\| < \lim_\gamma \|\Phi_{\alpha_\gamma} - v'\|$$

$$= \lim_\delta \|\Phi_{\alpha_\delta} - v'\| < \lim_\delta \|\Phi_{\alpha_\delta} - v\| = \lim_\alpha \|\Phi_\alpha - v\|\,.$$

This is a contradiction. So, we have that $v = v'$, which implies that $\{\Phi_\alpha\}$ converges weakly to some $y \in F(\mathcal{S})$. Next we prove that $\{\int u(h+t)d\mu_\alpha(t)\}$ converges weakly to y uniformly in h. In the above argument, take $\{p'_\alpha\} \subset G$ such that $p'_\alpha \geq p_\alpha$ for each $\alpha \in I$. Then, repeating the above argument, we see that $\{\Phi'_\alpha\}=\{\int u(p'_\alpha+t)d\mu_\alpha(t):\alpha\in I\}$ converges weakly to some $y'\in F(\mathscr{S})$. We show $y=y'$. From Lemma 3.2 and Remark 3.3, we know

$$\lim_\alpha\left\|\int u(p'_\alpha+t)d\mu_\alpha(t)-z\right\|=\lim_\alpha\left\|\int u(p_\alpha+t)d\mu_\alpha(t)-z\right\| \tag{6}$$

for every $z\in F(\mathscr{S})$. Suppose $y\neq y'$. Since y and y' are common fixed points of $T(t)$, $t\in G$, from (6) and Opial's condition, we have

$$\lim_\alpha\|\Phi_\alpha-y'\|=\lim_\alpha\|\Phi'_\alpha-y'\|<\lim_\alpha\|\Phi'_\alpha-y\|$$

$$=\lim_\alpha\|\Phi_\alpha-y\|<\lim_\alpha\|\Phi_\alpha-y'\|\ .$$

This is a contradiction. So, we have $y=y'\in F(\mathscr{S})$. Since $\{p'_\alpha\}$ is any subset in G such that $p'_\alpha\geq p_\alpha$ for each $\alpha\in I$, we have

$$w\text{-}\lim_\alpha\int u(h+p_\alpha+t)d\mu_\alpha(t)=y$$

uniformly in $h\in G$. Let $x^*\in E^*$ and $\varepsilon>0$. Then, there exists α_0 such that

$$\left|\int\langle u(h+p_\alpha+s),\,x^*\rangle d\mu_\alpha(s)-\langle y,\,x^*\rangle\right|<\frac{\varepsilon}{2} \tag{7}$$

for every $\alpha\geq\alpha_0$ and $h\in G$. Suppose

$$\mu_{\alpha_0}=\sum_{k=1}^m b_k\delta_{s_k}\qquad\left(b_k\geq 0,\ \sum_{k=1}^m b_k=1\right). \tag{8}$$

Put $\mu_0=\mu_{\alpha_0}$ and $p_0=p_{\alpha_0}$. From (7), we have

$$\begin{aligned}&\left|\int\int\langle u(h+t+p_0+s),\,x^*\rangle d\mu_0(s)d\lambda_\beta(t)-\langle y,\,x^*\rangle\right|\\&=\left|\int\left\langle\int u(h+t+p_0+s)d\mu_0(s),\,x^*\right\rangle d\lambda_\beta(t)-\int\langle y,\,x^*\rangle d\lambda_\beta(t)\right|\\&\leq\int\left|\left\langle\int u(h+t+p_0+s)d\mu_0(s)-y,\,x^*\right\rangle\right|d\lambda_\beta(t)<\frac{\varepsilon}{2}\end{aligned}$$

for every $h\in G$ and $\beta\in J$. Since $\{\lambda_\alpha\}$ satisfies $(*)$, there exists β_1 such that

$$\|\lambda_\beta-r^*_{p_0+s_k}\lambda_\beta\|<\frac{\varepsilon}{2\max\{1,\,M\|x^*\|\}}$$

for all $k=1,2,\cdots,m$ and $\beta\geq\beta_1$, where $M=\sup_{g\in G}\|u(g)\|$. Then, it follows that

$$\left|\int\langle u(h+t),x^*\rangle d\lambda_\beta(t)-\int\int\langle u(h+t+p_0+s),x^*\rangle d\mu_0(s)d\lambda_\beta(t)\right|$$
$$=\left|\int\langle u(h+t),x^*\rangle d\lambda_\beta(t)-\int\left\langle\sum_{k=1}^m b_k u(h+t+p_0+s_k),x^*\right\rangle d\lambda_\beta(t)\right|$$
$$\leq\sum_{k=1}^m b_k\left|\int\langle u(h+t),x^*\rangle d\lambda_\beta(t)-\int\langle u(h+t),x^*\rangle d(r^*_{p_0+s_k}\lambda_\beta)(t)\right|$$
$$\leq\sum_{k=1}^m b_k M\|x^*\|\,\|\lambda_\beta-r^*_{p_0+s_k}\lambda_\beta\|<\frac{\varepsilon}{2}$$

for every $\beta\geq\beta_1$ and $h\in G$. Therefore,

$$\left|\int\langle u(h+t),x^*\rangle d\lambda_\beta(t)-\langle y,x^*\rangle\right|$$
$$\leq\left|\int\langle u(h+t),x^*\rangle d\lambda_\beta(t)-\int\int\langle u(h+t+p_0+s),x^*\rangle d\mu_0(s)d\lambda_\beta(t)\right|$$
$$+\left|\int\int\langle u(h+t+p_0+s),x^*\rangle d\mu_0(s)d\lambda_\beta(t)-\langle y,x^*\rangle\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

for every $\beta\geq\beta_1$ and $h\in G$. Hence,

$$w\text{-}\lim_\beta\int u(h+t)d\lambda_\beta(t)=y$$

uniformly in $h\in G$. Since $\{\lambda_\beta\}$ is an arbitrary net of finite means on G such that

$$\lim_\beta\|\lambda_\beta-r_t^*\lambda_\beta\|=0 \qquad (*)$$

for every $t\in G$, we have that such an element y of $F(\mathscr{S})$ is independent of $\{\lambda_\beta\}$ and $\{\mu_\alpha\}$.

Finally, we prove that for any invariant mean μ on D, $y=u_\mu$. Since the set of all finite means is weak*-dense in the set of all means and as in the proof of [5, Theorem 1 in Section 5], we see that for any invariant mean μ on D, there exists a net $\{\mu_\beta\}$ of finite means on G such that $\lim_\beta\|\mu_\beta-r_s^*\mu_\beta\|=0$ for every $s\in G$ and μ_β converges to μ in the weak* topology. Then, we have

$$w\text{-}\lim_\beta\int u(t)d\mu_\beta(t)=\int u(t)d\mu(t)=u_\mu\,.$$

On the other hand, we obtain

$$\int u(t)d\mu_\beta(t)\rightharpoonup y\,.$$

Hence, we obtain $y=u_\mu$. □

4. Nonlinear ergodic theorem.

In this section, we prove a nonlinear ergodic theorem for a commutative semigroup of nonexpansive mappings in a uniformly convex Banach space which satisfies Opial's condition. Let D be a subspace of $m(G)$ containing constants and invariant under every r_s, $s \in G$. Then, according to Hirano, Kido and Takahashi [10], a net $\{\mu_\alpha : \alpha \in I\}$ of continuous linear functionals on D is called strongly regular if it satisfies the following conditions:

(a) $\sup_\alpha \|\mu_\alpha\| < +\infty$;

(b) $\lim_\alpha \mu_\alpha(1) = 1$;

(c) $\lim_\alpha \|\mu_\alpha - r_s^* \mu_\alpha\| = 0$ for every $s \in G$.

Using Lemma 3.5, we can now prove the following main theorem.

THEOREM 4.1. *Let E be a uniformly convex Banach space which satisfies Opial's condition, let C be a nonempty bounded closed convex subset of E and let $\mathscr{S} = \{T(t) : t \in G\}$ be a nonexpansive semigroup on C. Let $u: G \to C$ be an almost-orbit of $\mathscr{S} = \{T(t) : t \in G\}$ and let D be a subspace of $m(G)$ containing constants and invariant under every r_s, $s \in G$. Suppose that for each $x^* \in E^*$, the function $t \to \langle u(t), x^* \rangle$ is in D. If $\{\lambda_\alpha\}$ is a strongly regular net of continuous linear functionals on D, then $\int u(h+t) d\lambda_\alpha(t)$ converges weakly to some $y \in F(\mathscr{S})$ uniformly in $h \in G$. Further, such an element y of $F(\mathscr{S})$ is independent of $\{\lambda_\alpha\}$ and for any invariant mean μ on D, $y = u_\mu = \int u(t) d\mu(t)$. In this case, if*

$$Qu = w\text{-}\lim_\alpha \int u(t) d\lambda_\alpha(t) \qquad \textit{for each} \quad u \in AO(\mathscr{S}),$$

then Q is a mapping of $AO(\mathscr{S})$ onto $F(\mathscr{S})$ satisfying the following conditions:

(i) *Q is nonexpansive in the sense that*

$$\|Qu - Qv\| \le \sup_{t \in G} \|u(t) - v(t)\| = \|u - v\|_\infty$$

for every $u, v \in AO(\mathscr{S})$;

(ii) *$QT(t)u = T(t)Qu = Qu$ for every $t \in G$ and $u \in AO(\mathscr{S})$;*

(iii) *$Qu \in \bigcap_{s \in G} \overline{co}\{u(t) : t \ge s\}$ for every $u \in AO(\mathscr{S})$.*

PROOF. Let $\{\lambda_\alpha : \alpha \in I\}$ be a strongly regular net of continuous linear functionals on D and let $\{\mu_\beta : \beta \in J\}$ be a net of finite means on G such that

$$\lim_\beta \|\mu_\beta - r_t^* \mu_\beta\| = 0 \qquad \text{for every} \quad t \in G. \tag{$*$}$$

From Lemma 3.5, we have

$$\int u(h+t) d\mu_\beta(t) \rightharpoonup y \in F(\mathscr{S})$$

uniformly in $h \in G$. Let $x^* \in E^*$ and $\varepsilon > 0$. Let μ be an invariant mean on D. Then, from Lemma 3.5, we know $y = u_\mu$. Further, there exists β_1 such that

$$\left| \int \langle u(h+t), x^* \rangle d\mu_\beta(t) - \langle u_\mu, x^* \rangle \right| < \frac{\varepsilon}{\sup_\alpha \|\lambda_\alpha\|}$$

for every $\beta \geq \beta_1$ and $h \in G$. Suppose

$$\mu_{\beta_1} = \sum_{i=1}^{n} b_i \delta_{t_i}, \qquad \left(b_i \geq 0, \sum_{i=1}^{n} b_i = 1 \right) \tag{9}$$

and put $\mu_1 = \mu_{\beta_1}$. Then, we have

$$\begin{aligned} &\left| \int \langle u(h+t), x^* \rangle d\mu_1(t) - \langle u_\mu, x^* \rangle \right| \\ &= \left| \left\langle \sum_{i=1}^{n} b_i u(h+t_i), x^* \right\rangle - \langle u_\mu, x^* \rangle \right| < \frac{\varepsilon}{\sup_\alpha \|\lambda_\alpha\|} \end{aligned} \tag{10}$$

for every $h \in G$. We also know that

$$\begin{aligned} &\left\langle \iint u(h+s+t) d\mu_1(t) d\lambda_\alpha(s) - u_\mu, x^* \right\rangle \\ &= \int \left\langle \int u(h+s+t) d\mu_1(t), x^* \right\rangle d\lambda_\alpha(s) - \langle u_\mu, x^* \rangle \\ &= \int \left\langle \int u(h+s+t) d\mu_1(t) - u_\mu, x^* \right\rangle d\lambda_\alpha(s) + \int \langle u_\mu, x^* \rangle d\lambda_\alpha(s) - \langle u_\mu, x^* \rangle . \end{aligned}$$

Since $\{\lambda_\alpha\}$ is strongly regular, there exists α_0 such that

$$|1 - \lambda_\alpha(1)| < \frac{\varepsilon}{\max\{1, \|u_\mu\| \cdot \|x^*\|\}} \tag{11}$$

and

$$\|\lambda_\alpha - r_{t_i}^* \lambda_\alpha\| < \frac{\varepsilon}{\max\{1, M \cdot \|x^*\|\}} \tag{12}$$

for all $i = 1, \cdots, n$ and $\alpha \geq \alpha_0$, where $M = \sup_{g \in G} \|u(g)\|$. Then, we have

$$\left| \langle u_\mu, x^* \rangle - \int \langle u_\mu, x^* \rangle d\lambda_\alpha(s) \right| \leq |\langle u_\mu, x^* \rangle| \, |1 - \lambda_\alpha(1)| < \varepsilon$$

for every $\alpha \geq \alpha_0$ and from (10),

$$\left|\int\left\langle \int u(h+s+t)d\mu_1(t)-u_\mu,\ x^*\right\rangle d\lambda_\alpha(s)\right|$$
$$\leq \|\lambda_\alpha\|\left|\left\langle \int u(h+s+t)d\mu_1(t)-u_\mu,\ x^*\right\rangle\right|<\varepsilon$$

for every $h, s\in G$ and $\alpha\in I$. Thus, we obtain

$$\left|\left\langle \int\int u(h+s+t)d\mu_1(t)d\lambda_\alpha(s)-u_\mu,\ x^*\right\rangle\right|<\varepsilon+\varepsilon=2\varepsilon$$

for every $h\in G$ and $\alpha\geq\alpha_0$. On the other hand, we have from (9) and (12) that

$$\left|\left\langle \int u(h+s)d\lambda_\alpha(s),\ x^*\right\rangle-\left\langle \int\int u(h+s+t)d\mu_1(t)d\lambda_\alpha(s),\ x^*\right\rangle\right|$$
$$=\left|\int\left\langle u(h+s)-\sum_{i=1}^n b_i u(h+s+t_i),\ x^*\right\rangle d\lambda_\alpha(s)\right|$$
$$\leq\sum_{i=1}^n b_i\left|\int\langle u(h+s)-u(h+s+t_i),\ x^*\rangle d\lambda_\alpha(s)\right|$$
$$=\sum_{i=1}^n b_i\left|\int\langle u(h+s),\ x^*\rangle d(\lambda_\alpha-r_{t_i}^*\lambda_\alpha)(s)\right|$$
$$\leq\sum_{i=1}^n b_i\|\lambda_\alpha-r_{t_i}^*\lambda_\alpha\|M\|x^*\|<\varepsilon$$

for every $h\in G$ and $\alpha\geq a_0$. Therefore, we obtain

$$\left|\left\langle \int u(h+s)d\lambda_\alpha(s)-u_\mu,\ x^*\right\rangle\right|$$
$$\leq\left|\left\langle \int u(h+s)d\lambda_\alpha(s),\ x^*\right\rangle-\left\langle \int\int u(h+s+t)d\mu_1(t)d\lambda_\alpha(s),\ x^*\right\rangle\right|$$
$$+\left|\left\langle \int\int u(h+s+t)d\mu_1(t)d\lambda_\alpha(s)-u_\mu,\ x^*\right\rangle\right|<\varepsilon+2\varepsilon=3\varepsilon$$

for every $h\in G$ and $\alpha\geq\alpha_0$. Then, $\int u(h+t)d\lambda_\alpha(t)$ converges weakly to $y\in F(\mathscr{S})$ uniformly in h. Further, such an element y is independent of $\{\lambda_\alpha\}$ and $y=u_\mu$ for any invariant mean μ on D.

Next, put $Qu=w\text{-}\lim_\alpha\int u(t)d\lambda_\alpha(t)$ for every $u\in AO(\mathscr{S})$. Then, we show that (i), (ii) and (iii) hold.

Let us show (i). Let $u, v\in AO(\mathscr{S})$ and $\varepsilon>0$. Since $\{\lambda_\alpha\}$ is a strongly regular net of continuous linear functionals on D, there exists α' such that

$$|\lambda_\alpha(1)-1|<\frac{\varepsilon}{\|u-v\|_\infty}$$

for every $\alpha\geq\alpha'$. Then, we have, for $x^*\in E^*$ and $\alpha\geq\alpha'$,

$$\begin{aligned}
&\left\|\int u(t)d\lambda_\alpha(t)-\int v(t)d\lambda_\alpha(t)\right\| \\
&=\sup_{\|x^*\|\leq 1}\left|\int\langle u(t),x^*\rangle d\lambda_\alpha(t)-\int\langle v(t),x^*\rangle d\lambda_\alpha(t)\right| \\
&=\sup_{\|x^*\|\leq 1}\left|\int\langle u(t)-v(t),x^*\rangle d\lambda_\alpha(t)\right|\leq\left|\int\|u-v\|_\infty d\lambda_\alpha(t)\right| \\
&=\left|\int\|u-v\|_\infty d\lambda_\alpha(t)-\|u-v\|_\infty+\|u-v\|_\infty\right| \\
&\leq\|u-v\|_\infty|1-\lambda_\alpha(1)|+\|u-v\|_\infty<\varepsilon+\|u-v\|_\infty .
\end{aligned}$$

Since $x\mapsto\|x\|$ is weakly lower semicontinuous, we have

$$\begin{aligned}
\|Qu-Qv\|&\leq\liminf_\alpha\left\|\int u(t)d\lambda_\alpha(t)-\int v(t)d\lambda_\alpha(t)\right\| \\
&\leq\varepsilon+\|u-v\|_\infty .
\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we have $\|Qu-Qv\|\leq\|u-v\|_\infty$, which implies (i).

Let us show (ii). Let $u\in AO(\mathcal{S})$. Since $Qu\in F(\mathcal{S})$, we obtain $T(q)Qu=Qu$ for every $q\in G$. As in the proof of [23, Lemma 6], we can see that $s\mapsto T(q)u(s)$ is an almost-orbit of $\mathcal{S}=\{T(t):t\in G\}$ for every $q\in G$. So, $QT(q)u$ is well-defined for every $q\in G$. Let $\varepsilon>0$ and $x^*\in E^*$. Since u is an almost-orbit of $\mathcal{S}=\{T(t):t\in G\}$, there exists $t_0\in G$ such that

$$\sup_{p\in G}\|u(t+p)-T(p)u(t)\|<\frac{\varepsilon}{4\max\{\sup_\alpha\|\lambda_\alpha\|\cdot\|x^*\|,1\}}\tag{13}$$

for every $t\geq t_0$. Since $\{\lambda_\alpha\}$ is strongly regular, there exists α_1 such that

$$\|\lambda_\alpha-r_{t_0}^*\lambda_\alpha\|<\frac{\varepsilon}{4\max\{\sup\{|\langle x,x^*\rangle|:x\in C\},1\}}\tag{14}$$

for every $\alpha\geq\alpha_1$ and since

$$\int u(h+t)d\lambda_\alpha(t)\rightharpoonup u_\mu=Qu\in F(\mathcal{S})$$

uniformly in $h\in G$, there exists α_2 with $\alpha_2\geq\alpha_1$ such that

$$\sup_{h\in G}\left|\int\langle u(h+t),x^*\rangle d\lambda_\alpha(t)-\langle Qu,x^*\rangle\right|<\frac{\varepsilon}{4}\tag{15}$$

for every $\alpha \geq \alpha_2$. On the other hand, consider

$$I_1 = \left| \int \langle T(q)u(t), x^* \rangle d\lambda_\alpha(t) - \int \langle T(q)u(t_0+t), x^* \rangle d\lambda_\alpha(t) \right|,$$

$$I_2 = \left| \int \langle T(q)u(t_0+t) - u(q+t_0+t), x^* \rangle d\lambda_\alpha(t) \right|,$$

$$I_3 = \left| \int \langle u(q+t_0+t), x^* \rangle d\lambda_\alpha(t) - \int \langle u(q+t), x^* \rangle d\lambda_\alpha(t) \right|$$

and

$$I_4 = \left| \int \langle u(q+t), x^* \rangle d\lambda_\alpha(t) - \langle Qu, x^* \rangle \right|.$$

Then we have

$$\left| \int \langle T(q)u(t), x^* \rangle d\lambda_\alpha(t) - \langle Qu, x^* \rangle \right| \leq I_1 + I_2 + I_3 + I_4.$$

Then, from (14), we have

$$I_1 = \left| \int \langle T(q)u(t), x^* \rangle d\lambda_\alpha(t) - \int \langle T(q)u(t), x^* \rangle d(r_{t_0}^* \lambda_\alpha)(t) \right|$$

$$\leq \sup\{ | \langle x, x^* \rangle | : x \in C \} \cdot \| \lambda_\alpha - r_{t_0}^* \lambda_\alpha \| < \frac{\varepsilon}{4}$$

for every $q \in G$ and $\alpha \geq \alpha_2$. From (13), we obtain

$$I_2 \leq \left| \int \| T(q)u(t_0+t) - u(q+t_0+t) \| d\lambda_\alpha(t) \right| \cdot \| x^* \|$$

$$\leq \sup_{t \in G} \| T(q)u(t_0+t) - u(q+t_0+t) \| \cdot \| \lambda_\alpha \| \cdot \| x^* \| < \frac{\varepsilon}{4}$$

for every $q \in G$ and $\alpha \in I$. From (14), we have

$$I_3 = \left| \int \langle u(q+t), x^* \rangle d(r_{t_0}^* \lambda_\alpha)(t) - \int \langle u(q+t), x^* \rangle d\lambda_\alpha(t) \right|$$

$$\leq \sup_{g \in G} | \langle u(g), x^* \rangle | \cdot \| \lambda_\alpha - r_{t_0}^* \lambda_\alpha \| < \frac{\varepsilon}{4}$$

for every $q \in G$ and $\alpha \geq \alpha_2$. From (15), we obtain

$$I_4 = \left| \int \langle u(q+t), x^* \rangle d\lambda_\alpha(t) - \langle Qu, x^* \rangle \right| < \frac{\varepsilon}{4}$$

for every $g \in G$ and $\alpha \geq \alpha_2$. Therefore,

$$I=\left|\int\langle T(q)u(t), x^*\rangle d\lambda_\alpha(t)-\langle Qu, x^*\rangle\right|<\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\varepsilon$$

for all $q\in G$ and $\alpha\geq\alpha_2$. This implies that

$$w\text{-}\lim_\alpha\int T(q)u(t)d\lambda_\alpha(t)=QT(q)u=Qu$$

for every $q\in G$. Therefore, we have $QT(q)u=T(q)Qu=Qu$ for every $u\in AO(\mathcal{S})$ and $q\in G$, which implies (ii).

Let us show (iii). Since $Qu=u_\mu$ for every invariant mean on D, as in the proof of [21], we have

$$u_\mu=Qu\in\bigcap_{p\in G}\overline{co}\{u(t):t\geq p\},$$

which implies (iii). □

The following result is a generalization of Hirano [8, Theorems 3.1 and 3.2].

COROLLARY 4.2. *Let E, C, $\mathcal{S}=\{T(t):t\in G\}$ and u be as in Theorem* 4.1. *Then, $\{u(t):t\in G\}$ is weakly convergent if and only if*

$$u(s+t)-u(t)\rightharpoonup 0 \qquad \textit{for every} \quad s\in G. \tag{16}$$

In this case, the limit point of $\{u(t)\}$ is a common fixed point of $T(t)$, $t\in G$.

PROOF. We need only show the "if" part. Let $\{\mu_\alpha:\alpha\in I\}$ be a net of finite means on G such that

$$\lim_\alpha\|\mu_\alpha-r_t^*\mu_\alpha\|=0 \qquad \text{for every} \quad t\in G. \tag{$*$}$$

Then, by Lemma 3.5

$$w\text{-}\lim_\alpha\int u(h+t)d\mu_\alpha(t)=u_\mu\in F(\mathcal{S})$$

uniformly in $h\in G$, where μ is an invariant mean on $m(G)$. Let $\varepsilon>0$ and $x^*\in E^*$. Then, there exists α_0 such that

$$\left|\left\langle\int u(h+t)d\mu_\alpha(t)-u_\mu, x^*\right\rangle\right|<\frac{\varepsilon}{2}$$

for every $\alpha\geq\alpha_0$ and $h\in G$. Put

$$\mu_{\alpha_0}=\sum_{i=1}^n a_i\delta_{s_i} \qquad \left(a_i\geq 0, \sum_{i=1}^n a_i=1\right).$$

From (16), there exists $t_0\in G$ such that

$$|\langle u(t+s_i)-u(t), x^*\rangle|<\frac{\varepsilon}{2}$$

for every $t \geq t_0$ and $i=1, 2, \cdots, n$. Then, we obtain

$$\begin{aligned}
|\langle u(t)-u_\mu, x^*\rangle| &= \left|\left\langle \int u(t)d\mu_{\alpha_0}(s)-u_\mu, x^*\right\rangle\right| \\
&\leq \left|\left\langle \int u(t+s)d\mu_{\alpha_0}(s)-u_\mu, x^*\right\rangle\right| + \left|\left\langle \int [u(t+s)-u(t)]d\mu_{\alpha_0}(s), x^*\right\rangle\right| \\
&< \frac{\varepsilon}{2}+\sum_{i=1}^{n} a_i |\langle u(t+s_i)-u(t), x^*\rangle| < \varepsilon
\end{aligned}$$

for every $t \geq t_0$. This implies that $w\text{-}\lim_\alpha u(t)=\mu_\mu \in F(\mathscr{S})$. □

5. Applications.

In this section, using Theorem 4.1, we prove some nonlinear ergodic theorems in a uniformly convex Banach space which satisfies Opial's condition. Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of C into itself. Then, according to Bruck [4], a sequence $\{x_n\} \subset C$ is called an almost-orbit of T if

$$\lim_{n\to\infty}\left(\sup_{m\geq 0}\|x_{n+m}-T^m x_n\|\right)=0 .$$

THEOREM 5.1. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition. Let T be a nonexpansive mapping of C into itself and let $\{x_n\}$ be an almost-orbit of T. Then, $\frac{1}{n}\sum_{i=0}^{n-1} x_{i+k}$ converges weakly to some $y \in F(T)$, as $n \to \infty$, uniformly in $k=0, 1, 2, \cdots$.*

PROOF. Let $G=\{0, 1, 2, \cdots\}$, $\mathscr{S}=\{T^i : i \in G\}$, $D=m(G)$ and $\lambda_n(f)=\frac{1}{n}\sum_{i=0}^{n-1} f(i)$ for all $n=1, 2, \cdots$ and $f \in D$. Then, since

$$\begin{aligned}
\|\lambda_n - r_1^*\lambda_n\| &= \sup_{\|f\|\leq 1} |(\lambda_n - r_1^*\lambda_n)(f)| \\
&= \frac{1}{n}\sup_{\|f\|\leq 1} |f(0)-f(n)| \leq \frac{2}{n} \to 0 ,
\end{aligned}$$

as $n \to \infty$, $\{\lambda_n : n=1, 2, \cdots\}$ is strongly regular. Further, since $\{x_n\}$ is an almost-orbit of $\mathscr{S}=\{T^i : i \in G\}$, we obtain Theorem 5.1 using Theorem 4.1. □

Let $\mathbf{N}=\{0, 1, 2, \cdots\}$ and let $Q=\{q_{n,m}\}_{n,m\in\mathbf{N}}$ be a matrix satisfying the following conditions:

(a) $\sup_{n\geq 0}\sum_{m=0}^{\infty}|q_{n,m}|<\infty$;

(b) $\lim_{n\to\infty}\sum_{m=0}^{\infty}q_{n,m}=1$;

(c) $\lim_{n\to\infty}\sum_{m=0}^{\infty}|q_{n,m+1}-q_{n,m}|=0$.

Then, according to Lorentz [14], Q is called a strongly regular matrix. If Q is a strongly regular matrix, then for each $m\in\mathbf{N}$, we have that $|q_{n,m}|\to 0$, as $n\to\infty$ (see [10]).

THEOREM 5.2. *Let E and C be as in Theorem 5.1. Let T be a nonexpansive mapping of C into itself and let $\{x_n\}$ be an almost-orbit of T. If Q is a strongly regular matrix, then $\sum_{m=0}^{\infty}q_{n,m}x_{m+k}$ converges weakly to some $y\in F(T)$, as $n\to\infty$, uniformly in $k=0, 1, 2, \cdots$.*

PROOF. Let $G=\{0, 1, 2, \cdots\}$, $\mathscr{S}=\{T^n : n\in G\}$, $D=m(G)$ and $\lambda_n(f)=\frac{1}{n}\sum_{m=0}^{\infty}q_{n,m}f(m)$ for each $n=1, 2, \cdots$ and $f\in D$. Then, since Q is a strongly regular matrix, we have

$$\sup_{n\geq 0}\|\lambda_n\|=\sup_{n\geq 0}\sup_{\|f\|\leq 1}|\lambda_n(f)|\leq\sup_{n\geq 0}\sup_{\|f\|\leq 1}\left(\sum_{m=0}^{\infty}|q_{n,m}||f(m)|\right)$$
$$\leq\sup_{n\geq 0}\sum_{m=0}^{\infty}|q_{n,m}|<\infty$$

and

$$\lim_{n\to\infty}\lambda_n(1)=\lim_{n\to\infty}\sum_{m=0}^{\infty}q_{n,m}=1 .$$

We also have $\|\lambda_n-r_k^*\lambda_n\|\to 0$ for every $k=0, 1, 2, \cdots$. Indeed, we have that

$$\begin{aligned}\|\lambda_n-r_1^*\lambda_n\| &= \sup_{\|f\|\leq 1}|(\lambda_n-r_1^*\lambda_n)(f)|\\ &= \sup_{\|f\|\leq 1}\left|\sum_{m=0}^{\infty}q_{n,m}\{f(m)-f(m+1)\}\right|\\ &= \sup_{\|f\|\leq 1}\left|q_{n,0}f(0)+\sum_{m=0}^{\infty}q_{n,m+1}f(m+1)-\sum_{m=0}^{\infty}q_{n,m}f(m+1)\right|\\ &\leq\sum_{m=0}^{\infty}|q_{n,m+1}-q_{n,m}|+|q_{n,0}| ,\end{aligned}$$

as $n\to\infty$. Further for $k\geq 2$, we have

$$\begin{aligned}\|\lambda_n-r_k^*\lambda_n\| &\leq\|r_k^*\lambda_n-r_{k-1}^*\lambda_n\|+\cdots+\|r_1^*\lambda_n-\lambda_n\|\\ &\leq k\|\lambda_n-r_1^*\lambda_n\|\to 0 ,\end{aligned}$$

as $n \to \infty$. Therefore, $\{\lambda_n : n=1, 2, \cdots\}$ is strongly regular. So, using Theorem 4.1, we obtain Theorem 5.2. □

Let C be a bounded closed convex subset of a uniformly convex Banach space E and let $\mathcal{S}'=\{T(s) : 0 \leq s\}$ be a family of nonexpansive mappings of C into itself. Then, $\mathcal{S}'$ is called a nonexpansive semigroup on C if it satisfies the following conditions: $T(0)=I$, $T(t+s)=T(t)T(s)$ for all $t, s \in \mathbf{R}^+$ and $T(t)x$ is continuous in $t \in \mathbf{R}^+$ for each $x \in C$. According to Miyadera and Kobayasi [15], $u : \mathbf{R}^+ \to C$ is called an almost-orbit of $\mathcal{S}'=\{T(s) : 0 \leq s < +\infty\}$ if

$$\lim_{s \to \infty} \left(\sup_{t \geq 0} \|u(t+s) - T(t)u(s)\| \right) = 0 .$$

THEOREM 5.3. *Let E and C be as in Theorem 5.1. Let $\mathcal{S}'=\{T(t) : 0 \leq t < +\infty\}$ be a nonexpansive semigroup on C. Let u be an almost-orbit of $\mathcal{S}'=\{T(t) : 0 \leq t < +\infty\}$. Additionally, assume that $u : \mathbf{R}^+ \to C$ is continuous. Then, $\frac{1}{s}\int_0^s u(t+k)dt$ converges weakly to some $y \in F(\mathcal{S}')$, as $s \to \infty$, uniformly in $k \geq 0$.*

PROOF. Let $G=\mathbf{R}^+$, $\mathcal{S}=\{T(t) : 0 \leq t < \infty\}$ and let D be the Banach space $C(G)$ of all bounded continuous functions on G with the supremum norm. Define $\lambda_s(f)=\frac{1}{s}\int_0^s f(t)dt$ for every $s>0$ and $f \in D$. Then, we obtain

$$\|\lambda_s - r_k^* \lambda_s\| = \sup_{\|f\| \leq 1} \left| \frac{1}{s}\int_0^s f(t)dt - \frac{1}{s}\int_0^s f(t+k)dt \right|$$

$$= \frac{1}{s} \sup_{\|f\| \leq 1} \left| \int_0^s f(t)dt - \int_k^{s+k} f(t)dt \right| = \frac{1}{s} \sup_{\|f\| \leq 1} \left| \int_0^k f(t)dt - \int_s^{s+k} f(t)dt \right|$$

$$\leq \frac{1}{s} \sup_{\|f\| \leq 1} \left(\int_0^k |f(t)|dt + \int_s^{s+k} |f(t)|dt \right) = \frac{2k}{s} \to 0 ,$$

as $s \to \infty$. Therefore, using Theorem 4.1, we obtain Theorem 5.3. □

Using Theorem 5.3, we obtain the following theorem. However, using Theorem 4.1, we prove it.

THEOREM 5.4. *Let E and C be as in Theorem 5.1. Let $\mathcal{S}'=\{T(t) : 0 \leq t < +\infty\}$ be a nonexpansive semigroup on C. Let u be an almost-orbit of $\mathcal{S}'=\{T(t) : 0 \leq t < +\infty\}$. Additionally, assume that $u : \mathbf{R}^+ \to C$ is continuous. Then,*

$$r\int_0^\infty e^{-rt}u(t+k)dt \quad \textit{converges weakly to some } y \in F(\mathcal{S}') , \qquad \textit{as} \quad r \to 0 ,$$

uniformly in $k \geq 0$.

PROOF. Let $G=\mathbf{R}^+$, $\mathcal{S}=\{T(t) : 0 \leq t < \infty\}$ and $D=C(G)$. Define $\lambda_r(f)= r\int_0^\infty e^{-rt}f(t)dt$ for each $r>0$ and $f \in D$. Then, for each $s \in \mathbf{R}^+$, we have

$$\begin{aligned}\|\lambda_r - r_s^*\lambda_r\| &= \sup_{\|f\|\leq 1}\left|r\int_0^\infty e^{-rt}f(t)dt - r\int_0^\infty e^{-rt}f(s+t)dt\right| \\ &= \sup_{\|f\|\leq 1}\left|r\int_0^s e^{-rt}f(t)dt + r(1-e^{rs})\int_s^\infty e^{-rt}f(t)dt\right| \\ &\leq rs + |1-e^{rs}| \to 0,\end{aligned}$$

as $r\to 0$. Therefore, using Theorem 4.1, we obtain Theorem 5.4. □

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