TOKYO J. MATH.
VOL. 21, NO. 2, 1998

On Direct Sum Decomposition of Integers and Y. Ito's Conjecture

Masahito DATEYAMA and Teturo KAMAE

Osaka City University

(Communicated by Y. Maeda)

Let $\mathbf{N}=\{0, 1, 2, \cdots\}$. Every element $a\in\mathbf{N}$ can be expressed as

$$a=\sum_{i=0}^{n}\alpha_i 2^i \qquad \text{for some } n$$

where $\alpha_i\in\{0, 1\}$ for all i's.

We can identify a with $(\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n, 0, 0, \cdots)\in\{0, 1\}^{\mathbf{N}}$. We also identify $\{0, 1\}^{\mathbf{N}}$ in the usual way with $\mathbf{Z}_2$, the completion of $\mathbf{Z}$ in the 2-adic valuation norm.

Thus $\mathbf{N}$ is imbedded in $\mathbf{Z}_2$ as the 0-1-sequences with only finitely many 1's, while the negative integers are imbedded as those with finitely many 0's. For example, if a is a positive integer corresponding to the 0-1-sequence as above with $\alpha_i=0$ for any $i>n$, then $-a$ is identified with $(\underbrace{0, \cdots, 0, 1}_{m+1}, \overline{\alpha_{m+1}}, \overline{\alpha_{m+2}}, \cdots)$, where m is the smallest i with $a_i=1$ and we denote $\overline{0}=1, \overline{1}=0$.

We denote by $\bar{E}$ the closure of a subset E of $\mathbf{Z}_2$.

Let us denote

$$A=\left\{\sum_i \varepsilon_i 2^{2i+1}\ ;\ \varepsilon_i\in\{0, 1\} \text{ and } \varepsilon_i=1 \text{ for finitely many } i\text{'s}\right\}.$$

For $\omega=(\omega_0, \omega_1, \cdots)\in\{-1, 1\}^{\mathbf{N}}$ with $\omega_i=-1$ for infinitely many times, denote

$$B_\omega=\left\{\sum_i \varepsilon_i\omega_i 2^{2i}\ ;\ \varepsilon_i\in\{0, 1\} \text{ and } \varepsilon_i=1 \text{ for finitely many } i\text{'s}\right\}.$$

Let us denote

$$\mathscr{C}(A)=\{C\subset\mathbf{Z};\ 0\in C \text{ and } A\oplus C=\mathbf{Z}\}\ ,$$

where $A\oplus C=\mathbf{Z}$ implies that any element in $\mathbf{Z}$ can be written uniquely as a sum of elements in A and C.

Received March 24, 1997

THEOREM 1 (Y. Ito [1]). *Let C be a subset of* $\mathbf{Z}$ *containing* 0. *Then,* $C\in\mathscr{C}(A)$ *if and only if all of the following conditions are satisfied*:

(i) *For γ and δ in C, either $\gamma=\delta$ or the maximal number i such that 2^i divides $\gamma-\delta$ is even,*

(ii) *if a subset C' of* $\mathbf{Z}$ *satisfies the condition* (i) *and $C'\supset C$ then $C'=C$,*

(iii) *there exists an $\omega=(\omega_0, \omega_1, \cdots)\in\{-1, 1\}^{\mathbf{N}}$ with $\omega_i=-1$ for infinitely many times such that $A\oplus C\supset B_\omega$.*

Y. Ito [1] conjectured that B_ω in the above condition (iii) can be replaced by any $D\in\mathscr{C}(A)$. The aim of this paper is to generalize the above result a little bit toward the conjecture.

LEMMA 1. *Let C be a subset of* $\mathbf{Z}$ *with $0\in C$. Then C satisfies the conditions* (i) *and* (ii) *in Theorem* 1, *if and only if $C=\bar{C}\cap\mathbf{Z}$ and*

(*) *there exists a unique set of ϕ_n: $\{0, 1\}^n\to\{0, 1\}$ $(n=1, 2, \cdots)$ such that for any $n=1, 2, \cdots$ and $\xi=(\xi_0, \xi_1, \cdots, \xi_{n-1})\in\{0, 1\}^n$, there exists an element $c=(c_0, c_1, \cdots)\in C$ such that $c_{2i}=\xi_i$ $(0\le i\le n-1)$ and that for any c with this property, it holds that $c_{2i-1}=\phi_i(\xi_0, \xi_1, \cdots, \xi_{i-1})$ $(1\le i\le n)$.*

PROOF. Assume that C satisfies the conditions (i) and (ii). Suppose that every element in C takes 0 (resp. 1) at 0th coordinate. Let $n\in\mathbf{Z}$ be any odd (resp. even) number. Then $n-c$ is odd for any $c\in C$ and so $C\cup\{n\}$ satisfies the condition (i). It contradicts the condition (ii). Thus for any $a\in\{0, 1\}$ there is an element $c\in C$ whose 0th component is a.

If two elements in C have the same 0th coordinate and different values as the 1st coordinates, then the difference is divisible by 2^1 and is not divisible by 2^2. It contradicts the condition (i). Thus, ϕ_1 in the condition (*) is determined as the mapping from the value in the 0th coordinate to the value in the 1st coordinate of the elements in C.

For $n=1, 2, \cdots$, we define a condition

(P_n): there exists a unique ϕ_n such that for any $(\xi_0, \xi_1, \cdots, \xi_{n-1})\in\{0, 1\}^n$, there exists a $c=(c_0, c_1, \cdots)\in C$ such that $c_{2i}=\xi_i$ $(i=0, 1, \cdots, n-1)$ and that for any c with this property, it holds that $c_{2n-1}=\phi_n(\xi_0, \xi_1, \cdots, \xi_{n-1})$.

Assuming $(P_1), \cdots, (P_n)$, we prove (P_{n+1}). Take any $(\xi_0, \xi_1, \cdots, \xi_n)\in\{0, 1\}^{n+1}$. If there does not exist $c=(c_0, c_1, \cdots)\in C$ such that $c_{2i}=\xi_i$ $(0\le i\le n)$, then for $z=(\xi_0, z_1, \xi_1, z_3, \cdots, \xi_n, 0, 0, \cdots)\in\mathbf{Z}$ with $z_{2i-1}=\phi_i(\xi_0, \xi_1, \cdots, \xi_{i-1})$ $(1\le i\le n)$, $C\cup\{z\}$ satisfies the condition (i) by $(P_1), \cdots, (P_n)$, contradicting the condition (ii). Thus there is an element $c=(c_0, c_1, \cdots)\in C$ such that $c_{2i}=\xi_i$ $(0\le i\le n)$. Take any c like this. Then by the condition (i) and $(P_1), \cdots, (P_n)$, c_{2n+1} is determined by $(\xi_0, \xi_1, \cdots, \xi_n)$. Thus, we determine $\phi_{n+1}(\xi_0, \xi_1, \cdots, \xi_n)=c_{2n+1}$. Then, (P_{n+1}) is satisfied with this ϕ_{n+1}.

Hence we have (P_n) for any n. Thus the condition (*) holds.

Assume that $\xi=(\xi_0, \xi_1, \cdots)\in\bar{C}\cap\mathbf{Z}$. Then, it satisfies that $\xi_{2n-1}=\phi_n(\xi_0, \xi_2, \cdots, \xi_{2n-2})$ for all $n=1, 2, \cdots$ by the condition (*). Then for any $c\in C$, either $\xi=c$ or the first nonzero component of $c-\xi$ occurs at an even coordinate. This shows that $C\cup\{\xi\}$ satisfies the condition (i). By the condition (ii), $\xi\in C$. Thus $C=\bar{C}\cap\mathbf{Z}$.

Conversely, assume that C satisfies the condition $(*)$ together with $C=\bar{C}\cap\mathbf{Z}$. Then, (i) follows since for any $c\in C$, its odd coordinates are determined by its even coordinates before it.

To prove (ii), suppose that it is not satisfied. Then there exists $z\in\mathbf{Z}\backslash C$ satisfying that the maximum k such that 2^k divides $z-c$ is even for any $c\in C$. Let $z=(\xi_0, \xi_1, \cdots)$. Then, for any $n\in\mathbf{N}$, there exists an element $c=(c_0, c_1, \cdots)\in C$ such that $c_{2i}=\xi_{2i}$ $(0\le i\le n-1)$. Since the first k with $\xi_k\ne c_k$ is even, we have $k>2n-2$ for this k. Hence, $z\in\bar{C}\cap\mathbf{Z}=C$, which is a contradiction.

The following lemma follows immediately from Lemma 1:

LEMMA 2. *Let C be a subset of $\mathbf{Z}$ with $0\in C$ satisfying the conditions* (i) *and* (ii) *in Theorem* 1. *Let ϕ_n's be as in the condition* $(*)$. *Then it holds that*

$(**)$ $\xi=(\xi_0, \xi_1, \cdots)\in\bar{C}$ *if and only if* $\xi_{2n-1}=\phi_n(\xi_0, \xi_2, \cdots, \xi_{2n-2})$ *for any* $n=1, 2, \cdots$.

LEMMA 3. *Let C satisfy the condition* $(**)$ *in Lemma* 2 *for some set of ϕ_n's. Then*

$$\bar{A}\oplus\bar{C}=\mathbf{Z}_2$$

holds.

PROOF. Let ϕ_n's satisfy that for any $\xi=(\xi_0, \xi_1, \cdots)\in\mathbf{Z}_2$, $\xi\in\bar{C}$ if and only if

$$\xi_{2n-1}=\phi_n(\xi_0, \xi_1, \cdots, \xi_{2n-2}) \qquad (n=1, 2, \cdots).$$

For a given $z=(z_0, z_1, \cdots)\in\mathbf{Z}_2$, we construct $\alpha=(\alpha_0, \alpha_1, \cdots)\in\bar{A}$ and $\gamma=(\gamma_0, \gamma_1, \cdots)\in\bar{C}$ with $z=\alpha+\gamma$ as follows.

Since $\alpha_0=0$, $\gamma_0=z_0$. By $(**)$, γ_1 is determined by $\gamma_1=\phi_1(\gamma_0)$. So α_1 is determined by $\alpha_1+\gamma_1=z_1$ (mod 2), and the *carrier* c_2 is determined by $c_2=(\alpha_1+\gamma_1-z_1)/2$.

For $n\ge 2$ we can determine γ_n, α_n and c_{n+1} inductively by

$$\gamma_n+c_n=z_n \pmod 2, \quad \alpha_n=0, \quad \text{and} \quad c_{n+1}=(\gamma_n+c_n-z_n)/2$$

if n is even, and

$$\gamma_n=\phi_{(n+1)/2}(\gamma_0, \gamma_2, \cdots, \gamma_{n-1}),$$
$$\alpha_n+\gamma_n+c_n=z_n \pmod 2, \quad \text{and}$$
$$c_{n+1}=(\alpha_n+\gamma_n+c_n-z_n)/2$$

if n is odd.

Thus, we obtain α and γ in $\mathbf{Z}_2$ with $z=\alpha+\gamma$. It is clear that $\alpha\in\bar{A}$.

The uniqueness of the decomposition is proved as follows. If $\alpha+\gamma=\alpha'+\gamma'$ happens, then $\alpha-\alpha'=\gamma'-\gamma$. By $(**)$, $\gamma'-\gamma$ is either 0 or the first coordinate with different values in γ and γ' is even. However the first coordinate with different values in α and α' is odd if $\alpha\neq\alpha'$. Thus we have that $\alpha'=\alpha$ and $\gamma'=\gamma$.

THEOREM 2. *Let $C\subset\mathbf{Z}$ with $0\in C$. Then, $C\in\mathscr{C}(A)$ if and only if there exists a set of ϕ_n: $\{0, 1\}^n\to\{0, 1\}$ $(n=1, 2, \cdots)$ satisfying the condition $(**)$ in Lemma 2 such that*

($\sharp$) *for any $n\geq 1$ and for any $(\xi_0, \xi_1, \cdots, \xi_{n-1})\in\{0, 1\}^n$, there exists an $l_0\geq 0$ such that*

$$\phi_{n+l}(\xi_0, \xi_1, \cdots, \xi_{n-1}, \underbrace{1, 1, \cdots, 1}_{l})=1$$

for all $l\geq l_0$.

PROOF. Assume that $C\in\mathscr{C}(A)$. Then by Theorem 1, Lemmas 1 and 2, there exists a set of ϕ_n: $\{0, 1\}^n\to\{0, 1\}$ $(n=1, 2, \cdots)$ satisfying $(**)$ together with $(*)$. Suppose that ($\sharp$) does not hold for it. Then there exists an n and $(\xi_0, \xi_1, \cdots, \xi_{n-1})\in\{0, 1\}^n$ such that $\phi_{n+l}(\xi_0, \xi_1, \cdots, \xi_{n-1}, \underbrace{1, 1, \cdots, 1}_{l})=0$ for infinitely many l's.

Define $z=(z_0, z_1, \cdots)\in\mathbf{Z}_2$ so that $z_{2i}=\xi_i$, $z_{2i+1}=\phi_{i+1}(\xi_0, \xi_1, \cdots, \xi_i)$ $(0\leq i\leq n-1)$ and $z_i=1$ $(i\geq 2n)$. Then, z is a negative integer.

Define $\gamma=(\gamma_0, \gamma_1, \cdots)\in\bar{C}$ by $\gamma_{2i}=\xi_i$ $(0\leq i\leq n-1)$, $\gamma_{2i}=1$ $(i\geq n)$ and $\gamma_{2i-1}=\phi_i(\gamma_0, \gamma_2, \cdots, \gamma_{2i-2})$ for any $i=1, 2, \cdots$. Define $\alpha=(\alpha_0, \alpha_1, \cdots)\in\bar{A}$ by $\alpha_i=1$ for $i\geq 2n$ with $\gamma_i=0$, and $\alpha_i=0$ otherwise.

Then, $\alpha+\gamma=z$ and $\alpha\in\bar{A}\backslash A$. This is a contradiction by Lemma 3, since by our assumption, we have another decomposition of z into elements in A and C.

Conversely, assume that there exist ϕ_n's satisfying the conditions $(**)$ and ($\sharp$). By Lemma 3, we have $\bar{A}\oplus\bar{C}=\mathbf{Z}_2$. Therefore, for any $z=(z_0, z_1, \cdots)\in\mathbf{Z}$ there is an $\alpha=(\alpha_0, \alpha_1, \cdots)\in\bar{A}$ and a $\gamma=(\gamma_0, \gamma_1, \cdots)\in\bar{C}$ such that $\alpha+\gamma=z$. There are two cases for z.

Case 1. $z\geq 0$. There exists an n_0 such that $z_n=0$ for all $n\geq n_0$.

Assume that there exists an even n with $n\geq n_0$ such that there is a carrier to the n-th coordinate in the addition $\alpha+\gamma$. Since $z_n=0$ and $\alpha_n=0$, $\gamma_n=1$ and there is a carrier to the $(n+1)$-th coordinate. So γ_{n+1} must be $1-\alpha_{n+1}$ since $z_{n+1}=0$, and there is a carrier to the $(n+2)$-th coordinate. Thus $n+2$ satisfies the assumption again. Therefore $\gamma_m=1$ for all even $m\geq n$. By the condition ($\sharp$), there exists an $l_0\geq 0$ such that $\gamma_{2i-1}=1$ for all $i\geq n/2+l_0$. This shows that γ is a negative integer.

Now assume that there is no carrier to the n-th coordinate for all even n with $n\geq n_0$. Then, $\gamma_n=0$ and there is no carrier to the $(n+1)$-th coordinate for all even n with $n\geq n_0$, since $z_n=0$ and $\alpha_n=0$. Thus the fact $z_{n+1}=0$ implies $\alpha_{n+1}=\gamma_{n+1}$. If $\alpha_{n+1}=\gamma_{n+1}=1$, there must be a carrier to the $(n+2)$-th coordinate. Hence we have $\alpha_{n+1}=\gamma_{n+1}=0$. Therefore $\gamma_n=0$ for all $n\geq n_0$. This shows that γ is a nonnegative integer.

Case 2. $z<0$. There exists an n_0 such that $z_n=1$ for all $n\geq n_0$.

Let $n\geq n_0$ be even.

Assume that there is no carrier to the n-th coordinate in the addition $\alpha+\gamma$. Since

$z_n=1$, $\gamma_n=1$ and there is no carrier to the $(n+1)$-th coordinate. So γ_{n+1} must be $1-\alpha_{n+1}$ and there is no carrier to the $(n+2)$-th coordinate. Therefore, in this case, $\gamma_m=1$ for all even $m\geq n$. By the condition (♯), $\gamma_{2i-1}=1$ for all $i\geq n/2+l_0$. This shows that γ is a negative integer.

When there is a carrier to the n-th coordinate, $n+2$ satisfies the above assumption. Indeed, $\gamma_n=0$ since $\alpha_n=0$ and there is no carrier to the $(n+1)$-th coordinate. Since $z_{n+1}=1$, $\gamma_{n+1}=1-\alpha_{n+1}$ and there is no carrier to the $(n+2)$-th coordinate.

DEFINITION. Let $\psi=\{\psi_n\}_{n\geq 0}$ be a set of maps ψ_n: $\{-1, 0, 1\}^n\to\{-1, 1\}$ such that for any $(\varepsilon_0, \varepsilon_1, \cdots)\in\{-1, 0, 1\}^{\mathbf{N}}$, $\psi_n(\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_{n-1})=-1$ for infinitely many n's.

For a $\psi=\{\psi_n\}_{n\geq 0}$ as above, let $B_\psi\subset\mathbf{Z}$ be the set of

$$\beta=\sum_{n=0}^{\infty}\varepsilon_n 2^{2n}=(\beta_0, \beta_1, \beta_2, \cdots)$$

such that $\varepsilon_n\in\{-1, 0, 1\}$ satisfies that either $\varepsilon_n=0$ or $\varepsilon_n=\psi_n(\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_{n-1})$ where the constant ψ_0 can be ± 1.

THEOREM 3. *Suppose that C is an infinite set of $\mathbf{Z}$ containing* 0. *Then, $C\in\mathscr{C}(A)$ if and only if all of the following conditions are satisfied.*

(i) *For γ and δ in C, either $\gamma=\delta$ or the maximal number k such that 2^k divides $\gamma-\delta$ is even,*

(ii) *if a subset C' of $\mathbf{Z}$ satisfies the condition* (i) *and $C'\supset C$ then $C'=C$,*

(iii)′ *there is a B_ψ as in Definition such that $A\oplus C\supset B_\psi$.*

LEMMA 4. *Let B_ψ be as in Definition and $\beta=(\beta_0, \beta_1, \beta_2, \cdots)=\sum\varepsilon_i 2^{2i}\in\overline{B_\psi}$. Then, for any $n=0, 1, 2, \cdots$,*

(i) *$\beta_{2n}=\beta_{2n-1}$ if and only if $\varepsilon_n=0$, while $\varepsilon_n=0$ implies $\beta_{2n+1}=\beta_{2n}$,*

(ii) *$\beta_{2n}\neq\beta_{2n-1}$ if and only if $\varepsilon_n\neq 0$, while $\varepsilon_n\neq 0$ implies $\beta_{2n+1}=(1-\varepsilon_n)/2$, where we put $\beta_{-1}=0$.*

PROOF. Since $\beta\equiv 0, 1, 3 \pmod 4$ according to $\varepsilon_0=0, 1, -1$, respectively, we have (i) and (ii) for $n=0$. Let $n\geq 1$. Denote by $s(n)$, the maximal number i with $0\leq i\leq n-1$ such that $\varepsilon_i\neq 0$. Put $s(n)=-1$ if $\varepsilon_i=0$ for all $0\leq i\leq n-1$. Then $\beta_{2n-1}=1$ if and only if $s(n)\geq 0$ and $\varepsilon_{s(n)}=-1$. For, $\beta_{2n-1}=1$ is equivalent to the fact that β is congruent to a negative integer not less than -2^{2n-1} modulo 2^{2n}, and that $\beta_{2n-1}=0$ otherwise.

(i) If $\beta_{2n}=\beta_{2n-1}$, then β is congruent to an integer whose absolute value is not greater than 2^{2n-1} modulo 2^{2n+1}. Hence we have $\varepsilon_n=0$. The converse is also true. If $\varepsilon_n=0$, then $\beta_{2n+1}=\beta_{2n}$ since $s(n+1)=s(n)$.

(ii) By (i), $\beta_{2n}\neq\beta_{2n-1}$ if and only if $\varepsilon_n\neq 0$. Moreover $\varepsilon_n\neq 0$ implies $s(n+1)=n$, and so $\beta_{2n+1}=0$ if $\varepsilon_n=1$, and $\beta_{2n+1}=1$ if $\varepsilon_n=-1$.

LEMMA 5. *Let B_ψ be as in Definition. Then, B_ψ satisfies the condition* (∗∗).

PROOF. Let $\{\psi_n\}_{n\geq 0}$ be as in Definition. Let $\beta=\sum_{n=0}^{\infty}\varepsilon_n 2^{2n}=(\beta_0, \beta_1, \cdots)$

($\varepsilon_n \in \{-1, 0, 1\}$) be a number in $\overline{B_\psi}$.

If $\beta_0=0$, then $\varepsilon_0=0$ and so $\phi_1(0)=\beta_1=0$. If $\beta_0=1$, then $\varepsilon_0=1$ or $\varepsilon_0=-1$ according to the constant value ψ_0. Hence $\phi_1(1)=0$ if $\varepsilon_0=1$ and $\phi_1(1)=1$ if $\varepsilon_0=-1$.

Assume that $\phi_1, \phi_2, \cdots, \phi_n$ are already defined.

If $\beta_{2n}=\beta_{2n-1}$, then $\beta_{2n+1}=\beta_{2n}$ by Lemma 4. If $\beta_{2n}\neq\beta_{2n-1}$, then $\beta_{2n+1}=(1-\varepsilon_n)/2$ and $\varepsilon_n\neq 0$. In this case, $\varepsilon_n=\psi_n(\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_{n-1})$ by the definition of B_ψ. We define ϕ_{n+1} by $\phi_{n+1}(\beta_0, \beta_2, \cdots, \beta_{2n})=\beta_{2n}$ if $\beta_{2n}=\phi_n(\beta_0, \beta_2, \cdots, \beta_{2n-2})$ and $\phi_{n+1}(\beta_0, \beta_2, \cdots, \beta_{2n})$ $=(1-\psi_n(\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_{n-1}))/2$ otherwise.

Thus we have a set of maps $\phi_n: \{0, 1\}^n \to \{0, 1\}$ $(n=1, 2, \cdots)$.

By the construction of ϕ_n's, it is obvious that any $\beta=(\beta_0, \beta_1, \cdots)\in\overline{B_\psi}$ satisfies $\beta_{2n-1}=\phi_n(\beta_0, \beta_1, \cdots, \beta_{2n-2})$ $(n\geq 1)$.

Conversely assume that $\beta=(\beta_0, \beta_2, \cdots)\in \mathbf{Z}_2$ satisfies that $\beta_{2n-1}=\phi_n(\beta_0, \beta_2, \cdots, \beta_{2n-2})$ for any $n=1, 2, \cdots$. We define ε_n $(n=0, 1, 2, \cdots)$ by

$$\varepsilon_n=\begin{cases} 0 & (\beta_{2n}=\beta_{2n-1}) \\ 1-2\beta_{2n+1} & (\beta_{2n}\neq\beta_{2n-1}) . \end{cases}$$

Then, by the definition of ϕ_n's, $\varepsilon_n\neq 0$ implies that $\varepsilon_n=\psi_n(\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_{n-1})$.

Moreover, it is not difficult to prove that

$$\sum_{i=0}^{n} \varepsilon_i 2^{2i} \equiv \sum_{i=0}^{2n} \beta_i 2^i \qquad (\operatorname{mod} 2^{2n})$$

for any $n=1, 2, \cdots$.

Hence we have $\beta=\sum_{i=0}^{\infty}\varepsilon_i 2^{2i}\in\overline{B_\psi}$.

LEMMA 6. *For any B_ψ as in Definition, we have $B_\psi\in\mathscr{C}(A)$.*

PROOF. By Lemma 5, we can take ϕ_n's satisfying the condition (**) for this B_ψ. By Theorem 2, it is sufficient to prove the condition (#) for this ϕ_n's.

Take any n and $\xi=(\xi_0, \xi_1, \cdots, \xi_{n-1})\in\{0, 1\}^n$. Then by the definition for ϕ_n's, it holds that if there exists $l_0\geq 0$ such that

$$\phi_{n+l_0}(\xi_0, \xi_1, \cdots, \xi_{n-1}, \underbrace{1, 1, \cdots, 1}_{l_0})=1 ,$$

then $\phi_{n+l}(\xi_0, \xi_1, \cdots, \xi_{n-1}, \underbrace{1, 1, \cdots, 1}_{l})=1$ for any $l\geq l_0$ by Lemma 4 (i). In this case, the condition (#) holds. Suppose to the contrary that $\phi_{n+l}(\xi_0, \xi_1, \cdots, \xi_{n-1}, \underbrace{1, 1, \cdots, 1}_{l})=0$ for any $l\geq 0$. This implies the existence of $(\varepsilon_0, \varepsilon_1, \cdots)$ such that $\psi_{n+l}(\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_{n+l-1})=1$ for any $l\geq 0$, which contradicts the condition stated in Definition. Hence, we have the condition (#).

Proof of Theorem 3.

Assume that $C\subset \mathbf{Z}$ satisfies the conditions (i), (ii) and (iii)$'$. By Lemmas 2 and 3, there exist ϕ_n's such that C satisfies the condition ($**$) as well as $\bar{A}\oplus\bar{C}=\mathbf{Z}_2$. By Theorem 2, it remains only to show that ϕ_n's satisfy the condition (#).

Suppose that ϕ_n's do not satisfy the condition (#). Then there exist an m and $(\xi_0, \xi_1, \cdots, \xi_{m-1})\in\{0, 1\}^m$ such that $\phi_{m+l}(\xi_0, \xi_2, \cdots, \xi_{m-1}, \underbrace{1, 1, \cdots, 1}_{l})=0$ for infinitely many l's.

Let $\gamma=(\gamma_0, \gamma_1, \gamma_2\cdots)\in\bar{C}$ be the element defined by

$$\gamma_{2i}=\xi_i\ (i=0, 1, \cdots, m-1)\quad\text{and}\quad \gamma_{2i}=1\ (i\geq m).$$

Note that $\gamma\notin\mathbf{Z}$.

We denote τ_n $(n=1, 2, \cdots)$ in the place of ϕ_n in the condition ($**$) for B_ψ. We can define $\alpha=(\alpha_0, \alpha_1, \cdots)\in\bar{A}$ and $\beta=(\beta_0, \beta_1, \cdots)\in\overline{B_\psi}$ such that $\alpha+\gamma=\beta$ in a similar way as in the proof of Lemma 3 as follows.

Since $\alpha_0=0$, $\beta_0=\gamma_0$. Let $\beta_1=\tau_1(\beta_0)$ and define α_1 by $\alpha_1+\gamma_1=\beta_1 \pmod 2$, and the carrier c_2 is determined by $c_2=(\alpha_1+\gamma_1-\beta_1)/2$.

For $n\geq 2$, we can determine β_n, α_n and c_{n+1} inductively by

$$\gamma_n+c_n=\beta_n \pmod 2\,,\quad \alpha_n=0\,,\quad\text{and}\quad c_{n+1}=\frac{\gamma_n+c_n-\beta_n}{2}$$

if n is even, and

$$\beta_n=\tau_{(n+1)/2}(\beta_0, \beta_2, \cdots, \beta_{n-1})\,,$$

$$\alpha_n+\gamma_n+c_n=\beta_n \pmod 2\,,\quad\text{and}$$

$$c_{n+1}=\frac{\alpha_n+\gamma_n+c_n-\beta_n}{2}$$

if n is odd. We denote also $\beta=\sum_{n=0}^{\infty}\varepsilon_i 2^{2i}$ where $\varepsilon_i\in\{-1, 0, 1\}$ satisfying Definition and determined by β_{2i-1}, β_{2i}, and β_{2i+1} by Lemma 4.

We shall show that $\alpha+\gamma\in\mathbf{Z}$. Then, α and γ must be integers by the fact that $A\oplus C\supset B_\psi$. It contradicts the fact that $\gamma\notin\mathbf{Z}$.

For $i\geq m$, we call i of *type* (b, e, c) if $\beta_{2i-1}=b$, $\varepsilon_i=e$ and if $c_{2i}=c$. We use the symbol '$*$' to define a type which is indifferent to the values in the position of '$*$'. For example, i is of type $(*, 1, 1)$ if and only if $\varepsilon_i=1$ and $c_{2i}=1$.

Case 1. If there exists i $(i\geq m)$ of type $(0, *, 1)$ or of type $(*, 1, 1)$, then $\beta\in\mathbf{Z}$.

Indeed, $\beta_{2i}=0$ since there is a carrier to the $2i$-th coordinate, $\alpha_{2i}=0$ and $\gamma_{2i}=1$. By Lemma 4, we have that $\beta_{2i+1}=0$. Since there is a carrier to the $(2i+1)$-th coordinate, there must be a carrier to the $(2i+2)$-th coordinate. This shows that $i+1$ is of type

$(0, *, 1)$. Therefore we have $\beta_k=0$ for all $k\geq 2i$ inductively. In this case, β is a non-negative integer.

Case 2. If there exists i $(i\geq m)$ of type $(1, *, 0)$ or of type $(*, -1, 0)$, then $\beta\in\mathbf{Z}$.

Indeed, $\beta_{2i}=1$ since there is no carrier to the $2i$-th coordinate, $\alpha_{2i}=0$ and $\gamma_{2i}=1$. By Lemma 4, we have that $\beta_{2i+1}=1$. Since there is no carrier to the $(2i+1)$-th coordinate, there must be no carrier to the $(2i+2)$-th coordinate. This shows that $i+1$ is of type $(1, *, 0)$. Therefore we have $\beta_k=1$ for all $k\geq 2i$ inductively. In this case, β is a negative integer.

It remains to show that Case 1 or Case 2 always happen.

If i $(i\geq m)$ is of type $(1, -1, 1)$, then $\beta_{2i}=0$ since there is a carrier to the $2i$-th coordinate, $\alpha_{2i}=0$ and $\gamma_{2i}=1$. By Lemma 4, we have that $\beta_{2i+1}=1$. Since there is a carrier to the $(2i+1)$-th coordinate, $\alpha_{2i+1}+\gamma_{2i+1}=0$ (mod 2). There will be two cases.

If $\alpha_{2i+1}=\gamma_{2i+1}=0$, then there is no carrier to the $(2i+2)$-th coordinate. This shows that $i+1$ is of type $(1, *, 0)$ in the Case 2.

If $\alpha_{2i+1}=\gamma_{2i+1}=1$, then there is a carrier to the $(2i+2)$-th coordinate. This shows that $i+1$ is of type $(*, 1, 1)$ in the Case 1 or of type $(1, -1, 1)$. However, type $(1, -1, 1)$ cannot last indefinitely because of the assumption that $\gamma_{2m+2l-1}=\phi_{m+l}(\xi_0, \xi_1, \cdots, \xi_{m-1}, \underbrace{1, 1, \cdots, 1}_{l})=0$ for infinitely many l's.

If i $(i\geq m)$ is of type $(0, 1, 0)$, then $\beta_{2i}=1$ since there is no carrier to the $2i$-th coordinate. $\alpha_{2i}=0$ and $\gamma_{2i}=1$. By Lemma 4, we have that $\beta_{2i+1}=0$. Since there is no carrier to the $(2i+1)$-th coordinate, $\alpha_{2i+1}+\gamma_{2i+1}=0$ (mod 2). There will be two cases.

If $\alpha_{2i+1}=\gamma_{2i+1}=1$, then there is a carrier to the $(2i+2)$-th coordinate. This shows that $i+1$ is of type $(0, *, 1)$ in the Case 1.

If $\alpha_{2i+1}=\gamma_{2i+1}=0$, then there is no carrier to the $(2i+2)$-th coordinate. This shows that $i+1$ is of type $(*, -1, 0)$ in the Case 2 or of type $(0, 1, 0)$ again. However, type $(0, 1, 0)$ cannot last indefinitely because of the assumption that $\psi_i(\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_{i-1})=-1$ for infinitely many i's.

References

[1] Y. Ito, Direct sum decomposition of the integers, Tokyo J. Math. **18** (1995), 259–270.

Present Address:
Department of Mathematics, Osaka City University,
Sugimoto, Sumiyoshi-ku, Osaka, 558–8585 Japan.