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Complex Multiplication Formulae for Hyperelliptic Curves of Genus Three

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Introduction.

Let $\wp(u)$ be a Weierstrass elliptic function satisfying $\wp'(u)^2=4\wp(u)^3-1$. Let $\zeta:=e^{2\pi i/3}$. Then $\wp(u)$ has a property $\wp(-\zeta u)=\zeta\wp(u)$. If b is an element of $\mathbf{Z}[\zeta]$, the integer ring generated by ζ, we have a b-multiplication formula of $\wp(u)$. If b is a prime element and $b\equiv 1 \bmod 3$, the b-multiplication formula is of the form

$$\wp(bu)=\frac{\wp(u)(\wp(u)^{\mathrm{N}b-1}+\cdots+b)}{(b\wp(u)^{(\mathrm{N}b-1)/2}+\cdots\pm 1)^2}, \tag{0.1}$$

and all the coefficients belong to $\mathbf{Z}[\zeta]$. (These facts seem to be already known to Eisenstein [6]). Therefore the product of the roots $\{\wp(u)\}$ except for 0 of the numerator is equal to $\pm b$, and the product of reciprocals of the roots $\{\wp(u)\}$ of the denominator is equal to b^2. So we have factorization of b or b^2 in an extended integer ring of $\mathbf{Z}[\zeta]$. Analogous fact is known for a function $\wp(u)$ satisfying $\wp'(u)^2=4\wp(u)^3-\wp(u)$.

By using these facts essentially, the cubic and quartic Gauss sums were deeply investigated (see [12] and [13]). So it seems natural for us to expect the existence of formulae analogous to (0.1) for curves of higher genus. A remarkable formula was discovered by D. Grant for the curve of genus two defined by $y^2=x^5+1/4$ ([9]).

The purpose of this paper is to generalize his formula. Let C be a curve of genus g (≥ 1) defined by $y^2=f(x)$, where $f(x)$ is a polynomial of degree $2g+1$. Let J denote the Jacobian variety of the curve C, and $\iota: C\hookrightarrow J$ the canonical embedding. We identify J with a complex torus $\mathbf{C}^g/\Lambda$ where Λ is a lattice of $\mathbf{C}^g$. Let $u=(u_1,\cdots,u_g)$ be the canonical coordinate system of $\mathbf{C}^g$, and $\varphi(u)$ a meromorphic function on $\mathbf{C}^g/\Lambda$. We assume that $\varphi(u)$ satisfies $\varphi(-u)=-\varphi(u)$, because the Abelian functions $\varphi(u)$ we treat in this paper are odd or even functions. In the below, we denote by $x(u)$ and $y(u)$ the values of x-coordinate and y-coordinate, respectively, at u such that $u\in\iota(C)$. Then the restriction to $\iota(C)$ of the map $u\mapsto\varphi(bu)$ gives an algebraic function. Hence $\varphi\circ\iota$ has a

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rational expression of $x(u)$ and $y(u)$. Since $x(-u)=x(u)$ and $y(-u)=-y(u)$, we have an expression

$$\varphi(bu)=\frac{y(u)P(x(u))}{Q(x(u))} \tag{0.2}$$

with polynomials $P(X)$ and $Q(X)$. Here we do not mention the irreducibility of right-hand side of the expression. We regard (0.2) as a generalization of (0.1). We also call such formula a *b-multiplication formula*. However, our aim is, as mentioned about (0.1), to find a nice Abelian function $\varphi(u)$ such that every root of its numerator $P(X)$ (or its denominator) is an algebraic integer and the product of the roots gives a factorization of b or of a product of conjugates of b, in a certain integer ring.

The author found several such functions $\varphi(u)$ in the family of polynomials of hyperelliptic $\wp$-functions constructed by H. F. Baker ([2], [3] and [4]) as Grant did, because the author believes all the roots of $P(X)$ and $Q(X)$ or all of their reciprocals are algebraic integers. We will prove that the numerator of the complex multiplication formula of each our function has required properties. Our functions $\varphi(u)$ are Abelian functions associated to the following curves: curves of genus two defined by $y^2=x^5+1/4$ (Grant's case) and by $y^2=x^5-x$, and those of genus three defined by $y^2=x^7+1/4$ and by $y^2=x^7-x$ (see Theorems 6.1.6, 6.2.5, 7.1.6 and 7.2.5, respectively). Unfortunately it is generally unknown the existence of such nice functions. So the author do not explain how to find such functions.

In Section 1, we recall the fundamental facts about hyperelliptic functions from [2], [3] and [4]. We introduce a well-tuned theta series $\sigma(u)$ called the sigma function, and define Abelian functions called (hyperelliptic) $\wp$-functions as second derivatives of $\log\sigma(u)$. They are nice generalization of sigma function and $\wp$-function of Weierstrass. So our function $\varphi(u)$ is a rational function of $\sigma(u)$ and its (higher) derivatives. Dividing its numerator and denominator by certain power of $\sigma(u)$ or of its derivative yields the expression just obtained by rewriting (0.2) in terms of $\sigma(u)$ and its derivatives. In this expression, the denominator is the so-called psi function. We can prove the psi function is a polynomial of $x(u)$ or polynomial of $x(u)$ multiplied by $y(u)$ when $u\in\iota(C)$.

Now we have a rational expression of $P(x(u))$ in terms of $\sigma(u)$, $\sigma(bu)$, and their derivatives. We investigate $P(x(u))$ by using Taylor expansions of $\sigma(u)$. Such expansions are given by using differential equations of the sigma function after investigation of singularity of the theta divisor. Let $u=P_0\in\iota(C)$ be a point such that $x(P_0)=0$. For each of our curves such point P_0 is a torsion point in J. For instance, in the case of $\wp(u)$ used in (0.1), a point P_0 such that $\wp(P_0)=0$ is $(1-\zeta)$-torsion. Suppose P_0 be a c-torsion point for a non-trivial endomorphism c. Assume $b\in\mathrm{End}(J)$, the ring of endomorphisms of J, satisfies $b\equiv 1 \bmod c^2$ in $\mathrm{End}(J)$. Then we can obtain very explicitly first several terms of the Taylor expansion of $\sigma(u)$ at the image of ∞ and P_0 of C by the embedding ι. This expansion at ∞ gives the expansion of $\varphi(bu)$ on $\iota(C)$ at $\iota(\infty)$.

Hence we can determine the highest term of $P(X)$.

The most difficult part is to give the Taylor expansion of $\sigma(bu)$ at $u=P_0$. Since $\sigma(bu)=\sigma(b(u-P_0)+P_0+(b-1)P_0)$ and $(b-1)P_0\in\Lambda$ by the assumption of b, we may first use the expansion of $\sigma(v+P_0)$ at $v=0$. However, we need an explicit relation of the leading coefficients of expansions of $\sigma(v+P_0)$ and $\sigma(bv+P_0)$ at $v=0$. We can express $\sigma(P_0)$ as a special value of exponential of the linear form associated with the translational formula. The final form of the expansion of $\sigma(bu)$ at $u=P_0$ is obtained in Part II by using this expression. Thus we can determine the lowest term of $P(X)$ by this expansion. Grant determines the lowest term of $P(X)$ for the curve $y^2=x^5+1/4$ by induction on b. Since the author can not generalize such induction to other curves, he determines it by using the Taylor expansion at P_0.

In Sections 6 and 7, we prove the main results for our curves of genus two and three, respectively. As an instruction, we give proofs of original formulae for elliptic curves in Section 5 by the method of ours.

We do not discuss the integrality of the coefficients of $P(X)$ in this paper. For the curve $y^2=x^5+1/4$, the integrality of the coefficients of $P(X)$ is essentially proved by Grant (see [17]), and for the curve $y^2=x^5-x$, such thing seems to be proved similarly. The author is now preparing tools to investigate the coefficients for curves of genus three.

If we use most of the results up to Section 4, we can investigate lower and higher terms of the polynomial expression in terms of $x(u)$ of the numerator of an arbitrary Abelian function which is a polynomial of Baker's $\wp$-functions. Furthermore, if we take a 2-torsion point Q_0 instead of P_0 (then $y(Q_0)=0$) and y instead of x, we can find many Abelian functions such that their coefficients have similar properties like the above $\varphi(u)$. The reason that the author does not discuss such minor formulae is that he wants to find a formula which gives a non-canonical way to give certain power-root of b or of a product of conjugates of b as a partial product as in [12] and [13].

Convention. We denote, as usual, by **Z**, **Q**, **R** and **C** the ring of rational integers, the field of rational numbers, the field of real numbers and the field of complex numbers, respectively. The imaginary unit is denoted by i. For a variety V, the global sections of a sheaf $\mathscr{F}$ on V is denoted by $\Gamma(V,\mathscr{F})$. The sheaf associated to a divisor D is denoted by $\mathcal{O}(D)$. In an expression of the Laurent expansion of a function, the symbol $(d^\circ(z_1,\cdots,z_j)\geq n)$ means the terms of total degree at least n with respect to the variables $z_1,\cdots,z_j$. When the member of variables or the least total degree are clear from the context, we simply use the symbol $(d^\circ\geq n)$ or the dots "$\cdots$".

For cross references, we indicate a formula as (1.2.3), and each of Lemmas, Propositions, Theorems and Remarks as 4.5.6 for example.

Contents

I. Hyperelliptic Abelian Functions and Theta Divisor

In this part we treat fundamentals of the theory of hyperelliptic functions.

1. Generalities.

1.1. Differential forms and period matrices. Let C be a smooth projective model of a curve of genus $g>0$ defined over $\mathbf{C}$ whose affine equation is given by $y^2=f(x)$, where

$$f(x)=\lambda_0+\lambda_1 x+\lambda_2 x^2+\cdots+\lambda_{2g+1}x^{2g+1}\,.$$

In this paper, we keep the agreement $\lambda_{2g+1}=1$. We use, however, the letter λ_{2g+1} too when this notation makes easy to read an equation of homogeneous weight (for example, 1.5.1 below). The roots of the equation $f(x)=0$ are denoted by

$$(1.1.1)\qquad c_1, a_1, c_2, a_2, \cdots, c_g, a_g, c,$$

according to their positions (cf. Figure 1). We denote by ∞ the point of C at infinity. It is known that the set of

$$\omega^{(j)} := \frac{x^{j-1}dx}{2y} \qquad (j=1, \cdots, g)$$

makes a basis of the vector space $\Gamma(C, \Omega^1)$, where Ω^1 is the sheaf of differential forms of the first kind (see [13, p. 3.77]). Let

$$\eta^{(j)} := \frac{1}{2y} \sum_{k=j}^{2g-j} (k+1-j)\lambda_{k+1+j} x^k dx \qquad (j=1, \cdots, g),$$

which are differential forms of the second kind without poles except at ∞ (see [2, p. 195, Ex. i] or [3, p. 314]). We fix generators $\alpha^{(i)}$, $\beta^{(i)}$ $(i=1, \cdots, g)$ of the fundamental group of C such that their intersections are $\alpha^{(i)} \cdot \alpha^{(j)} = \beta^{(i)} \cdot \beta^{(j)} = 0$, $\alpha^{(i)} \cdot \beta^{(j)} = \delta_{ij}$ for $i, j=1, \cdots, g$ as illustrated in Figure 1.

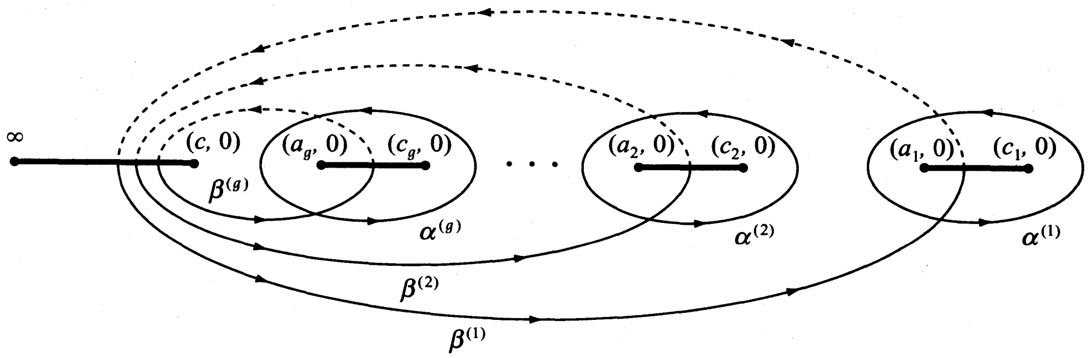

FIGURE 1

As usual we let

$$\omega' := \begin{bmatrix} \int_{\alpha^{(1)}} \omega^{(1)} & \cdots & \int_{\alpha^{(g)}} \omega^{(1)} \\ \vdots & \ddots & \vdots \\ \int_{\alpha^{(1)}} \omega^{(g)} & \cdots & \int_{\alpha^{(g)}} \omega^{(g)} \end{bmatrix}, \qquad \omega'' := \begin{bmatrix} \int_{\beta^{(1)}} \omega^{(1)} & \cdots & \int_{\beta^{(g)}} \omega^{(1)} \\ \vdots & \ddots & \vdots \\ \int_{\beta^{(1)}} \omega^{(g)} & \cdots & \int_{\beta^{(g)}} \omega^{(g)} \end{bmatrix}$$

be the period matrices. Then the modulus of C is given by $Z := \omega'^{-1}\omega''$. The lattice of periods is denoted by Λ, that is

$$\Lambda := \omega' {}^t[\mathbf{Z}\ \mathbf{Z}\ \cdots\ \mathbf{Z}] + \omega'' {}^t[\mathbf{Z}\ \mathbf{Z}\ \cdots\ \mathbf{Z}] \qquad (\subset \mathbf{C}^g).$$

We also introduce the matrices of quasi-period:

$$\eta' := \begin{bmatrix} \int_{\alpha^{(1)}} \eta^{(1)} & \cdots & \int_{\alpha^{(g)}} \eta^{(1)} \\ \vdots & \ddots & \vdots \\ \int_{\alpha^{(1)}} \eta^{(g)} & \cdots & \int_{\alpha^{(g)}} \eta^{(g)} \end{bmatrix}, \qquad \eta'' := \begin{bmatrix} \int_{\beta^{(1)}} \eta^{(1)} & \cdots & \int_{\beta^{(g)}} \eta^{(1)} \\ \vdots & \ddots & \vdots \\ \int_{\beta^{(1)}} \eta^{(g)} & \cdots & \int_{\beta^{(g)}} \eta^{(g)} \end{bmatrix}.$$

1.2. The Jacobian variety, the theta divisor. Let J be the Jacobian variety of the curve C. We identify J with the Picard group $\mathrm{Pic}^\circ(C)$ of the linearly equivalent classes of divisors of degree zero of C. Let $\mathrm{Sym}^g(C)$ be the g-th symmetric product of C. Then we have a birational map

$$\begin{aligned} \mathrm{Sym}^g(C) &\rightarrow \mathrm{Pic}^\circ(C) = J \\ (P_1, \cdots, P_g) &\mapsto \text{the class of } P_1 + \cdots + P_g - g\cdot\infty . \end{aligned}$$

As an analytic manifold, J is identified with $\mathbf{C}^g/\Lambda$. We denote by κ the canonical map $\mathbf{C}^g \rightarrow \mathbf{C}^g/\Lambda = J$. We embed C into J by $\iota : Q \mapsto Q - \infty$. Let Θ be the theta divisor, that is, the divisor of J determined by the set of classes of the form $P_1 + \cdots + P_{g-1} - (g-1)\cdot\infty$.

1.3. The hyperelliptic sigma Function $\sigma(u)$. We let

$$\delta'' := {}^t\!\left[\frac{1}{2}\ \frac{1}{2} \cdots \frac{1}{2}\right], \qquad \delta' := {}^t\!\left[\frac{g}{2}\ \frac{g-1}{2} \cdots \frac{1}{2}\right] \quad \text{and} \quad \delta := \begin{bmatrix} \delta'' \\ \delta' \end{bmatrix}.$$

For a and b in $(\frac{1}{2}\mathbf{Z})^g$, we let

$$\begin{aligned} \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z) &= \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z ; Z) \\ &= \sum_{n \in \mathbf{Z}^g} \exp[2\pi i\{\tfrac{1}{2}{}^t(n+a)Z(n+a) + {}^t(n+a)(z+b)\}] . \end{aligned}$$

Then the hyperelliptic sigma function on $\mathbf{C}^g$ associated with C is defined by

$$\tilde{\sigma}(u) = \exp(-\tfrac{1}{2}u\eta'\omega'^{-1}\,{}^tu)\vartheta[\delta](\omega'^{-1}\,{}^tu ; Z)$$

up to a multiplicative constant. We fix the constant as follows:

LEMMA 1.3.1. (1) *The lowest terms of the Taylor expansion of $\tilde{\sigma}(u)$ at $u=0$ contain the term $\gamma u_1 u_3 \cdots u_g$ if g is odd, or $\gamma u_1 u_3 \cdots u_{g-1}$ if g is even, with a non-zero constant γ independent of $u_1, \cdots, u_g$;*

(2) *The function $\tilde{\sigma}(u)$ is an odd function if $g \equiv 1, 2 \bmod 4$, and is an even one if $g \equiv 3, 0 \bmod 4$;*

(3) *The divisor of $\tilde{\sigma}(u)$ is the pull-back of Θ by the map $\kappa : \mathbf{C}^g \rightarrow \mathbf{C}^g/\Lambda = J$.*

PROOF. For a proof of (1), see [3, p. 353]. The statements (2) and (3) are given in [15, p. 3.97, p. 3.100], Proposition 6.3(c), respectively. □

In this paper, we make the following normalization: we let

$$\sigma(u):=\gamma^{-1}\tilde{\sigma}(u).$$

The constant γ in 1.3.1 for curves of genus two is studied in [7]. For more details on $\sigma(u)$, we refer the reader to [1] and [3].

1.4. Hyperelliptic Abelian functions $\wp_{jk}(u)$. For $j, k, \cdots, r\in\{1, \cdots, g\}$, let

$$\text{(1.4.1)}\qquad \begin{gathered} \sigma_j(u)=\frac{\partial}{\partial u_j}\sigma(u), \qquad \sigma_{jk\cdots r}(u)=\frac{\partial}{\partial u_j}\sigma_{k\cdots r}(u), \\ \wp_{jk}(u)=-\frac{\partial^2}{\partial u_j\partial u_k}\log\sigma(u), \qquad \wp_{jk\cdots r}(u)=\frac{\partial}{\partial u_j}\wp_{k\cdots r}(u). \end{gathered}$$

Then the functions $\wp_{jk\cdots r}(u)$ are Abelian functions on the Jacobian variety J of C. We call each of these functions, simply, a $\wp$-function when we talk about their uniform properties. In the genus one case, the function $\wp_{11}(u)$ is essentially the Weierstrass elliptic function.

Let $(u_1, \cdots, u_g)$ be the system of variables of $\sigma(u)$. Then we can find a set of g points $(x_1, y_1), \cdots, (x_g, y_g)$ on C such that

$$\text{(1.4.2)}\qquad u_j=\int_\infty^{(x_1,y_1)}\omega^{(j)}+\cdots+\int_\infty^{(x_g,y_g)}\omega^{(j)} \qquad (j=1, \cdots, g)$$

with certain paths of integrals. In this situation, the $\wp$-functions are characterized as follows ([3, p. 377]).

LEMMA 1.4.1. *Assume that the variables $u_1, \cdots, u_g$ of $\sigma(u)$ depend on g variable points $(x_1, y_1), \cdots, (x_g, y_g)$ of C by the equation* (1.4.2). *Let*

$$F(X_1, X_2)=\sum_{j=0}^{g} X_1^jX_2^j(\lambda_{2j+1}(X_1+X_2)+2\lambda_{2j}).$$

Then the functions $\wp_{jk}(u)$ are characterized by the equations

$$\sum_{j=1}^{g}\sum_{k=1}^{g}\wp_{jk}(u)x_r^{j-1}x_s^{k-1}=\frac{F(x_r, x_s)-2y_ry_s}{(x_r-x_s)^2},$$

$$x_r^g-\sum_{j=1}^{g}\wp_{jg}(u)x_r^{j-1}=0$$

for $r, s=1, \cdots, g$ with $r\neq s$. Especially, the functions $\wp_{ij}(u)$ are defined over the field $\mathbf{Q}(\lambda_0, \cdots, \lambda_{2g+1})$, and $(-1)^{g-j}\wp_{gj}(u)$ is the elementary symmetric function of degree $g-j+1$ of $x_1, \cdots, x_g$.

For more details on $\wp$-functions, we refer the reader to [2] and [3].

By Lemma 1.3.1(3), we know that

$$(1.4.3)\qquad \wp_{ij}(u)\in\Gamma(J,\mathcal{O}(2\Theta)),\quad \wp_{ijk}(u)\in\Gamma(J,\mathcal{O}(3\Theta)),\quad \wp_{ijkl}(u)\in\Gamma(J,\mathcal{O}(4\Theta)),$$

where $\Gamma(J,\mathcal{O}(n\Theta))$ denotes the functions on J having poles, only along Θ, with at most n-th order.

1.5. Algebraic relations for $\wp$-functions. Here we recall relations among the functions $\wp_{ij}(u)$ and $\wp_{ijkl}(u)$.

PROPOSITION 1.5.1. *Let $\wp_{ijkl}:=\wp_{ijkl}(u)$ and $\wp_{ij}:=\wp_{ij}(u)$ for simplicity. The following equations hold for $g=1$, 2 and 3:*

(1) $\wp_{3333}-6\wp_{33}^2=2\lambda_5\lambda_7+4\lambda_6\wp_{33}+4\lambda_7\wp_{32}$,

(2) $\wp_{3332}-6\wp_{33}\wp_{32}=4\lambda_6\wp_{32}+2\lambda_7(3\wp_{31}-\wp_{22})$,

(3) $\wp_{3331}-6\wp_{31}\wp_{33}=4\lambda_6\wp_{31}-2\lambda_7\wp_{21}$,

(4) $\wp_{3322}-4\wp_{32}^2-2\wp_{33}\wp_{22}=2\lambda_5\wp_{32}+4\lambda_6\wp_{31}-2\lambda_7\wp_{21}$,

(5) $\wp_{3321}-2\wp_{33}\wp_{21}-4\wp_{32}\wp_{31}=2\lambda_5\wp_{31}$,

(6) $\wp_{3311}-4\wp_{31}^2-2\wp_{33}\wp_{11}=2\Delta$,

(7) $\wp_{3222}-6\wp_{32}\wp_{22}=-4\lambda_2\lambda_7-2\lambda_3\wp_{33}+4\lambda_4\wp_{32}+4\lambda_5\wp_{31}-6\lambda_7\wp_{11}$,

(8) $\wp_{3221}-4\wp_{32}\wp_{21}-2\wp_{31}\wp_{22}=-2\lambda_1\lambda_7+4\lambda_4\wp_{31}-2\Delta$,

(9) $\wp_{3211}-4\wp_{31}\wp_{21}-2\wp_{32}\wp_{11}=-4\lambda_0\lambda_7+2\lambda_3\wp_{31}$,

(10) $\wp_{3111}-6\wp_{31}\wp_{11}=4\lambda_0\wp_{33}-2\lambda_1\wp_{32}+4\lambda_2\wp_{31}$,

(11) $\wp_{2222}-6\wp_{22}^2=-8\lambda_2\lambda_6+2\lambda_3\lambda_5-6\lambda_1\lambda_7$
$\qquad -12\lambda_2\wp_{33}+4\lambda_3\wp_{32}+4\lambda_4\wp_{22}+4\lambda_5\wp_{21}-12\lambda_6\wp_{11}+12\Delta$,

(12) $\wp_{2221}-6\wp_{22}\wp_{21}=-4\lambda_1\lambda_6-8\lambda_0\lambda_7-6\lambda_1\wp_{33}+4\lambda_3\wp_{31}+4\lambda_4\wp_{21}-2\lambda_5\wp_{11}$,

(13) $\wp_{2211}-4\wp_{21}^2-2\wp_{22}\wp_{11}=-8\lambda_0\lambda_6-8\lambda_0\wp_{33}-2\lambda_1\wp_{32}+4\lambda_2\wp_{31}+2\lambda_3\wp_{21}$,

(14) $\wp_{2111}-6\wp_{21}\wp_{11}=-2\lambda_0\lambda_5-8\lambda_0\wp_{32}+2\lambda_1(3\wp_{31}-\wp_{22})+4\lambda_2\wp_{21}$,

(15) $\wp_{1111}-6\wp_{11}^2=-4\lambda_0\lambda_4+2\lambda_1\lambda_3+4\lambda_0(4\wp_{31}-3\wp_{22})+4\lambda_1\wp_{21}+4\lambda_2\wp_{11}$,

where

$$\Delta=\wp_{32}\wp_{21}-\wp_{31}\wp_{22}+\wp_{31}^2-\wp_{33}\wp_{11}\,.$$

These equations are presented under the convention that if $g=1$ or 2 then λ_i with $i>2g+1$ and $\wp$-functions whose suffix contain j bigger than g are all zero.

Note that when $g=1$ the equation (15) above is a well-known equation derived from $\wp'(u)^2=4f(\wp(u))$. We refer the reader to [4] for the proof of this proposition.

1.6. The algebraic addition formulae. Here we present algebraic addition formulae which express each function $\wp_{kl}(u+v)$ as a rational function of $\{\wp_{ij}(u)\}$, $\{\wp_{ij}(v)\}$, $\{\wp_{hij}(u)\}$ and $\{\wp_{hij}(v)\}$ with $1\le h\le g$, $1\le i\le g$ and $1\le j\le g$.

PROPOSITION 1.6.1. *$\sigma(u+v)\sigma(u-v)/(\sigma(u)^2\sigma(v)^2)$ can be expressed as a polynomial in* *the* $g(g+1)$ *functions* $\{\wp_{ij}(u)\}$ *and* $\{\wp_{ij}(v)\}$ *with coefficients in* $\mathbf{Q}$.

For a proof of this proposition we refer the reader to [3].

COROLLARY 1.6.2. *Each function* $\wp_{ij\cdots r}(u+v)$ *has a rational expression in terms of the functions* $\{\wp_{ij}(u)\}$, $\{\wp_{ij}(v)\}$, $\{\wp_{hij}(u)\}$ *and* $\{\wp_{hij}(v)\}$ *with coefficients in* $\mathbf{Q}(\lambda_0, \cdots, \lambda_{2g+1})$.

PROOF. After logarithmically differentiating the expression of 1.6.1 by u_i and v_i, respectively, by adding the obtained two equations, we have a rational expression of $2\frac{\partial}{\partial u_i}\log\sigma(u+v)-4\frac{\partial}{\partial u_i}\log\sigma(u)-4\frac{\partial}{\partial v_i}\log\sigma(v)$ in the functions $\{\wp_{ij}(u)\}$, $\{\wp_{ij}(v)\}$, $\{\wp_{hij}(u)\}$ and $\{\wp_{hij}(v)\}$. We operate $\partial/\partial u_j$ to this expression. Then we have a rational expression of $\wp_{ij}(u+v)$ in the functions $\{\wp_{ij}(u)\}$, $\{\wp_{ij}(v)\}$, $\{\wp_{hij}(u)\}$, $\{\wp_{hij}(v)\}$, $\{\wp_{ijkl}(u)\}$, $\{\wp_{ijkl}(v)\}$, $\{\wp_{ijklm}(u)\}$ and $\{\wp_{ijklm}(v)\}$. We can obtain the desired expression by using the equations in 1.5.1. □

1.7. Geometry of the theta divisor. To make clear the subsequent argument, we define and fix the local parameter t at each point P of C by

$$(1.7.1)\qquad t=\begin{cases} x-x(P) & \text{if } P \text{ is an ordinary point}\,, \\ y & \text{if } P \text{ is a branch point different from } \infty\,, \\ 1/\sqrt{x} & \text{if } P=\infty\,. \end{cases}$$

Here we call P a branch point if $y(P)=0$ or ∞, and an ordinary point otherwise.

We determine the singular locus of the theta divisor Θ by using certain matrix attached to a positive divisor of C. Here our argument is based on [5, pp. 85–86]. For a point P of C, let t be the local parameter defined above. We denote by P_t the point of C such that the value of t at P_t is t. Then we define for $\mu\in\Gamma(C, \Omega^1)$

$$D^l\mu(P)=\frac{d^{l+1}}{dt^{l+1}}\int_{\infty}^{P_t}\mu\,\bigg|_{t=0}\,.$$

Since μ is a holomorphic form, $D^l\mu(P)$ takes finite value at every point P. Let $D:=\sum_{j=1}^{k} n_jP_j$ be a positive divisor. We define by $B(D)$ the matrix with g rows and $\deg D:=\sum n_j$ columns whose $(i, n_1+\cdots+n_{j-1}+l)$-entry is $D^l\omega^{(i)}(P_j)$. This matrix $B(D)$ informs us of singularity of Θ in J at the point determined by the divisor $D-(\deg D)\infty$. For $\mu\in\Gamma(C, \Omega^1)$, we can find uniquely $c_1, \cdots, c_g\in\mathbf{C}$ such that $\mu=c_1\omega^{(1)}+\cdots+c_g\omega^{(g)}$. In this situation, the three statements

(1) $\mu\in\Gamma(C, \Omega^1(-D))$,

(2) $D^l\mu(P_j)=0$ for all j and l with $1\le j\le k$ and $0\le l\le n$, and

(3) $B(D)\,{}^t[c_1 \cdots c_g]={}^t[0 \cdots 0]$

are equivalent. So $\dim\Gamma(C, \mathcal{O}(D))=g-\operatorname{rank}B(D)$. The Riemann-Roch theorem states $\dim\Gamma(C, \mathcal{O}(D))=\deg D-g+1+\dim\Gamma(C, \Omega^1(-D))$. Hence

$$\dim\Gamma(C, \mathcal{O}(D)) = \deg D + 1 - \operatorname{rank} B(D)\,. \tag{1.7.2}$$

However, by [1, p. 190, (4.5)], the singularity of Θ is known as follows.

LEMMA 1.7.1. *The singular locus of Θ is the points determined by the elements of* $\{P_1 + \cdots + P_{g-1} - (g-1)\infty \mid \dim\Gamma(C, \mathcal{O}(P_1 + \cdots + P_{g-1})) > 1\}$.

By (1.7.2), $\dim\Gamma(C, \mathcal{O}(D)) = 1$ if and only if $\operatorname{rank} B(D) = \deg D$. So we can determine the singular locus of Θ by calculating $\operatorname{rank} B(D)$. The result is

LEMMA 1.7.2. (1) *If $g=2$, Θ is non-singular.*
(2) *If $g=3$, Θ has only one non-singular point at the origin $O=(0, \cdots, 0)$.*

PROOF. We first show (2). For two points P_1 and P_2, we calculate $B(D)$ and its rank in each case that $P_1 = P_2$ or $P_1 \neq P_2$, and that each P_i is ∞, a branch point different from ∞, or an ordinary point. Then we see $\operatorname{rank} B(D)$ is 1 only when $P_1 = P_2 = \infty$ and is 2 ($= \deg(P_1 + P_2)$) otherwise. According to 1.7.1 and the statement above this lemma, we conclude the assertion (2). The assertion (1) is shown by a similar explicit calculation of $B(P)$ for each point P. □

2. Taylor and Laurent expansions.

In this section, we give lower terms of the Taylor expansion of $\sigma(u)$ at each point on the curve C.

2.1. Taylor expansion of $\sigma(u)$ at O. Let $O = (0, \cdots, 0) \in \mathbf{C}^g$.

PROPOSITION 2.1.1. (1) *If $g=1$, then the Taylor expansion of $\sigma(u)$ is of the following form*:

$$\sigma(u) = u + (d^\circ \geq 1)\,.$$

(2) *If $g=2$, then the Taylor expansion of $\sigma(u)$ is of the following form*:

$$\sigma(u) = u_1 + \tfrac{1}{6}\lambda_2 u_1^3 - \tfrac{1}{3}\lambda_5 u_2^3 + (d^\circ \geq 5)\,, \qquad (\lambda_5 = 1)\,.$$

(3) *If $g=3$, then the Taylor expansion of $\sigma(u)$ is of the form*

$$\sigma(u) = u_1u_3 - u_2^2 - \frac{\lambda_0}{3}u_1^4 - \frac{\lambda_1}{3}u_1^3u_2 - \lambda_2 u_1^2u_2^2 - \frac{\lambda_3}{3}u_1u_2^3 - \frac{\lambda_4}{3}u_2^4$$
$$- \frac{2\lambda_2}{3}u_1^3u_3 - \frac{\lambda_5}{3}u_2^3u_3 - \frac{\lambda_6}{2}u_2^2u_3^2 + \frac{\lambda_6}{6}u_1u_3^3 - \frac{\lambda_7}{3}u_2u_3^3 + (d^\circ \geq 6)\,, \qquad (\lambda_7 = 1)\,,$$

and the coefficient of the term u_3^6 is $\lambda_7/45$.

Proposition 2.1.1 will be used in Sections 5, 6 and 7. The last statement about a term of degree six is used only in 3.2.3.

PROOF OF 2.1.1. (1) is well-known. The proof of (2) was given by Baker, and is reproduced in [7, pp. 129–130]. Let us prove (3). Since $\sigma(-u)=\sigma(u)$, the terms of odd total degree vanish. From [3, p. 353], we know that the constant term vanishes, and the form of terms of second order is $u_1u_3-u_2^2$. Hence $\sigma_{22}(O)\neq 0$, $\sigma_{31}(O)\neq 0$ and the other partial derivatives of second order vanish. The method to compute the terms of higher degree is essentially the same as in the proof of (2) in [7]. We set $u=O$, after operating $\partial^2/\partial u_1\partial u_3$ or $\partial^2/\partial u_2^2$ to the equations, of $\sigma(u)$ and its partial derivatives, obtained from (6), (8) and (11) of 1.5.1 by multiplying $\sigma(u)^2$. Then we have the following six equations:

$$(\sigma^2\Delta)_{31}(O)=-\tfrac{1}{2}\sigma_{3311}(O)\,,\qquad (\sigma^2\Delta)_{31}(O)=\sigma_{3311}(O)\,,$$
$$(\sigma^2\Delta)_{31}(O)=-\tfrac{2}{3}\lambda_4+(-\tfrac{1}{12}\sigma_{2222}+\sigma_{3221})(O)\,,\qquad (\sigma^2\Delta)_{22}(O)=(\sigma_{3311}-2\sigma_{3221})(O)\,,$$
$$(\sigma^2\Delta)_{22}(O)=4\lambda_4+(\tfrac{1}{2}\sigma_{2222}+2\sigma_{3221})(O)\,,\qquad (\sigma^2\Delta)_{22}(O)=-\tfrac{4}{3}\lambda_4-\tfrac{1}{6}\sigma_{2222}(O)\,.$$

These equations yield $\sigma_{2222}(O)=-8\lambda_4$, $\sigma_{3221}(O)=\sigma_{3311}(O)=0$. Furthermore, we rewrite the leftover eleven equations in 1.5.1 by $\sigma(u)$ and its partial derivatives by the definition of $\wp$-functions. Multiplying $\sigma(u)^2$ to, for instance, the equation obtained from 1.5.1(1) yields

$$\begin{aligned}&\sigma_{3333}(u)\sigma(u)+4\sigma_{333}(u)\sigma_3(u)-\sigma_{33}(u)^2\\&\quad=2\lambda_5\lambda_7\sigma(u)^2+4\lambda_6(\sigma_3(u)^2-\sigma_{33}(u)\sigma(u))+4\lambda_7(\sigma_3(u)\sigma_2(u)-\sigma_{32}(u)\sigma(u))\,.\end{aligned}$$

After operating $\partial^2/\partial u_2^2$ on this, by plugging $u=O$, we have

$$-\sigma_{3333}(O)\sigma_{22}(O)=0\,. \tag{2.1.1}$$

Since $\sigma_{22}(O)\neq 0$, we obtain $\sigma_{3333}(O)=0$. This shows that the term of u_3^4 vanishes. Similarly, the leftover equations (2), (3), (4), (5), (7), (9), (10), (12), (13), (14) and (15) of 1.5.1 give rise to the coefficients of the terms of $u_2u_3^3$, $u_1u_3^3$, $u_2^2u_3^2$, $u_1u_2u_3^2$, $u_2^3u_3$, $u_1^2u_2u_3$, $u_1^3u_3$, $u_1u_2^3$, $u_1^2u_2^2$, $u_1^3u_2$ and u_1^4, respectively. We can show $\sigma_{333333}(O)=16\lambda_7$ by operating $\partial^4/\partial u_2^4$ on the equation 1.5.1(1) multiplied by $\sigma(u)^2$. □

2.2. Taylor expansion of $\sigma(u)$ at each point of C other than O.

PROPOSITION 2.2.1. *Let P be an arbitrary point of $\kappa^{-1}\iota(C)$ different from points in Λ. Then the following statements hold.*

(1) *If $g=1$ then $\sigma(P)\neq 0$.*

(2) *If $g=2$ then $\sigma_2(P)\neq 0$ and $\sigma_1(P)=-x(P)\sigma_2(P)$. Furthermore the partial derivatives at P of third degree are written by ones of first and second degree as in the following*:

$$\sigma_{111}(P)=\left(3\frac{\sigma_{21}\sigma_{11}}{\sigma_2}-\frac{3}{2}\frac{\sigma_{22}\sigma_{11}\sigma_1}{\sigma_2^2}-3\frac{\sigma_{21}^2\sigma_1}{\sigma_2^2}+3\frac{\sigma_{21}\sigma_{22}\sigma_1^2}{\sigma_2^3}\right.$$

$$-\frac{3}{4}\frac{\sigma_{22}^2\sigma_1^3}{\sigma_2^4}-2\lambda_1\sigma_2+4\lambda_2\sigma_1-3\lambda_3\frac{\sigma_1^2}{\sigma_2}-3\lambda_4\frac{\sigma_1^3}{\sigma_2^2}-3\lambda_5\frac{\sigma_1^4}{\sigma_2^3}\Bigg)(P),$$

$$\sigma_{112}(P)=\left(\frac{1}{2}\frac{\sigma_{22}\sigma_{11}}{\sigma_2}+\frac{\sigma_{21}^2}{\sigma_2}-\frac{\sigma_{22}\sigma_{21}\sigma_1}{\sigma_2^2}+\frac{1}{4}\frac{\sigma_{22}^2\sigma_1^2}{\sigma_2^3}+\lambda_3\sigma_1+\lambda_4\frac{\sigma_1^2}{\sigma_2}+\lambda_5\frac{\sigma_1^3}{\sigma_2^2}\right)(P),$$

$$\sigma_{122}(P)=\left(\frac{\sigma_{22}\sigma_{21}}{\sigma_2}-\frac{1}{4}\frac{\sigma_{22}^2\sigma_1}{\sigma_2}-\lambda_4\sigma_1-\lambda_5\frac{\sigma_1^2}{\sigma_2}\right)(P),$$

$$\sigma_{222}(P)=\left(\frac{3}{4}\frac{\sigma_{22}^2}{\sigma_2}+\lambda_4\sigma_2+\lambda_5\sigma_1\right)(P).$$

(3) *If $g=3$ then $\sigma_3(P)=0$, $\sigma_2(P)\neq 0$, $\sigma_1(P)=-x(P)\sigma_2(P)$ and $(\sigma^2\Delta)(P)=(\lambda_7\sigma_1^3/\sigma_2)(P)$. Furthermore, the partial derivatives at P of third degree are written by ones of first and second degree as in the following form*:

$$\sigma_{111}(P)=\Bigg(-3\frac{\sigma_{21}\sigma_{11}}{\sigma_2}-\frac{3}{2}\frac{\sigma_{22}\sigma_{11}\sigma_1}{\sigma_2^2}-3\frac{\sigma_{21}^2\sigma_1}{\sigma_2^2}+3\frac{\sigma_{21}\sigma_{22}\sigma_1^2}{\sigma_2^3}-\frac{3}{4}\frac{\sigma_{22}^2\sigma_1^3}{\sigma_2^4}-2\lambda_1\sigma_2$$
$$+4\lambda_2\sigma_1-3\lambda_3\frac{\sigma_1^2}{\sigma_2}-3\lambda_4\frac{\sigma_1^3}{\sigma_2^2}-3\lambda_5\frac{\sigma_1^4}{\sigma_2^3}+3\lambda_6\frac{\sigma_1^5}{\sigma_2^4}+\frac{3}{4}\lambda_7\frac{\sigma_1^6}{\sigma_2^5}\Bigg)(P),$$

$$\sigma_{112}(P)=\Bigg(\frac{1}{2}\frac{\sigma_{22}\sigma_{11}}{\sigma_2}+\frac{\sigma_{21}^2}{\sigma_2}-\frac{\sigma_{22}\sigma_{21}\sigma_1}{\sigma_2^2}+\frac{1}{4}\frac{\sigma_{22}^2\sigma_1^2}{\sigma_2^3}+\lambda_3\sigma_1$$
$$+\lambda_4\frac{\sigma_1^2}{\sigma_2}+\lambda_5\frac{\sigma_1^3}{\sigma_2^2}-\lambda_6\frac{\sigma_1^4}{\sigma_2^3}-\lambda_7\frac{\sigma_1^5}{\sigma_2^4}\Bigg)(P),$$

$$\sigma_{122}(P)=\left(\frac{\sigma_{22}\sigma_{21}}{\sigma_2}-\frac{1}{4}\frac{\sigma_{22}^2\sigma_1}{\sigma_2}-\lambda_4\sigma_1-\lambda_5\frac{\sigma_1^2}{\sigma_2}+\lambda_6\frac{\sigma_1^3}{\sigma_2^2}+\lambda_7\frac{\sigma_1^4}{\sigma_2^3}\right)(P),$$

$$\sigma_{222}(P)=\left(\frac{3}{4}\frac{\sigma_{22}^2}{\sigma_2}+\lambda_4\sigma_2+\lambda_5\sigma_1-3\lambda_6\frac{\sigma_1^2}{\sigma_2}+3\lambda_7\frac{\sigma_1^3}{\sigma_2^2}\right)(P),$$

$$\sigma_{113}(P)=\left(\frac{\sigma_{32}\sigma_{11}}{\sigma_2}\right)(P),\qquad \sigma_{123}(P)=\left(\frac{\sigma_{32}\sigma_{21}}{\sigma_2}\right)(P),$$

$$\sigma_{133}(P)=\left(\frac{\sigma_{32}\sigma_{21}}{\sigma_2}-\lambda_7\frac{\sigma_1^2}{\sigma_2}\right)(P),\qquad \sigma_{223}(P)=\left(\frac{\sigma_{32}\sigma_{22}}{\sigma_2}-2\lambda_7\frac{\sigma_1^2}{\sigma_2}\right)(P),$$

$$\sigma_{233}(P)=\left(\frac{\sigma_{32}^2}{\sigma_2}-\lambda_7\sigma_1\right)(P).\qquad \sigma_{333}(P)=-2\lambda_7\sigma_2(P).$$

PROOF. The assertion (1) is well-known. We show (3). Since

$$\frac{\sigma_1}{\sigma_2}(P)=\frac{\sigma_1\sigma_2-\sigma_{12}\sigma}{\sigma_2^2-\sigma_{22}\sigma}(P)=\frac{\wp_{12}}{\wp_{22}}(P)=\frac{x_1x_2x_3}{-x_1x_2-x_2x_3-x_3x_1}\Bigg|_{\substack{x_1=x_2=\infty\\ x_3=x(P)}}=-x(P),$$

$$\frac{\sigma_3}{\sigma_2}(P)=\frac{\sigma_3\sigma_2-\sigma_{32}\sigma}{\sigma_2^2-\sigma_{22}\sigma}(P)=\frac{\wp_{32}}{\wp_{22}}(P)=\left.\frac{x_1+x_2+x_3}{-x_1x_2-x_2x_3-x_3x_1}\right|_{\substack{x_1=x_2=\infty\\ x_3=x(P)}}=0\,,$$

and $P\neq O$ it must be $\sigma_1(P)=\sigma_3(P)=0$ and so $\sigma_2(P)\neq 0$ by virtue of 1.7.2. We get $\sigma_{33}^2(P)=0$ by setting $u=P$ in the equation which is obtained from 1.5.1(1) by writing it in terms of $\sigma(u)$ and its partial derivatives and multiplying it by $\sigma(u)^2$. Hence

$$\sigma_{33}(P)=0\,. \tag{2.2.1}$$

We note that $\Delta(u)\in\Gamma(J,\mathcal{O}(2\Theta))$ by (6), (8) or (11) of 1.5.1. So we get

$$(\sigma_1\sigma_{32})(P)=(\sigma_2\sigma_{31})(P) \tag{2.2.2}$$

by plugging $u=P$ in the equation

$$(\sigma^3\Delta)(u)=(\sigma_3\sigma_2\sigma_{21}-\sigma_2\sigma_1\sigma_{32}+\sigma_3\sigma_1\sigma_{22}+\sigma_2^2\sigma_{31}-2\sigma_3\sigma_1\sigma_{31}+\sigma_3^2\sigma_{11}-\sigma_1^2\sigma_{33}$$
$$+\sigma_{32}\sigma_{21}\sigma-\sigma_{31}\sigma_{22}\sigma+\sigma_{31}^2\sigma-\sigma_{11}\sigma_{33}\sigma)(u)\,.$$

Here we have used (2.2.1), $\sigma_3(P)=0$ and $\sigma(P)=0$. The rest of our proof are also done by repeating the same operation as above. Though the facts (2.2.1), (2.2.2) and $\sigma_2(P)\neq 0$ are used often in the following, we do not mention it in the proof when they used. The equation 1.5.1(6) gives rise to

$$(\sigma^2\Delta)(P)=\lambda_7(\sigma_1^3/\sigma_2)(P)\,. \tag{2.2.3}$$

Then the equations (8) and (11) of 1.5.1 give rise to the formulae for $\sigma_{321}(P)$ and $\sigma_{222}(P)$ by (2.2.3). The equations (3) and (15) of 1.5.1 are not necessary here. The leftover equations (2), (4), (5), (7), (9), (10), (12), (13) and (14) of 1.5.1 give rise to the formulae for $\sigma_{333}(P)$, $\sigma_{332}(P)$, $\sigma_{331}(P)$, $\sigma_{322}(P)$, $\sigma_{311}(P)$, $\sigma_{221}(P)$, $\sigma_{211}(P)$ and $\sigma_{111}(P)$, respectively. The assertion (2) is obtained by similar calculations. □

2.3. The Laurent expansions of analytic coordinates on C. There are two different coordinates which identify a point of $\kappa^{-1}\iota(C)$ or $\iota(C)$, the analytic coordinate $u=(u_1,\cdots,u_g)$ and a pair of solution (x,y) of the algebraic affine equation defining C. This subsection is used to make relate these coordinates. If $u\in\kappa^{-1}\iota(C)$ and $\kappa(u)=\iota(x,y)$, then, by (1.4.2),

$$u_j=\int_\infty^{(x,y)}\omega^{(j)}\qquad(j=1,\cdots,g) \tag{2.3.1}$$

with certain paths of integrals.

LEMMA 2.3.1. *The Laurent expansion of* $x(u)$ *and* $y(u)$ *at* $u=O$ *on the pull-back* $\kappa^{-1}\iota(C)$ *of* C *to* $\mathbf{C}^g$ *are*

$$x(u)=\frac{1}{u_g^2}+(d^\circ(u_g)\geq -1)\,,\qquad y(u)=-\frac{1}{u_g^{2g+1}}+(d^\circ(u_g)\geq -2g)\,.$$

PROOF. We take $t=1/\sqrt{x}$ as a local parameter at O along $\kappa^{-1}\iota(C)$. If u is in $\kappa^{-1}\iota(C)$ and sufficiently near O, we agree to that t, $u=(u_1, \cdots, u_g)$ and (x, y) are coordinates of the same point on C. Then

$$u_g=\int_{\infty}^{(x,y)}\frac{x^{g-1}dx}{2y}=\int_{\infty}^{(x,y)}\frac{x^{-3/2}dx}{2\sqrt{1+\lambda_6(1/x)+\cdots+\lambda_0(1/x^{2g+1})}}$$
$$=\int_0^t\frac{t^3\cdot(-2/t^3)dt}{2+(d^\circ\geq 1)}=-t+(d^\circ(t)\geq 2)\,.$$

Hence $x(u)=1/u_g^2+(d^\circ(u_g)\geq -1)$ and our assertion is proved. □

LEMMA 2.3.2. *If $u\in\kappa^{-1}\iota(C)$, then the following statements hold.*

(1) *If $g=2$ then*

$$u_1=\tfrac{1}{3}u_2^3+(d^\circ(u_2)\geq 4)\,.$$

(2) *If $g=3$ then*

$$u_1=\tfrac{1}{5}u_3^5+(d^\circ(u_3)\geq 6)\,,\qquad u_2=\tfrac{1}{3}u_3^3+(d^\circ(u_3)\geq 4)\,.$$

PROOF. Similar argument as we had in deriving $u_g=-t+(d^\circ(t)\geq 2)$ in 2.3.1 gives

$$u_{g-1}=-\tfrac{1}{3}t^3+(d^\circ(t)\geq 4)\,,\qquad u_{g-2}=-\tfrac{1}{5}t^5+(d^\circ(t)\geq 6)\,.$$

Hence we have the desired formulae. □

The following lemma gives an expression of the Taylor expansion of the analytic coordinates with respect to the local parameter y at branch points different from ∞ along $\kappa^{-1}\iota(C)$.

LEMMA 2.3.3. *Let $(a, 0)$ be a branch point of C different from ∞, that is, $f(a)=0$, and let P denote a point of $\mathbf{C}^g$ such that $\kappa(P)=\iota(a, 0)$. Choose $v=(v_1, \cdots, v_g)$ such that $\kappa(v+P)=\iota(x, y)$. Then the Taylor expansion of v_i as a function of y is of the following form:*

$$v_1=\frac{1}{f'(a)}y+\frac{f''(a)}{3f'(a)^2}y^3+(d^\circ(y)\geq 5)\qquad \textit{if}\quad g\geq 1\,,$$

$$v_2=\frac{a}{f'(a)}y+\frac{1+af''(a)}{3f'(a)^2}y^3+(d^\circ(y)\geq 5)\qquad \textit{if}\quad g\geq 2\,,$$

$$v_3=\frac{a^2}{f'(a)}y+\frac{a(2+af''(a))}{3f'(a)^2}y^3+(d^\circ(y)\geq 5)\qquad \textit{if}\quad g\geq 3\,.$$

PROOF. Let $g=3$. Since $f'(a)\neq 0$ and $y^2=f(x)=f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\cdots$,

$$x=a+\frac{1}{f'(a)}y^2+\frac{f''(a)}{2f'(a)^2}y^4+(d^\circ\geq 6)\,.$$

Therefore we have

$$\begin{aligned}
v_3&=\int_{(0,0)}^{(x,y)}\frac{x^2dx}{2y}\\
&=\int_0^y\left(a+\frac{1}{f'(a)}y^2+\frac{f''(a)}{2f'(a)^2}y^4+(d^\circ\geq 6)\right)^2\left(\frac{1}{f'(a)}+\frac{f''(a)}{f'(a)^2}y^2+(d^\circ\geq 4)\right)dy\\
&=\int_0^t\left(\frac{a^2}{f'(a)}+\left(\frac{a(2+a^2f''(a))}{f'(a)^2}\right)y^2+(d^\circ\geq 4)\right)dy\\
&=\frac{a^2}{f'(a)}y+\frac{a(2+f''(a))}{3f'(a)^2}y^3+(d^\circ\geq 5)\,.
\end{aligned}$$

The formulae for v_1 and v_2 are obtained in the same way. For $g=1$ or $g=2$, the formulae are also shown similarly. □

3. The translational formula of $\sigma(u)$.

In this section, we discuss the translational formula and the Riemann form of $\sigma(u)$. We also give a generalization of Weber's psi function ([20, p. 150] or [19, p. 146]) to higher genus case. Our generalization of the psi function is based on Grant [9].

3.1. The translational formula of $\sigma(u)$. For $u\in\mathbf{C}^g$ we conventionally denote by u' and u'' such elements of $\mathbf{R}^g$ that $u=\omega'u'+\omega''u''$, where ω' and ω'' are those defined in Section 1. We define a $\mathbf{C}$-valued $\mathbf{R}$-bilinear form $L(\ ,\ )$ by $L(u,v)={}^tu(\eta'v'+\eta''v'')$ for $u, v\in\mathbf{C}^g$. For l in Λ, the lattice of periods as defined in Section 1, let

$$\chi(l)=\exp[2\pi i({}^tl'\delta''-{}^tl''\delta')-\pi i{}^tl'l''] ,$$

where δ' and δ'' are those defined in Section 1.

LEMMA 3.1.1 (the translational formula). *The function $\sigma(u)$ satisfies*

$$\sigma(u+l)=\chi(l)\sigma(u)\exp L(u+\tfrac{1}{2}l, l)$$

for all $u\in\mathbf{C}^g$ and $l\in\Lambda$.

For a proof of this formula we refer the reader to [2, p. 286].

Let

$$E(u,v)=L(u,v)-L(v,u)\,,\qquad (u,v\in\mathbf{C}^g)\,. \tag{3.1.1}$$

Then, $E(\ ,\ )$ is a $\mathbf{C}$-valued $\mathbf{R}$-bilinear form satisfying $E(u,v)=-E(v,u)$.

LEMMA 3.1.2. *The linear form $E(\ ,\ )$ has the following properties*:

(1) $E(iu, v)=E(iv, u)$,

(2) $E(u, v)=2\pi i({}^t u' v'' - {}^t u'' v')$.

Especially, $E(\ ,\ )$ *is an* $i\mathbf{R}$*-valued form and* $2\pi i\mathbf{Z}$*-valued on* $\Lambda\times\Lambda$.

PROOF. Statement (1) is proved in [10, p. 85, Theorem 1.2]. Let us prove (2). In the theory of curves, it is a basic fact that ${}^t\omega'\eta'$ and ${}^t w''\eta''$ are symmetric. So

$$\begin{aligned}E(u, v)&=L(u, v)-L(v, u)={}^t u(\eta' v'+\eta'' v'')-{}^t v(\eta' u'+\eta'' u'')\\&={}^t v'\,{}^t\omega'\eta'\omega'^{-1}u+{}^t v'\,{}^t\omega''\eta''\omega''^{-1}u-{}^t u'\,{}^t\omega'\eta'\omega'^{-1}v-{}^t u''\,{}^t\omega''\eta''\omega''^{-1}v\\&={}^t v'\,{}^t\omega'\eta'(u'+Zu'')+{}^t v'\,{}^t\omega''\eta''(Z^{-1}u'+u'')\\&\quad-{}^t u'\,{}^t\omega'\eta'(v'+Zv'')-{}^t u''\omega''\eta''(Z^{-1}v'+v'')\,.\end{aligned}$$

Since ${}^t\omega'\eta'$ and Z are symmetric, it follows that

$$\begin{aligned}{}^t\omega'\eta' Z&={}^t Z\,{}^t\omega'\eta'={}^t\omega''\,{}^t\omega'^{-1}\,{}^t\omega'\eta'={}^t\omega''\eta'\,,\\{}^t\omega''\eta'' Z&={}^t Z\,{}^t\omega''\eta''={}^t\omega'\,{}^t\omega''^{-1}\,{}^t\omega''\eta''={}^t\omega'\eta''\,.\end{aligned}$$

Therefore, by using the symmetricity of ${}^t\omega'\eta'$ and ${}^t\omega''\eta''$ once more, we have

$$\begin{aligned}E(u, v)&={}^t u'\,{}^t\omega'\eta' v'+{}^t u''\,{}^t\omega''\eta' v'+{}^t u'\,{}^t\omega'\eta'' v''+{}^t u''\,{}^t\omega''\eta'' v''\\&\quad-{}^t v'\,{}^t\omega'\eta' u'-{}^t v''\,{}^t\omega''\eta' u'-{}^t v'\,{}^t\omega'\eta'' u''-{}^t v''\,{}^t\omega''\eta'' u''\,.\end{aligned}$$

The generalized Legendre relation ${}^t\omega'\eta''-{}^t w''\eta'=2\pi i 1_g$ shows our assertion. □

3.2. Functions $\psi_n(u)$. In this subsection, we review the original and generalized Weber's psi functions defined for the (hyper)elliptic curve C. For the case that J has complex multiplication, we will treat them more extensively in 4.4.

DEFINITION 3.2.1. Let $n\in\mathbf{Z}$. We let

$$\psi_n(u)=\frac{\sigma(nu)}{\sigma(u)^{n^2}}\qquad\text{when}\quad g=1\,,$$

$$\psi_n(u)=\frac{\sigma(nu)}{\sigma_2(u)^{n^2}}\qquad\text{when}\quad g=2\ \text{or}\ 3\,.$$

PROPOSITION 3.2.2. *The function* $\psi_n(u)$ *is a function on* C *if* $g=1$ *and on* Θ *if* $g\geq 2$. *In other words, as a function on* $\mathbf{C}=\kappa^{-1}(C)$ *if* $g=1$ *and on* $\kappa^{-1}(\Theta)$ *if* $g\geq 2$, *it is periodic with respect to the lattice* Λ. *Furthermore* $\psi_n(u)$ *restricted to* $u\in\kappa^{-1}\iota(C)$ *is a polynomial of* $x(u)$ *if* $g=1$, 2 *with* n *odd or* $g=3$ *with* n *even, and is a polynomial of* $x(u)$ *multiplied by* $y(u)$ *if* $g=1$, 2 *with* n *even or* $g=3$ *with* n *odd.*

PROOF. We follow [9, p. 126, Lemma 1]. We have $(-1)^*\Theta=\Theta$, because our theta divisor is comming from a hyperelliptic curve. So $n^*\Theta=n^2\Theta$ ([16, p. 59]). Hence the function $\phi_n(u):=\sigma(nu)/\sigma(u)^{n^2}$ is a trivial theta function. On the other hand, by 3.1.1, we have $\sigma(n(u+l))=\chi(nl)\sigma(nu)\exp[n^2L(u+\tfrac{1}{2}l, l)]$. By the definition of $\chi(\ )$, $\chi(nl)$ is equal to

$\chi(l)$ or 1 if n is odd or even, respectively. So we have $\phi_n(u+l)=\phi_n(u)$ for all $u\in\mathbf{C}^g$ and $l\in\Lambda$. Hence the first statement for $g=1$. Now assume $g=2$ or 3. Because of

$$\wp_{22}(u)=\left(\frac{\sigma_2^2-\sigma_{22}\sigma}{\sigma^2}\right)(u), \qquad \wp_{222}(u)=\left(\frac{-2\sigma_2^3+3\sigma_2\sigma_{22}\sigma+\sigma_{222}\sigma^2}{\sigma^3}\right)(u)$$

and of $\sigma(u)=0$ for all $u\in\kappa^{-1}\Theta$, we have

$$\frac{\phi_n(u)}{\wp_{22}(u)^{n^2/2}}=\frac{\sigma(nu)}{\sigma_2(u)^{n^2}} \quad\text{or}\quad \frac{\phi_n(u)}{-\frac{1}{2}\wp_{222}(u)\wp_{22}(u)^{(n^2-3)/2}}=\frac{\sigma(nu)}{\sigma_2(u)^{n^2}}$$

for all $u\in\kappa^{-1}(\Theta)$ if n is even or odd, respectively. Thus $\psi_n(u)$ is a function on Θ. Hence the first statement. For $u\in\kappa^{-1}\iota(C)$, $u=O$ if and only if $\sigma_2(u)=0$ by 2.2.1. Therefore $\psi_n(u)$ has, as a function on C, only pole at $u=0$. So it must be a polynomial of $x(u)$ and $y(u)$. The last statement is shown by $x(-u)=x(u)$, $y(-u)=-y(u)$ and 1.3.1(3). □

We compute ψ_n for $n=2$, 3 and 4 in 3.2.4 below. To do so we give the following

LEMMA 3.2.3. *Let C be the hyperelliptic curve of genus g (≥ 2) defined in* 1.1. *Let P be a point of C different from ∞. If $nP\in\Theta$ with $n=g$ or $g+1$, then P is a branch point, that is, $y(P)=0$.*

PROOF. Since $nP\in\Theta$, we have $g-1$ points $Q_1,\cdots,Q_{g-1}$ such that, as divisors, nP is linearly equivalent to $Q_1+\cdots+Q_{g-1}+(n-g+1)\infty$ ([15, pp. 3.28–29]).

For a point Q of C, we here denote by $\bar{Q}$ the point $(x(Q), -y(Q))$. We first assume $n=g$. In this case, there exists a function G on C whose divisor is $(Q_1+\cdots+Q_{g-1}+\infty)-nP$. Since $P\neq\infty$, G may not be a constant function. However, there is no non-constant function whose poles are bounded by a divisor $\sum_{j=1}^{g}P_j$ such that $P_j\neq\infty$ and $P_j\neq\bar{P}_j$ for every i and j with $i\neq j$ ([15, pp. 3.30]). Since $P\neq\infty$, it must be $P=\bar{P}$, and hence $y(P)=0$.

Secondly, we assume $n=g+1$. Then there exists a function G on C whose divisor is $(Q_1+\cdots+Q_{g-1}+2\infty)-nP$. The divisor of the function $(x-x(P))/G$ is $Q_1+\cdots+Q_{g-1}-(n-2)P$. This function may not be a constant. So, by the same argument as in the case $n=g$, we have $P=\bar{P}$, and hence $y(P)=0$. □

LEMMA 3.2.4. (1) *If $g=1$ then $\psi_2(u)=-2y(u)$ and if $g=2$ then $\psi_2(u)=2y(u)$.*
(2) *If $g=2$ or $g=3$ then $\psi_3(u)=-8y(u)^3$.*
(3) *If $g=3$ then $\psi_4(u)=64y(u)^4$.*

PROOF. (1) When $g=1$, 2.1.1(1) implies

$$\psi_2(u)=\frac{\sigma(2u)}{\sigma(u)^4}=\frac{2u+(d^\circ\geq 2)}{(u+(d^\circ\geq 2))^4}=\frac{2}{u^3}+\cdots.$$

Thus 2.3.1 and 3.2.2 imply $\psi_2(u)=-2y(u)$ for $u\in C$. When $g=2$, 2.1.1(2) and 2.3.2(1) imply

$$\psi_2(u)\,|_{u\in\kappa^{-1}\iota(C)}=\frac{\sigma(2u)}{\sigma_2(u)^4}=\frac{2u_1+\frac{1}{6}\lambda_2 8u_1^3-\frac{1}{3}\lambda_5 8u_2^3+(d^\circ\geq 5)}{(-u_2^2+(d^\circ\geq 4))^4}$$
$$=\frac{-2u_2^3+(d^\circ\geq 5)}{(-u_2^2+(d^\circ\geq 4))^4}=-\frac{2}{u_5^2}+\cdots.$$

Thus 2.3.1 and 3.2.2 imply $\psi_2(u)=2y(u)$ for $u\in C$.

(2) When $g=2$, we have

$$\psi_3(u)\,|_{u\in\kappa^{-1}\iota(C)}=\frac{\sigma(3u)}{\sigma_2(u)^9}=\frac{3u_1+\frac{1}{6}\lambda_2 27u_1^3-\frac{1}{3}27u_2^3+(d^\circ\geq 5)}{(-u_2^2+(d^\circ\geq 4))^9}=8\frac{1}{u_2^{15}}+\cdots$$

by 2.1.1(2) and 2.3.2(1). Let $P=(x(u), y(u))$ and assume $\psi_3(u)=0$. Then we have $\sigma(3u)=0$ because $\sigma_2(u)=0$ if and only if $u=O$ as seen in 2.2.1(2). So $3P\in\Theta$. By 3.2.3, it must be $P=\infty$ or $P=\bar{P}$. This means $y(u)=\infty$ or $y(u)=-y(u)$. Hence we have known, for $u\in\kappa^{-1}\iota(C)$, that $\psi_3(u)=0$ is equivalent to $y(u)=0$. So $\psi_3(u)$ must be of the form

$$\psi_3(u)\,|_{u\in\kappa^{-1}\iota(C)}=-8y(u)\prod_{y(P)=0}(x(u)-x(P)) \tag{3.2.1}$$

by 3.2.2. To determine the product for points P, we look at the vanishing order at each P such as $y(P)=0$. Let $P=(a, 0)$. Assume $u=v+P\in\kappa^{-1}\iota(C)$. Then $y=y(v+P)$ is a local parameter at P. Since

$$\psi_3(v+P)\,|_{v+P\in\kappa^{-1}\iota(C)}$$
$$=\frac{\sigma(3(v+P))}{\sigma_2(v+P)^9}=\frac{\sigma(3v+P)\chi(2P)\exp L(3v+P+P, 2P)}{\sigma_2(v+P)^9}$$
$$=\frac{(3\sigma_1(P)v_1+3\sigma_2(P)v_2+(d^\circ\geq 3))\exp 4L(P, P)(1+(d^\circ(v_1, v_2)\geq 1))}{(\sigma_2(P)+(d^\circ(v_1, v_2)\geq 1))^9},$$

it follows from the first statement of 2.2.1(2) and 2.3.3 that

$$\psi(v+P)\,|_{v+P\in\kappa^{-1}\iota(C)}=(d^\circ(y)\geq 3).$$

This argument is independent of the choice of a. So the factors of the product in (3.2.1) contain $x(v+P)-a$ for all a with $f(a)=0$. Thus the product must be equal to $y(u)^2$. Hence $\psi_3(u)\,|_{u\in\kappa^{-1}\iota(C)}=-8y(u)^3$.

When $g=3$, we have

$$\psi_3(u)\,|_{u\in\kappa^{-1}\iota(C)}=\frac{\sigma(3u)}{\sigma_2(u)^9}=\frac{9u_1u_3-9u_2^2-81\frac{\lambda_7}{3}u_2u_3^3+3^6\frac{\lambda_7}{45}u_3^6+\cdots}{(-2u_2-\frac{\lambda_7}{3}u_3^3+\cdots)^9}$$
$$=8\frac{u_3^6+(d^\circ(u_3)\geq 8)}{(-u_3^3+(d^\circ(u_3)\geq 5))^9}=-\frac{8}{u_3^{21}}+\cdots$$

for $u\in\kappa^{-1}\iota(C)$ by 2.1.1(3) and 2.3.2(2). Let $P=(x(u), y(u))$ and assume $\psi_3(u)=0$. Then we have $\sigma(3u)=0$ because $\sigma_2(u)=0$ if and only if $u=O$ as seen in 2.2.1(3). Therefore

$3P\in\Theta$. By 3.2.3, it must be $P=\infty$ or $P=\bar{P}$. This means $y(u)=\infty$ or $y(u)=-y(u)$. Hence we have known, for $u\in C$, that $\psi_3(u)=0$ is equivalent to $y(u)=0$. So $\psi_3(u)$ must be of the form

$$(3.2.2)\qquad \psi_3(u)\,|_{u\in\kappa^{-1}\iota(C)}=-8y(u)\prod_{y(P)=0}(x(u)-x(P))$$

by 3.2.2. As in the case $g=2$, we look at the vanishing order at a point $P=(a,0)\in C$. By using the Taylor expansion 2.2.1(3) we have

$$\psi_3(v+P)\,|_{v+P\in\kappa^{-1}\iota(C)}=\frac{\sigma(3(v+P))}{\sigma_2(v+P)^9}=\frac{\sigma(3v+P)\chi(2P)\exp L(3v+P+P,2P)}{\sigma_2(v+P)^9}$$
$$=\frac{(3(\sigma_1(P)v_1+\sigma_2(P)v_2+\sigma_3(P)v_3)+(d^\circ\geq 3))\exp 4L(P,P)(1+(d^\circ(v_1,v_2,v_3)\geq 1))}{(\sigma_2(P)+(d^\circ(v_1,v_2,v_3)\geq 1))^9}.$$

So 2.3.3 and the first statement of 2.2.1(3) give

$$\psi_3(v+P)\,|_{v+P\in\kappa^{-1}\iota(C)}=(d^\circ(y)\geq 3)\,.$$

This argument is independent of the choice of a with $f(a)=0$. So the factors of the product in (3.2.2) contain $x(v+P)-a$ for all a with $f(a)=0$. Thus the product must be equal to $y(u)^2$. Hence $\psi_3(u)\,|_{u\in\kappa^{-1}\iota(C)}=-8y(u)^3$.

(3) We have

$$\psi_4(u)\,|_{u\in\kappa^{-1}\iota(C)}=\frac{\sigma(4u)}{\sigma_2(u)^{16}}=\frac{16u_1u_3-16u_2^2-4^4\frac{\lambda_7}{3}u_2u_3^3+4^6\frac{\lambda_7}{45}u_3^6+\cdots}{(-2u_2-\frac{\lambda_7}{3}u_3^3+\cdots)^{16}}$$
$$=\frac{64u_3^6+(d^\circ(u_3)\geq 8)}{(-u_3^3+(d^\circ(u_3)\geq 5))^{16}}=\frac{64}{u_3^{42}}+\cdots$$

for $u\in\kappa^{-1}\iota(C)$ by 2.1.1(3). Let $P=(x(u),y(u))$ and assume $\psi_4(u)=0$. Then we have $\sigma(4u)=0$ because $\sigma_2(u)=0$ if and only if $u=O$ as seen in 2.2.1(3). Hence $4P\in\Lambda$. By 3.2.3, it must be $P=\infty$ or $P=\bar{P}$. This means $y(u)=\infty$ or $y(u)=-y(u)$. Hence we have shown, for $u\in\kappa^{-1}\iota(C)$, that $\psi_4(u)=0$ is equivalent to $y(u)=0$. So $\psi_4(u)$ must be of the form

$$(3.2.3)\qquad \psi_4(u)\,|_{u\in\kappa^{-1}\iota(C)}=64\prod_{y(P)=0}(x(u)-x(P))$$

by 3.2.2. As in the proof of (2), we look at the vanishing order of $\psi_4(u)$ at a point $P=(a,0)\in C$. We take $y=y(u)$ as a local parameter at P along $\kappa^{-1}\iota(C)$. Let $u=v+P$ on $\kappa^{-1}\iota(C)$. We first show that $\sigma(4v)=(d^\circ(y(u))\geq 6)$. By 2.3.3, we have

$$v_1v_3-v_2^2=\left(\frac{1}{f'}y+\frac{f''}{3f'^2}y^2+(d^\circ\geq 5)\right)\left(\frac{a^2}{f'}y+\frac{a(2+af'')}{3f'^2}y^3+(d^\circ\geq 5)\right)$$

$$-\left(\frac{a}{f'}y+\frac{1+af''}{f'^2}y^3+(d^\circ\geq 5)\right)^2$$
$$=(d^\circ(y)\geq 6)\,,$$

$$-\frac{\lambda_0}{3}v_1^4-\frac{\lambda_1}{3}v_1^3v_2-\lambda_2 v_1^2v_2^2-\frac{\lambda_3}{3}v_1v_2^3$$
$$-\frac{\lambda_4}{3}v_2^4+\frac{2\lambda_2}{3}v_1^3v_3-\frac{\lambda_5}{3}v_2^3v_3-\frac{\lambda_6}{2}v_2^2v_3^2+\frac{\lambda_6}{6}v_1v_3^3-\frac{\lambda_7}{3}v_2v_3^3$$
$$=\frac{1}{f'^4}\left(-\frac{\lambda_0}{3}-\frac{\lambda_1 a}{3}-\lambda_2 a^2+\frac{2\lambda_2 a^2}{3}-\frac{\lambda_3 a^3}{3}-\frac{\lambda_4 a^4}{3}\right.$$
$$\left.-\frac{\lambda_5 a^5}{3}-\frac{\lambda_6 a^6}{2}+\frac{\lambda_6 a^6}{6}-\frac{\lambda_7 a^7}{3}\right)y^4+(d^\circ\geq 6)$$
$$=(d^\circ(y)\geq 6)\,,$$

where we simply write f' and f'' instead of $f'(a)$ and $f''(a)$, respectively. By 3.1.1, we have

$$\psi_4(v+P)\,|_{v+P\in\kappa^{-1}\iota(C)}=\frac{\sigma(4(v+P))}{\sigma_2(v+P)^{16}}=\frac{\sigma(4v)\chi(4P)\exp L(4v+2P,2P)}{\sigma_2(v+P)^{16}}\,.$$

Therefore

$$\psi_4(v+P)\,|_{v+P\in\kappa^{-1}\iota(C)}=(d^\circ(y)\geq 6)\,.$$

This argument is independent of the choice of a with $f(a)=0$. So the factors of the product in (3.2.3) contain $x(v+P)-a$ for all $a, f(a)=0$, with multiplicity at least three. Hence the product must be equal to $y(u)^6$. Therefore we have shown $\psi_4(u)\,|_{v+P\in\kappa^{-1}\iota(C)}=64y(u)^6$, and we have established the proof. □

4. Curves of cyclotomic type.

4.1. Automorphisms of C and endomorphisms of J. In this subsection, we treat the case when the affine equation of the curve C is given by $y^2=x^m+1/4$ or $y^2=x^n-x$ with m and n odd. In this paper we say such a curve to be *of cyclotomic type*. In the latter case, if $n-1$ is a power of 2, then we call such a curve to be *of 2-primary (cyclotomic) type*.

In the first case, we let $\zeta=\exp(2\pi i/m)$. Then there are automorphisms

$$\ulcorner\pm\zeta^j\urcorner: C\to C\,,\qquad (x,y)\mapsto(\zeta^j x,\pm y)$$

for $j=0,\cdots,m-1$. Especially, $\ulcorner\pm\zeta^j\urcorner\infty=\infty$, $\ulcorner\zeta^j\urcorner(0,1/2)=(0,1/2)$ and $\ulcorner-1\urcorner(-4^{-1/m},0)=(-4^{-1/m},0)$.

In the second case, we let $\zeta=\exp(\pi i/(n-1))$. Then there are automorphisms

$$\ulcorner\zeta^j\urcorner : C \to C, \qquad (x, y) \mapsto (\zeta^{2j}x, \zeta^j y)$$

for $j=0, \cdots, n-1$. We have $\ulcorner\zeta^j\urcorner\infty=\infty$ and $\ulcorner\zeta^j\urcorner(0,0)=(0,0)$.

In each of the cases, the automorphism extends to an endomorphism

$$\ulcorner\pm\zeta^j\urcorner : P_1+\cdots+P_g-g\infty \mapsto \ulcorner\pm\zeta^j\urcorner P_1+\cdots+\ulcorner\pm\zeta^j\urcorner P_g-g\infty$$

of $\mathrm{Pic}^\circ(C)$, hence, of J, where $P_1, \cdots, P_g$ are points of C. We denote by $\mathbf{Z}[\ulcorner\zeta\urcorner]$ the subring of $\mathrm{End}(J)$ generated by $\{\ulcorner\zeta^j\urcorner\}$. The ring $\mathbf{Z}[\ulcorner\zeta\urcorner]$ also acts on $\mathbf{C}^g$ with Λ being stable, that is, $\alpha\Lambda\subset\Lambda$ for all $\alpha\in\mathbf{Z}[\ulcorner\zeta\urcorner]$. We have obvious relations $\ulcorner 1\urcorner=1$, $\ulcorner\zeta^j\urcorner\ulcorner\zeta^k\urcorner=\ulcorner\zeta^{j+k}\urcorner$ and $\ulcorner-\zeta^j\urcorner=-\ulcorner\zeta^j\urcorner$. In each case, since $\ulcorner\pm\zeta^j\urcorner\iota(C)=\iota(C)$, it is obvious that $\ulcorner\pm\zeta^j\urcorner\Theta=\Theta$.

LEMMA 4.1.1. (1) *If C is defined by $y^2=x^{2g+1}+1/4$, then $\mathbf{Z}[\ulcorner\zeta\urcorner]\cong\mathbf{Z}[X]/(X^{2g}+\cdots+X+1)$ by $\ulcorner\zeta\urcorner\mapsto X$.*

(2) *If C is defined by $y^2=x^{2g+1}-x$, then $\mathbf{Z}[\ulcorner\zeta\urcorner]\cong\mathbf{Z}[X]/(X^{2g}+1)$ by $\ulcorner\zeta\urcorner\mapsto X$.*

PROOF. The isomorphism of (1) (resp. (2)) is easily obtained from the action

$$\ulcorner\zeta\urcorner(u_1, u_2, \cdots, u_g)=(\zeta u_1, \zeta^2 u_2, \cdots, \zeta^g u_g)$$
$$(\text{resp. } \ulcorner\zeta\urcorner(u_1, u_2, \cdots, u_g))=((\zeta u_1, \zeta^3 u_2, \cdots, \zeta^{2g-1}u_g)). \qquad \square$$

Let b be an element of $\mathbf{Z}[\ulcorner\zeta\urcorner]$. In the following, we will investigate the b-multiplication $\sigma(bu)$ for $\sigma(u)$, and pull-back $b^*\Theta$ of b-multiplication for Θ. If $b\in\mathbf{Z}$ then most results of this section are quite simple. However, for our main results, one of the most important cases would be when b is an "imaginary" number in $\mathbf{Z}[\ulcorner\zeta\urcorner]$.

4.2. The Riemann form for a curve of cyclotomic type.

DEFINITION 4.2.1. The function $\sigma(u)=\sigma(u;Z)$ is said to be a *normalized* theta function (in the sense of [10, p. 87] or [18, p. 20]) if the form $L(u,v)$ defined in 3.1 is hermitian, i.e., $L(v,u)=\overline{L(u,v)}$ where the bar means the complex conjugate. If that is so,

$$L(u,v)=\frac{1}{2i}[E(iu,v)+iE(u,v)]$$

for all $u, v\in\mathbf{C}^g$.

LEMMA 4.2.2. *Let η' and η'' be the period matrix of differential forms of second kind as is defined in 1.1. If $\eta'^{-1}\eta''=\overline{Z}$ then $\sigma(u)$ is a normalized theta function.*

PROOF. By the definition of $L(,)$, $L(iu,v)=iL(u,v)$. We will show that $L(u,iv)=-iL(u,v)$. Let us define w' and $w''\in\mathbf{R}^g$ by $i\omega'^{-1}v=w'+Zw''$. Then $-i\overline{\omega'^{-1}v}=w'+\overline{Z}w''$. Since $\omega'^{-1}v=v'+Zv''$ and $\overline{\omega'^{-1}v}=v'+\overline{Z}v''$, we have

$$L(u,iv)=u(\eta'w'+\eta''w'')=u\eta'(w'+\overline{Z}w'')=u\eta'(\overline{i\omega'^{-1}v})$$
$$=u\eta'(-i)(v'+\overline{Z}v'')=-iu(\eta'v'+\eta''v'')=-iL(u,v).$$

Since $E(\,,\,)$ is $\mathbf{R}$-valued, we have $L(u, v)=\overline{L(v, u)}$ by 3.1.2(1) and the relation of $L(\,,\,)$ and $E(\,,\,)$ in 4.2.1. Therefore we have the assertion. □

PROPOSITION 4.2.3. *If C is of cyclotomic type, then $\eta'^{-1}\eta''=\overline{Z}$. Hence $\sigma(u\,;Z)$ is normalized because of* 4.2.2.

PROOF. In our case, the differential forms $\eta^{(1)}, \cdots, \eta^{(g)}$ defined in 1.1 are

$$\eta^{(1)}=(2g-1)\frac{x^{2g-1}}{2y}dx\,, \quad \eta^{(2)}=(2g-3)\frac{x^{2g-2}}{2y}dx\,, \quad \cdots, \quad \eta^{(g)}=\frac{x^{g}}{2y}dx\,.$$

Let C be the curve defined by $y^2=x^{2g+1}+1/4$ (resp. $y^2=x^{2g+1}-x$) and let

$$K_i=\int_{(0,1/2)}^{(-4^{-1/(2g+1)},0)}\omega^{(i)}\,, \qquad H_i=\int_{(0,1/2)}^{(-4^{-1/(2g+1)},0)}\eta^{(i)}$$

$$\left(\text{resp. } K_i=\int_{(0,0)}^{(1,0)}\omega^{(i)}\,, \qquad H_i=\int_{(0,0)}^{(1,0)}\eta^{(i)}\right)$$

be integrals along the real axis. Then we have

$$\int_{(0,1/2)}^{(-4^{-1/(2g+1)}\zeta^k,0)}\omega^{(i)}=\int_{(0,1/2)}^{(-4^{-1/(2g+1)},0)}\lceil\zeta^k\rceil\omega^{(i)}=\zeta^{ki}K_i\,,$$

$$\int_{(0,1/2)}^{(-4^{-1/(2g+1)}\zeta^k,0)}\eta^{(i)}=\int_{(0,1/2)}^{(-4^{-1/(2g+1)},0)}\lceil\zeta^k\rceil\eta^{(i)}=\zeta^{(2g-i+1)k}H_i=\zeta^{-ki}H_i$$

$$\left(\text{resp. } \int_{(0,0)}^{(\zeta^k,0)}\omega^{(i)}=\int_{(0,0)}^{(1,0)}\lceil\zeta^k\rceil\omega^{(i)}=\zeta^{(2i-1)k}K_i\,,\right.$$

$$\left.\int_{(0,1/2)}^{(\zeta^k,0)}\eta^{(i)}=\int_{(0,1/2)}^{(1,0)}\lceil\zeta^k\rceil\eta^{(i)}=\zeta^{(2(2g-i)+1)k}H_i=\zeta^{(-2i+1)k}H_i\right),$$

where each of integrals is along the segment with a constant argument. Let us compute

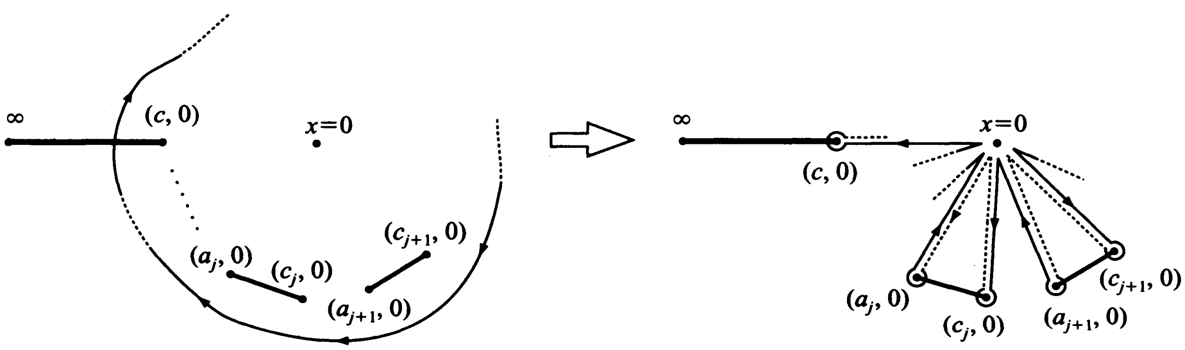

FIGURE 2

the periods matrices η' and η'' by choosing paths $\alpha^{(j)}$ and $\beta^{(j)}$ as a join of segments of line in x-plane with constant $\arg x$ as in Figure 2. Then we are led to the following relations:

$$\int_{\alpha^{(j)}} \eta^{(i)} = \frac{H_i}{K_i} \overline{\int_{\alpha^{(j)}} \omega^{(i)}}, \qquad \int_{\beta^{(j)}} \eta^{(i)} = \frac{H_i}{K_i} \overline{\int_{\beta^{(j)}} \omega^{(i)}}$$

for all i and j. Hence

$$\eta' = \begin{bmatrix} H_1/K_1 & & \\ & \ddots & \\ & & H_g/K_g \end{bmatrix} \overline{\omega'}, \qquad \eta'' = \begin{bmatrix} H_1/K_1 & & \\ & \ddots & \\ & & H_g/K_g \end{bmatrix} \overline{\omega''}.$$

So we have $\eta'^{-1}\eta'' = \overline{\omega'^{-1}\omega''} = \overline{Z}$. □

For each $b \in \mathbf{Z}[\ulcorner\zeta\urcorner]$ we denote by $\bar{b}$ the involution in $\mathbf{Z}[\ulcorner\zeta\urcorner]$ induced by $\overline{\ulcorner\zeta^j\urcorner} = \ulcorner\zeta^{-j}\urcorner$.

PROPOSITION 4.2.4. *If C is of cyclotomic type, then*

$$E(bu, v) = E(u, \bar{b}v), \qquad L(bu, v) = L(u, \bar{b}v),$$

for all $u, v \in \mathbf{C}^g$ and $b \in \mathbf{Z}[\ulcorner\zeta\urcorner]$.

PROOF. Since $\ulcorner\zeta\urcorner$ is an automorphism of Λ, there exists a matrix $M(\xi^j)$ with entries in $\mathbf{Z}$ such that

$$\ulcorner\zeta^j\urcorner u = [\omega'\omega''] M(\zeta^j) \begin{bmatrix} u' \\ u'' \end{bmatrix}.$$

Since $\ulcorner\zeta^j\urcorner$ is an automorphism of C over $\mathbf{Q}$, it induces an automorphism of the fundamental group of C. Hence ${}^tM(\zeta^j) I M(\zeta^j) = I$ with $I = \begin{bmatrix} 0 & 1_g \\ -1_g & 0 \end{bmatrix}$ and $M(\zeta^j)M(\zeta^{-j}) = 1_{2g}$. Thus we have ${}^tM(\zeta^j)I = IM(\zeta^j)^{-1} = IM(\zeta^{-j})$. We define U' and U'' by $\ulcorner\zeta^j\urcorner u = \omega'U' + \omega''U''$ or equivalently by $\begin{bmatrix} U' \\ U'' \end{bmatrix} = M(\zeta^j)\begin{bmatrix} u' \\ u'' \end{bmatrix}$, and let $\begin{bmatrix} V' \\ V'' \end{bmatrix} = M(\zeta^{-j})\begin{bmatrix} v' \\ v'' \end{bmatrix}$, where the letters u', u'', v' and v'' are used under the convention of 3.1. Then 3.1.2(2) and the above equation give

$$\begin{aligned} E(\ulcorner\zeta^j\urcorner u, v) &= 2\pi i({}^tU'v'' - {}^tU''v') = 2\pi i[{}^tU' \; {}^tU'']I\begin{bmatrix} v' \\ v'' \end{bmatrix} \\ &= 2\pi i[{}^tu' \; {}^tu'']{}^tM(\zeta^j)I\begin{bmatrix} v' \\ v'' \end{bmatrix} = 2\pi i[{}^tu' \; {}^tu'']I{}^tM(\zeta^{-j})\begin{bmatrix} v' \\ v'' \end{bmatrix} \\ &= 2\pi i[{}^tu' \; {}^tu'']I\begin{bmatrix} V' \\ V'' \end{bmatrix} = 2\pi i({}^tu'V'' - {}^tu''V') = E(u, \overline{\ulcorner\zeta^j\urcorner}v). \end{aligned}$$

By linearity the proof of the first equation is completed. The second is obtained by the relation in 4.2.1. □

LEMMA 4.2.5. *If C is of cyclotomic type, then there is $j \in \mathbf{Z}$ such that*

$$\sigma(\ulcorner\zeta\urcorner u) = \zeta^j \sigma(u) .$$

In particular,

(1) *If the genus of C is 1 or 2, that is, C is defined by $y^2 = y^3 + 1/4$, $y^2 = y^3 - x$, $y^2 = y^5 + 1/4$ or $y^2 = y^5 - x$, then $\sigma(\ulcorner\zeta\urcorner u) = \zeta\sigma(u)$;*

(2) *If C is defined by $y^2 = x^7 + 1/4$, then $\sigma(\ulcorner\zeta\urcorner u) = \zeta^4\sigma(u)$;*

(3) *If C is defined by $y^2 = x^7 - x$, then $\sigma(\ulcorner\zeta\urcorner u) = \zeta^6\sigma(u)$.*

PROOF. Since $\ulcorner\zeta\urcorner^*\Theta = \Theta$, the two functions $\sigma(\ulcorner\zeta\urcorner u)$ and $\sigma(u)$ have the same divisor of zeros. So $\sigma(\ulcorner\zeta\urcorner u)/\sigma(u)$ is an entire function, i.e., a trivial theta function. On the other hand, by 3.1.1, we have

$$\frac{\sigma(\ulcorner\zeta\urcorner(u+l))}{\sigma(u+l)} = \frac{\chi(\ulcorner\zeta\urcorner l)}{\chi(l)} \frac{\sigma(\ulcorner\zeta\urcorner u)}{\sigma(u)} \frac{\exp L(\ulcorner\zeta\urcorner(u+\frac{1}{2}l), \ulcorner\zeta\urcorner l)}{\exp L(u+\frac{1}{2}l, l)} .$$

Since $\chi(\ )$ is 1 or -1, the above quotient is equal to $\pm\sigma(\ulcorner\zeta\urcorner u)/\sigma(u)$ by virtue of 4.2.4. Therefore the function $\sigma(\ulcorner\zeta\urcorner u)/\sigma(u)$ is bounded. In fact, if M is the maximum of absolute values of this function on the domain

$$\left\{ u = \omega' \begin{bmatrix} u_1' \\ \vdots \\ u_g' \end{bmatrix} + \omega'' \begin{bmatrix} u_1'' \\ \vdots \\ u_g'' \end{bmatrix} ;\ 0 \le u_j' \le 1,\ 0 \le u_j'' \le 1 \text{ for } j = 1, \cdots g \right\},$$

then $\sigma(\ulcorner\zeta\urcorner u)/\sigma(u) \le M$ for all $u \in \mathbf{C}^g$. Liouville's theorem says such function is a constant function, say $\sigma(\ulcorner\zeta\urcorner u)/\sigma(u) = c$. Consequently, if $\zeta^k = 1$, then

$$c^k = \frac{\sigma(\ulcorner\zeta\urcorner u)}{\sigma(u)} \frac{\sigma(\ulcorner\zeta^2\urcorner u)}{\sigma(\ulcorner\zeta\urcorner u)} \cdots \frac{\sigma(\ulcorner\zeta^{k-1}\urcorner u)}{\sigma(\ulcorner\zeta^{k-2}\urcorner u)} \frac{\sigma(u)}{\sigma(\ulcorner\zeta^{k-1}\urcorner u)} = 1 .$$

So $c = \zeta^j$ for some $j \in \mathbf{Z}$. If g is 1, 2 or 3, by looking at the Taylor expansion 2.1.1 at O, we get the desired formulae. □

The following Lemma is used in 4.2.8 below.

LEMMA 4.2.6. *Let C be of cyclotomic type. Let c and b be elements of $\mathbf{Z}[\ulcorner\zeta\urcorner]$ such that $\bar{c} = c$ and $\bar{b} \equiv b \bmod c^2$. Let P be a point in $\mathbf{C}^g$ such that $cP \in \Lambda$. Then*

$$L(bP, P) \equiv L(P, bP) \quad \bmod 2\pi i\mathbf{Z} .$$

PROOF. Since $\bar{b} - b \equiv 0 \bmod c^2$, we can write $\bar{b} - b = ac^2$ with $a \in \mathbf{Z}[\ulcorner\zeta\urcorner]$. Then $E(P, (\bar{b} - b)P) = E(P, ac^2P) = E(\bar{c}P, acP) = E(cP, acP) \in 2\pi i\mathbf{Z}$ by 4.2.1, because of $cP \in \Lambda$ and 3.1.2(2). Therefore[1], by 4.2.1,

[1] Incidentally, since $-E(P, bP) = E(bP, P)$, we have $2E(P, bP) \equiv 0 \bmod 2\pi i\mathbf{Z}$.

$$(4.2.1)\qquad \begin{aligned} E(bP, P) &= E(P, \bar{b}P) = E(P, (\bar{b}-b)P + bP) \\ &= E(P, (\bar{b}-b)P) + E(P, bP) \equiv E(P, bP) \mod 2\pi i\mathbf{Z}. \end{aligned}$$

Furthermore, since

$$(4.2.2)\qquad E(i \cdot bP, P) = E(iP, bP)$$

by 3.1.2(1), we obtain that

$$L(bP, P) = \frac{1}{2i}(E(i \cdot bP, P) + iE(bP, P)) \equiv L(P, bP) \mod 2\pi i\mathbf{Z}.$$

by 4.2.1, (4.2.1) and (4.2.2). □

DEFINITION 4.2.7. Let $\rho : \zeta \mapsto \zeta^{-1}$ be the complex conjugate. Let T be an element of $\mathbf{Z}[\mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})]$. If $T + \rho T$ is the norm from $\mathbf{Q}(\zeta)$ to $\mathbf{Q}$, then T is called a *type norm* ([11, p. 22]).

LEMMA 4.2.8. *Let C be of cyclotomic type.*

(1) *Let c and b be elements of $\mathbf{Z}[\ulcorner\zeta\urcorner]$ such that, as ideals, $(c^\gamma) = (c)$ for all $\gamma \in \mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})$ and such that $b \equiv 1 \mod c^2$. Let P be a point of $\mathbf{C}^g$ such that $cP \in \Lambda$. If T is a type norm, then for all $v \in \mathbf{C}^g$,*

$$\sigma(b^T(v+P)) = \sigma(b^Tv + P)\exp[\tfrac{1}{2}(\mathrm{N}b - 1)L(P, P) + \tfrac{1}{2}L(b^Tv, (b^T - 1)P)]\chi((b^T - 1)P).$$

(2) *For all $v \in \mathbf{C}^g$,*

$$\begin{aligned} \sigma(v + \ulcorner\zeta\urcorner P_0) = \sigma(v + P_0) \exp[&L(v, (\ulcorner\zeta\urcorner - 1)P_0) \\ &+ \tfrac{1}{2}L((\overline{\ulcorner\zeta\urcorner} - \ulcorner\zeta\urcorner)P_0, P_0)]\chi((\ulcorner\zeta\urcorner - 1)P). \end{aligned}$$

PROOF. The assumption on b and c implies $b^T \equiv 1 \mod c^2$. So $(b^T - 1)P \in \Lambda$ and 3.1.1 gives

$$\begin{aligned} \sigma(b^T(v+P)) &= \sigma(b^Tv + P + (b^T - 1)P) \\ &= \sigma(b^Tv + P)\exp(L(b^Tv + P + \tfrac{1}{2}(b^T - 1)P, (b^T - 1)P)\chi((b^T - 1)P). \end{aligned}$$

Here

$$\begin{aligned} &L(b^Tv + P + \tfrac{1}{2}(b^T - 1)P, (b^T - 1)P) \\ &\quad = L(\tfrac{1}{2}(b^T + 1)P, (b^T - 1)P) + L(b^Tv, (b^T - 1)P) \\ &\quad = \tfrac{1}{2}L((b^T + 1)P, (b^T - 1)P) + L(b^Tv, (b^T - 1)P) \\ &\quad \equiv \tfrac{1}{2}(L(b^TP, b^TP) - L(P, P)) + L(b^Tv, (b^T - 1)P) \mod 2\pi i\mathbf{Z} \qquad \text{by } 4.2.6 \\ &\quad = \tfrac{1}{2}(L(b^T\overline{b^T}P, P) - L(P, P)) + L(b^Tv, (b^T - 1)P) \qquad \text{by } 4.2.4 \\ &\quad = \tfrac{1}{2}(\mathrm{N}b - 1)L(P, P) + L(b^Tv, (b^T - 1)P). \end{aligned}$$

Hence we have (1). The formula (2) is obtained analogously. □

4.3. Action for the theta divisor. In this subsection, the curve C is still assumed to be of cyclotomic type. For $b \in \mathbf{Z}[\ulcorner \zeta \urcorner]$ we denote by $b^*\Theta$ the pull-back of Θ with respect to the endomorphism b. Therefore $\kappa^{-1}(b^*\Theta)$ is just the divisor of zeros of $\sigma(bu)$, and $E(bu, bv)$ is the Riemann form associated to this divisor.

The following proposition seems to be true for all C of cyclotomic type. But the author has no proof of it for 2-primary type (see 4.1) except for the curve defined by $y^2 = x^5 - x$. We denote by $\approx$ algebraic equivalence and by $\sim$ linear equivalence.

PROPOSITION 4.3.1. *Assume $g \geq 2$ and C is not of 2-primary type. Let $\varepsilon_1, \cdots, \varepsilon_n$ and b be elements of $\mathbf{Z}[\ulcorner \zeta \urcorner]$ and $l_0, l_1, \cdots, l_n$ be rational integers. Let $\rho : \zeta \mapsto \zeta^{-1}$ be the complex conjugate. If $b^{1+\rho} = l_0 + l_1\varepsilon_1^2 + \cdots + l_n\varepsilon_n^2$, then*

$$b^*\Theta \sim l_0 \cdot \Theta + l_1 \cdot \varepsilon_1^*\Theta + \cdots + l_n \cdot \varepsilon_n^*\Theta \,.$$

If C is the curve defined by $y^2 = x^5 + 1/4$, 4.3.1 is proved in [9, p. 127, Prop. 1]. We first prove the following lemmas as in [9].

LEMMA 4.3.2. *Assume $g \geq 2$ and C is not of 2-primary type. Let D be a divisor of J. If $D \approx 0$ and $\ulcorner \pm \zeta \urcorner^* D \sim D$, then $D \sim 0$.*

PROOF. We prove by using the dual Abelian variety of J. Since Θ gives a principal polarization of J and $D \approx 0$, $D \sim \Theta_u - \Theta$ for some $u \in J$, where Θ_u denotes the translation of Θ by u ([16, p. 77, Theorem 1]). Since $\ulcorner \zeta^j \urcorner(\Theta) = \Theta$, we have $\ulcorner \pm\zeta^j \urcorner(\Theta_u) = \Theta_{\ulcorner \pm\zeta^j \urcorner u} \sim \Theta_u$. Hence $\ulcorner \pm\zeta^j \urcorner u = u$ by [14, p. 186, 6.6]. Because $n-1$ is not a power of 2, there is an integer v such that $1 - \ulcorner \zeta^v \urcorner$ and 2 are coprime in $\mathbf{Z}[\ulcorner \zeta \urcorner]$. The above linear equivalences imply that u is 2-torsion and $1 - \ulcorner \zeta^v \urcorner$-torsion. Hence $u = O$ and so $D \sim 0$. □

PROOF OF 4.3.1. For a divisor D in J, we denote by $E_D(,)$ the Riemann form associated to D which takes values in $2\pi i\mathbf{Z}$ on $\Lambda \times \Lambda$ ([11, p. 68]). Then

$$\begin{aligned} E_{b^*\Theta}(u, v) &= E(bu, bv) = E(b\bar{b}u, v) \qquad \text{(by 4.2.4)} \\ &= E(b^{1+\rho}u, v) = E((l_0 + l_1\varepsilon_1^2 + \cdots + l_n\varepsilon_n^2)u, v) \\ &= l_0 E(u, v) + l_1 E(\varepsilon_1^2 u, v) + \cdots + l_n E(\varepsilon_n^2 u, v) \\ &= l_0 E(u, v) + l_1 E(\varepsilon_1 u, \varepsilon_1 v) + \cdots + l_n E(\varepsilon_n u, \varepsilon_n v) \\ &= E_{l_0 \cdot \Theta + l_1 \cdot \varepsilon_1^*\Theta + \cdots + l_n \cdot \varepsilon_n^*\Theta}(u, v) \,. \end{aligned}$$

Thus $b^*\Theta \approx l_0 \cdot \Theta + l_1 \cdot \varepsilon_1^*\Theta + \cdots + l_n \cdot \varepsilon_n^*\Theta$. Since the both divisors are invariant by the action $\ulcorner \pm\zeta \urcorner^*$, 4.3.2 implies they are linearly equivalent. □

For a curve of 2-primary type, the proof above cannot be applied. Here we give a proof only for the curve defined by $y^2 = x^5 - x$, for the case ε_1 of 4.3.1 is a certain special element. Note that, for this curve, the map $\mathbf{Z}[\ulcorner \zeta \urcorner] \to \mathrm{End}(J)$ is known to be injective and the image is isomorphic to $\mathbf{Z}[\zeta]$ by $\ulcorner \zeta^j \urcorner \mapsto \zeta$ (see also 6.2).

PROPOSITION 4.3.3. *Assume C is defined by $y^2=x^5-x$. Let $\varepsilon_1=1+\sqrt{2}=1+\zeta-\zeta^3$ and b be an element of $\mathbf{Z}[\zeta]$. Let $\rho:\zeta\mapsto\zeta^{-1}$ be the complex conjugate. If $b^{1+\rho}=l_0+l_1\varepsilon_1^2$ with rational integers l_0 and l_1, then $b^*\Theta\sim l_0\cdot\Theta+l_1\cdot\varepsilon_1^*\Theta$.*

PROOF. We prove the statement in somewhat extended form. First of all, we note the following. Let $b\in\mathbf{Z}[\zeta]$ and $b=p+q\zeta+r\zeta^2+s\zeta^3$ with integers p, q, r and s. Then

$$b^{1+\rho}=(p^2+q^2+r^2+s^2+\tfrac{3}{2}(-pq+ps-rs+qr))+(\tfrac{1}{2}(pq-ps+rs-qr))\varepsilon_1^2\,.$$

So, in the expression $b^{1+\rho}=l_0+l_1\varepsilon_1^2$ for arbitrary $b\in\mathbf{Z}[\zeta]$ with l_0 and $l_1\in\mathbf{Q}$, it is actually $2l_0$ and $2l_1\in\mathbf{Z}$. Now let us prove that, for every $b\in\mathbf{Z}[\zeta]$, if $2b^{1+\rho}=2l_0+2l_1\varepsilon_1^2$ then $2(b^*\Theta)\sim 2l_0\cdot\Theta+2l_1\cdot\varepsilon_1^*\Theta$, and if moreover l_0, $l_1\in\mathbf{Z}$ then $(b^*\Theta)\sim l_0\cdot\Theta+l_1\cdot\varepsilon_1^*\Theta$ by induction with respect to p, q, r and s. In the following we note that $\ulcorner\zeta^j\urcorner^*\Theta=\Theta$. If four or three of p, q, r and s are 0, the statement is trivial. We frequently apply [16, p. 58, Corollary 2]. We get that

$$\begin{aligned}\Theta&=(1+i-i)^*\Theta\sim(1+i)^*\Theta+(1-i)^*\Theta+0^*\Theta-3\Theta\\&=(1+i)^*\Theta+((i+1)(-i))^*\Theta-3\Theta=2\cdot(1+i)^*\Theta-3\Theta\,.\end{aligned}$$

Hence $(1+i)^*\Theta\sim 2\cdot\Theta$ and $(\zeta-\zeta^3)^*\Theta=((1+i)(-\zeta^3))^*\Theta\sim 2\cdot\Theta$. For the pull-back of $1+\zeta$, from

$$\begin{aligned}\varepsilon_1^*\Theta&=(1+\zeta-\zeta^3)^*\Theta\sim(1+\zeta)^*\Theta+(1-\zeta^3)^*\Theta+(\zeta-\zeta^3)^*\Theta-3\Theta\\&\sim(1+\zeta)^*\Theta+((\zeta+1)(-\zeta^3))^*\Theta+2\Theta-3\Theta=2\cdot(1+\zeta)^*\Theta-\Theta\,,\end{aligned}$$

we have $2\cdot(1+\zeta)^*\Theta\sim-\Theta+\varepsilon_1^*\Theta$ and $(1-\zeta)^*\Theta=(1+\zeta-\zeta-\zeta)^*\Theta\sim 4\Theta-(1+\zeta)^*\Theta$. These are a part of the desired results since $(1+\zeta)^{1+\rho}=\frac{1}{2}(-1+\varepsilon_1^2)$. Therefore the statement is shown for $1+\zeta^3=\zeta^3(1-\zeta)$, $\zeta+\zeta^2=\zeta(1+i)$ and $\zeta^2+\zeta^3=\zeta(1+\zeta)$. By using these results, we can check easily the statement for b with three or four of p, q, r and s being 1. The rest of the proof is completed by induction as follows. If the statement is true for b and $b-\zeta^j$ then it is true for $b+\zeta^j$. In fact, let $b^{1+\rho}=l_0+l_1\varepsilon_1^2$ and $(b-\zeta^j)^{1+\rho}=b^{1+\rho}-(\zeta^{-j}b+\zeta^jb^\rho)+1=m_0+m_1\varepsilon_1^2$. Then $(\zeta^{-j}b+\zeta^jb^\rho)=(l_0-m_0+1)+(l_1-m_1)\varepsilon_1^2$. Thus $(b+\zeta^j)^{1+\rho}=(2l_0-m_0+2)+(2l_1-m_1)\varepsilon_1^2$. Note that the coefficients $2l_0-m_0+2$ and m_0, $2l_1-m_1$ and m_1 are of the same parity. On the other hand,

$$b^*\Theta\sim(b+\zeta^j)^*\Theta+(b-\zeta^j)^*\Theta+0^*\Theta-b^*\Theta-2\cdot\Theta$$

yields

$$(b+\zeta^j)^*\Theta\sim 2\cdot b^*\Theta-(b-\zeta^j)^*\Theta+2\cdot\Theta\,.$$

So we have

$$2\cdot(b+\zeta^j)^*\Theta\sim 2(2l_0-m_0+2)\cdot\Theta+2(2l_1-m_1)\cdot\varepsilon_1^*\Theta\,.$$

Furthermore, if $2l_0-m_0+2$ and $2l_1-m_1\in\mathbf{Z}$, then we have

$$(b+\zeta^j)^*\Theta\sim(2l_0-m_0+2)\cdot\Theta+(2l_1-m_1)\cdot\varepsilon_1^*\Theta\,.$$

Hence the statement is also true for $b+\zeta^j$. Similarly, if the statement is true for b and $b+\zeta^j$ then it is true for $b-\zeta^j$. Thus we have shown the assertion for all b. □

4.4. Further generalization of psi functions. Here we construct a generalized Weber's psi function.

LEMMA 4.4.1. *Let b be an element of* $\mathbf{Z}[\ulcorner\zeta\urcorner]$. *Under the notation of* 4.3.1 *or* 4.3.3, *the function*

$$\phi_b(u)=\frac{\sigma(bu)}{\sigma(u)^{l_0}\sigma(\varepsilon_1 u)^{l_1}\cdots\sigma(\varepsilon_n u)^{l_n}}$$

on $\mathbf{C}^g$ *satisfies*

$$\phi_b(u+l)=\pm\phi_b(u)$$

for all $u\in\mathbf{C}^g$ *and* $l\in\Lambda$. *Here the signature* $\pm$ *is independent of u. Moreover if C is not of* 2-*primary type or is defined by* $y^2=x^5-x$, *then*

$$\phi_b(u+l)=\phi_b(u)$$

for all $u\in\mathbf{C}^g$ *and* $l\in\Lambda$.

The author can not follow the proof of [9, Section 3] for C defined by $y^2=x^5+1/4$. Here we give another proof.

PROOF. As is shown in 4.3.1 or 4.3.3,

$$E(bu,bv)=l_0E(u,v)+l_1E(\varepsilon_1u,\varepsilon_1v)+\cdots+l_nE(\varepsilon_nu,\varepsilon_nv)\,.$$

Because of this and $i(bu)=b(iu)$ for all $u\in\mathbf{C}^g$, we have

$$L(bu,bv)=l_0L(u,v)+l_1L(\varepsilon_1u,\varepsilon_1v)+\cdots+l_nL(\varepsilon_nu,\varepsilon_nv)\,.$$

Hence

$$\begin{aligned}\sigma(b(u+l))&=\sigma(bu+bl)=\chi(bl)\sigma(bu)\exp[L(b(u+\tfrac{1}{2}l),bl)]\\&=\chi(bl)\sigma(bu)\exp[l_0L(u+\tfrac{1}{2}l,l)]\exp[l_1L(\varepsilon_1(u+\tfrac{1}{2}l),\varepsilon_1l)]\\&\quad\cdots\exp[l_nL(\varepsilon_n(u+\tfrac{1}{2}l),\varepsilon_nl)]\end{aligned}$$

by 3.1.1. On the other hand, we have

$$\sigma(\varepsilon_j(u+l))=\chi(\varepsilon_jl)\sigma(\varepsilon_ju)\exp[L(\varepsilon_j(u+\tfrac{1}{2}l),\varepsilon_jl)]$$

for $j=1,\cdots,n$ by 3.1.1. Since $\chi(\lambda)=\pm1$ for $\lambda\in\Lambda$, we get $\phi_b(u+l)=\pm\phi_b(u)$ for all $u\in\mathbf{C}^g$ and $l\in\Lambda$. As $\phi_b(u+l)/\phi_b(u)$ is a meromorphic function, the signature $\pm$ must be determined by l. Now we assume that C is not of 2-primary type or the curve defined by $y^2=x^5-x$. Then 4.3.1 and 4.3.3 imply that the divisor of $\phi_b(u)$ is the pull-back of a divisor of a function with respect to the map $\kappa:\mathbf{C}^g\to\mathbf{C}^g/\Lambda$. Thus we can write $\phi_b(u)=f(u)e(u)$, where $f(u)$ is periodic with the periods Λ and $e(u)$ is a trivial theta function with respect to the lattice Λ (see [10, p. 82]). Then we have $e(u+l)=\pm e(u)$. As in the proof of 4.2.5, if M is the maximum of $e(u)$ on the domain

$$\left\{u=\omega'\begin{bmatrix}u_1'\\ \vdots\\ u_g'\end{bmatrix}+\omega''\begin{bmatrix}u_1''\\ \vdots\\ u_g''\end{bmatrix};\ 0\le u_j'\le 1, 0\le u_j''\le 1 \text{ for } j=1,\cdots,g\right\},$$

then $e(u)\le M$ for all $u\in\mathbf{C}^g$. Thus Liouville's theorem says that $e(u)$ is a constant function. Hence the signature must be $+$. So we have completed the proof. □

Since $\phi_b(u)$ has poles along the pull-back of Θ, we modify it as in [9].

Definition-Proposition 4.4.2. *Let* $b\in\mathbf{Z}[\ulcorner\zeta\urcorner]$. *Let*

$$\psi_b(u)=\frac{\sigma(bu)}{\sigma(u)^{l_0}} \quad \text{if} \quad g=1 \quad \text{and}$$

$$\psi_b(u)=\frac{\sigma(bu)}{\sigma_2(u)^{l_0}\sigma(\varepsilon_1 u)^{l_1}\cdots\sigma(\varepsilon_n u)^{l_n}} \quad \text{if} \quad g=2 \text{ or } 3,$$

in the same situation as 4.3.1 *or* 4.3.2. *Then* $\psi_b(u+l)=\pm\psi_b(u)$ *for all* $u\in\kappa^{-1}\iota(C)$ *and* $l\in\Lambda$. *Here the signature* $\pm$ *is independent of* u. *Moreover, if* C *is of genus* 1 *or not of* 2*-primary type except the curve defined by* $y^2=x^5-x$, *then*

$$\psi_b(u+l)=\psi_b(u)$$

for all $u\in\kappa^{-1}\iota(C)$ *and* $l\in\Lambda$.

The proof is given by a similar fashion as in 3.2.2 by looking at the parity of l_0.

Remark 4.4.3. In the rest of this paper we treat only the case $b^{1+\rho}=l_0\in\mathbf{Z}$. So we need not choose $\{\varepsilon_j\}$ explicitly. We see that, in this case, $\psi_b(u)$ is a polynomial of $x(u)$ or a such multiplied by $y(u)$ due to 3.2.2.

II. Complex Multiplication Formulae

We mention here conventions for the following three sections. We freely use the notation of Part I. Let $\varphi(u)$ be an element of the ring

$$\mathbf{Q}[\wp_{ij}(u), \wp_{ij}(v), \wp_{ijk}(u), \wp_{ijk}(v) \mid i, j, k=1,\cdots,g].$$

Let $b\in\mathbf{Z}[\ulcorner\zeta\urcorner]$. Then 1.4.1 and 1.6.2 show that $\varphi(bu)|_{u\in\iota(C)}$ can be expressed as

$$\varphi(bu)|_{u\in\iota(C)}=\frac{P(x(u), y(u))}{Q(x(u), y(u))},$$

where $P(X, Y)$ and $Q(X, Y)\in\mathbf{Q}(\zeta)[X, Y]$. Especially we have shown the assertions about the coefficients in Theorems 5.1.3, 5.2.3, 6.1.6, and 7.1.6 below.

From now on we assume C is a curve of cyclotomic type. We fix a special point P_0 such that $x(P_0)=0$: if C is defined by the affine equation $y^2=x^{2g+1}+1/4$ then P_0 is the point $(0, 1/2)$, if C is defined by $y^2=x^{2g+1}-x$ then without saying P_0 is the point $(0, 0)$.

Suppose we have labeled the roots of $f(x)$ as in (1.1.1). Such labels are described at the beginning of each subsection below. By applying the argument in our proof of 4.2.3, with the same notation, for the integrals along the paths $\alpha^{(1)}$ and $\beta^{(1)}$, we can write the entries of ω', ω'', η', and η'' by K_j's and H_j's.

We choose and fix a point in $\mathbf{C}^g$ whose image of the map $\kappa:\mathbf{C}\to\mathbf{C}^g/\Lambda=J$ is P_0. We denote such a point also by P_0. Throughout Sections 5, 6 and 7 such a point is assumed to be given by taking the integral (2.3.1) along the line on which the x-coordinate is real negative (resp. positive) and the y-coordinate has negative imaginary part or is real positive if the curve C is defined by $y^2=x^{2g+1}+1/4$ (resp. $y^2=x^{2g+1}-x$). Then the coordinates of $\ulcorner\zeta^j\urcorner P_0$'s can be written explicitely, as we describe in each of the following subsections, in the form $\ulcorner\zeta^j\urcorner P_0=\omega' u'+\omega'' u''$ by taking care that the integral from ∞ to $(-4^{1/(2g+1)}, 0)$ (resp. to $(1, 0)$) along negative (resp. positive) part of the real axis of x is half of the one along $\alpha^{(1)}$.

In these sections, we give explicitly the highest and lowest term of $P(X, Y)$ for each of the special functions $\varphi(u)$.

In Section 5 we write u_1 as u, K_1 as K, and H_1 as H.

5. Elliptic curves of cyclotomic type.

5.1. The curve defined by $y^2=x^3+1/4$. We here give a version of the product formula of Eisenstein (see Section 8) for the curve C defined by $y^2=x^3+1/4$. According to 4.1.1(1) the ring $\mathbf{Z}[\ulcorner\zeta\urcorner]$ is isomorphic to the ring $\mathbf{Z}[\zeta]$ by $\ulcorner\zeta\urcorner\mapsto\zeta$. So we may identify $\mathbf{Z}[\ulcorner\zeta\urcorner]$ and $\mathbf{Z}[\zeta]$.

We let $c=-4^{-1/3}$, $a_1=-4^{-1/3}\zeta$, $c_1=-4^{-1/3}\zeta^2$ in (1.1.1). Then we have

$$\omega'=2K(\zeta-\zeta^2)\,,\quad \omega''=2K(\zeta-1)\,,\quad \eta'=2H(\zeta^2-\zeta)\,,\quad \eta''=2H(\zeta^2-1)\,,$$

and

$$P_0=K(-\zeta^2+\zeta)-K=\tfrac{1}{3}\omega'+\tfrac{1}{3}\omega''\,. \tag{5.1.1}$$

PROPOSITION 5.1.1. $\sigma(P_0)^3=-\exp\frac{3}{2}L(P_0, P_0)$.

PROOF. Because of $y(P_0)=1/2$, it is obtained from 3.2.4(1) and (2) that $\sigma(2P_0)=-\sigma(P_0)^4$. On the other hand, from 3.1.1, we get

$$\sigma(2P_0)=\sigma(-P_0+3P_0)=-\exp[\tfrac{3}{2}L(P_0, P_0)]\sigma(-P_0)=\exp[\tfrac{3}{2}L(P_0, P_0)]\sigma(P_0)\,.$$

Here we used that $\sigma(-u)=-\sigma(u)$ and that $\chi(3P_0)=-1$ which is calculated by (5.1.1). Hence the statement. □

PROPOSITION 5.1.2. *Let b be an element of* $\mathbf{Z}[\zeta]$. *If* $b\equiv 1 \bmod (1-\zeta)^2$, *then*

$$\sigma(b(v+P_0))=(-1)^{(Nb-1)/3}\chi((b-1)P_0)\sigma(P_0)^{Nb-1}\sigma(bv+P_0)(1+(d^\circ\geq 1)).$$

PROOF. Since $Nb-1\equiv 0 \bmod 3$, the statement follows from 4.2.8(1) and

$$\exp[\tfrac{1}{2}(Nb-1)L(P_0, P_0)]=\sigma(P_0)^{Nb-1}$$

which is a result of 5.1.1. □

THEOREM 5.1.3 (Eisenstein). *Let* $b\in\mathbf{Z}[\zeta]$ *and assume* $b\equiv 1 \bmod (1-\zeta)^2$. *Then* $\psi_b(u)^2\wp(bu)$ *is of the form*

$$\psi_b(u)^2\wp(bu)=x(u)\sum_{\substack{0\leq j\leq Nb-1\\ j\equiv 0 \bmod 3}}\gamma_j x(u)^j$$

with $\gamma_j\in\mathbf{Q}(\zeta)$. *Moreover* $\gamma_0=b$ *and* $\gamma_{Nb-1}=1$.

PROOF. First, we look at the Laurent expansion at $u=O$. By 2.1.1(1), we have

$$\wp(bu)(\psi_b(u))^2=\frac{\sigma'(bu)^2-\sigma''(bu)\sigma(bu)}{\sigma(u)^{2Nb}}$$
$$=\frac{(1+\cdots)^2-(d^\circ\geq 0)(bu+\cdots)}{(u+\cdots)^{2Nb}}=\frac{1}{u^{2Nb}}+\cdots.$$

Since $\sigma(u)$ is an odd function and has only zeros at $u\in\Lambda$ by 2.2.1(1), we know that $\wp(bu)\psi_b(u)^2$ is a polynomial of $x(u)$. Thus we have $\wp(bu)\psi_b(u)^2=x(u)^{Nb}+\cdots$ by 2.3.1. Secondly, we look at the Laurent expansion at $u=P_0$. Since $b-1\equiv 0 \bmod (1-\zeta)^2$, by using 5.1.2, we have

$$\wp(b(v+P_0))(\psi_b(v+P_0))^2/\wp(v+P_0)=\frac{\sigma(b(v+P_0))^2}{\sigma(v+P_0)^{2Nb}}\cdot\frac{\wp(b(v+P_0))}{\wp(v+P_0)}$$
$$=\frac{(-1)^{2(Nb-1)/3}\chi((b-1)P_0)^2\sigma(P_0)^{2Nb-2}\sigma(bv+P_0)^2}{\sigma(v+P_0)^{2Nb-2}\sigma(v+P_0)^2}\cdot\frac{\wp(bv+P_0)}{\wp(v+P_0)}+(d^\circ\geq 1)$$
$$=\frac{\sigma(bv+P_0)^2}{\sigma(v+P_0)^2}\cdot\frac{b\wp'(bv+P_0)}{\wp'(v+P_0)}+(d^\circ\geq 1)\qquad(\text{since } \sigma(P_0)\neq 0)$$
$$=b+(d^\circ\geq 1)\qquad(\text{since } \wp'(bP_0)=\wp'(P_0)\neq 0).$$

Because 2.2.1(1) states the function $\psi_b(u)^2\wp(bu)$ has only pole at $u=O$ the coefficient of the lowest term must be b. Since $\psi_b(-\zeta u)^2\wp(-\zeta bu)=\zeta^{Nb}\psi_b(u)^2\wp(bu)$ because of 4.2.5(1), the function must be a polynomial of $x(u)^3$ multiplied by $x(u)$. □

5.2. The curve defined by $y^2=x^3-x$. Here we assume that the curve C is defined by $y^2=x^3-x$. For this curve the ring $\mathbf{Z}[\ulcorner i\urcorner]$ is also isomorphic to the ring $\mathbf{Z}[i]$ by $\ulcorner i\urcorner\mapsto i$. So we identify $\mathbf{Z}[\ulcorner i\urcorner]$ and $\mathbf{Z}[i]$. In this subsection, we write u_1 as u. We let $c=1$, $a_1=0$, and $c_1=-1$, in the notation of 1.1.1. The argument in the proof of 4.2.3

applied to the integrals along the paths $\alpha^{(1)}$ and $\beta^{(1)}$ gives

$$\omega'=2K\,,\quad \omega''=2Ki\,,\quad \eta'=2H\,,\quad \eta''=-2Hi\,.$$

As in the previous subsection we take a point in $\mathbf{C}$ whose image of the map $\kappa:\mathbf{C}\to\mathbf{C}/\Lambda=J=C$ is P_0 and denote it also by P_0.

Similar path as in (5.1.1) gives

$$P_0=iK-K=-\tfrac{1}{2}\omega'+\tfrac{1}{2}\omega''\,. \tag{5.2.1}$$

PROPOSITION 5.2.1. $\sigma(P_0)^4=\exp[2L(P_0,P_0)]$.

PROOF. After differentiating the formula of 3.2.4(1), by setting $u=P_0$, we have $-2\sigma(P_0)^4=2\sigma'(2P_0)$. On the other hand, we get $\sigma(u+2P_0)=\chi(2P_0)\sigma(u)\exp[L(u+P_0,2P_0)]$ from 3.1.1. After differentiating this, by setting $u=0$, we have $\sigma'(2P_0)=-\exp(2L(P_0,P_0))$ because of $\sigma'(0)=1$ and $\sigma(0)=0$. Here we have used the fact $\chi(2P_0)=-1$ which is obtained by (5.2.1). Hence $\sigma(P_0)^4=\exp[2L(P_0,P_0)]$. □

PROPOSITION 5.2.2. *Let b be an element of $\mathbf{Z}[i]$. If $b\equiv 1 \bmod 4$, then*

$$\sigma(b(v+P_0))=\chi((b-1)P_0)\sigma(P_0)^{\mathrm{N}b-1}\sigma(bv+P_0)(1+(d^\circ\geq 1))\,.$$

PROOF. Since $b\equiv 1 \bmod 4$, we have $\mathrm{N}b\equiv 1 \bmod 4$. The statement follows from 4.2.8(1) and

$$\exp[\tfrac{1}{2}(\mathrm{N}b-1)L(P_0,P_0)]=\sigma(P_0)^{\mathrm{N}b-1}$$

which is given by 5.2.1. □

THEOREM 5.2.3 (Eisenstein). *Let $b\in\mathbf{Z}[i]$ and assume $b\equiv 1 \bmod 4$. Then $\psi_b(u)^2\wp(bu)$ is of the form*

$$\psi_b(u)^2\wp(bu)=x(u)\sum_{\substack{0\leq j\leq \mathrm{N}b-1\\ j\equiv 0 \bmod 2}}\gamma_j x(u)^j$$

with $\gamma_j\in\mathbf{Q}(i)$. Moreover $\gamma_0=b^2$ and $\gamma_{\mathrm{N}b-1}=1$.

PROOF. As in the proof of 5.1.3 we have that $\wp(bu)\psi_b(u)^2=1/u^{2\mathrm{N}b}+\cdots$, that $\wp(bu)\psi_b(u)^2$ is a polynomial of $x(u)$ with coefficients in $\mathbf{Q}(i)$, and that $\wp(bu)\psi_b(u)^2=x(u)^{\mathrm{N}b}+\cdots$. For the Laurent expansion at $u=P_0$, since $b-1\equiv 0 \bmod 4$ and $\wp(u)$ has a double order zero at P_0,

$$\begin{aligned}\wp(b(v+P_0))(\psi_b(v+P_0))^2/\wp(v+P_0)&=\frac{\sigma(b(v+P_0))^2}{\sigma(v+P_0)^{2\mathrm{N}b}}\cdot\frac{\wp(b(v+P_0))}{\wp(v+P_0)}\\ &=\frac{\chi((b-1)P_0)^2\sigma(P_0)^{2\mathrm{N}b-2}\sigma(bv+P_0)^2}{\sigma(v+P_0)^{2\mathrm{N}b-2}\sigma(v+P_0)^2}\cdot\frac{\wp(bv+P_0)}{\wp(v+P_0)}+(d^\circ\geq 1)\qquad\text{(by 5.2.2)}\\ &=\frac{\sigma(bv+P_0)^2}{\sigma(v+P_0)^2}\cdot\frac{b^2\wp''(bv+P_0)}{\wp''(v+P_0)}+(d^\circ\geq 1)\qquad(\text{since }\sigma(P_0)\neq 0)\end{aligned}$$

$$=b^2+(d^\circ\geq 1) \qquad (\text{since } \wp'(bP_0)=\wp'(P_0)\neq 0)\,.$$

Since $\psi_b(iu)^2\wp(ibu)=(-1)^{\mathrm{N}b}\psi_b(u)^2\wp(bu)$ because of 4.2.5(1), the function must be a polynomial of $x(u)^2$ multiplied by $x(u)$. □

6. Genus two curves of cyclotomic type.

6.1. The curve defined by $y^2=x^5+1/4$. Now let us consider Grant's original case. So the curve C is defined by $y^2=x^5+1/2$. According to the isomorphism of 4.1.1(1) the ring $\mathbf{Z}[\ulcorner\zeta\urcorner]$ can be identified with $\mathbf{Z}[\zeta]$ by $[\zeta]\mapsto\zeta$. The endomorphism $\ulcorner -\zeta^j\urcorner$ on $\mathbf{C}^2$ is described as

(6.1.1)
$$\ulcorner -\zeta^j\urcorner(u_1, u_2)=(-\zeta^j u_1,\ -\zeta^{2j}u_2)\,.$$

We let $c=-4^{-1/5}$, $a_1=-4^{-1/5}\zeta$, $c_1=-4^{-1/5}\zeta^2$, $a_2=-4^{-1/5}\zeta^3$, $c_2=-4^{-1/5}\zeta^4$, in (1.1.1). Then we have

$$\omega'=\begin{bmatrix} 2K_1(\zeta^3-\zeta^4) & 2K_1(\zeta-\zeta^2) \\ 2K_2(\zeta-\zeta^3) & 2K_2(\zeta^2-\zeta^4)\end{bmatrix},$$

$$\omega''=\begin{bmatrix} 2K_1(-1+\zeta-\zeta^2+\zeta^3) & 2K_1(\zeta-1) \\ 2K_2(-1+\zeta^2-\zeta^4+\zeta) & 2K_2(\zeta^2-1)\end{bmatrix},$$

$$\eta'=\begin{bmatrix} 2H_1(\zeta^2-\zeta) & 2H_1(\zeta^4-\zeta^3) \\ 2H_2(\zeta^4-\zeta^2) & 2H_2(\zeta^3-\zeta)\end{bmatrix},$$

$$\eta''=\begin{bmatrix} 2H_1(-1+\zeta^4-\zeta^3+\zeta^2) & 2H_1(\zeta^4-1) \\ 2H_2(-1+\zeta^3-\zeta+\zeta^4) & 2H_2(\zeta^3-1)\end{bmatrix}.$$

The point P_0 is

(6.1.2)
$$P_0=\begin{bmatrix} K_1(\zeta-\zeta^2+\zeta^3-\zeta^4)-K_1 \\ K_2(\zeta^2-\zeta^4+\zeta-\zeta^3)-K_2\end{bmatrix}=\omega'\begin{bmatrix}2/5\\1/5\end{bmatrix}+\omega''\begin{bmatrix}1/5\\1/5\end{bmatrix}.$$

Then $\ulcorner\zeta^j\urcorner P_0$ are given by (6.1.1) as follows:

(6.1.3)
$$\ulcorner\zeta\urcorner P_0=\begin{bmatrix} \zeta K_1(\zeta-\zeta^2+\zeta^3-\zeta^4-1) \\ \zeta^2K_2(\zeta^2-\zeta^4+\zeta-\zeta^3-1)\end{bmatrix}=\omega'\begin{bmatrix}-3/5\\-4/5\end{bmatrix}+\omega''\begin{bmatrix}1/5\\1/5\end{bmatrix},$$

$$\ulcorner\zeta^2\urcorner P_0=\begin{bmatrix} \zeta^2K_1(\zeta-\zeta^2+\zeta^3-\zeta^4-1) \\ \zeta^4K_2(\zeta^2-\zeta^4+\zeta-\zeta^3-1)\end{bmatrix}=\omega'\begin{bmatrix}2/5\\1/5\end{bmatrix}+\omega''\begin{bmatrix}1/5\\-4/5\end{bmatrix},$$

$$\ulcorner\zeta^3\urcorner P_0=\begin{bmatrix} \zeta^3K_1(\zeta-\zeta^2+\zeta^3-\zeta^4-1) \\ \zeta K_2(\zeta^2-\zeta^4+\zeta-\zeta^3-1)\end{bmatrix}=\omega'\begin{bmatrix}-3/5\\1/5\end{bmatrix}+\omega''\begin{bmatrix}1/5\\1/5\end{bmatrix},$$

$$\lceil\zeta^4\rceil P_0=\begin{bmatrix}\zeta^4K_1(\zeta-\zeta^2+\zeta^3-\zeta^4-1)\\ \zeta^3K_2(\zeta^2-\zeta^4+\zeta-\zeta^3-1)\end{bmatrix}=\omega'\begin{bmatrix}2/5\\ -4/5\end{bmatrix}+\omega''\begin{bmatrix}1/5\\ 1/5\end{bmatrix}.$$

Now let us compute the Taylor expansion at $u=P_0$ explicitly. Since

(6.1.4) $$(\lceil\zeta\rceil-1)P_0=\omega'\begin{bmatrix}-1\\ -1\end{bmatrix}+\omega''\begin{bmatrix}0\\ 0\end{bmatrix}$$

by (6.1.2) and (6.1.3), we have $\chi((\lceil\zeta\rceil-1)P_0)=1$. After substituting this to 4.2.8(2) and differentiating it by v_2, by setting $v=O$, we have

$$\sigma_2(\lceil\zeta\rceil P_0)=\sigma_2(P_0)\exp\tfrac{1}{2}L((\overline{\lceil\zeta\rceil}-\lceil\zeta\rceil)P_0,P_0)\,,$$

since $\sigma(P_0)=0$. Because of 4.2.5(1) and $\sigma_2(P_0)\neq 0$ (see 2.2.1(2)), it must be

$$\exp\tfrac{1}{2}L((\overline{\lceil\zeta\rceil}-\lceil\zeta\rceil)P_0,P_0)=\zeta^4\,.$$

Therefore 4.2.8(2) gives rise to

(6.1.5) $$\sigma(v+\lceil\zeta\rceil P_0)=\zeta^4\sigma(v+P_0)\exp[L(v,(\lceil\zeta\rceil-1)P_0)]\,.$$

After operating $\partial^2/\partial u_i\partial u_j$ to (6.1.5), by setting $v=O$, we have

(6.1.6) $$\sigma_{ij}(\lceil\zeta\rceil P_0)=\zeta^4\sigma_{ij}(P_0)+\sigma_i(P_0)(-\eta'_{1j}-\eta'_{2j})\zeta^4+\sigma_j(P_0)(-\eta'_{1i}-\eta'_{2i})\zeta^4$$

by (6.1.4). For the case $i=j=1$, (6.1.6) is of no use because $\sigma_1(P_0)=0$. But 2.2.1 gives $\sigma_{11}(P_0)=2\sqrt{\lambda_0}\sigma_2(P_0)=\sigma_2(P_0)$. Set $i=1$ and $j=2$ in (6.1.6), then $\sigma_{12}(P_0)=2H_1(\zeta^2+\zeta^4)\sigma_2(P_0)$. By a similar fashion, we get $\sigma_{22}(P_0)=4H_2(\zeta^4+\zeta^3)\sigma_2(P_0)$. Although these explicit values are unnecessary to prove 6.1.6 below, we mention this here to make 6.1.1 below clean. The Taylor expansion at O is given by 2.1.1(2). Thus we have arrived at

Proposition 6.1.1. *Assume C is defined by $y^2=x^5+1/4$. Let P_0 be the point whose coordinate is given by* (6.1.2). *Then*

(1) $$\sigma(u)=u_1-\tfrac{1}{3}u_2^3+(d^\circ\geq 5)\,,$$

(2) $$\sigma(v+P_0)=\sigma_2(P_0)\bigg(v_2+\frac{1}{2}v_1^2+\gamma_{12}v_1v_2+\frac{\gamma_{22}}{2}v_2^2+\frac{\gamma_{12}}{2}v_1^3+\left(\frac{\gamma_{22}}{4}+\frac{\gamma_{12}^2}{2}\right)v_1^2v_2+\frac{\gamma_{12}\gamma_{22}}{2}v_1v_2^2+\frac{\gamma_{22}^2}{8}v_2^3+(d^\circ\geq 3)\bigg)\,,$$

where $\gamma_{12}=2H_1(\zeta^2+\zeta^4)$ and $\gamma_{22}=4H_2(\zeta^3+\zeta^4)$.

Proposition 6.1.2. $\sigma_2(P_0)^5=\exp\frac{5}{2}L(P_0,P_0)$.

Proof. Because of $y(P_0)=1/2$, it is obtained from 3.2.4(1) and (2) that $\sigma(2P_0)=\sigma_2(P_0)^4$, $\sigma(3P_0)=\sigma_2(P_0)^9$. On the other hand, from 3.1.1, we get

$$\sigma(3P_0)=\sigma(-2P_0+5P_0)=-\exp[\tfrac{5}{2}L(P_0,P_0)]\sigma(2P_0).$$

Here we used that $\sigma(-u)=-\sigma(u)$ and that $\chi(5P_0)=1$ which is given by (6.1.2). Therefore we obtain

$$-\sigma_2(P_0)^9=-\exp[\tfrac{5}{2}L(P_0,P_0)]\sigma_2(P_0)^4$$

and hence the statement. □

We denote by τ the element of Gal($\mathbf{Q}(\zeta)/\mathbf{Q}$) such that $\zeta^\tau=\zeta^2$. Then $1+\tau$ is a type norm (see 4.2.7) in $\mathbf{Z}[\mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})]$.

LEMMA 6.1.3. *If $b\in\mathbf{Z}[\zeta]$, then $\chi((b^{1+\tau^{-1}}-1)P_0)^{Nb-1}=1$.*

PROOF. If Nb is odd, the statement is trivial. So we assume Nb is even. For $l\in\Lambda$, it is easily verified from the definition that the value $\chi(l)$ is determined only by $l \bmod 2\Lambda$. By the assumption on b, we may write $b^{1+\tau^{-1}}=(a_1\zeta+a_2\zeta^2+a_3\zeta^3+a_4\zeta^4)(1-\zeta)+1$. Since 2 is a prime in $\mathbf{Z}[\zeta]$, we have $b\equiv 0 \bmod 2$ and hence $b^{1+\tau^{-1}}\equiv 0 \bmod 2$. By simple calculation, we see that $a_1\equiv a_3\equiv 1 \bmod 2$ and $a_2\equiv a_4\equiv 0 \bmod 2$. Therefore

$$\chi((b^{1+\tau^{-1}}-1)P_0)=\chi((\zeta+\zeta^3)(1-\zeta)P_0)=\chi((\zeta-\zeta^2+\zeta^3-\zeta^4)P_0)=1$$

because of

$$(\zeta-\zeta^2+\zeta^3-\zeta^4)P_0=\omega'\begin{bmatrix}-2\\-1\end{bmatrix}+\omega''\begin{bmatrix}1\\1\end{bmatrix}$$

which is obtained from (6.1.3.). □

PROPOSITION 6.1.4. *Let b be an element of $\mathbf{Z}[\zeta]$. If $b\equiv 1 \bmod (1-\zeta)^2$, then*

$$\sigma(b^{1+\tau^{-1}}(v+P_0))=\sigma_2(P_0)^{Nb-1}\chi(b^{1+\tau^{-1}}-1)\sigma(b^{1+\tau^{-1}}v+P_0)(1+(d^\circ\geq 1)).$$

PROOF. The statement follows from 4.2.8(1) and

$$\exp[\tfrac{1}{2}(Nb-1)L(P_0,P_0)]=\sigma_2(P_0)^{Nb-1}$$

which is given by 6.1.2. □

LEMMA 6.1.5. *Let $\varphi(u)$ denote the function $(\wp_{12}^2-\wp_{22}\wp_{11})(u)$. Then*
(1) $\varphi(\ulcorner\zeta\urcorner u)=\zeta^4\varphi(u)$,
(2) $\varphi(u)\in\Gamma(J,\mathcal{O}(3\Theta))$,
(3) *the Taylor expansions of $\sigma(u)^3\varphi(u)$ at O and P_0 are of the form*

$$\sigma(u)^3\varphi(u)=2u_2+(d^\circ(u_1,u_2)\geq 2)\quad \textit{and}$$

$$\sigma(v+P_0)^3\varphi(v+P_0)=\sigma_2(P_0)^3(-1+(d^\circ(v_1,v_2)\geq 1)).$$

PROOF. (1) follows from 4.2.5 and the definition of $\wp$-functions. (2) follows from

$$(\sigma^3\varphi)(u)=-\sigma_2(u)^2\sigma_{11}(u)-\sigma_1(u)^2\sigma_{22}(u)+2\sigma_1(u)\sigma_2(u)\sigma_{12}(u)+\sigma_{12}(u)^2\sigma(u).$$

The statement (3) is easily derived from the equation above and 6.1.1. □

THEOREM 6.1.6 (Grant [9]). *Let* $\varphi(u):=(\wp_{12}^2-\wp_{22}\wp_{11})(u)$. *Let* $b\in\mathbf{Z}[\zeta]$ *and assume* $b\equiv 1 \bmod (1-\zeta)^2$. *Then* $\psi_{b^{1+\tau^{-1}}}(u)^3\varphi(b^{1+\tau^{-1}}u)$ *is of the form*

$$\psi_{b^{1+\tau^{-1}}}(u)^3\varphi(b^{1+\tau^{-1}}u)=2y(u)\sum_{\substack{0\leq j\leq 3(Nb-1)\\ j\equiv 0 \bmod 5}}\gamma_j x(u)^j$$

with every $\gamma_j\in\mathbf{Q}(\zeta)$. *Moreover,* $\gamma_{3(Nb-1)}=(-1)^{Nb}b^{1+\tau}$ *and* $\gamma_0=-1$.

PROOF. First, we look at the Laurent expansion at $u=O$. By 6.1.5(3) and 6.1.1(1), we have

$$\begin{aligned}
&\psi_{b^{1+\tau^{-1}}}(u)\varphi(b^{1+\tau^{-1}}u)\big|_{u\in\kappa^{-1}\iota(C)}\\
&=\frac{\sigma(b^{1+\tau^{-1}}u)^3\varphi(b^{1+\tau^{-1}}u)}{\sigma_2(u)^{3Nb}}=\frac{2(b^{1+\tau^{-1}})^{\tau}u_2+(d^\circ(u_2)\geq 2)}{(-u_2^2+(d^\circ(u_2)\geq 4))^{3Nb}}\\
&=(-1)^{Nb}2b^{\tau+1}\frac{1}{u_2^{6Nb-1}}+\cdots=(-1)^{Nb}2b^{\tau+1}\frac{-1}{u_2^5}\left(\frac{1}{u_2^2}\right)^{3(Nb-1)}+\cdots\\
&=(-1)^{Nb}2b^{\tau+1}y(u)(x(u)^{3(Nb-1)}+\text{“lower terms of power of } x(u)^5\text{”}).
\end{aligned}$$

Here we used 2.3.1 and the fact that the above function is a polynomial of $x(u)$ multiplied by $y(u)$, which is deduced from that this function is odd and σ_2 has only zeroes at $u\in\Lambda$ by the first statement of 2.2.1(2). Secondly, we look at the Laurent expansion at $u=P_0$ ($\kappa(P_0)=\iota(0,1/2)$). Since $b\equiv 1 \bmod (1-\zeta)^2$ we have $b^{\tau+1}\equiv 1 \bmod (1-\zeta)^2$. Because of $(1-\zeta)P_0\in\Lambda$ and $\varphi(u)$ being periodic, we have $\varphi(b^{1+\tau^{-1}}(v+P_0))=\varphi(b^{1+\tau^{-1}}v+P_0)$. Consequently, 6.1.4, 6.1.1, 6.1.5 and 6.1.3 imply

$$\begin{aligned}
&\psi_{b^{1+\tau^{-1}}}(v+P_0)^3\varphi(b^{1+\tau^{-1}}(v+P_0))\big|_{v+P_0\in\kappa^{-1}\iota(C)}\\
&=\frac{\sigma(b^{1+\tau^{-1}}(v+P_0))^3\varphi(b^{1+\tau^{-1}}(v+P_0))}{\sigma_2(b^{1+\tau^{-1}}(v+P_0))^{3Nb}}\Bigg|_{v+P_0\in\kappa^{-1}\iota(C)}\\
&=\frac{\sigma_2(P_0)^{3(Nb-1)}\sigma(b^{1+\tau^{-1}}v+P_0)^3\chi((b^{1+\tau^{-1}}-1)P_0)^3(1+(d^\circ(v_2)\geq 1))\varphi(b^{1+\tau^{-1}}v+P_0)}{[\sigma_2(b^{1+\tau^{-1}}v+P_0)\chi((b^{1+\tau^{-1}}-1)P_0)(1+(d^\circ(v_2)\geq 1))]^{3Nb}}\\
&=\frac{\sigma_2(P_0)^{3(Nb-1)}\sigma_2(P_0)^3(-1+(d^\circ(v_1)\geq 1))}{\sigma_2(b^{1+\tau^{-1}}v+P_0)^{3Nb}}\chi((b^{1+\tau^{-1}}-1)P_0)^{3(1-Nb)}\\
&=-1+(d^\circ(v_1)\geq 1)=-2y(u)(1+(d^\circ(x(u))\geq 2)).
\end{aligned}$$

Furthermore, since

$$\psi_{b^{1+\tau^{-1}}}(\ulcorner-\zeta\urcorner u)^3\varphi(b^{1+\tau^{-1}}\ulcorner-\zeta\urcorner u)=-\zeta^{2Nb-2}\psi_{b^{1+\tau^{-1}}}(u)^3\varphi(b^{1+\tau^{-1}}u)$$

by 4.2.5(1), the function must be a polynomial of $x(u)^5$ multiplied by $y(u)$. □

6.2. The curve defined by $y^2=x^5-x$. We treat here the other genus two curve C defined by $y^2=x^5-x$. The ring $\mathbf{Z}[\ulcorner\zeta\urcorner]$ can also be identified with $\mathbf{Z}[\zeta]$ by 4.1.1(2).

The endomorphism $\lceil -\zeta^j \rceil$ acts as

$$\lceil -\zeta^j \rceil (u_1, u_2) = (-\zeta^j u_1,\ -\zeta^{3j} u_2)$$

because $\lceil \zeta \rceil \omega^{(1)} = \zeta \omega^{(1)}$ and $\lceil \zeta \rceil \omega^{(2)} = \zeta^2 \omega^{(2)}$. We let $c=1$, $a_1=i$, $c_1=-1$, $a_2=-i$, $c_2=0$, in (1.1.1). In this case

$$\omega' = \begin{bmatrix} -2K_1\zeta^3 & 2K_1(\zeta^2-\zeta) \\ -2K_2\zeta & 2K_2(\zeta^6-\zeta^3) \end{bmatrix},$$

$$\omega'' = \begin{bmatrix} 2K_1(\zeta^2-\zeta+1) & 2K_1(-\zeta+1) \\ 2K_2(\zeta^6-\zeta^3+1) & 2K_2(-\zeta^3+1) \end{bmatrix},$$

$$\eta' = \begin{bmatrix} -2H_1\zeta^5 & 2H_1(\zeta^6-\zeta^7) \\ -2H_2\zeta^7 & 2H_2(\zeta^2-\zeta^5) \end{bmatrix},$$

$$\eta'' = \begin{bmatrix} 2H_1(\zeta^6-\zeta^7+1) & 2H_1(-\zeta^7+1) \\ 2H_2(\zeta^2-\zeta^5+1) & 2H_2(-\zeta^5+1) \end{bmatrix}.$$

Our choice of P_0 in $\mathbf{C}^2$ gives

$$P_0 = \begin{bmatrix} K_1(\zeta-\zeta^2+\zeta^3)-K_1 \\ K_2(\zeta^3-\zeta^6+\zeta)-K_2 \end{bmatrix} = \omega' \begin{bmatrix} -1/2 \\ 0 \end{bmatrix} + \omega'' \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}, \tag{6.2.1}$$

and

$$\lceil \zeta \rceil P_0 = \begin{bmatrix} \zeta K_1(\zeta-\zeta^2+\zeta^3-1) \\ \zeta^3 K_2(\zeta^3-\zeta^6+\zeta-1) \end{bmatrix} = \omega' \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} + \omega'' \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}. \tag{6.2.2}$$

Then our arguments go in parallel with the previous subsection. Instead of (6.1.4) we derive

$$(\lceil \zeta \rceil - 1)P_0 = \omega' \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \omega'' \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6.2.3}$$

from (6.2.1) and (6.2.2), and then

$$\sigma(v+\lceil \zeta \rceil P_0) = \zeta^6 \sigma(v+P_0) \exp[L(v, (\lceil \zeta \rceil - 1)P_0)] \tag{6.2.4}$$

and

$$\sigma_{ij}(\lceil \zeta \rceil P_0) = \zeta^6 \sigma_{ij}(P_0) + \sigma_i(P_0)(\eta'_{1j}+\eta'_{2j})\zeta^6 + \sigma_j(P_0)(\eta'_{1i}+\eta'_{2i})\zeta^6 \tag{6.2.5}$$

instead of (6.1.5) and (6.1.6), respectively. Then we have $\sigma_{12}(P_0) = H_1(-1-(\sqrt{2}-1)i)\sigma_2(P_0)$ and $\sigma_{22}(P_0) = 2H_2(-1+\sqrt{2}+i)\sigma_2(P_0)$. From 2.2.1(2) we have $\sigma_{11}(P_0) = 2\sqrt{\lambda_0}\sigma_2(P_0) = 0$. Thus, we arrive at

PROPOSITION 6.2.1. *Assume C is defined by* $y^2 = x^5 - x$. *Let* P_0 *be the point whose coordinate is given by* (6.2.1). *Then*

(1) $$\sigma(u)=u_1-\tfrac{1}{3}u_2^3+(d^\circ\geq 5)\,,$$

(2) $$\sigma(v+P_0)=\sigma_2(P_0)(v_2+\gamma_{12}v_1v_2+\frac{\gamma_{22}}{2}v_2^2-\frac{1}{3}v_1^3+\frac{\gamma_{12}^2}{2}v_1^2v_2$$
$$+\frac{\gamma_{12}\gamma_{22}}{2}v_1v_2^2+\frac{\gamma_{22}^2}{8}v_2^3+(d^\circ\geq 3))\,,$$

where $\gamma_{12}=H_1(-1-(\sqrt{2}-1)i)$ *and* $\gamma_{22}=2H_2(-1+\sqrt{2}+i)$.

PROPOSITION 6.2.2. $\sigma_2(P_0)^4=\exp 2L(P_0,P_0)$.

PROOF. Take $y=y(u)$ as a local parameter at P_0 along $\kappa^{-1}\iota(C)$. By 3.2.2(1), we have $2y(u)\sigma_2(u)^4=\sigma(2u)$. After differentiating this with respect to y, by setting $u=P_0$, we get $2\sigma_2(P_0)^4=2\sigma_1(2P_0)$ because of $y(P_0)=0$ and $\sigma(2P)=0$ which is led from the fact $2P_0\in\Lambda$. Similarly, by differentiating $\sigma(u+2P_0)=\chi(2P_0)\sigma(u)\exp L(u+P_0,2P_0)$ with respect to u_1 and setting $u=P_0$, we get $\sigma_1(2P_0)=\exp 2L(P_0,P_0)$ because $\sigma(O)=0$, $\sigma_1(O)=1$ and $\chi(2P_0)=1$ where the last is obtained from (6.2.1) and the definition of $\chi(\ )$. Hence the statement. □

We denote by τ the element of $\mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})$ such that $\zeta^\tau=\zeta^3$. Then $1+\tau$ is a type norm (see 4.2.7) in $\mathbf{Z}[\mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})]$.

PROPOSITON 6.2.3. *Let b be an element of $\mathbf{Z}[\zeta]$. If $b\equiv 1 \bmod 4$, then*

$$\sigma(b^{1+\tau}(v+P_0))=\sigma_2(P_0)^{\mathrm{N}b-1}\sigma(b^{1+\tau}v+P_0)(1+(d^\circ\geq 1))\,.$$

PROOF. By the assumption, $b^{1+\tau}-1\equiv 0 \bmod 4$ and hence $(b^{1+\tau}-1)P_0\in 2\Lambda$. So $\chi((b^{1+\tau}-1)P_0)=1$. Moreover $\mathrm{N}b-1\equiv 0 \bmod 4$ and $2P_0\in\Lambda$. Then the statement follows from 4.2.8(1) and

$$\exp[\tfrac{1}{2}(\mathrm{N}b-1)L(P_0,P_0)]=\sigma_2(P_0)^{\mathrm{N}b-1}$$

which is given by 6.2.2. □

LEMMA 6.2.4. *Let* $\varphi(u):=(\tfrac{1}{8}(\wp_{2222}-6\wp_{22}^2)\wp_{111}+\tfrac{1}{4}(\wp_{1112}-6\wp_{11}\wp_{12})\wp_{222})(u)$. *Then it has the following properties.*

(1) $\varphi(\ulcorner\zeta\urcorner u)=\zeta^3\varphi(u)$,

(2) $\varphi(u)\in\Gamma(J,\mathcal{O}(5\Theta))$,

(3) *the Taylor expansions of $\sigma(u)^5\varphi(u)$ at O and P_0 are of the form*

$$\sigma(u)^5\varphi(u)=u_2^2+c_1u_1^2+c_2u_1u_2+(d^\circ(u_1,u_2)\geq 4)$$

for some constants c_1 and c_2,

$$\sigma(v+P_0)^5\varphi(v+P_0)=\sigma_2(P_0)^5(1+(d^\circ(v_1,v_2)\geq 1))\,.$$

PROOF. (1) follows from 4.2.5(1) and the definition of $\wp$-functions. Since

$$\sigma(u)^2(\wp_{2222}-6\wp_{22}^2)(u)=(-\sigma_{2222}\sigma+4\sigma_{222}\sigma_2-3\sigma_{22}^2)(u)\,,$$

$$\sigma(u)^3\wp_{111}(u)=(-2\sigma_1^3+3\sigma_1\sigma_{11}\sigma-\sigma_{111}\sigma^2)(u)\,,$$
$$\sigma(u)^2(\wp_{1112}-6\wp_{11}\wp_{12}^2)(u)=(-\sigma_{1112}\sigma+3\sigma_{112}\sigma_1+\sigma_{111}\sigma_2-3\sigma_{11}\sigma_{12})(u)\,,$$
$$\sigma(u)^3\wp_{222}(u)=(-2\sigma_2^3+3\sigma_2\sigma_{22}\sigma-\sigma_{222}\sigma^2)(u)$$

the statement (2) holds. The expansion in 6.2.1(1) gives

$$\begin{aligned}&(-\sigma_{2222}\sigma+4\sigma_{222}\sigma_2-3\sigma_{22}^2)(u)\\&\quad=-(d^\circ\geq 1)(u_1+(d^\circ\geq 3))+4(-2+(d^\circ\geq 1))(-u_2^2+(d^\circ\geq 4))-3(-2u_2+(d^\circ\geq 3))^2\\&\quad=-4u_2^2+c_2'u_1^2+c_2'u_1u_2+(d^\circ\geq 4)\quad\text{for some constants } c_1' \text{ and } c_2',\end{aligned}$$

$$\begin{aligned}&(-2\sigma_1^3+3\sigma_1\sigma_{11}\sigma-\sigma_{111}\sigma^2)(u)\\&\quad=-2(1+(d^\circ\geq 1))^3+3(1+(d^\circ\geq 1))(d^\circ\geq 3)(d^\circ\geq 1)-(d^\circ\geq 3)(d^\circ\geq 1)^2\\&\quad=-2+(d^\circ\geq 2)\,,\end{aligned}$$

$$\begin{aligned}&(-\sigma_{1112}\sigma+3\sigma_{112}\sigma_1-3\sigma_{111}\sigma_2-3\sigma_{11}\sigma_{12})(u)\\&\quad=-(d^\circ\geq 1)(u_1+(d^\circ\geq 3))+3(d^\circ\geq 2)(d^\circ\geq 4)+(d^\circ\geq 2)(d^\circ\geq 2)-3(d^\circ\geq 3)(d^\circ\geq 3)\\&\quad=(d^\circ\geq 2)\,,\end{aligned}$$

$$\begin{aligned}&(-2\sigma_2^3+3\sigma_2\sigma_{22}\sigma-\sigma_{222}\sigma^2)(u)\\&\quad=-2(d^\circ\geq 2)^3+3(d^\circ\geq 2)(d^\circ\geq 1)(d^\circ\geq 1)-(d^\circ\geq 0)(d^\circ\geq 1)^2=(d^\circ\geq 2)\,.\end{aligned}$$

Therefore

$$\sigma(u)^5\varphi(u)=u_2^2+c_1u_1^2+c_2u_1u_2+(d^\circ\geq 4)$$

for some constants c_1 and c_2. Similarly, 6.2.1(2) gives

$$\begin{aligned}&(-2\sigma_1^3+3\sigma_1\sigma_{11}\sigma-\sigma_{111}\sigma^2)(v+P_0)\\&\quad=\sigma_2(P_0)^3[-2(d^\circ\geq 2)^3+3(d^\circ\geq 2)(d^\circ\geq 1)(d^\circ\geq 1)-(d^\circ\geq 0)(d^\circ\geq 1)^2]=(d^\circ\geq 2)\,,\end{aligned}$$

$$\begin{aligned}&(-\sigma_{1112}\sigma+3\sigma_{112}\sigma_1+\sigma_{111}\sigma_2-3\sigma_{11}\sigma_{12})(v+P_0)\\&\quad=\sigma_2(P_0)^2[-(d^\circ\geq 0)(d^\circ\geq 1)+3(d^\circ\geq 0)(d^\circ\geq 2)\\&\qquad+(-2+(d^\circ\geq 1))(1+(d^\circ\geq 1))-3(d^\circ\geq 1)(d^\circ\geq 0)]=\sigma_2(P_0)^2(-2+(d^\circ\geq 1))\,,\end{aligned}$$

$$\begin{aligned}&(-2\sigma_2^3+3\sigma_2\sigma_{22}\sigma-\sigma_{222}\sigma^2)(v+P_0)\\&\quad=\sigma_2(P_0)^3[-2(1+(d^\circ\geq 1))^3+3(1+(d^\circ\geq 1))(d^\circ\geq 0)(d^\circ\geq 1)-(d^\circ\geq 0)(d^\circ\geq 1)^2\\&\quad=\sigma_2(P_0)^3(-2+(d^\circ\geq 1))\,.\end{aligned}$$

Hence $\sigma(u)^5\varphi(v+P_0)=\sigma_2(P_0)^5(1+(d^\circ\geq 1))$. This is (3). □

THEOREM 6.2.5. *Let* $\varphi(u)$ *be as in* 6.2.4. *Let* $b\in\mathbf{Z}[\zeta]$ *and assume* $b\equiv 1 \bmod 4$. *Then* $\psi_{b^{1+\tau}}(u)^5\varphi(b^{1+\tau}u)$ *is of the form*

$$\psi_{b^{1+\tau}}(u)^5\varphi(b^{1+\tau}u)=\sum_{\substack{0\le j\le 5Nb-1\\ j\equiv 0 \bmod 4}}\gamma_j x(u)^j$$

with $\gamma_j\in\mathbf{Q}(\zeta)$. *Moreover*, $\gamma_{5Nb-1}=b^{2(1+\tau)}$ *and* $\gamma_0=1$.

PROOF. We follow the same arguments as in the proof of 6.1.6. We look at the Laurent expansion at $u=O$. By 6.2.4(3) and 6.2.1(1), we have

$$\begin{aligned}\psi_{b^{1+\tau}}(u)\varphi(b^{1+\tau}u)\big|_{u\in\kappa^{-1}\iota(C)}&=\frac{\sigma(b^{1+\tau}u)^5\varphi(b^{1+\tau}u)}{\sigma_2(u)^{5Nb}}=\frac{-(b^{1+\tau})^{2\tau}u_2^2+(d^\circ(u_2)\ge 4)}{(-u_2^2+(d^\circ(u_2)\ge 4))^{5Nb}}\\&=b^{2(\tau+1)}\frac{1}{u_2^{10Nb-2}}+\cdots=b^{2(\tau+1)}\left(\frac{1}{u_2^2}\right)^{5Nb-1}+\cdots\\&=b^{2(\tau+1)}x(u)^{5Nb-1}+\text{“lower terms of power of } x(u)\text{”},\end{aligned}$$

since the above function is even and so a polynomial of $x(u)$. Then we look at the Laurent expansion at $u=P_0$ ($\kappa(P_0)=\iota(0,0)$). Since $b\equiv 1 \bmod 4$ and so $b^{\tau+1}\equiv 1 \bmod 4$, we have $\varphi(b^{1+\tau}(v+P_0))=\varphi(b^{1+\tau}v+P_0)$. Therefore, 6.2.3, 6.2.1 and 6.2.4 imply

$$\begin{aligned}&\psi_{b^{1+\tau}}(v+P_0)^5\varphi(b^{1+\tau}(v+P_0))\big|_{v+P_0\in\kappa^{-1}\iota(C)}\\&=\frac{\sigma(b^{1+\tau}(v+P_0))^5\varphi(b^{1+\tau}(v+P_0))}{\sigma_2(b^{1+\tau}(v+P_0))^{5Nb}}\Bigg|_{v+P_0\in\kappa^{-1}\iota(C)}\\&=\frac{\sigma_2(P_0)^{5(Nb-1)}\sigma(b^{1+\tau}v+P_0)^5\chi((b^{1+\tau}-1)P_0)^5(1+(d^\circ(v_1)\ge 1))\varphi(b^{1+\tau}v+P_0)}{\sigma_2(P_0)^{5Nb}(1+(d^\circ\ge 1))\chi((b^{1+\tau}-1)P_0)^{5Nb}}\\&=\frac{\sigma_2(P_0)^{5(Nb-1)}\sigma_2(P_0)^5(1+(d^\circ(v_1)\ge 1))\chi((b^{1+\tau}-1)P_0)^{5(1-Nb)}}{\sigma_2(P_0)^{5Nb}(1+(d^\circ\ge 1))}\\&=\chi((b^{1+\tau}-1)P_0)^{5(1-Nb)}(1+(d^\circ(v_1)\ge 1))=1+(d^\circ(x(u))\ge 2)\,.\end{aligned}$$

Here the last equality follows from the fact that $b^{1+\tau}-1$ is divisible by 4 and so $\chi((b^{1+\tau}-1)P_0)=1$. According to 4.2.5(1),

$$\psi_{b^{1+\tau}}(\ulcorner\zeta\urcorner u)^5\varphi(b^{1+\tau}\ulcorner\zeta\urcorner u)=\zeta^{3(5Nb-1)}\psi_{b^{1+\tau}}(u)^5\varphi(b^{1+\tau}u)\,,$$

and hence the function must be a polynomial of $x(u)^4$. □

7. Genus three curves of cyclotomic type.

7.1. The curve defined by $y^2=x^7+1/4$. Let us treat the genus three case. First example is the curve C defined by $y^2=x^7+1/4$. As in Sections 5 and 6 the ring $\mathbf{Z}[\ulcorner\zeta\urcorner]$ is isomorphic to $\mathbf{Z}[\zeta]$. Then $\ulcorner-\zeta^j\urcorner$ acts as

$$\ulcorner-\zeta^j\urcorner(u_1,u_2)=(-\zeta^j u_1,-\zeta^{2j}u_2,-\zeta^{3j}u_3)\,.$$

We let $c=-4^{-1/7}$, $a_1=-4^{-1/7}\zeta$, $c_1=-4^{-1/7}\zeta^2$, $a_2=-4^{-1/7}\zeta^3$, $c_2=-4^{-1/7}\zeta^4$, $a_3=$

$-4^{-1/7}\zeta^5$, $c_3=-4^{-1/7}\zeta^6$, in (1.1.1). Then

$$\omega'=\begin{bmatrix} 2K_1(\zeta^5-\zeta^4) & 2K_1(\zeta^3-\zeta^4) & 2K_1(\zeta-\zeta^2) \\ 2K_2(\zeta^3-\zeta) & 2K_2(\zeta^6-\zeta) & 2K_2(\zeta^2-\zeta^4) \\ 2K_3(\zeta-\zeta^5) & 2K_3(\zeta^2-\zeta^5) & 2K_3(\zeta^3-\zeta^6) \end{bmatrix},$$

$$\omega''=\begin{bmatrix} 2K_1(\zeta^5-\zeta^4+\zeta^3-\zeta^2+\zeta-1) & 2K_1(\zeta^3-\zeta^2+\zeta-1) & 2K_1(\zeta-1) \\ 2K_2(\zeta^3-\zeta+\zeta^6-\zeta^4+\zeta^2-1) & 2K_2(\zeta^6-\zeta^4+\zeta^2-1) & 2K_2(\zeta^2-1) \\ 2K_3(\zeta-\zeta^5+\zeta^2-\zeta^6+\zeta^3-1) & 2K_3(\zeta^2-\zeta^6+\zeta^3-1) & 2K_3(\zeta^3-1) \end{bmatrix},$$

$$\eta'=\begin{bmatrix} 2H_1(\zeta^2-\zeta^3) & 2H_1(\zeta^4-\zeta^3) & 2H_1(\zeta^6-\zeta^5) \\ 2H_2(\zeta^4-\zeta^6) & 2H_2(\zeta-\zeta^6) & 2H_2(\zeta^5-\zeta^3) \\ 2H_3(\zeta^6-\zeta^2) & 2H_3(\zeta^5-\zeta^2) & 2H_3(\zeta^4-\zeta) \end{bmatrix},$$

$$\eta''=\begin{bmatrix} 2H_1(\zeta^2-\zeta^3+\zeta^4-\zeta^5+\zeta^6-1) & 2H_1(\zeta^4-\zeta^5+\zeta^6-1) & 2H_1(\zeta^6-1) \\ 2H_2(\zeta^4-\zeta^6+\zeta-\zeta^3+\zeta^5-1) & 2H_2(\zeta-\zeta^3+\zeta^5-1) & 2H_2(\zeta^5-1) \\ 2H_3(\zeta^6-\zeta^2+\zeta^5-\zeta+\zeta^4-1) & 2H_3(\zeta^5-\zeta+\zeta^4-1) & 2H_3(\zeta^4-1) \end{bmatrix}.$$

The point P_0 in $\mathbf{C}^3$ is given by

(7.1.1) $$P_0\begin{bmatrix} K_1(\zeta-\zeta^2+\zeta^3-\zeta^4+\zeta^5-\zeta^6)-K_1 \\ K_2(\zeta^2-\zeta^4+\zeta^6-\zeta+\zeta^3-\zeta^5)-K_2 \\ K_3(\zeta^3-\zeta^6+\zeta^2-\zeta^5+\zeta-\zeta^4)-K_3 \end{bmatrix}=\omega'\begin{bmatrix} 3/7 \\ 2/7 \\ 1/7 \end{bmatrix}+\omega''\begin{bmatrix} 1/7 \\ 1/7 \\ 1/7 \end{bmatrix}.$$

Then

$$\ulcorner\zeta\urcorner P_0=\omega'\begin{bmatrix} -4/7 \\ -5/7 \\ -6/7 \end{bmatrix}+\omega''\begin{bmatrix} 1/7 \\ 1/7 \\ 1/7 \end{bmatrix},\quad \ulcorner\zeta^2\urcorner P_0=\omega'\begin{bmatrix} 3/7 \\ 2/7 \\ 1/7 \end{bmatrix}+\omega''\begin{bmatrix} 1/7 \\ 1/7 \\ -6/7 \end{bmatrix},$$

(7.1.2) $$\ulcorner\zeta^3\urcorner P_0=\omega'\begin{bmatrix} -4/7 \\ -5/7 \\ 1/7 \end{bmatrix}+\omega''\begin{bmatrix} 1/7 \\ 1/7 \\ 1/7 \end{bmatrix},\quad \ulcorner\zeta^4\urcorner P_0=\omega'\begin{bmatrix} 3/7 \\ 2/7 \\ 1/7 \end{bmatrix}+\omega''\begin{bmatrix} 1/7 \\ -6/7 \\ 1/7 \end{bmatrix},$$

$$\ulcorner\zeta^5\urcorner P_0=\omega'\begin{bmatrix} -4/7 \\ 2/7 \\ 1/7 \end{bmatrix}+\omega''\begin{bmatrix} 1/7 \\ 1/7 \\ 1/7 \end{bmatrix},\quad \ulcorner\zeta^6\urcorner P_0=\omega'\begin{bmatrix} 3/7 \\ 2/7 \\ 1/7 \end{bmatrix}+\omega''\begin{bmatrix} -6/7 \\ 1/7 \\ 1/7 \end{bmatrix}.$$

To compute the Taylor expansion at $u=P_0$, we again follow the arguments in 6.1. Instead of (6.1.4) we have

(7.1.3) $$(\ulcorner\zeta\urcorner-1)P_0=\omega'\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}+\omega''\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which is given by (7.1.1) and (7.1.2), and then

(7.1.4) $$\sigma(v+\lceil\zeta\rceil P_0)=\zeta^2\sigma(v+P_0)\exp[L(v,(\lceil\zeta\rceil-1)P_0)]$$

and

(7.1.5) $$\begin{aligned}\sigma_{ij}(\lceil\zeta\rceil P_0)=\zeta^4\sigma_{ij}(P_0)+\sigma_i(P_0)(-\eta'_{1j}-\eta'_{2j}-\eta'_{3j})\zeta^2\\+\sigma_j(P_0)(-\eta'_{1i}-\eta'_{2i}-\eta'_{3i})\zeta^2\end{aligned}$$

instead of (6.1.5) and (6.1.6), respectively. Then we have

$$\begin{aligned}&\sigma_{11}(P_0)=2\sqrt{\lambda_0}\sigma_2(P_0)=\sigma_2(P_0)\neq 0\,, && \sigma_{12}(P_0)=2H_1(-\zeta^3+\zeta^4+\zeta^6)\sigma_2(P_0)\,,\\ &\sigma_{22}(P_0)=4H_2(\zeta^4+\zeta+\zeta^5)\sigma_2(P_0)\,, && \sigma_{13}(P_0)=0\,,\\ &\sigma_{23}(P_0)=2H_2(-\zeta^2+\zeta^5+\zeta^4)\sigma_2(P_0)\,, && \sigma_{33}(P_0)=0\,.\end{aligned}$$

Since the Taylor expansion at O is given by 2.1.1(3), we have obtained the following.

Proposition 7.1.1. *Assume C is defined by $y^2=x^7+1/4$. Let P_0 be the point whose coordinate is given by* (7.1.1). *Then*

(1) $$\sigma(u)=u_1u_3-u_2^2-\tfrac{1}{12}u_1^4-\tfrac{1}{3}u_2u_3^3+(d^\circ\geq 6)\,,$$

(2) $$\begin{aligned}\sigma(v+P_0)=\sigma_2(P_0)\Big(&v_2+\frac{1}{2}v_1^2+\gamma_{12}v_1v_2+\frac{\gamma_{22}}{2}v_2^2+\Big(\frac{\gamma_{12}}{6}-\frac{1}{3}\Big)v_1^3\\&+\Big(\frac{\gamma_{22}}{8}+\frac{\gamma_{12}^2}{4}\Big)v_1^2v_2+\frac{\gamma_{12}\gamma_{22}}{4}v_1v_2^2+\frac{\gamma_{22}^2}{8}v_2^3+\frac{\gamma_{23}}{4}v_1^2v_3\\&+\gamma_{22}\gamma_{12}v_1v_2v_3+\gamma_{22}\gamma_{23}v_2^2v_3+\frac{\gamma_{23}^2}{4}v_2v_3^2-\frac{1}{3}v_3^3+(d^\circ\geq 4)\Big)\,,\end{aligned}$$

where $\gamma_{12}=2H_1(-\zeta^3+\zeta^4+\zeta^6)$, $\gamma_{22}=4H_2(\zeta^4+\zeta+\zeta^5)$ and $\gamma_{23}=2H_2(-\zeta^2+\zeta^5+\zeta^4)$.

Proposition 7.1.2. $\sigma_2(P_0)^7=\exp\frac{7}{2}L(P_0,P_0)$.

Proof. Because of $y(P_0)=1/2$, it is obtained from 3.2.4(2) and (3) that

$$\sigma(3P_0)=\sigma_2(P_0)^9 \qquad\text{and}\qquad \sigma(4P_0)=\sigma_2(P_0)^{16}\,,$$

respectively. On the other hand, from 3.1.1 we get

$$\sigma(4P_0)=\sigma(-3P_0+7P_0)=-\exp[\tfrac{7}{2}L(P_0,P_0)]\sigma(3P_0)\,.$$

Here we have used that $\sigma(-3P_0)=\sigma(3P_0)$ and (7.1.1) which implies $\chi(7P_0)=1$. Therefore we obtain

$$\sigma_2(P_0)^{16}=\exp[\tfrac{7}{2}L(P_0,P_0)]\sigma_2(P_0)^9$$

and hence the statement. □

We denote by τ the element of $\mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})$ such that $\zeta^\tau=\zeta^3$. Then $1+\tau$ is a type norm (see 4.2.7) in $\mathbf{Z}[\mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})]$.

LEMMA 7.1.3. *If* $b \in \mathbf{Z}[\zeta]$, *then* $\chi((b^{1+\tau^{-1}+\tau^{-2}}-1)P_0)^{Nb-1}=1$.

PROOF. The proof is almost the same as that of 6.1.3. We may assume Nb is even. By the assumption on b, we may write $b^{1+\tau^{-1}+\tau^{-2}}=(a_1\zeta+a_2\zeta^2+a_3\zeta^3+a_4\zeta^4+a_5\zeta^5+a_6\zeta^6)(1-\zeta)+1$ with integers a_j. Since 2 is a prime in $\mathbf{Z}[\zeta]$, we have $b\equiv 0 \bmod 2$ and hence $b^{1+\tau^{-1}+\tau^{-2}}\equiv 0 \bmod 2$. Then we see $a_1\equiv a_3=a_5\equiv 1 \bmod 2$ and $a_2\equiv a_4\equiv a_6\equiv 0 \bmod 2$. Therefore

$$\begin{aligned}\chi((b^{1+\tau^{-1}+\tau^{-2}}-1)P_0)&=\chi(((\zeta+\zeta^3+\zeta^5)(1-\zeta)P_0)\\&=\chi(((\zeta-\zeta^2+\zeta^3-\zeta^4+\zeta^5-\zeta^6)P_0)=1\end{aligned}$$

because of

$$(\zeta-\zeta^2+\zeta^3-\zeta^4+\zeta^5-\zeta^6)P_0=\omega'\begin{bmatrix}-3\\-2\\-1\end{bmatrix}+\omega''\begin{bmatrix}1\\1\\1\end{bmatrix}$$

which is obtained from (7.1.2). □

PROPOSITION 7.1.4. *Let b be an element of* $\mathbf{Z}[\zeta]$. *If* $b\equiv 1 \bmod (1-\zeta)^2$, *then*

$$\begin{aligned}&\sigma(b^{1+\tau^{-1}+\tau^{-2}}(v+P_0))\\&\quad=\chi((b^{1+\tau^{-1}+\tau^{-2}}-1)P_0)\sigma_2(P_0)^{Nb-1}\sigma(b^{1+\tau^{-1}+\tau^{-2}}v+P_0)(1+(d^\circ\geq 1)).\end{aligned}$$

PROOF. The statement follows from 4.2.8 and

$$\exp[\tfrac{1}{2}(Nb-1)L(P_0,P_0)]=\sigma_2(P_0)^{Nb-1}$$

which is given by 7.1.2. □

LEMMA 7.1.5. *Let* $\varphi(u):=(\wp_{12}^2-\wp_{22}\wp_{11})(u)$. *Then*

(1) $\varphi(\ulcorner\zeta\urcorner u)=\zeta^6\varphi(u)$,

(2) $\varphi(u)\in\Gamma(J,\mathcal{O}(3\Theta))$,

(3) *the Taylor expansions of* $\sigma(u)^3\varphi(u)$ *at O and* P_0 *are of the form*

$$\sigma(u)^3\varphi(u)=2u_3^2+(d^\circ(u_1,u_2,u_3)\geq 4)\qquad \textit{and}$$

$$\sigma(v+P_0)^3\varphi(v+P_0)=-\sigma_2(P_0)^3(1+(d^\circ(v_1,v_2,v_3)\geq 1)).$$

The proof is similar to that of 6.1.5 and we omit the details.

THEOREM 7.1.6. *Let* $\varphi(u)=(\wp_{12}^2-\wp_{22}\wp_{11})(u)$ *as above. Let* $b\in\mathbf{Z}[\zeta]$ *and assume* $b\equiv 1 \bmod (1-\zeta)^2$.

(1) *If* Nb *is odd, then* $\varphi_{b^{1+\tau^{-1}+\tau^{-2}}}(u)^3\varphi(b^{1+\tau^{-1}}u)$ *is of the form*

$$\psi_{b^{1+\tau^{-1}+\tau^{-2}}}(u)^3\varphi(b^{1+\tau^{-1}+\tau^{-2}}u)=2y(u)\sum_{\substack{0\leq j\leq 9(Nb-1)/2\\ j\equiv 0 \bmod 7}}\gamma_j x(u)^j$$

with $\gamma_j\in\mathbf{Q}$. *Moreover*, $\gamma_{9(Nb-1)/2}=b^{2(\tau+1+\tau^{-1})}$ *and* $\gamma_0=-1$.

(2) *If* Nb *is even, then* $\varphi_{b^{1+\tau^{-1}+\tau^{-2}}}(u)^3\varphi(b^{1+\tau^{-1}}u)$ *is of the form*

$$\psi_{b^{1+\tau^{-1}+\tau^{-2}}}(u)^3\varphi(b^{1+\tau^{-1}+\tau^{-2}}u)=\sum_{\substack{0\le j\le(9\mathrm{N}b-2)/2\\ j\equiv 0 \bmod 7}}\gamma_j x(u)^j$$

with $\gamma_j\in\mathbf{Q}(\zeta)$. *Moreover,* $\gamma_{(9\mathrm{N}b-2)/2}=2b^{2(\tau+1+\tau^{-1})}$ *and* $\gamma_0=-1$.

PROOF. First, we look at the Laurent expansion at $u=O$. By 7.1.5(3) and 7.1.1(1), we have

$$\begin{aligned}
(7.1.6)\quad &\psi_{b^{1+\tau^{-1}+\tau^{-2}}}(u)^3\varphi(b^{1+\tau^{-1}+\tau^{-2}}u)\big|_{u\in\kappa^{-1}\iota(C)}\\
&=\frac{\sigma(b^{1+\tau^{-1}+\tau^{-2}}u)^3\varphi(b^{1+\tau^{-1}+\tau^{-2}}u)}{\sigma_2(u)^{3\mathrm{N}b}}=\frac{2(b^{1+\tau^{-1}+\tau^{-2}})^{2\tau}u_3^2+(d^\circ\ge 4)}{(-2u_2-\frac{1}{3}u_3^3+(d^\circ\ge 5))^{3\mathrm{N}b}}\\
&=\frac{2(b^{2(\tau+1+\tau^{-1})})u_3^2+(d^\circ\ge 4)}{(-u_3^3+(d^\circ\ge 5))^{3\mathrm{N}b}}=(-1)^{\mathrm{N}b}2b^{2(\tau+1+\tau^{-1})}\frac{1}{u_3^{9\mathrm{N}b-2}}+\cdots.
\end{aligned}$$

This function is odd or even and σ_2 has only zeroes at $u\in\Lambda$ by the first statement of 2.2.1(3), and accordingly is a polynomial of $x(u)$ multiplied by $y(u)$ or a polynomial of $x(u)$, respectively. If Nb is odd, then the last of (7.1.6) is

$$\begin{aligned}
&=2b^{2(\tau+1+\tau^{-1})}\frac{-1}{u_3^7}\left(\frac{1}{u_3^2}\right)^{9(\mathrm{N}b-1)/2}+\cdots\\
&=2y(u)(b^{2(\tau+1+\tau^{-1})}x(u)^{9(\mathrm{N}b-1)/2}+\text{“lower terms of power of } x(u)\text{”)}
\end{aligned}$$

by 2.3.1 and 2.3.2. Similarly, if Nb is even, then the last of (7.1.6) is

$$\begin{aligned}
&=2b^{2(\tau+1+\tau^{-1})}\left(\frac{1}{u_3^2}\right)^{(9\mathrm{N}b-2)/2}+\cdots\\
&=2b^{2(\tau+1+\tau^{-1})}x(u)^{(9\mathrm{N}b-2)/2)}+\text{“lower terms of power of } x(u)\text{”}.
\end{aligned}$$

Secondly, we look at the Laurent expansion at $u=P_0$ ($\kappa(P_0)=\iota(0,1/2)$). Since $b\equiv 1 \bmod (1-\zeta)^2$ we have $b^{1+\tau^{-1}+\tau^{-2}}\equiv 1 \bmod (1-\zeta)^2$. Because of $(1-\zeta)P_0\in\Lambda$ and $\varphi(u)$ being periodic, we have $\varphi(b^{1+\tau^{-1}+\tau^{-2}}(v+P_0))=\varphi(b^{1+\tau^{-1}+\tau^{-2}}v+P_0)$. Therefore, 7.1.3, 7.1.1, 7.1.4 and 7.1.5 imply

(7.1.7)

$$\begin{aligned}
&\psi_{b^{1+\tau^{-1}+\tau^{-2}}}(v+P_0)^3\varphi(b^{1+\tau^{-1}+\tau^{-2}}(v+P_0))\big|_{v+P_0\in\kappa^{-1}\iota(C)}\\
&=\frac{\sigma(b^{1+\tau^{-1}+\tau^{-2}}(v+P_0))^3\varphi(b^{1+\tau^{-1}+\tau^{-2}}(v+P_0))}{\sigma_2(b^{1+\tau^{-1}+\tau^{-2}}(v+P_0))^{3\mathrm{N}b}}\Bigg|_{v+P_0\in\kappa^{-1}\iota(C)}\\
&=\{\sigma_2(P_0)^{3(\mathrm{N}b-1)}\sigma(b^{1+\tau^{-1}+\tau^{-2}}v+P_0)^3\chi((b^{1+\tau^{-1}+\tau^{-2}}-1)P_0)^3(1+(d^\circ(v_1)\ge 1))\\
&\qquad\varphi(b^{1+\tau^{-1}+\tau^{-2}}v+P_0)\}/\{\sigma_2(b^{1+\tau^{-1}+\tau^{-2}}(v+P_0)^{3\mathrm{N}b}\chi((b^{1+\tau^{-1}+\tau^{-2}}-1)P_0)^{3\mathrm{N}b}\}\\
&=\frac{\sigma_2(P_0)^{3(\mathrm{N}b-1)}\sigma_2(P_0)^3(-1+(d^\circ(v_1)\ge 1))}{\sigma_2(P_0)^{3\mathrm{N}b}(1+(d^\circ\ge 1))}\qquad\text{(by 7.1.3)}
\end{aligned}$$

$$=-1+(d^\circ(v_1)\geq 1)=\begin{cases} -2y(u)(1+(d^\circ(x(u))\geq 1)) & \text{if } \mathrm{N}b \text{ is odd}, \\ -1+(d^\circ(x(u))\geq 1) & \text{if } \mathrm{N}b \text{ is even}. \end{cases}$$

Furthermore, since

$$\psi_{b^{1+\tau^{-1}+\tau^{-2}}}(\ulcorner -\zeta \urcorner u)\varphi(b^{1+\tau^{-1}+\tau^{-2}}\ulcorner -\zeta \urcorner u)=(-1)^{\mathrm{N}b}\psi_{b^{1+\tau^{-1}+\tau^{-2}}}(u)\varphi(b^{1+\tau^{-1}+\tau^{-2}}u)$$

by 4.2.5(2), the function must be a polynomial of $x(u)^7$ if Nb is even, or a such multiplied by $y(u)$ if Nb is odd. □

7.2. The curve defined by $y^2=x^7-x$. Second example of genus three is the curve C defined by $y^2=x^7-x$. The ring $\mathbf{Z}[\ulcorner \zeta \urcorner]$ is isomorphic to the ring $\mathbf{Z}[i]\oplus\mathbf{Z}[\zeta]$ by $\ulcorner \zeta^j \urcorner \mapsto i^j\oplus\zeta^j$ by 4.1.1(2). The endomorphism $\ulcorner \zeta^j \urcorner$ acts such as

(7.2.1) $$\ulcorner \zeta^j \urcorner(u_1,u_2,u_3)=(\zeta^j u_1, i^j u_2, \zeta^{5j}u_3)$$

because $\ulcorner \zeta \urcorner\omega^{(j)}=\zeta^j\omega^{(j)}$ for $j=1,2,3$. We let $c=1$, $a_1=\zeta^2$, $c_1=\zeta^4$, $a_2=\zeta^6$, $c_2=\zeta^8$, $a_3=\zeta^{10}$, $c_3=0$, in (1.1.1).

As in the previous subsections, we have

$$\omega'=\begin{bmatrix} -2K_1\zeta^5 & 2K_1(\zeta^4-\zeta^3) & 2K_1(\zeta^2-\zeta) \\ -2K_2\zeta^3 & 2K_2(\zeta+\zeta^3) & 2K_2(-1-\zeta^3) \\ -2K_3\zeta & 2K_3(-\zeta^2-\zeta^3) & 2K_3(-\zeta^4-\zeta^5) \end{bmatrix},$$

$$\omega''=\begin{bmatrix} 2K_1(\zeta^4-\zeta^3+\zeta^2-\zeta+1) & 2K_1(-\zeta^3+\zeta^2-\zeta+1) & 2K_1(-\zeta+1) \\ 2K_2 & 0 & 2K_2(-\zeta^3+1) \\ 2K_3(-\zeta^2-\zeta^3-\zeta^4-\zeta^5+1) & 2K_3(-\zeta^3-\zeta^4-\zeta^5+1) & 2K_3(-\zeta^5+1) \end{bmatrix},$$

$$\eta'=\begin{bmatrix} 2H_1\zeta & 2H_1(-\zeta^2+\zeta^3) & 2H_1(-\zeta^3+\zeta^5) \\ 2H_2\zeta^3 & 2H_2(-\zeta^5-\zeta^3) & 2H_2(-1+\zeta^3) \\ 2H_3\zeta^5 & 2H_3(\zeta^4+\zeta^3) & 2H_3(\zeta^2+\zeta) \end{bmatrix},$$

$$\eta''=\begin{bmatrix} 2H_1(-\zeta^2+\zeta^3-\zeta^4+\zeta^5+1) & 2H_1(\zeta^3-\zeta^4+\zeta^5+1) & 2H_1(\zeta^5+1) \\ 2H_2 & 0 & 2H_2(\zeta^3+1) \\ 2H_3(\zeta^4+\zeta^3+\zeta^2+\zeta+1) & 2H_3(-\zeta^3+\zeta^2+\zeta+1) & 2H_3(\zeta+1) \end{bmatrix}.$$

Furthermore,

(7.2.2) $$P_0=\begin{bmatrix} K_1(\zeta-\zeta^2+\zeta^3-\zeta^4+\zeta^5)-K_1 \\ K_2(\zeta^3+1-\zeta^3-1+\zeta^3)-K_2 \\ K_3(\zeta^5+\zeta^4+\zeta^3+\zeta^2+\zeta)-K_3 \end{bmatrix}=\omega'\begin{bmatrix} -1/2 \\ 0 \\ 0 \end{bmatrix}+\omega''\begin{bmatrix} -1/2 \\ 0 \\ 0 \end{bmatrix}.$$

and, by (7.2.1),

(7.2.3)
$$\lceil\zeta\rceil P_0=\begin{bmatrix}\zeta K_1(-1+\zeta-\zeta^2+\zeta^3-\zeta^4+\zeta^5-1)\\ \zeta^3K_2(-1+\zeta^3+1-\zeta^3-1+\zeta^3-1)\\ \zeta^5K_3(-1-\zeta^5+\zeta^4+\zeta^3+\zeta^2+\zeta-1)\end{bmatrix}=\omega'\begin{bmatrix}1/2\\1\\1\end{bmatrix}+\omega''\begin{bmatrix}-1/2\\0\\0\end{bmatrix}.$$

Let us compute the Taylor expansion at $u=P_0$ explicitly. Again, the method is the same as in 6.1. In a similar fashion as we derived (6.1.4), (6.1.5) and (6.1.6), we have

(7.2.4)
$$(\lceil\zeta\rceil-1)P_0=\omega'\begin{bmatrix}1\\1\\-1\end{bmatrix}+\omega''\begin{bmatrix}0\\0\\0\end{bmatrix},$$

(7.2.5)
$$\sigma(v+\lceil\zeta\rceil P_0)=\zeta^3\sigma(v+P_0)\exp[L(v,(\lceil\zeta\rceil-1)P_0)],$$

and

(7.2.6)
$$\sigma_{ij}(\lceil\zeta\rceil P_0)=\zeta^3\sigma_{ij}(P_0)+\sigma_i(P_0)(\eta'_{j1}+\eta'_{j2}+\eta'_{j3})\zeta^3+\sigma_j(P_0)(\eta'_{i1}+\eta'_{i2}+\eta'_{i3})\zeta^3,$$

respectively. Then

$$\sigma_{11}(P_0)=0, \qquad \sigma_{12}(P_0)=H_1(-1-(2-\sqrt{3})i)\sigma_2(P_0),$$
$$\sigma_{22}(P_0)=H_2(-1-i)\sigma_2(P_0), \qquad \sigma_{13}(P_0)=0,$$
$$\sigma_{23}(P_0)=H_1(-1-(\sqrt{3}+2)i)\sigma_2(P_0), \qquad \sigma_{33}(P_0)=0.$$

Thus we arrive at

PROPOSITION 7.2.1. *Assume C is defind by $y^2=x^7-x$. Let P_0 be the point whose coordinate is given by* (7.2.1). *Then*

(1)
$$\sigma(u)=u_1u_3-u_2^2-\tfrac{1}{12}u_1^4-\tfrac{1}{3}u_2u_3^3+(d^\circ\ge 6),$$

(2)
$$\sigma(v+P_0)=\sigma_2(P_0)\Big(v_2+\gamma_{12}v_1v_2+\frac{\gamma_{22}}{2}v_2^2+\gamma_{13}v_1v_3+\gamma_{23}v_2v_3-\frac{1}{3}v_1^3$$
$$+\frac{\gamma_{12}^2}{4}v_1^2v_2+\frac{\gamma_{12}\gamma_{22}}{4}v_1v_2^2+\gamma_{12}\gamma_{23}v_1v_2v_3+\frac{\gamma_{22}^2}{8}v_2^3$$
$$+\frac{\gamma_{22}\gamma_{23}}{4}v_2^2v_3+\frac{\gamma_{23}^2}{4}v_2v_3^2-\frac{1}{3}v_3^3+(d^\circ\ge 4)\Big),$$

where $\gamma_{12}=H_1(-1-(2-\sqrt{3})i)$, $\gamma_{22}=H_2(-1-i)$ and $\gamma_{23}=H_1(-1-(\sqrt{3}+2)i)$.

PROPOSITION 7.2.2. $\sigma_2(P_0)^8=\exp 4L(P_0,P_0)$.

PROOF. Take $y=y(u)$ as a local parameter at P_0 along $\kappa^{-1}\iota(C)$. By 3.2.4(2), we have $8y(u)\sigma_2(u)^9=\sigma(3u)$. Operating d^3/dy^3 to this and setting $u=P_0$, we get

$$(7.2.7)\qquad 27\sigma_{111}(3P_0)+6\sigma_2(3P_0)=-48\sigma_2(P_0)^9$$

because of $y(P_0)=0$ and 2.3.2(3). Moreover, we have the equation

$$(7.2.8)\qquad \sigma(u+3P_0)=\chi(2P_0)\sigma(u+P_0)\exp L(u+P_0+P_0,2P_0)$$

given by 3.1.1. Operating $\partial^3/\partial u_1^3$ to (7.2.8) and putting $u=0$, we get $\sigma_{111}(3P_0)=-\sigma_{111}(P_0)\exp 2L(P_0,P_0)$ because of $\sigma(O)=0$, $\sigma_1(O)=1$ and $\chi(2P_0)=1$. Similarly, differentiating (7.2.8) with respect to u_2 and setting $u=0$, we get

$$(7.2.9)\qquad \sigma_2(3P_0)=\sigma_2(P_0)\exp 4L(P_0,P_0)\,.$$

Summing up (7.2.7), (7.2.8) and (7.2.9), we arrive at the statement. □

We denote by τ the element of $\mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})$ such that $\zeta^\tau=\zeta^5$. Then $1+\tau$ is a type norm (see 4.2.7) in $\mathbf{Z}[\mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})]$.

PROPOSITION 7.2.3. *Let b be an element of $\mathbf{Z}[\ulcorner\zeta\urcorner]$. If $b\equiv 1 \bmod 8$, then*

$$\sigma(b^{1+\tau}(v+P_0))=\sigma_2(P_0)^{\mathrm{N}b-1}\sigma(b^{1+\tau}v+P_0)(1+(d^\circ\geq 1))\,.$$

PROOF. Note that $2P_0\in\Lambda$. By the assumption, $b^{1+\tau}-1$ is divisible by 8. So $\chi((b^{1+\tau}-1)P_0)=1$. Moreover $\mathrm{N}b-1$ is divisible by 8. The statement follows from 4.2.8 and

$$\exp[\tfrac{1}{2}(\mathrm{N}b-1)L(P_0,P_0)]=\sigma_2(P_0)^{\mathrm{N}b-1}$$

which is given by 7.2.2. □

LEMMA 7.2.4. *Let*

$$\varphi(u):=[\tfrac{1}{24}(\wp_{2222}-6\wp_{22}^2)\wp_{111}+\tfrac{1}{2}(\wp_{1112}-6\wp_{11}\wp_{22})\wp_{222}](u)\,.$$

Then it has the following properties.

(1) $\varphi(\ulcorner\zeta\urcorner u)=\zeta^3\varphi(u)$,

(2) $\varphi(u)\in\Gamma(J,\mathcal{O}(5\Theta))$,

(3) *the Taylor expansions of $\sigma(u)^5\varphi(u)$ at O and P_0 are of the form*

$$\sigma(u)^5\varphi(u)=-u_3^3+(d^\circ(u_1,u_2,u_3)\geq 5)\,,$$

$$\sigma(v+P_0)^5\varphi(v+P_0)=\sigma_2(P_0)^5(-1+(d^\circ(v_1,v_2,v_3)\geq 1))\,.$$

PROOF. (1) follows from 4.2.5(3) and the definition of $\wp$-functions. Since

$$\sigma(u)^2(\wp_{2222}-6\wp_{22}^2)(u)=(-\sigma_{2222}\sigma+4\sigma_{222}\sigma_2-3\sigma_{22}^2)(u)\,,$$

$$\sigma(u)^3\wp_{111}(u)=(-2\sigma_1^3+3\sigma_1\sigma_{11}\sigma-\sigma_{111}\sigma^2)(u)\,,$$

$$\sigma(u)^2(\wp_{1112}-6\wp_{11}\wp_{22}^2)(u)=(-\sigma_{1112}\sigma+3\sigma_{112}\sigma_1-\sigma_{111}\sigma_2-3\sigma_{11}\sigma_{12})(u)\,,$$

$$\sigma(u)^3\wp_{222}(u)=(-2\sigma_2^3+3\sigma_2\sigma_{22}\sigma-\sigma_{222}\sigma^2)(u),$$

the statement (2) holds. The expansion in 7.2.1(1) gives

$$\begin{aligned}&(-\sigma_{2222}\sigma+4\sigma_{222}\sigma_2-3\sigma_{22}^2)(u)\\&=-(d^\circ\geq 2)(d^\circ\geq 2)+4(d^\circ\geq 2)(d^\circ\geq 2)-3(-2+(d^\circ\geq 2))^2=12+(d^\circ\geq 2),\\&(-2\sigma_1^3+3\sigma_1\sigma_{11}\sigma-\sigma_{111}\sigma^2)(u)\\&=-2(u_3+(d^\circ\geq 3))^3+3(d^\circ\geq 1)(d^\circ\geq 2)(d^\circ\geq 2)-(d^\circ\geq 1)(d^\circ\geq 2)^2\\&=-2u_3^3+(d^\circ\geq 4),\\&(-\sigma_{1112}\sigma+3\sigma_{112}\sigma_1-\sigma_{111}\sigma_2-3\sigma_{11}\sigma_{12})(u)\\&=-(d^\circ\geq 0)(d^\circ\geq 2)+3(d^\circ\geq 1)(d^\circ\geq 1)+(d^\circ\geq 0)(d^\circ\geq 1)-3(d^\circ\geq 2)(d^\circ\geq 2)\\&=(d^\circ\geq 2),\\&(-2\sigma_2^3+3\sigma_2\sigma_{22}\sigma-\sigma_{222}\sigma^2)(u)\\&=-2(d^\circ\geq 1)^3+3(d^\circ\geq 1)(d^\circ\geq 1)(d^\circ\geq 2)-(d^\circ\geq 3)(d^\circ\geq 2)^2=(d^\circ\geq 3).\end{aligned}$$

Therefore $\sigma(u)^5\varphi(u)=-u_3^3+(d^\circ\geq 5)$. Similarly, 7.2.1(2) gives

$$\begin{aligned}&(-2\sigma_1^3+3\sigma_1\sigma_{11}\sigma-\sigma_{111}\sigma^2)(v+P_0)\\&=-2(d^\circ\geq 1)^3+3(d^\circ\geq 1)(d^\circ\geq 1)(d^\circ\geq 1)-(d^\circ\geq 0)(d^\circ\geq 1)^2=(d^\circ\geq 2),\\&(-\sigma_{1112}\sigma+3\sigma_{112}\sigma_1+\sigma_{111}\sigma_2-3\sigma_{11}\sigma_{12})(v+P_0)\\&=\sigma_2(P_2)^2[-(d^\circ\geq 0)(d^\circ\geq 1)+3(d^\circ\geq 0)(d^\circ\geq 1)\\&\quad+(1+(d^\circ\geq 1))(1+(d^\circ\geq 1))-3(d^\circ\geq 1)(d^\circ\geq 0)]\\&=\sigma_2(P_0)^2(1+(d^\circ\geq 1)),\\&(-2\sigma_2^3+3\sigma_2\sigma_{22}\sigma-\sigma_{222}\sigma^2)(v+P_0)\\&=\sigma_2(P_0)^3[-2(1+(d^\circ\geq 1))^3+3(d^\circ\geq 0)(d^\circ\geq 0)(d^\circ\geq 1)-(d^\circ\geq 0)(d^\circ\geq 1)^2]\\&=\sigma_2(P_0)^3(-2+(d^\circ\geq 1)).\end{aligned}$$

Hence $\sigma(u)^5\varphi(v+P_0)=\sigma_2(P_0)^5(1+(d^\circ\geq 1))$. So (3) is proved. □

THEOREM 7.2.5. *Let $\varphi(u)$ be as in 7.2.4. Let $b\in\mathbf{Z}[\lceil\zeta\rceil]$ and assume $b\equiv 1 \bmod 8$. Then $\varphi_{b^{1+\tau}}(u)^5\varphi(b^{1+\tau}u)$ is of the form*

$$\psi_{b^{3(1+\tau)}}(u)^5\varphi(b^{1+\tau}u)=\sum_{\substack{0\leq j\leq(15Nb-3)/2\\ j\equiv 0 \bmod 6}}\gamma_j x(u)^j$$

with $\gamma_j\in\mathbf{Q}(\zeta)$. Moreover $\gamma_{(15Nb-3)/2}=b^{3(1+\tau)}$ and $\gamma_0=1$.

PROOF. The proof is almost the same as that of 6.2.5. First, similarly as in 6.2.5, we have

$$\begin{aligned}\psi_{b^{1+\tau}}(u)^5\varphi(b^{1+\tau}u)\big|_{u\in\kappa^{-1}\iota(C)}&=\frac{\sigma(b^{1+\tau}u)^5\varphi(b^{1+\tau}u)}{\sigma_2(u)^{5Nb}}=\frac{-(b^{1+\tau})^{3\tau}u_3^3+(d^\circ(u_2)\geq 4)}{(-2u_2-\frac{1}{3}u_3^3+(d^\circ(u_3)\geq 4))^{5Nb}}\\&=b^{3(\tau+1)}\frac{1}{u_3^{15Nb-3}}+\cdots=b^{3(\tau+1)}\left(\frac{1}{u_3^2}\right)^{(15Nb-3)/2}+\cdots\\&=b^{3(\tau+1)}(x(u)^{(15Nb-3)/2}+\text{"lower terms of power of } x(u)\text{"}).\end{aligned}$$

Secondly, since $b\equiv 1 \bmod 8$, we have $b^{\tau+1}\equiv 1 \bmod 8$. Because of $2P_0\in\Lambda$ and $\varphi(u)$ being periodic, we have $\varphi(b^{1+\tau}(v+P_0))=\varphi(b^{1+\tau}v+P_0)$. Consequently, 7.2.3, 7.2.1 and 7.2.4 imply

$$\begin{aligned}&\psi_{b^{1+\tau}}(v+P_0)^5\varphi(b^{1+\tau}(v+P_0))\big|_{v+P_0\in\kappa^{-1}\iota(C)}\\&=\frac{\sigma(b^{1+\tau}(v+P_0))^5\varphi(b^{1+\tau}(v+P_0))}{\sigma_2(b^{1+\tau}(v+P_0))^{5Nb}}\Bigg|_{v+P_0\in\kappa^{-1}\iota(C)}\\&=\frac{\sigma_2(P_0)^{5(Nb-1)}\sigma(b^{1+\tau}v+P_0)^5(1+(d^\circ(v_1)\geq 1))\varphi(b^{1+\tau}v+P_0)}{\sigma_2(P_0)^{5Nb}(-1+(d^\circ\geq 1))}\\&=\frac{\sigma_2(P_0)^{5(Nb-1)}\sigma_2(P_0)^5(1+(d^\circ(v_1)\geq 1))}{\sigma_2(P_0)^{5Nb}(1+(d^\circ\geq 1))}\\&=-1+(d^\circ(v_1)\geq 1)=-1+(d^\circ(x(u))\geq 1).\end{aligned}$$

Furthermore, since $\psi_{b^{1+\tau}}(\ulcorner-\zeta\urcorner u)^5\varphi(b^{1+\tau}\ulcorner-\zeta\urcorner u)=-\zeta^{3(Nb-1)}\psi_{b^{1+\tau}}(u)^5\varphi(b^{1+\tau}u)$ by 4.2.5(3), the function must be a polynomial of $x(u)^6$. □

8. Some remarks and comments.

1. As is mentioned in the beginning of Part II, in each formula in 5.1.3, 5.2.3, 6.1.6, 6.2.6, 7.1.6 and 7.2.6, the coefficients of the right hand side, which side is a polynomial expression in $x(u)$ and $y(u)$, are contained in the field $\mathbf{Q}(\zeta)$. Furthermore we can prove that the coefficients of the right hand side of each formula of 5.1.3 and 5.2.3 are contained in $\mathbf{Z}[e^{2\pi i/3}]$ and $\mathbf{Z}[i]$, respectively. The coefficients of the right hand side of the formula in 6.1.6 are also contained in $\mathbf{Z}[e^{2\pi i/5}]$ (see [9] or [17, p. 46]). For each of the other three formulae, its coefficients seem also to be contained in the ground integer ring.

2. Theorem 5.1.3 implies

$$\prod_{\substack{P\in b^*(\wp)_0,\\(1-\zeta)P_0\neq O\\/\pm 1}} x(P)=(-1)^{Nb-1}b\,.$$

Theorem 5.2.3 implies

$$\prod_{\substack{P\in b^*(\wp)_0,\\ (1+i)P_0\neq O\\ /\pm 1}} x(P)=(-1)^{Nb-1}b^2 .$$

These are versions of the product formula of Eisenstein.

3. Theorem 6.1.6 (Grant's formula) implies

$$\prod_{\substack{P\in \iota(C)\cdot(b^{1+\tau^{-1}})^*(\varphi)\\ 2P\neq O\\ /\pm 1}} x(P)=\frac{1}{b^{1+\tau}},$$

where $\cdot$ denotes an intersection of cycles in J. In fact, the cycle $\iota(C)\cdot(b^{1+\tau^{-1}})^*(\varphi)_0$ contains only five 2-torsion points $(-4^{1/5}\zeta^j, 0)$ with $j=0,\cdots,4$ (see [9, p. 131]). Theorem 6.2.6 also implies that the product of roots $x(u)$ of the right hand side of the formula of 6.2.6 is equal to $1/b^{2(1+\tau)}$. Similarly Theorem 7.1.6 states that the product of roots $x(u)$ of the right hand side of the formula in 7.1.6 is equal to $\pm 1/b^{2(\tau+1+\tau^{-1})}$ or $\pm 1/2b^{2(\tau+1+\tau^{-1})}$, and Theorem 7.2.6 states that the product of roots $x(u)$ of the right hand side of the formula above is equal to $1/b^{3(1+\tau)}$. These are generalizations of the product formula of Eisenstein.

4. The polynomial of $x(u)$ in the right hand side of each of the formula of 5.1.3 and 5.2.3 is known to be irreducible over the ground ring when b is a prime element. It is unknown whether the other polynomials of 6.1.6, 6.2.6, 7.1.6 and 7.2.6 are irreducible.

5. The roots of each polynomial of $x(u)$ generate a finite algebraic extension over the ground field. For the genus one case, such extensions are known to be abelian. Contrarily, the extensions in higher genus cases seem not to be abelian but to have very large Galois groups. For Grant's original formula, some numerical examples are given in [17].

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