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The Relationship between Entropy and Strong Orbit Equivalence for the Minimal Homeomorphisms (II)

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Abstract. Every minimal homeomorphism of a Cantor set is strongly orbit equivalent to a homeomorphism of infinite entropy.

1. Introduction.

In [Su], we showed the claim is valid for the finite entropy case. That is to say, suppose (X, T) is any Cantor system and α is any positive number, and fix them. Then there exists a Cantor system (Y, S) such that (Y, S) is strong orbit equivalent to (X, T) and topological entropy of (Y, S) is equal to α. In this paper we show the claim for the infinite entropy case. The difference of construction between finite case and infinite case is that the simple ordered Bratteli diagram made in infinite case has no distinct orderings (Definition 2.3). Therefore the calculation of entropy differs from each other in the two cases. In the main theorem we shall show this calculation. The construction, except for distinct ordering, of diagram in infinite case is basically the same as for the finite case, i.e. we construct by induction the base diagram $\mathscr{D}'$ and the sequence of the set of numbers $\{\{\# \tilde{V}_j^{(k)}\}_{j=1}^{\# V'^{(k)}}\}_{k=1}^{\infty}$, which are numbers chosen from some equivalence class on vertex set defined in Definition 2.2. We apply Lemma 4.1 to the diagram which is obtained by $\mathscr{D}'$ and $\{\{\# \tilde{V}_j^{(k)}\}_{j=1}^{\# V'^{(k)}}\}_{k=1}^{\infty}$, so we get the simple unordered Bratteli diagram. Moreover we take a definite partial order in it so as to have simple ordered Bratteli diagram yielding the associated Cantor system having an infinite entropy.

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2. On the order of Bratteli diagram.

In this section we shall consider some partial orders on the edge set of Bratteli diagram in order to be able to calculate infinite entropy.

First, we give definitions and notations. Given $\mathscr{B}=\{M^{(k)}\}_{k=1}^{\infty}$ which is a sequence of positive incidence matrices of (V, E), we define for $1\leq i\leq \# V^{(k-1)}$, $1\leq j\leq \# V^{(k)}$,

$$\bar{m}_j^{(k)}\equiv\sum_{i=1}^{\# V^{(k-1)}} m_{i,j}^{(k)}\,,\qquad \hat{m}_{i,j}^{(k)}\equiv\frac{m_{i,j}^{(k)}}{\bar{m}_j^{(k)}}$$

$$\bar{m}^{(k)}\equiv\max_{1\leq j\leq \# V^{(k)}}\bar{m}_j^{(k)}\,,\qquad M_j^{(k)}\equiv(m_{1,j}^{(k)}, m_{2,j}^{(k)}, \cdots m_{\# V^{(k-1)},j}^{(k)})^t\,.$$

For $p\leq q\in\mathbf{N}$, let

$$M^{(p,q)}\equiv M^{(p)}M^{(p+1)}\cdots M^{(q-1)}M^{(q)}\,,\qquad \text{which we write } M^{(p,q)}=(m_{i,j}^{(p,q)})\,.$$

In the same way we define $\bar{m}_j^{(p,q)}$, $\hat{m}_{ij}^{(p,q)}$, $\bar{m}^{(p,q)}$, $M_j^{(p,q)}$ $(1\leq i\leq \# V^{(p-1)},\ 1\leq j\leq \# V^{(q)})$.

DEFINITION 2.1 (ordered vector). We define $\tau\colon V\backslash V^{(0)}\times\mathbf{N}\to V$ which indicates the order of edges on the diagram. For any $v\in V\backslash V^{(0)}$ and k $(k\geq 1)$,

$$\tau(v, j)=v^{(k-1)}\ \overset{\text{def}}{\Longleftrightarrow}\ \begin{array}{l} j\text{-th edge whose range vertex is } v\in V^{(k)} \text{ is connected} \\ \text{with } v^{(k-1)}\in V^{(k-1)} \text{ as a source vertex}\,. \end{array}$$

And we say an *ordered vector* of v if it consists of vertices defined by

$$(\tau(v, 1), \tau(v, 2), \cdots \tau(v, \# r^{-1}(v)-1), \tau(v, \# r^{-1}(v))\subset(V^{(k-1)})^{\# r^{-1}(v)}\,,$$

where $r\colon E^{(k)}\to V^{(k)}$ is range map. Let $O.vec(v)$ denote the ordered vector of v.

DEFINITION 2.2 (equivalence relations on the vertex set). Given a simple unordered Bratteli diagram (V, E) and the incidence matrix $M^{(k)}$, we decompose $V^{(k)}$ $(k\geq 2)$ using the equivalence relation $\sim$ on $V\backslash V^{(0)}\cup V^{(1)}$ defined as follows:

$$i, j\in V^{(k)}\,,\qquad i\sim j\ \overset{\text{def}}{\Longleftrightarrow}\ M_i^{(k)}=M_j^{(k)}\,.$$

Denote the equivalence class of $v\in V_j^{(k)}$ by $[V_j^{(k)}]$. For an incidence matrix $M^{(k)}=(m_{i,j}^{(k)})$ and $V_j^{(k)}$, let $M_{[j]}^{(k)}=(m_{1,[j]}^{(k)}, m_{2,[j]}^{(k)}, \cdots, m_{\# V^{(k-1)},[j]}^{(k)})^t$ denote the incidence vector of $[V_j^{(k)}]$ (cf. Definition 3.8).

DEFINITION 2.3 (distinct orderings). We say edges to vertices of $V^{(k)}$ have *distinct orderings* (to put it simply that $V^{(k)}$ has distinct orderings) if for $v, v'\in V^{(k)}$, $O.vec(v)=O.vec(v')$ implies $v=v'$. And we say a subset $A\subset V^{(k)}$ has *partial distinct orderings* if for $v, v'\in A$, $O.vec(v)=O.vec(v')$ implies $v=v'$.

Now we consider some simple unordered Bratteli diagram $(\tilde{V}, \tilde{E})$ satisfying the following assumption:

ASSUMPTION 2.4. For all $k \geq 2$ and $1 \leq j \leq \#\tilde{V}^{(k)}$,

(i) $\tilde{m}_{1,j}^{(k)} = \tilde{m}_{2,j}^{(k)} = 3$,

(ii) $\tilde{m}_{i,j}^{(k)} \geq 2$ for all $1 \leq i \leq \#\tilde{V}^{(k-1)}$.

Next we make the following four assumptions with respect to the (partial) order of edges.

ASSUMPTION 2.5 (minimal/maximal property). For each $k \geq 2$, $1 \leq j \leq \#\{\tilde{V}^{(k)}/\sim\}$ and $v \in \tilde{V}_j^{(k)}$, there exist $v_{\min}^{(k-1)}, v_{\max}^{(k-1)} \in \tilde{V}^{(k-1)}$ such that

$$\tau(v, 1) = v_{\min}^{(k-1)}, \qquad \tau(v, \#r^{-1}(v)) = v_{\max}^{(k-1)}.$$

ASSUMPTION 2.6. For each $k \geq 1$, $v, v' \in \tilde{V}^{(k)}$ and $1 \leq i \leq \#\{(\tilde{V}^{(k+1)}/\sim\}$, there exist $v_i^{(k+1)} \in \tilde{V}_i^{(k+1)}$ and $1 \leq t_i < \#r^{-1}(v_i^{(k+1)})$ such that

$$\tau(v_i^{(k+1)}, t_i) = v \qquad \tau(v_i^{(k+1)}, t_i + 1) = v'.$$

ASSUMPTION 2.7. For each $k \geq 2$,

(1) $v_{\min}^{(k)}, v_{\max}^{(k)} \in \tilde{V}_1^{(k)}$,

(2) $\{v \in \tilde{V}^{(k)} \mid \exists t : \tau(v, t) = \tau(v, t+1) = \tau(v, t+2) = v_{\min}^{(k-1)}, \tau(v, t+3) = \tau(v, t+4) = v_{\max}^{(k-1)}\} = \{v_{\min}^{(k)}\}$ and
$\{v \in \tilde{V}^{(k)} \mid \exists t : \tau(v, t) = \tau(v, t+1) = v_{\min}^{(k-1)}, \tau(v, t+2) = \tau(v, t+3) = \tau(v, t+4) = v_{\max}^{(k-1)}\} = \{v_{\max}^{(k)}\}$.

ASSUMPTION 2.8. For each $k \geq 2$ and $1 \leq i \leq \#\{\tilde{V}^{(k)}/\sim\}$,

(1) there exists $\eta_k \in \mathbf{N}$ such that

(i) for $i = 1$, $\tilde{V}_1^{(k)} \setminus \{v_{\min}^{(k)}, v_{\max}^{(k)}\} = \bigcup_{l=1}^{\eta_k} \tilde{V}_{1,l}^{(k)}$ as a disjoint set and $\#\tilde{V}_{1,l}^{(k)} = (\#\tilde{V}_1^{(k)} - 2)/\eta_k$ for all l,

(ii) for $i \neq 1$, $\tilde{V}_i^{(k)} = \bigcup_{l=1}^{\eta_k} \tilde{V}_{i,l}^{(k)}$ as a disjoint set and $\#\tilde{V}_{i,l}^{(k)} = \#\tilde{V}_i^{(k)}/\eta_k$ for all l,

(2) $\tilde{V}_{i,l}^{(k)}$ has partial distinct orderings for all i and l,

(3) if we write $\tilde{V}_{i,l}^{(k)} = \{v_{i,l,p}^{(k)}\}_{p=1}^{(\#\tilde{V}_i^{(k)} - 2\delta(i))/\eta_k}$, then $O.vec(v_{i,l,p}^{(k)}) = O.vec(v_{i,l',p}^{(k)})$ for all $1 \leq l, l' \leq \eta_k$ and $1 \leq p \leq (\#\tilde{V}_i^{(k)} - 2\delta(i))/\eta_k$, where $\delta: \mathbf{N} \to \{0, 1\}$ is defined by

$$\delta(i) \equiv \begin{cases} 0 & \text{if } i \neq 1, \\ 1 & \text{if } i = 1, \end{cases}$$

(4) there exists unique $v_{i,l}^* \in \tilde{V}_{i,l}^{(k)}$ such that $\tau(v_{i,l}^*, 2) = v_{\max}^{(k-1)}$ and $\tau(v_{i,l}^*, 3) = v_{\min}^{(k-1)}$,

(5) for any $v \in \tilde{V}^{(k)}$, $\tau(v, \#r^{-1}(v) - 1) \neq v_{\min}^{(k-1)}$,

(6) $\{v \in \tilde{V}^{(k)} \mid \exists t : \tau(v, t) = v_{\max}^{(k-1)}, \tau(v, t+1) = v_{\min}^{(k-1)}\} = \{v_{i,l}^* \mid \forall i, \forall l\}$.

For any set of equivalence class $\tilde{V}_j^{(k)}$, define $\mathrm{Dist}(\tilde{V}_j^{(k)}) \in \mathbf{N}$ by

$$\mathrm{Dist}(\tilde{V}_j^{(k)}) = \left(\sum_{i=3}^{\#\tilde{V}^{(k-1)}} \tilde{m}_{i,[j]}^{(k)} \right)! \bigg/ \prod_{i=3}^{\#\tilde{V}^{(k-1)}} \tilde{m}_{i,[j]}^{(k)}! \, . \tag{2.1}$$

REMARKS. (1) Assumption 2.5 implies that we can accommodate the unique maximal and minimal path condition.

(2) Later when we calculate the topological entropy of some simple ordered Bratteli diagram, we need Assumption 2.6 as a technical convenience (Lemma 3.7, 3.9).

(3) Assumption 2.7 and 2.8 are the conditions in order that $(\tilde{V}, \tilde{E}, \tilde{\leq})$ satisfies Condition 3.2 in §3. Later we shall show in Lemma 3.9, the reason why Condition 3.2 holds if $(\tilde{V}, \tilde{E}, \tilde{\leq})$ satisfies Assumption 2.7 and 2.8.

(4) If $\tilde{V}^{(k)}$ satisfies Assumption 2.4, 2.7 and 2.8 and $\tilde{V}_{j,l}^{(k)}$ has partial distinct orderings for all l, then there is a maximum possible value for $\#\tilde{V}_{j,l}^{(k)}$, for all j. Let $\mathrm{Max}(\tilde{V}_j^{(k)})$ be this maximum value. Then $\mathrm{Dist}(\tilde{V}_j^{(k)})$ is a maximum possible value of arranging all $\tilde{m}_{i,[j]}^{(k)}$ vertices of $i\in\tilde{V}^{(k-1)}$ for $3\leq i\leq\#\tilde{V}^{(k-1)}$. It is easily seen that the relation between $\mathrm{Max}(\tilde{V}_j^{(k)})$ and $\mathrm{Dist}(\tilde{V}_j^{(k)})$ is $\mathrm{Max}(\tilde{V}_j^{(k)})\geq\mathrm{Dist}(\tilde{V}_j^{(k)})$.

Let $\mathrm{c.d}(a_1, a_2, \cdots, a_n)$ denote the set of common divisors of $a_1, a_2, \cdots, a_n\in\mathbf{N}$. The following Lemma guarantees the existence of the above simple ordered Bratteli diagram satisfying Assumptions 2.5, 2.6, 2.7 and 2.8.

LEMMA 2.9. *Suppose* $(\tilde{V}, \tilde{E})$ *is the unordered Bratteli diagram satisfying Assumption* 2.4. *And suppose* $\binom{\#\tilde{V}^{(k)}}{2}<(\#\tilde{V}_j^{(k+1)}-2\delta(j))/\eta_{k+1}\leq\mathrm{Dist}(\tilde{V}_j^{(k+1)})+2\#\tilde{V}^{(k)}-3$ *for all* $k\geq1$ *and* $1\leq j\leq\{\tilde{V}^{(k+1)}/\sim\}$, *where* $\eta_{k+1}\in\mathrm{c.d}(\#\tilde{V}_j^{(k+1)}-2\delta(j)\mid 1\leq j\leq\#\{\tilde{V}^{(k+1)}/\sim\})$. *Then there exists a partial order* $\tilde{\leq}$ *on* $\tilde{E}$ *such that* $(\tilde{V}, \tilde{E}, \tilde{\leq})$ *satisfies Assumption* 2.5, 2.6, 2.7 *and* 2.8.

PROOF. Take any $k\geq1$ and fix it. We write $\tilde{V}^{(k)}=\{v_i^{(k)}\}_{i=1}^{\#\tilde{V}^{(k)}}$ and put

$$v_1^{(k)}=v_{\min}^{(k)}=1\in\tilde{V}^{(k)}\,,\qquad v_{\#\tilde{V}^{(k)}}^{(k)}=v_{\max}^{(k)}=2\in\tilde{V}^{(k)}\,. \tag{2.2}$$

From Assumption 2.4 and (2.2), we see for any $v\in\tilde{V}^{(k+1)}$,

$$\#\{e\in\tilde{E}^{(k+1)}\mid s(e)=v_{\min}^{(k)}, r(e)=v\}=\#\{e\in\tilde{E}^{(k+1)}\mid s(e)=v_{\max}^{(k)}, r(e)=v\}=3\,,$$

where $S\colon\tilde{E}^{(k+1)}\to\tilde{V}^{(k)}$ is source map.

Now we fix any $\tilde{V}_j^{(k+1)}$. Let $e_{i,j}$ be the number of edges connecting $v_i^{(k)}$ and $[\tilde{V}_j^{(k+1)}]$. We define a basic order $\hat{\tau}_j\colon\{1, 2, \cdots, \sum_{i=1}^{\#\tilde{V}^{(k)}}e_{i,j}\equiv\bar{e}_j\}\to\tilde{V}^{(k)}$ as follows:

$$\hat{\tau}_j(t)=v_i^{(k)}\quad\overset{\text{def}}{\Longleftrightarrow}\quad 1+\sum_{l=0}^{i-1}e_{l,j}\leq t\leq\sum_{l=0}^{i}e_{l,j}\quad(e_{0,j}\equiv0)\,,$$

i.e.

$$\prod_{t=1}^{\bar{e}_j}\hat{\tau}_j(t)=(v_{\min}^{(k)}, v_{\min}^{(k)}, v_{\min}^{(k)}, \underbrace{v_2^{(k)}, \cdots, v_2^{(k)}}_{e_{2,j}\text{ times}}, \cdots, \underbrace{v_{\#\tilde{V}^{(k)}-1}^{(k)}, \cdots, v_{\#\tilde{V}^{(k)}-1}^{(k)}}_{e_{\#\tilde{V}^{(k)}-1,j}\text{ times}}, v_{\max}^{(k)}, v_{\max}^{(k)}, v_{\max}^{(k)})\,.$$

Since $\eta_{k+1}\in\mathrm{c.d}(\#\tilde{V}_j^{(k+1)}-2\delta(j)\mid1\leq j\leq\#\{\tilde{V}^{(k+1)}/\sim\})$, we can decompose $\tilde{V}_j^{(k+1)}$ so as to satisfy Assumption 2.8 (1). Take any $1\leq l\leq\eta_{k+1}$ and fix it. We pick up any $\binom{\#\tilde{V}^{(k)}}{2}$ vertices in $\tilde{V}_{j,l}^{(k+1)}$ and write $\{v_{i,i'}^{(k+1)}\mid1\leq i<i'\leq\#\tilde{V}^{(k)}\}\equiv\hat{V}_j^{(k+1)}$. We assign the following order τ to edges connecting these vertices.

(1) If $i' \neq \# \tilde{V}^{(k)}$,

$$\tau(v_{i,i'}^{(k+1)}, t) \equiv \begin{cases} v_i^{(k)} & \text{if} \quad t = \sum_{l=0}^{i'} e_{l,j} \\ v_{i'}^{(k)} & \text{if} \quad t = \sum_{l=0}^{i} e_{l,j} \\ \hat{\tau}_j(t) & \text{otherwise}, \end{cases}$$

(2) If $i \neq 1$, $i' = \# \tilde{V}^{(k)}$,

$$\tau(v_{i,\#\tilde{V}^{(k)}}^{(k+1)}, t) \equiv \begin{cases} v_i^{(k)} & \text{if} \quad t = \bar{e}_j - 2 \\ v_{\max}^{(k)} & \text{if} \quad t = 1 + \sum_{l=0}^{i-1} e_{l,j} \\ \hat{\tau}_j(t) & \text{otherwise}, \end{cases} \tag{2.3}$$

(3) If $i = 1$, $i' = \# \tilde{V}^{(k)}$,

$$\tau(v_{\#\tilde{V}^{(k)}}^{(k+1)}, t) \equiv \begin{cases} v_{\min}^{(k)} & \text{if} \quad t = 1, 3, 4 \\ \hat{\tau}_j(t-1) & \text{if} \quad 1 + e_{1,j} \leq t \leq 1 + \sum_{l=0}^{\#\tilde{V}^{(k)}-1} e_{l,j} \\ v_{\max}^{(k)} & \text{if} \quad t = 2, \bar{e}_j - 1, \bar{e}_j . \end{cases}$$

(1) $\Leftrightarrow O.vec(v_{i,i'}^{(k+1)}) = (\cdots, \underbrace{v_i^{(k)}, \cdots, v_i^{(k)}}_{e_{i,j}-1 \text{ times}}, v_{i'}^{(k)}, \underbrace{v_{i+1}^{(k)}, \cdots, v_{i+1}^{(k)}}_{e_{i+1,j} \text{ times}}, \cdots,$

$\underbrace{v_{i'}^{(k)}, \cdots, v_{i'}^{(k)}}_{e_{i',j}-1 \text{ times}}, v_i^{(k)}, \underbrace{v_{i'+1}^{(k)}, \cdots, v_{i'+1}^{(k)}}_{e_{i'+1,j} \text{ times}} \cdots),$

(2) $\Leftrightarrow O.vec(v_{i,\#\tilde{V}^{(k)}}^{(k+1)}) = (\cdots, \underbrace{v_{i-1}^{(k)}, \cdots, v_{i-1}^{(k)}}_{e_{i-1,j} \text{ times}}, v_{\max}^{(k)}, \underbrace{v_i^{(k)}, \cdots, v_i^{(k)}}_{e_{i,j}-1 \text{ times}}, \cdots,$

$\underbrace{v_{\#\tilde{V}^{(k)}-1}^{(k)}, \cdots, v_{\#\tilde{V}^{(k)}-1}^{(k)}}_{e_{\#\tilde{V}^{(k)}-1,j} \text{ times}}, v_i^{(k)}, v_{\max}^{(k)}, v_{\max}^{(k)}),$

(3) $\Leftrightarrow O.vec(v_{1,\#\tilde{V}^{(k)}}^{(k+1)}) = (v_{\min}^{(k)}, v_{\max}^{(k)}, v_{\min}^{(k)}, v_{\min}^{(k)}, \underbrace{v_2^{(k)}, \cdots, v_2^{(k)}}_{e_{2,j} \text{ times}}, \cdots,$

$\underbrace{v_{\#\tilde{V}^{(k)}-1}^{(k)}, \cdots, v_{\#\tilde{V}^{(k)}-1}^{(k)}}_{e_{\#\tilde{V}^{(k)}-1,j} \text{ times}}, v_{\max}^{(k)}, v_{\max}^{(k)}).$

Moreover define the order on $r^{-1}(v_{\min}^{(k+1)})$ and $r^{-1}(v_{\max}^{(k+1)})$ as follows:

(4)

$$\tau(v_{\min}^{(k+1)}, t) \equiv \begin{cases} v_{\min}^{(k)} & \text{if} \quad t=1, 2, 3 \\ \hat{\tau}_1(t-2) & \text{if} \quad 6 \leq t \leq \bar{e}_1 - 1 \\ v_{\max}^{(k)} & \text{if} \quad 4, 5, \bar{e}_1 , \end{cases}$$

(5)

$$\tau(v_{\max}^{(k+1)}, t) \equiv \begin{cases} v_{\min}^{(k)} & \text{if} \quad t=1, \bar{e}_1 - 4, \bar{e}_1 - 3 \\ \hat{\tau}_1(t+2) & \text{if} \quad 2 \leq t \leq \bar{e}_1 - 5 \\ v_{\max}^{(k)} & \text{if} \quad t = \bar{e}_1 - 2, \bar{e}_1 - 1, \bar{e}_1 . \end{cases}$$

$$(4) \quad \Leftrightarrow O.vec(v_{\min}^{(k+1)}) = (v_{\min}^{(k)}, v_{\min}^{(k)}, v_{\min}^{(k)}, v_{\max}^{(k)}, v_{\max}^{(k)}, \underbrace{v_2^{(k)}, \cdots, v_2^{(k)}}_{e_{2,1} \text{ times}}, \cdots, \underbrace{v_{\#\tilde{V}^{(k)}-1}^{(k)}, \cdots, v_{\#\tilde{V}^{(k)}-1}^{(k)}}_{e_{\#\tilde{V}^{(k)}-1,1} \text{ times}}, v_{\max}^{(k)}) ,$$

$$(5) \quad \Leftrightarrow O.vec(v_{\max}^{(k+1)}) = (v_{\min}^{(k)}, \underbrace{v_2^{(k)}, \cdots, v_2^{(k)}}_{e_{2,1} \text{ times}}, \cdots, \underbrace{v_{\#\tilde{V}^{(k)}-1}^{(k)}, \cdots, v_{\#\tilde{V}^{(k)}-1}^{(k)}}_{e_{\#\tilde{V}^{(k)}-1,1} \text{ times}}, v_{\min}^{(k)}, v_{\min}^{(k)}, v_{\max}^{(k)}, v_{\max}^{(k)}, v_{\max}^{(k)}) .$$

Let $\hat{V}_{jc}^{(k+1)} \equiv \{v \in \hat{V}_{j,l}^{(k+1)} \mid \tau(v, 3) \neq v_{\min}^{(k)} \text{ or } \tau(v, \bar{e}_j - 2) \neq v_{\max}^{(k)}\}$. Then by the above construction we see

$$\hat{V}_{jc}^{(k+1)} = \{v_{i,i'}^{(k+1)} \in \hat{V}_{j,l}^{(k+1)} \mid i=1 \text{ or } i' = \#\tilde{V}^{(k)}\} , \qquad \#\hat{V}_{jc}^{(k+1)} = 2\#\tilde{V}^{(k)} - 3 .$$

Since $\#(\tilde{V}_{j,l}^{(k+1)} \setminus \hat{V}_{jc}^{(k+1)}) \leq \text{Dist}(\tilde{V}_j^{(k+1)})$ holds, we can assign an order on $\tilde{V}_{j,l}^{(k+1)} \setminus \hat{V}_j^{(k+1)}$ satisfying the following conditions:

(i) for any $v \in \tilde{V}_{j,l}^{(k+1)} \setminus \hat{V}_j^{(k+1)}$ and any $1 < i < i' < \#\tilde{V}^{(k)}$, $O.vec(v) \neq O.vec(v_{i,i'}^{(k)})$,

(ii) for any $u, v \in \tilde{V}_{j,l}^{(k+1)} \setminus \hat{V}_j^{(k+1)}$ with $u \neq v$, $O.vec(u) \neq O.vec(v)$,

(iii) for any $v \in \tilde{V}_{j,l}^{(k+1)} \setminus \hat{V}_j^{(k+1)}$, $\tau(v, 1) = \tau(v, 2) = \tau(v, 3) = v_{\min}^{(k)}$, $\tau(v, \bar{e}_j - 2) = \tau(v, \bar{e}_j - 1) = \tau(v, \bar{e}_j) = v_{\max}^{(k)}$.

Then the above conditions imply that $\tilde{V}_{j,l}^{(k+1)}$ has partial distinct orderings. And it is easily seen that Assumption 2.5, 2.7 and 2.8 hold. (In this case, $v^* = v_{1,\#\tilde{V}^{(k)}}^{(k+1)}$ in Assumption 2.8.)

Finally we shall show Assumption 2.6 holds. Since $e_{i,j} \geq 2$ for all $1 \leq i \leq \#\tilde{V}^{(k)}$ by Assumption 2.4, for any $v_i^{(k)}$ and $v_{i'}^{(k)}$, we can choose $v_{i,i'}^{(k+1)}$ if $i \neq i'$ or $v_{t,t'}^{(k+1)}$ if $i = i'$ where t, t' are any pair of number with $t \neq i$, $t' \neq i$. Therefore Assumption 2.6 holds. So we are done. □

3. Calculation of topological entropy.

The aim of this section is to calculate the topological entropy of a lexicographic map on a Bratteli compactum in a special case.

First, we calculate the topological entropy of subshift in a special case. Suppose that A is a finite set, which will be called an alphabet. Let $A^{\mathbf{Z}}$ be the set of all bisequences $\mathbf{x}=\cdots x_{-1}x_0x_1\cdots$ (with each x_i in A), equipped with the product topology. Then $A^{\mathbf{Z}}$ is a compact metrizable totally disconnected space, and the shift map $\sigma: A^{\mathbf{Z}}\to A^{\mathbf{Z}}$ given by $(\sigma x)_i=x_{i+1}$ is a homeomorphism. The restriction of σ to a closed invariant subset Y of $A^{\mathbf{Z}}$ is called a subshift. If S is the restriction of σ to Y, then the topological entropy of S, $h(S)$, is called the growth rate of the number of words of length n occurring in points of Y:

$$h(S)=\limsup_{n\to\infty}\frac{\log\#\mathscr{W}_n(S)}{n}, \tag{3.1}$$

where

$$\mathscr{W}_n(S)=\{\dot{y}_1\dot{y}_2\dot{y}_3\cdots\dot{y}_n \mid \text{there exists } \mathbf{y}=(y_i)_{i\in\mathbf{Z}} \text{ in } Y \text{ with } \dot{y}_i=y_i \text{ for } 1\le i\le n\}$$

(cf. [Wa], [DGS]). This definition is given in [Wa: Theorem 7.13] or [DGS: Proposition 16.11].

Let $\mathbf{W}$ be a set of words with alphabet A. Here we give the following definition.

DEFINITION 3.1 (concatenating point). We say $w=w_1w_2\cdots w_n\in\mathbf{W}$ ($w_i\in A$ for all $1\le i\le n$) has *concatenating points in* $\mathbf{W}$ if there exist more than two words $w^j=w_1^jw_2^j\cdots w_{n_j}^j\in\mathbf{W}$ ($1\le j\le d$, $d\ge 2$) and integers $0=p_0<p_1<\cdots<p_{d-1}<p_d=n$ such that

$$w_{p_{i-1}+1}w_{p_{i-1}+2}\cdots w_{p_i}=\begin{cases} w^1_{n_1-p_1+1}w^1_{n_1-p_1+2}\cdots w^1_{n_1} & \text{if } i=1\\ w^i & \text{if } 1<i<d\\ w_1^dw_2^d\cdots w^d_{n-p_{d-1}} & \text{if } i=d.\end{cases}$$

Here we say a word $w=w_1w_2\cdots w_n$ (w_j is an alphabet for all j) is the *sub-word of* $u=u_1u_2\cdots u_m$ if $n\le m$ and there exists i with $1\le i\le m-n+1$ such that

$$w=u_iu_{i+1}\cdots u_{i+n-1} \qquad (w_j=u_{i+j-1} \text{ for all } 1\le j\le n).$$

Now let $\mathbf{W}$ be a set of words with alphabet A satisfying the following conditions:

CONDITION 3.2. (1) $\#\mathbf{W}<\infty$,

(2) $\mathbf{W}$ has no word which has concatenating points in $\mathbf{W}$.

EXAMPLE. Let A be an alphabet $\{a, b, c, d\}$.

(1) $\mathbf{W}_1=\{\alpha=abcd,\ \beta=acbd,\ \gamma=aabcdd,\ \delta=acbdbabcd\}$. Then $\mathbf{W}_1$ satisfies Condition 3.2. Because any word in $\mathbf{W}_1$ does not contain a sub-word "da" and any concatenated word always contains it. (But α is a sub-word of γ and δ. β is also a sub-word of δ.)

(2) $\mathbf{W}_2=\{\alpha=abcd,\ \beta=abc,\ \gamma=dba,\ \delta=aab,\ \varepsilon=cdb\}$. Then $\mathbf{W}_2$ does not satisfy Condition 3.2. In fact, α is a sub-word of concatenated word $\beta\gamma$ and $\delta\varepsilon$. Therefore α

has concatenating points in $\mathbf{W}_2$.

DEFINITION 3.3 (subshift generated by $\mathbf{W}$). Let $Y_{\mathbf{W}}$ be the set of all bisequences formed by concatenating of words in $\mathbf{W}$, i.e.

$$Y_{\mathbf{W}} \equiv \{\mathbf{y}=(y_i)_{i=-\infty}^{\infty} \in A^{\mathbf{Z}} \mid \exists \{n_i \in \mathbf{Z} \mid n_j < n_{j+1} \forall j \in \mathbf{Z}\}_{i=-\infty}^{\infty} \exists \{w_i \mid w_i \in \mathbf{W}\}_{i=-\infty}^{\infty}$$
$$\text{s.t. } y_{n_i} y_{n_i+1} \cdots y_{n_{i+1}-2} y_{n_{i+1}-1} = w_i \ (\forall i \in \mathbf{Z})\} .$$

Let $S_{\mathbf{W}}$ be the restriction of shift to $Y_{\mathbf{W}}$. We call $(Y_{\mathbf{W}}, S_{\mathbf{W}})$ *the subshift generated by* $\mathbf{W}$.

For $w \in \mathbf{W}$, let $|w|$ be the length of w. Then there exist $I \in \mathbf{N}$ and J: $\{1, 2, \cdots, I\} \to \mathbf{N}$ such that

$$\mathbf{W} = \{w_{i,j} \mid 1 \le i \le I, 1 \le j \le J(i), w_{i,j} \ne w_{i',j'} \text{ if } (i,j) \ne (i',j')$$
$$|w_{i,j}| = |w_{i,j'}| \text{ for } 1 \le \forall j, \forall j' \le J(i)\} .$$

Put $l_i = |w(i,j)|$. The next lemma is a calculation of topological entropy $(Y_{\mathbf{W}}, S_{\mathbf{W}})$.

LEMMA 3.4 ([Su]: Lemma 4.4). *Suppose* $(Y_{\mathbf{W}}, S_{\mathbf{W}})$ *is a subshift generated by* $\mathbf{W}$ *and* $\mathbf{W}$ *satisfies Condition* 3.2. *Let* α_0 *be the positive solution of the following equation*:

$$\sum_{i=1}^{I} \frac{J(i)}{x^{l_i}} = 1 . \tag{3.2}$$

Then

$$h(S_{\mathbf{W}}) = \log \alpha_0 .$$

PROOF. First without loss of generality, we may assume $1 \le l_1 \le l_2 \le \cdots \le l_I$. Let L_n be a set of concatenated words of length n which are made up from words in $\mathbf{W}$. We take any $n > l_I$, $w \in L_n$ and fix them. Then from Condition 3.2 (2), there exist "unique" words $w_1, w_2, \cdots w_t \in \mathbf{W}$ such that w is the concatenated word $w_1 w_2 \cdots w_t$. This uniqueness of concatenated words is very important, because this uniqueness guarantees that for all $n > l_I$, the following linear homogeneous difference equation holds:

$$\sharp L_n = \sum_{i=1}^{I} J(i) \sharp L_{n-l_i} . \tag{3.3}$$

(If $\mathbf{W} = \{\alpha = abc, \beta = abcabc, \gamma = abcab, \delta = cabc\}$ and $w = abcabcabc \in L_9$, then in this case $\mathbf{W}$ does not satisfy Condition 3.2, so (3.3) does not hold. In fact, w has four cases of concatenation: $\alpha\alpha\alpha$, $\alpha\beta$, $\beta\alpha$, $\gamma\delta$. In this case it is easy to check that (3.3) does not hold.)

Here consider $\mathcal{W}_n(S_{\mathbf{W}})$ for $n \ge l_I$. Take any $w \in \mathcal{W}_n(S_{\mathbf{W}})$ and fix it. Then there exists non negative integer i with $0 \le i \le 2l_I - 2$ and $w' \in L_{n+i}$ such that w is the sub-word of w'. Therefore the relation between $\mathcal{W}_n(S_{\mathbf{W}})$ and $\{L_{n+i}\}_{i=0}^{2l_I-2}$ is as follows:

$$\#L_n \leq \#\mathscr{W}_n(S_{\mathbf{W}}) \leq \sum_{i=0}^{2l_I-2} \#L_{n+i} . \tag{3.4}$$

By the way, we also consider the following indicial equation (3.5) associated with (3.3):

$$x^{l_I} = \sum_{i=1}^{I} J(i) x^{l_I - l_i} . \tag{3.5}$$

Immediately we see that the equation (3.5) is equivalent to (3.2). Let $\alpha_1, \alpha_2, \cdots, \alpha_d$ be all the solutions to (3.2) with $\alpha_i \neq \alpha_j$ if $i \neq j$ and $\alpha_j \neq \alpha_0$ for $1 \leq j \leq d$.

Now we claim $\alpha_0 \geq |\alpha_j|$ for all $1 \leq j \leq d$.

Define $f(x) \equiv \sum_{i=1}^{l_I} J(i)/x^{l_i}$. Since $f(x)$ is a monotone decreasing function if $x > 0$ and $f(1) = \sum_{i=1}^{I} J(i) \geq 1$, we see that $\alpha_0 \geq 1$ is unique positive solution. Moreover for any i, we see that

$$1 = f(\alpha_0) = f(\alpha_i) = \sum_{k=1}^{I} \frac{J(k)}{\alpha_i^{l_k}} = \left| \sum_{k=1}^{I} \frac{J(k)}{\alpha_i^{l_k}} \right| \leq \sum_{k=1}^{I} \left| \frac{J(k)}{\alpha_i^{l_k}} \right| = \sum_{k=1}^{I} \frac{J(k)}{|\alpha_i|^{l_k}} = f(|\alpha_i|) .$$

Therefore the claim holds.

Next, let m_i be the multiplicity of α_i. For $n > l_I$, using Lagrange's method of variation of constants (see [EDM]), we get the general solution to (3.3) by $\#L_n = \sum_{i=0}^{d} P_i(n) \alpha_i^n$, where P_i is a polynomial and $\deg P_i = m_i - 1$ $(0 \leq i \leq d)$. Let $M \equiv \max_{0 \leq i \leq d} m_i$. For each sufficiently large n, $1 \leq \sum_{i=0}^{d} |P_i(n)| < n^M$ holds. Since $|\alpha_i/\alpha_0| \leq 1$ and $\sum_{i=0}^{d} P_i(n) \alpha_i^n$ takes a positive integer value, we can calculate

$$\left| \frac{1}{n} \log \#L_n - \log \alpha_0 \right| = \left| \frac{1}{n} \log \left\{ \sum_{i=0}^{d} P_i(n) \left(\frac{\alpha_i}{\alpha_0} \right)^n \right\} \right| < \frac{1}{n} \left| \log \sum_{i=0}^{d} |P_i(n)| \right|$$

$$< \frac{M}{n} \log n \to 0 \qquad (\text{as } n \to \infty) .$$

So we get $\lim_{n \to \infty} \frac{1}{n} \log \#L_n = \log \alpha_0$. For $n > l_I$, define $l(n)$ by $\#L_{n+l(n)} = \max_{0 \leq k \leq 2l_I - 2} \#L_{n+k}$. From (3.4), $\#\mathscr{W}_n(S_{\mathbf{W}}) < 2l_I \#L_{n+l(n)}$ holds. And since $0 \leq l(n) \leq 2l_I - 2$, the following inequality holds:

$$\left| \frac{1}{n} \log \{ 2l_I \#L_{n+l(n)} \} - \log \alpha_0 \right|$$

$$< \frac{1}{n} \log 2l_I + \frac{n + l(n)}{n} \left| \frac{1}{n + l(n)} \log \#L_{n+l(n)} - \log \alpha_0 \right| + \frac{l(n)}{n} \log \alpha_0$$

$$\to 0 \qquad (\text{as } n \to \infty) .$$

Therefore we have

$$h(S_{\mathbf{W}}) = \limsup_{n \to \infty} \frac{\log \#\mathscr{W}_n(S_{\mathbf{W}})}{n} = \log \alpha_0 .$$

So we are done. □

We give some notations of an ordered Bratteli diagram $\mathscr{D}=(V, E, \leq)$. For $v\in V$, let $\mathscr{P}(v)$ be the set of all paths connecting $V^{(0)}$ and v. And define $\#\mathscr{P}[V_i^{(k)}]\equiv\#\mathscr{P}(v)$ where $v\in V_i^{(k)}$.

Let $\mathscr{B}=\{M^{(k)}\}_{k=1}^{\infty}$ be a sequence of incidence matrices of $\mathscr{D}$. Then it is easy to see that the relation between $\mathbf{P}_k\equiv(\#\mathscr{P}(1), \#\mathscr{P}(2), \cdots, \#\mathscr{P}(\# V^{(k)}))$ and $\mathscr{B}$ is as follows:

$$\mathbf{P}_k=M^{(1)}M^{(2)}\cdots M^{(k-1)}M^{(k)} \qquad \text{for all} \quad k\geq 1 .$$

Let $\mathscr{P}$ be the Bratteli compactum associated with $\mathscr{D}$.

Definition 3.5 (subshift associated with an ordered Bratteli diagram). We shall give the two types of subshift associated with an ordered Bratteli diagram:

(1) Let $S:\mathscr{P}\to\mathscr{P}$ be the lexicographic map of $\mathscr{D}$. For $k\in\mathbf{N}$ and $n\in\mathbf{N}\cup\{\infty\}$ with $n\geq k$, we define $\pi_k:\mathscr{P}(V^{(n)}) \to \mathscr{P}(V^{(k)})$ by

$$\pi_k(p)\equiv(e_1, e_2, \cdots, e_k)\in\mathscr{P}(V^{(k)}) ,$$

where $p=(e_1, e_2, \cdots, e_n)\in\mathscr{P}(V^{(n)})$ and $\mathscr{P}(V^{(\infty)})\equiv\mathscr{P}$. And define the shift invariant closed subset Y_k of $\mathscr{P}(V^{(k)})^{\mathbf{Z}}$ by

$$Y_k\equiv\{\{\pi_k(S^n p)\}_{n=-\infty}^{\infty}\in\mathscr{P}(V^{(k)})^{\mathbf{Z}} \mid p\in\mathscr{P}\} . \tag{3.6}$$

Let S_k be the restriction of shift to Y_k. We call (Y_k, S_k) *the subshift of* $\mathscr{P}(V^{(k)})^{\mathbf{Z}}$ *whose domain is* Y_k.

(2) For fixed $k\geq 1$ and $v\in V_i^{(k)}$, we write $\mathscr{P}(v)=\{p_1, p_2, \cdots, p_{\#\mathscr{P}[V_i^{(k)}]}\}$ with $p_1<p_2<\cdots<p_{\#\mathscr{P}[V_i^{(k)}]}$. ("$<$" is the partial order of $\mathscr{P}(V^{(k)})$ associated with the order of edges.) We write $Con(v)\equiv p_1p_2\cdots p_{\#\mathscr{P}[V_i^{(k)}]}$ as a concatenation of $p_1, p_2, \cdots, p_{\#\mathscr{P}[V_i^{(k)}]}$.

For $t\geq 0$ and $v_i^{(k+t)}\in V_i^{(k+t)}$, define $\boldsymbol{\pi}_k$ by

$$\boldsymbol{\pi}_k(Con(v_i^{(k+t)}))\equiv\pi_k(p_1)\pi_k(p_2)\cdots\pi_k(p_{\#\mathscr{P}[V_i^{(k+t)}]}) ,$$

where $\{p_1, p_2, \cdots, p_{\#\mathscr{P}[V_i^{(k+t)}]} \mid p_1<p_2<\cdots<p_{\#\mathscr{P}[V_i^{(k+t)}]}\}=\mathscr{P}(v_i^{(k+t)})$. And define $\mathbf{W}_k(t)$ by

$$\mathbf{W}_k(t)\equiv\{\boldsymbol{\pi}_k(Con(v^{(k+t)})) \mid v^{(k+t)}\in V^{(k+t)}\} .$$

Moreover we define the shift invariant closed subset $Y_k(t)$ of $P(V^{(k)})^{\mathbf{Z}}$ by

$$\begin{aligned} Y_k(t)\equiv\{\mathbf{y}=(y_i)_{i\in\mathbf{Z}}\in\mathscr{P}(V^{(k)})^{\mathbf{Z}} \mid \exists\{w_i \mid w_j<w_{j+1}\forall j\in\mathbf{Z}\}_{i=-\infty}^{\infty}\subset\mathbf{Z} \\ \exists\{v_i\}_{i=-\infty}^{\infty}\subset V^{(k+t)} \text{ s.t. } y_{w_i}y_{w_i+1}\cdots y_{w_{i+1}-1}=\boldsymbol{\pi}_k(Con(v_i))\forall i\in\mathbf{Z}\} . \end{aligned}$$

Let $S_k(t)$ be the restriction of the shift to $Y_k(t)$. We call $(Y_k(t), S_k(t))$ *the subshift of* $P(V^{(k)})^{\mathbf{Z}}$ *whose domain is* $Y_k(t)$.

The next lemma is the relation of entropy between $(\mathscr{P}, S)$ and (Y_k, S_k).

Lemma 3.6 ([Su]: Lemma 4.6). *Suppose S is a lexicographic map on the Bratteli* *compactum associated with a simple ordered Bratteli diagram* $\mathscr{D}=(V, E, \leq)$ *and* S_k *is the subshift defined as above. Then*

$$h(S)=\lim_{k\to\infty} h(S_k)\,. \tag{3.7}$$

PROOF. This proof follows [Wa]: §7.1 and §7.2. For $k\geq 1$, let $\mathscr{C}_k$ be the family of cylinder sets in $\mathscr{P}$ whose length is k. That is, $\mathscr{C}_k\equiv\{U(e_1, e_2, \cdots, e_k)\mid(e_1, e_2, \cdots, e_k)\in\mathscr{P}(V^{(k)})\}$. Then $\mathscr{C}_k$ is a finite clopen cover of $\mathscr{P}$. Let d denote the metric on $\mathscr{P}$. We remark that since $\mathscr{P}$ is compact set, the entropy of S does not depend on the metric chosen on $\mathscr{P}$. Define $\operatorname{diam}(\mathscr{C}_k)\equiv\sup_{A\in\mathscr{C}_k}\operatorname{diam}(A)$, where $\operatorname{diam}(A)$ denotes the diameter of the set A measured by d. Then $\operatorname{diam}(\mathscr{C}_k)\to 0$ as $k\to\infty$. Therefore from [Wa]: Theorem 7.6, we can get

$$h(S)=\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\log N\left(\bigvee_{i=0}^{n-1} S^{-i}\mathscr{C}_k\right), \tag{3.8}$$

where $\bigvee_{i=0}^{n-1} S^{-i}\mathscr{C}_k$ is a clopen cover, consisting of sets of the form $A_0\cap S^{-1}A_1\cap S^{-2}A_2\cap\cdots\cap S^{-(n-1)}A_{n-1}$ with $A_i\in\mathscr{C}_k$ $(i=0, 1, 2, \cdots, n-1)$ and $N(\bigvee_{i=0}^{n-1} S^{-i}\mathscr{C}_k)$ is the number of sets in a finite subcover of $\bigvee_{i=0}^{n-1} S^{-2}\mathscr{C}_k$ with the smallest cardinality. Now we write $\mathscr{C}_k=\{A_i\}_{i=1}^{\#\mathscr{P}(V^{(k)})}$. Since A_i's are disjoint, we see if $(m_i)_{i=0}^{n-1}$, $(m_i')_{i=0}^{n-1}\in\{1, 2, \cdots\#\mathscr{P}(V^{(k)})\}^n$ with $(m_1, m_2, \cdots, m_{n-1})\neq(m_1', m_2', \cdots, m_{n-1}')$, then

$$\left(\bigcap_{i=0}^{n-1} S^{-i}A_{m_i}\right)\cap\left(\bigcap_{i=0}^{n-1} S^{-i}A_{m_i'}\right)=\varnothing\,.$$

Therefore the following equality holds:

$$N\left(\bigvee_{i=0}^{n-1} S^{-i}\mathscr{C}_k\right)=\#\left\{A=\bigcap_{i=0}^{n-1} S^{-i}A_{m_i}\;\middle|\;A\neq\varnothing, m_i\in\{1, 2, \cdots, \#\mathscr{P}(V^{(k)})\}, \forall i\right\}. \tag{3.9}$$

We see that from (3.6) and (3.9), $\#\mathscr{W}_n(S_k)=N(\bigvee_{i=0}^{n-1} S^{-i}\mathscr{C}_k)$ holds. Therefore from (3.1) and (3.8), we can obtain (3.7). So we finish the proof. □

The next lemma is the relation of entropy betweeen (Y_k, S_k) and $(Y_k(t), S_k(t))$.

LEMMA 3.7 ([Su]: Lemma 4.7). *Suppose* $(V, E, \leq)$ *is a simple ordered Bratteli diagram satisfying Assumption* 2.5 *and Assumption* 2.6. *Then for* $k\geq 1$,

$$h(S_k)=\lim_{t\to\infty} h(S_k(t))\,. \tag{3.10}$$

PROOF. This proof follows [BH: Lemma 2.5]. Clearly $Y_k(0)\supset Y_k(1)\supset\cdots$ and $Y_k\subset\bigcap_{t\geq 0} Y_k(t)$. Conversely, suppose $\mathbf{y}\in\bigcap_{t\geq 0} Y_k(t)$. By compactness, $\mathbf{y}\in Y_k$ if for any $n\in\mathbf{N}$, the word $y_{-n}\cdots y_n$ appears in a point of Y_k. It suffices to show $y_{-n}\cdots y_n$ occurs in the subshift generated by $\pi_k(Con(v))$ for some vertex v. Choose $m\geq k$ such that $\#\mathscr{P}[V_i^{(m)}]>2n$ for any $1\leq i\leq\#\{V^{(m)}/\sim\}$. Since $\mathbf{y}\in Y_k(m-k)$, $y_{-n}\cdots y_n$ is a sub-word of $\pi_k(Con(v'))\pi_k(Con(v''))$ for some vertices $v', v'' \in V^{(m)}$. By Assumption 2.6, $\pi_k(Con(v'))\pi_k(Con(v''))$ is a sub-word of $\pi_k(Con(v))$ for some $v \in V^{(m+1)}$. Therefore S_k is the nested intersection of the subshifts $S_k(t)$. From [DGS; p. 113, Corollary to Propositon 16.12], (3.10) holds. So we are done. □

Now we define another equivalence relation on vertex sets.

DEFINITION 3.8 (equivalence relations on the vertex set). We shall define two types of equivalence relation on the vertex set (cf. Definition 2.2).

(1) For a simple ordered Bratteli diagram $(\tilde{V}, \tilde{E}, \tilde{\leq})$ and $k, n \in \mathbf{N}$ with $k \leq n$, we decompose $\tilde{V}^{(n)}$ using the equivalence relation $\approx_k$ defined as follows:

$$i, j \in \tilde{V}^{(n)} \qquad i \approx_k j \overset{\text{def}}{\iff} \pi_k(Con(i)) = \pi_k(Con(j)) .$$

Let $K_j^{(k:n)}$ be the set of the equivalence classes of $\tilde{V}^{(n)}$ for $\approx_k$. i.e. we decompose $\tilde{V}^{(n)} = \bigcup_{j=1}^{\#\{\tilde{V}^{(n)}/\approx_k\}} K_j^{(k:n)}$ as a disjoint union and denote the equivalence class of $v \in K_j^{(k:n)}$ by $[K_j^{(k:n)}]$.

(2) For a simple ordered Bratteli diagram $(\tilde{V}, \tilde{E}, \tilde{\leq})$, a sequence of incidence matrices $\{\tilde{M}^{(n)}\}_{n=1}^{\infty}$ of $(\tilde{V}, \tilde{E}, \tilde{\leq})$ and $k, n \in \mathbf{N}$ with $k < n$, we define $\sim_k$ using $\approx_k$, by

$$i, j \in \tilde{V}^{(n)} \qquad i \sim_k j \overset{\text{def}}{\iff} \sum_{t \in K_l^{(k:n-1)}} \tilde{m}_{t,i}^{(n)} = \sum_{t \in K_l^{(k:n-1)}} \tilde{m}_{t,j}^{(n)} ,$$

for all $1 \leq l \leq \#\{\tilde{V}^{(n-1)}/\approx_k\}$. Let $\tilde{V}_i^{(k:n)}$ be the set of the equivalence classes of $\tilde{V}^{(n)}$ for $\sim_k$, $[\tilde{V}_i^{(k:n)}]$ be the equivalence class of $v \in \tilde{V}_i^{(k:n)}$.

REMARKS. (1) If we take $n = k$, we see that $K_i^{(k:k)} = \{i\} \subset \tilde{V}^{(k)}$ and $\#\{\tilde{V}^{(k)}/\approx_k\} = \#\tilde{V}^{(k)}$.

(2) If we take $n = k+1$, we can easily verify by (1) that the equivalence relation $\sim_k$ on $\tilde{V}^{(k+1)}$ corresponds to $\sim$ on $\tilde{V}^{(k+1)}$. So $\tilde{V}_i^{(k:k+1)} = \tilde{V}_i^{(k+1)}$ holds for any $1 \leq i \leq \#\{\tilde{V}^{(k+1)}/\sim_k\} = \#\{\tilde{V}^{(k+1)}/\sim\}$.

(3) If we take $n > k+1$, it is easy to check that for $i, j \in \tilde{V}^{(n)}$, $i \sim j$ implies $i \sim_k j$. Therefore we can define the map $\xi_{k:n} : \{1, 2, \cdots, \#\{\tilde{V}^{(n)}/\sim\}\} \to \{1, 2, \cdots, \#\{\tilde{V}^{(n)}/\sim_k\}\}$ by

$$\xi_{k:n}(i) = j \underset{\text{def}}{\iff} \tilde{V}_i^{(n)} \subset \tilde{V}_j^{(k:n)} . \tag{3.11}$$

(4) The relation between $\approx_k$ and $\sim_k$ is that for $i, j \in \tilde{V}^{(n)}$, $i \approx_k j$ implies $i \sim_k j$. Therefore we can define the map $\kappa_{k:n} : \{1, 2, \cdots, \#\{\tilde{V}^{(n)}/\approx_k\}\} \to \{1, 2, \cdots, \#\{\tilde{V}^{(n)}/\sim_k\}\}$ by

$$\kappa_{k:n}(i) = j \underset{\text{def}}{\iff} K_i^{(k:n)} \subset \tilde{V}_j^{(k:n)} . \tag{3.12}$$

(5) For any $k, l, n \in \mathbf{N}$ with $k < l < n$, we see that

$$i, j \in V^{(n)} , \qquad i \approx_l j \text{ implies } i \approx_k j . \tag{3.13}$$

In fact for i and j, there exist $q \in \mathbf{N}$ and $\{v_t \in \tilde{V}^{(l)}\}_{t=1}^{q}$ such that

$$\pi_l(Con(i)) = \pi_l(Con(j)) = Con(v_1)Con(v_2)\cdots Con(v_q)\,.$$

In addition, for any t, there exist $s_t \in \mathbf{N}$ and $\{u_{t,t'} \in \tilde{V}^{(k)}\}_{t'=1}^{s_t}$ such that

$$\pi_k(Con(v_t)) = Con(u_{t,1})Con(u_{t,2})\cdots Con(u_{t,s_t})\,.$$

So we have

$$\begin{aligned}\pi_k(Con(i)) &= Con(u_{1,1})\cdots Con(u_{1,s_1})Con(u_{2,1})\cdots Con(u_{2,s_2})\cdots Con(u_{q,1})\cdots Con(u_{q,s_q})\\ &= \pi_k(Con(j))\,.\end{aligned}$$

Hence we get (3.13).

(6) Moreover for any $k, l, n \in \mathbf{N}$ with $k < l < n$, we see that

$$i, j \in \tilde{V}^{(n)}, \qquad i \sim_l j \quad \text{implies} \quad i \sim_k j\,. \tag{3.14}$$

In fact, by (3.13) we define $\psi : \{1, 2, \cdots, \#\{\tilde{V}^{(n-1)}/\approx_l\}\} \to \{1, 2, \cdots, \#\{\tilde{V}^{(n-1)}/\approx_k\}\}$ by

$$\psi(p') = p \underset{\text{def}}{\Longleftrightarrow} K_{p'}^{(l:n-1)} \subset K_p^{(k:n-1)}\,.$$

So we have

$$\begin{aligned}&\sum_{t \in K_{p'}^{(l:n-1)}} \tilde{m}_{t,i}^{(n)} = \sum_{t \in K_{p'}^{(l:n-1)}} \tilde{m}_{t,j}^{(n)} \qquad \text{for all} \quad 1 \le p' \le \#\{\tilde{V}^{(n-1)}/\approx_l\}\,,\\ \Longrightarrow\ &\sum_{p' \in \psi^{-1}(p)} \sum_{t \in K_{p'}^{(l:n-1)}} \tilde{m}_{t,i}^{(n)} = \sum_{p' \in \psi^{-1}(p)} \sum_{t \in K_{p'}^{(l:n-1)}} \tilde{m}_{t,j}^{(n)} \qquad \text{for all} \quad 1 \le p \le \#\{\tilde{V}^{(n-1)}/\approx_k\}\,,\\ \Longleftrightarrow\ &\sum_{t \in K_p^{(k:n-1)}} \tilde{m}_{t,j}^{(n)} = \sum_{t \in K_p^{(k:n-1)}} \tilde{m}_{t,j}^{(n)} \qquad \text{for all} \quad 1 \le p \le \#\{\tilde{V}^{(n-1)}/\approx_k\}\,.\end{aligned}$$

So we get (3.14).

(7) For any $i, j \in \tilde{V}^{(n)}$,

$$i \sim_k j \quad \text{implies} \quad \#\mathscr{P}(i) = \#\mathscr{P}(j)\,. \tag{3.15}$$

Because if $n = k+1$, it is clear by remark (2) that (3.15) holds. If some $n \ge k+1$, we assume (3.15) holds. Then we may define $\#\mathscr{P}[\tilde{V}_j^{(k:n)}]$ by

$$\#\mathscr{P}[\tilde{V}_j^{(k:n)}] \equiv \#\mathscr{P}(v) \qquad \text{where } v \in \tilde{V}_j^{(k:n)}\,. \tag{3.16}$$

For any $i, j \in \tilde{V}^{(n+1)}$ with $i \sim_k j$,

$$\begin{aligned}\#\mathscr{P}(i) = \sum_{t \in \tilde{V}^{(n)}} \#\mathscr{P}(t)\tilde{m}_{t,i}^{(n+1)} &= \sum_{l=1}^{\#\{\tilde{V}^{(n)}/\approx_k\}} \sum_{t \in K_l^{(k:n)}} \#\mathscr{P}[\tilde{V}_{\kappa_{k:n}(l)}^{(k:n)}]\tilde{m}_{t,i}^{(n+1)}\\ &= \sum_{l=1}^{\#\{\tilde{V}^{(n)}/\approx_k\}} \sum_{t \in K_l^{(k:n)}} \#\mathscr{P}[\tilde{V}_{\kappa_{k:n}(l)}^{(k:n)}]\tilde{m}_{t,j}^{(n+1)} = \#\mathscr{P}(j)\,.\end{aligned}$$

By induction, (3.15) holds for any $n \ge k+1$. Therefore we can define $\#\mathscr{P}[\tilde{V}_j^{(k:n)}]$ as (3.16).

Now we define $W_j^{(k:n)}$ by

$$W_j^{(k:n)} \equiv \{\pi_k(Con(v)) \mid v \in \tilde{V}_j^{(k:n)}\}\,.$$

Next we calculate the topological entropy of $(Y_k(t), S_k(t))$.

LEMMA 3.9. *Suppose* $(Y_k(t), S_k(t))$ *is the subshift associated with* $\tilde{\mathscr{D}}=(\tilde{V}, \tilde{E}, \tilde{\lesssim})$ *satisfying Assumption* 2.4, 2.5, 2.6, 2.7 *and* 2.8. *And suppose* $\alpha_{k,k+t}$ *is the postive solution of the following equality for* x:

$$\sum_{j=1}^{\#\{\tilde{V}^{(k:k+t)}/\sim_k\}} \frac{\# W_j^{(k:k+t)}}{x^{\#\mathscr{P}[\tilde{V}_j^{(k:k+t)}]}}=1 .$$

Then

$$h(S_k(t))=\log\alpha_{k,k+t} .$$

PROOF. First, we shall show the following claims:

CLAIMS. (i) *For any* $k\in\mathbf{N}$, $\mathbf{W}_k(0)$ *and* $\mathbf{W}_k(1)$ *satisfy Condition* 3.2,
(ii) *for any* $k, t\in\mathbf{N}$,

$$\begin{aligned} &\{v\in\tilde{V}^{(k+t)} \mid \pi_k(Con(v))=\pi_k(Con(v_{\min}^{(k+t)}))\}=\{v_{\min}^{(k+t)}\} \\ &\{v\in\tilde{V}^{(k+t)} \mid \pi_k(Con(v))=\pi_k(Con(v_{\max}^{(k+t)}))\}=\{v_{\max}^{(k+t)}\} , \end{aligned} \tag{3.17}$$

(iii) *for any* $k\in\mathbf{N}$ *and* $t\geq 2$, $\mathbf{W}_k(t)$ *satisfies Condition* 3.2.

THE PROOF OF (i). If we take any two words w, w' in $\mathbf{W}_k(0)$, then there is no common alphabet $(=\mathscr{P}(\tilde{V}^{(k)}))$ between w and w'. So it is clear that $\mathbf{W}_k(0)$ satisfies Condition 3.2. From this reason, we can identify the word $Con(v)$ in $\mathbf{W}_k(0)$ with $v\in\tilde{V}^{(k)}$.

Take any l with $1\leq l\leq\eta_{k+1}$, where η_{k+1} is positive integer satisfying Assumption 2.8 (1) and fix it. And define $\tilde{V}_{*,l}^{(k+1)}\equiv\bigcup_{i=1}^{\#\{\tilde{V}^{(k+1)}/\sim\}}\tilde{V}_{i,l}^{(k+1)}$. We can easily see from Assumption 2.8 (2) that $\tilde{V}_{*,l}^{(k+1)}\cup\{v_{\min}^{(k+1)}, v_{\max}^{(k+1)}\}$ has also partial distinct orderings. And since $\mathbf{W}_k(1)$ consists of the concatenated words made up from $\mathbf{W}_k(0)$, for any $v, v'\in\tilde{V}_{*,l}^{(k+1)}\cup\{v_{\min}^{(k+1)}, v_{\max}^{(k+1)}\}$ with $v\neq v'$, $\pi_k(Con(v))\neq\pi_k(Con(v'))$ holds. Moreover we see

$$\mathbf{W}_k(1)=\{\pi_k(Con(v^{(k+1)})) \mid v^{(k+1)}\in\tilde{V}_{*,l}^{(k+1)}\cup\{v_{\min}^{(k+1)}, v_{\max}^{(k+1)}\}\}$$

holds for all l. Now, we also identify the word $\pi_k(Con(v))\in W_k(1)$ with the ordered vector $O.vec(v)$ $(v\in\tilde{V}_{*,l}^{(k+1)}\cup\{v_{\min}^{(k+1)}, v_{\max}^{(k+1)}\})$. From Assumption 2.5, we see that the first vertex of all words in $\mathbf{W}_k(1)$ is $v_{\min}^{(k)}$ and the last vertex of them is $v_{\max}^{(k)}$. Assumption 2.8 (4) and (6) imply that there is no word in $\mathbf{W}_k(1)$ having the sub-word "$v_{\max}^{(k)}v_{\min}^{(k)}$" except the words $\pi_k(Con(v_{j,l}^{*(k+1)}))$ for all $1\leq j\leq\#\{\tilde{V}^{(k+1)}/\sim\}$. Moreover Assumption 2.8 (6) implies that all $\pi_k(Con(v_{j,l}^{*(k+1)}))$'s always have an alphabet (vertex) "$v_{\min}^{(k)}$" just before the sub-word "$v_{\max}^{(k)}v_{\min}^{(k)}$". We note that any concatenated word made up from $\mathbf{W}_k(1)$ always has the sub-word "$v_{\max}^{(k)}v_{\min}^{(k)}$" at the points of concatenation among words in $\mathbf{W}_k(1)$. And from Assumption 2.8 (5), we see that at these points of concatenation, there is no alphabet (vertex) "$v_{\min}^{(k)}$" just before the sub-word "$v_{\max}^{(k)}v_{\min}^{(k)}$". Therefore $\mathbf{W}_k(1)$ satisfies Condition 3.2. We finish the proof.

THE PROOF OF (ii) AND (iii). We shall do them by the induction with respect to

$t\in\mathbf{Z}^+$. For all $k\in\mathbf{N}$, Assumption 2.7 (2) is equivalent to

$$\{v\in\tilde{V}^{(k+1)}\mid \pi_k(Con(v))=\pi_k(Con(v_{\min}^{(k+1)}))\}=\{v_{\min}^{(k+1)}\}\,,$$
$$\{v\in\tilde{V}^{(k+1)}\mid \pi_k(Con(v))=\pi_k(Con(v_{\max}^{(k+1)}))\}=\{v_{\max}^{(k+1)}\}\,.$$

So (ii) holds for $t=1$. And from (i), we see $\mathbf{W}_k(t)$ satisfies Condition 3.2 for $t=0$, 1. So (iii) holds for $t=0$, 1.

Suppose for all $k\in\mathbf{N}$, (ii) holds for some $t\geq 1$ and $\mathbf{W}_k(t)$ satisfies Condition 3.2. First we shall show (ii) holds in the case of $t+1$.

Take any $v\in\tilde{V}^{(k+t+1)}$ with $\pi_k(Con(v))=\pi_k(Con(v_{\min}^{(k+t+1)}))$ and fix it. For v, there exist unique $\{v_i^{(k+t)}\}_{i=1}^{s}\subset\tilde{V}^{(k+1)}$ ($s\equiv\sharp r^{-1}(v)$) such that

(1) $O.vec(v)=(v_1^{(k+t)}, v_2^{(k+t)}, \cdots, v_s^{(k+t)})$,

(2) $\pi_k(Con(v))=\pi_k(Con(v_1^{(k+t)}))\pi_k(Con(v_2^{(k+t)}))\cdots\pi_k(Con(v_s^{(k+t)}))$.

Similarly, for $v_{\min}^{(k+t+1)}$ there exist unique $\{u_i^{(k+t)}\}_{i=1}^{s'}\subset\tilde{V}^{(k+t)}$ ($s'\equiv\sharp r^{-1}(v_{\min}^{(k+t+1)})$) such that

(3) $O.vec(v_{\min}^{(k+t+1)})=(u_1^{(k+t)}, u_2^{(k+t)}, \cdots, u_{s'}^{(k+t)})$,

(4) $\pi_k(Con(v_{\min}^{(k+t+1)}))=\pi_k(Con(u_1^{(k+t)}))\pi_k(Con(u_2^{(k+t)}))\cdots\pi_k(Con(u_{s'}^{(k+t)}))$.

Since $\mathbf{W}_k(t)$ satisfies Condition 3.2, for any concatenated word in $\mathbf{W}_k(t)$, we can distinguish the points of concatenation. That is to say, suppose w is any concatenated word made up from words in $\mathbf{W}_k(t)$, there exist unique $w_1, w_2, \cdots, w_p\in\mathbf{W}_k(t)$ such that $w=w_1w_2\cdots w_p$. Therefore from (2) and (4), we have $s=s'$ and for each $1\leq i\leq s$,

$$\pi_k(Con(v_i^{(k+t)}))=\pi_k(Con(u_i^{(k+t)}))\,.$$

Especially by the hypothesis of (ii), for any $g\in\{i\mid u_i^{(k+t)}=v_{\min}^{(k+t)}\}$ and $h\in\{i\mid u_i^{(k+t)}=v_{\max}^{(k+t)}\}$,

$$\begin{aligned}\pi_k(Con(v_g^{(k+t)}))=\pi_k(Con(u_g^{(k+t)})) &\iff v_g^{(k+t)}=v_{\min}^{(k+t)}\,,\\ \pi_k(Con(v_h^{(k+t)}))=\pi_k(Con(u_h^{(k+t)})) &\iff v_h^{(k+t)}=v_{\max}^{(k+t)}\,.\end{aligned}\tag{3.18}$$

We note that Assumption 2.7 (2) and (3.18) imply $v=v_{\min}^{(k+t+1)}$. Similarly we can show the case of $v_{\max}^{(k+t+1)}$. Therefore for $t+1$, (ii) holds.

Next, we shall show a contradiction if $\mathbf{W}_k(t+1)$ does not satisfy Condition 3.2. Suppose $w\in\mathbf{W}_k(t+1)$ has concatenating points in $\mathbf{W}_k(t+1)$. We write $w=w_1w_2\cdots w_n$ ($w_i\in\mathscr{P}(\tilde{V}^{(k)})$ $1\leq i\leq n$). Then there exist more than two words $\{w^j=w_1^jw_2^j\cdots w_{n_j}^j\in\mathbf{W}_k(t+1)\mid w_i^j\in\mathscr{P}(\tilde{V}^{(k)})$ for $1\leq i\leq n_j\}_{j=1}^{d}$ $(d\geq 2)$ and integers $0=p_0<p_1<\cdots<p_{d-1}<p_d=n$ such that

$$w_{p_{i-1}+1}w_{p_{i-1}+2}\cdots w_{p_i}=\begin{cases} w_{n_1-p_1+1}^1w_{n_1-p_1+2}^1\cdots w_{n_1}^1 & \text{if } i=1\\ w^i & \text{if } 1<i<d\\ w_1^dw_2^d\cdots w_{n-p_{d-1}}^d & \text{if } i=d\,.\end{cases}$$

For w, there exists $v\in\tilde{V}^{(k+t+1)}$ and for v, there exist unique $\{v_i^{(k+t)}\}_{i=1}^{s}\subset\tilde{V}^{(k+t)}$ such that

(5) $O.vec(v)=(v_1^{(k+t)}, v_2^{(k+t)}, \cdots, v_s^{(k+t)})$,

(6) $w=\pi_k(Con(v))=\pi_k(Con(v_1^{(k+t)}))\pi_k(Con(v_2^{(k+t)}))\cdots\pi_k(Con(v_s^{(k+t)}))$.

Similarly, for w^j, there exists $v^j\in\tilde{V}^{(k+t+1)}$ and for v^j, there exist unique $\{v_i^{j(k+t)}\}_{i=1}^{s_j}\subset\tilde{V}^{(k+t)}$ such that

(7) $O.vec(v^j)=(v_1^{j(k+t)}, v_2^{j(k+t)}, \cdots, v_{s_j}^{j(k+t)})$,

(8) $w^j=\pi_k(Con(v^j))=\pi_k(Con(v_1^{j(k+t)}))\pi_k(Con(V_2^{j(k+t)}))\cdots\pi_k(Con(v_{s_j}^{j(k+t)}))$.

As any element of $\mathbf{W}_k(t+1)$ is concatenated words made up from $\mathbf{W}_k(t)$ and $\mathbf{W}_k(t)$ satisfies Condition 3.2, the above condition (6) and (8) imply that there are unique integers $0=m_1<m_2<\cdots<m_d<s$ such that

$$\pi_k((Con(v_{m_j+i}^{(k+t)}))=\begin{cases}\pi_k((Con(v_{s_1-m_2+i}^{1(k+t)})) & \text{if } j=1 \text{ and } 1\le i\le m_2\\ \pi_k((Con(v_i^{j(k+t)})) & \text{if } 1<j<d \text{ and } 1\le i\le m_{j+1}-m_j\ (=s_j)\\ \pi_k((Con(v_i^{d(k+t)})) & \text{if } j=d \text{ and } 1\le i\le s-m_d\,.\end{cases} \tag{3.19}$$

Since we know

$$\{u\in\tilde{V}^{(k+t)}\mid\pi_k(Con(u))=\pi_k(Con(v_{\min}^{(k+t)}))\}=\{v_{\min}^{(k+t)}\}\,,$$

$$\{u\in\tilde{V}^{(k+t)}\mid\pi_k(Con(u))=\pi_k(Con(v_{\max}^{(k+t)}))\}=\{v_{\max}^{(k+t)}\}\,,$$

immediately we see (3.19) is equivalent to the following:

$$v_{m_j}^{(k+t)}=v_{s_j}^{j(k+t)}=v_{\max}^{(k+t)}\,,\quad v_{m_j+1}^{(k+t)}=v_1^{j+1(k+t)}=v_{\min}^{(k+t)}\qquad\text{for}\quad 2\le j\le d\,. \tag{3.20}$$

From Assumption 2.8 (5), $v_{s_j-1}^{j(k+t)}\ne v_{\min}^{(k+t)}$. This implies that

$$v_{m_j-1}^{(k+t)}\ne v_{\min}^{(k+t)}\qquad\text{for}\quad 2\le j\le d\,. \tag{3.21}$$

So (3.20) and (3.21) imply that $O.vec(v)$ has at least $d-1$ (≥ 1) sub-words "$v_{\max}^{(k+t)}v_{\min}^{(k+t)}$" which have no word "$v_{\min}^{(k+t)}$" just before. By the way, from Assumption 2.8 (6) all $v_{i,l}^*$'s (in $\tilde{V}^{(k+t+1)}$) have the sub-word "$v_{\max}^{(k+t)}v_{\min}^{(k+t)}$", but from Assumption 2.8 (4) they always have "$v_{\min}^{(k+t)}$" just before the sub-word "$v_{\max}^{(k+t)}v_{\min}^{(k+t)}$". So v does not coincide with any element in $\{v_{i,l}^*\mid\forall i, \forall l\}$. This is a contradiction. Therefore $\mathbf{W}_k(t+1)$ has no word which has concatenating points in $\mathbf{W}_k(t+1)$. By induction, we finish the proof of (iii).

Finally we calculate the topological entropy of $(Y_k(t), S_k(t))$. The lengths of all words in $W_j^{(k:k+t)}$ are all $\#\mathscr{P}[\tilde{V}_j^{(k:k+t)}]$ and since we decompose $\tilde{V}^{(k+t)}=\bigcup_{j=1}^{\#\{\tilde{V}^{(k+t)}/\sim_k\}}\tilde{V}_j^{(k:k+t)}$ as a disjoint union, we can also decompose $\mathbf{W}_k(t)=\bigcup_{j=1}^{\#\{\tilde{V}^{(k+t)}/\sim_k\}}W_j^{(k:k+t)}$ as a disjoint union. Therefore if we take $\mathbf{W}_k(t)$ in Lemma 3.4, we have $h(S_k(t))=\log\alpha_{k,k+t}$. So we finish the proof of Lemma 3.9. □

4. The modification of dimension group preserving order isomorphism.

In this section, we construct the sequence of incidence matrices, which we can get by modifying the given one so as to preserve order isomorphism.

Let $\mathscr{D}'=(V', E')$ be the unordered Bratteli diagram, $\{M'^{(k)}=(m_{i,j}'^{(k)})\}_{k\ge 1}$ be the sequence of incidence matrices of $\mathscr{D}'$ and $\{\{\#\tilde{V}_i^{(k-1)}\}_{i=1}^{\#V'^{(k-1)}}\}_{k\geq 2}$ be the sequence of subsets of $\mathbf{N}$ satisfying $m_{i,j}'^{(k)} \geq \#\tilde{V}_i^{(k-1)}$ for all $k\geq 2$, $1\leq i\leq \#V'^{(k-1)}$, $1\leq j\leq \#V'^{(k)}$. Define $\#\tilde{V}^{(k)} \equiv \sum_{l=1}^{\#V'^{(k)}} \#\tilde{V}_l^{(k)}$. Now we define the sequence of incidence matrices $\tilde{\mathscr{B}}=\{\tilde{M}^{(k)}=(\tilde{m}_{i,j}^{(k)}) \in \mathbf{N}^{\#\tilde{V}^{(k-1)} \times \#\tilde{V}^{(k)}}\}_{k\geq 1}$ of $\tilde{\mathscr{D}}=(\tilde{V}, \tilde{E})$ by

$$\tilde{M}^{(k)} \equiv (\underbrace{\tilde{M}_{[1]}^{(k)}, \cdots, \tilde{M}_{[1]}^{(k)}}_{\#\tilde{V}_1^{(k)} \text{ times}}, \underbrace{\tilde{M}_{[2]}^{(k)}, \cdots, \tilde{M}_{[2]}^{(k)}}_{\#\tilde{V}_2^{(k)} \text{ times}}, \cdots, \underbrace{\tilde{M}_{[\#V'^{(k)}]}^{(k)}, \cdots, \tilde{M}_{[\#V'^{(k)}]}^{(k)}}_{\#\tilde{V}_{\#V'^{(k)}}^{(k)} \text{ times}}),$$

$$\tilde{M}_{[j]}^{(k)} \equiv (\tilde{M}_{1,j}^{(k)}, \tilde{M}_{2,j}^{(k)}, \cdots, \tilde{M}_{\#V'^{(k-1)},j}^{(k)})^t \in \mathbf{N}^{\#\tilde{V}^{(k-1)}},$$

where $\{\{\tilde{M}_{i,j}^{(k)} \in \mathbf{N}^{\#\tilde{V}_i^{(k-1)}}\}_{i=1}^{\#V'^{(k-1)}}\}_{j=1}^{\#V'^{(k)}}$ satisfies the following condition:

$$\tilde{M}_{i,j}^{(k)} \in \left\{ (n_1, n_2, \cdots, n_{\#\tilde{V}_i^{(k-1)}}) \in \mathbf{N}^{\#\tilde{V}_i^{(k-1)}} \;\middle|\; \sum_{l=1}^{\#\tilde{V}_i^{(k-1)}} n_l = m_{i,j}'^{(k)} \right\}.$$

Similarly, we define the dimension group $K_0(\tilde{V}, \tilde{E})$ by

$$K_0(\tilde{V}, \tilde{E}) \equiv \varinjlim (\mathbf{Z}^{\#\tilde{V}^{(k-1)}}, \tilde{\Phi}_k) = Z^{\#\tilde{V}^{(0)}} \xrightarrow{\tilde{\Phi}_1} Z^{\#\tilde{V}^{(1)}} \xrightarrow{\tilde{\Phi}_2} \mathbf{Z}^{\#\tilde{V}^{(2)}} \xrightarrow{\tilde{\Phi}_3} \cdots$$

and a distinguished order unit $\tilde{u} \equiv [\tilde{u}_0, 0]$, where $\tilde{\Phi}_k(\mathbf{x}) \equiv \mathbf{x}\tilde{M}^{(k)}$, $\mathbf{x} \in \mathbf{Z}^{\#\tilde{V}^{(k-1)}}$ and $\tilde{u}_0 = 1 \in \mathbf{Z}^{\#\tilde{V}^{(0)}}$ $(=\mathbf{Z})$.

EXAMPLE. If $M'^{(k)} = \begin{bmatrix} 10 & 10 & 20 & 30 \\ 20 & 20 & 35 & 45 \\ 30 & 30 & 40 & 50 \end{bmatrix}$ and $\#\tilde{V}_1^{(k-1)}=2$, $\#\tilde{V}_2^{(k-1)}=3$, $\#\tilde{V}_3^{(k-1)}=4$, $\#\tilde{V}_1^{(k)}=3$, $\#\tilde{V}_2^{(k)}=2$, $\#\tilde{V}_3^{(k)}=3$, $\#\tilde{V}_4^{(k)}=4$, then, for example, we can define $\{\tilde{M}_{i,j}^{(k)}\}$ by

$\tilde{M}_{1,1}^{(k)}=[4, 6]^t$ $\tilde{M}_{1,2}^{(k)}=[7, 3]^t$ $\tilde{M}_{1,3}^{(k)}=[9, 11]^t$ $\tilde{M}_{1,4}^{(k)}=[17, 13]^t$
$\tilde{M}_{2,1}^{(k)}=[6, 7, 7]^t$ $\tilde{M}_{2,2}^{(k)}=[7, 2, 11]^t$ $\tilde{M}_{2,3}^{(k)}=[30, 2, 3]^t$ $\tilde{M}_{2,4}^{(k)}=[20, 12, 13]^t$
$\tilde{M}_{3,1}^{(k)}=[9, 5, 10, 6]^t$ $\tilde{M}_{3,2}^{(k)}=[3, 2, 23, 2]^t$ $\tilde{M}_{3,3}^{(k)}=[14, 4, 6, 16]^t$ $\tilde{M}_{3,4}^{(k)}=[13, 10, 20, 7]^t$.

Therefore

$$\tilde{M}^{(k)} = \begin{bmatrix} 4 & 4 & 4 & 7 & 7 & 9 & 9 & 9 & 17 & 17 & 17 & 17 \\ 6 & 6 & 6 & 3 & 3 & 11 & 11 & 11 & 13 & 13 & 13 & 13 \\ 6 & 6 & 6 & 7 & 7 & 30 & 30 & 30 & 20 & 20 & 20 & 20 \\ 7 & 7 & 7 & 2 & 2 & 2 & 2 & 2 & 12 & 12 & 12 & 12 \\ 7 & 7 & 7 & 11 & 11 & 3 & 3 & 3 & 13 & 13 & 13 & 13 \\ 9 & 9 & 9 & 3 & 3 & 14 & 14 & 14 & 13 & 13 & 13 & 13 \\ 5 & 5 & 5 & 2 & 2 & 4 & 4 & 4 & 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 23 & 23 & 6 & 6 & 6 & 20 & 20 & 20 & 20 \\ 6 & 6 & 6 & 2 & 2 & 16 & 16 & 16 & 7 & 7 & 7 & 7 \end{bmatrix}.$$

LEMMA 4.1 ([Su]: Lemma 5.4). *Let $\mathscr{D}'$ be an unordered Bratteli diagram and $\tilde{\mathscr{D}}$ be the unordered Bratteli diagram constructed above. Then*

(1) *$K_0(V', E') \cong K_0(\tilde{V}, \tilde{E})$ as dimension groups via an isomorphism preserving distinguished order units,*

(2) *for each $k\geq 1$ and $i \in V'^{(k)}$, $\#\mathscr{P}[\tilde{V}_i^{(k)}] = \#\mathscr{P}(i)$.*

PROOF. We construct sequence of maps $\{\varphi_k\}_{k=0}^{\infty}$ where for $k \geq 0$, $\varphi_k\colon \mathbf{Z}^{\# V'(k)} \to \mathbf{Z}^{\# \tilde{V}(k)}$ will be order isomorphisms. For $\mathbf{x}=(x_1, x_2, \cdots, x_{\# V'(k)}) \in \mathbf{Z}^{\# V'(k)}$, define φ_k as follows:

$$\varphi_k(\mathbf{x}) \equiv (\underbrace{x_1, \cdots, x_1}_{\# \tilde{V}_1^{(k)} \text{ times}}, \underbrace{x_2, \cdots, x_2}_{\# \tilde{V}_2^{(k)} \text{ times}}, \cdots, \underbrace{x_{\# V'(k)}, \cdots, x_{\# V'(k)}}_{\# \tilde{V}_{\# V'(k)}^{(k)} \text{ times}}) \in \mathbf{Z}^{\# \tilde{V}(k)}$$

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{\Phi'_k} & \mathbf{Z}^{\# V'(k)} & \xrightarrow{\Phi'_{k+1}} & \mathbf{Z}^{\# V'(k+1)} & \xrightarrow{\Phi'_{k+2}} & \cdots \\
 & & \Big\downarrow \varphi_k & & \Big\downarrow \varphi_{k+1} & & \\
\cdots & \xrightarrow{\tilde{\Phi}_k} & \mathbf{Z}^{\# \tilde{V}(k)} & \xrightarrow{\tilde{\Phi}_{k+1}} & \mathbf{Z}^{\# \tilde{V}(k+1)} & \xrightarrow{\tilde{\Phi}_{k+2}} & \cdots
\end{array}$$

Now we calculate

$$\begin{aligned}
\varphi_{k+1} \circ \Phi'_{k+1}(\mathbf{x}) &= \varphi_{k+1}(x'_1, x'_2, \cdots, x'_{\# V'(k+1)}) \\
&= (\underbrace{x'_1, \cdots, x'_1}_{\# \tilde{V}_1^{(k+1)} \text{ times}}, \underbrace{x'_2, \cdots, x'_2}_{\# \tilde{V}_2^{(k+1)} \text{ times}}, \cdots, \underbrace{x'_{\# V'(k+1)}, \cdots, x'_{\# V'(k+1)}}_{\# \tilde{V}_{\# V'(k+1)}^{(k+1)} \text{ times}}),
\end{aligned}$$

where $x'_j = \sum_{i=1}^{\# V'(k)} x_i m'^{(k+1)}_{i,j}$.

$$\begin{aligned}
\tilde{\Phi}_{k+1} \circ \varphi_k(\mathbf{x}) &= \tilde{\Phi}_{k+1}(\underbrace{x_1, \cdots, x_1}_{\# \tilde{V}_1^{(k)} \text{ times}}, \underbrace{x_2, \cdots, x_2}_{\# \tilde{V}_2^{(k)} \text{ times}}, \cdots, \underbrace{x_{\# V'(k)}, \cdots, x_{\# V'(k)}}_{\# \tilde{V}_{\# V'(k)}^{(k)} \text{ times}}) \\
&= (\underbrace{\tilde{x}_1, \cdots, \tilde{x}_1}_{\# \tilde{V}_1^{(k+1)} \text{ times}}, \underbrace{\tilde{x}_2, \cdots, \tilde{x}_2}_{\# \tilde{V}_2^{(k+1)} \text{ times}}, \cdots, \underbrace{\tilde{x}_{\# V'(k+1)}, \cdots, \tilde{x}_{\# V'(k+1)}}_{\# \tilde{V}_{\# V'(k+1)}^{(k+1)} \text{ times}})
\end{aligned}$$

where $\tilde{x}_j$ is calculated by the following:

$$\tilde{x}_j = \sum_{i=1}^{\# V'(k)} \{(\underbrace{x_i, \cdots, x_i}_{\# \tilde{V}_i^{(k)} \text{ times}}) \tilde{M}_{i,j}^{(k+1)}\} = \sum_{i=1}^{\# V'(k)} x_i m'^{(k+1)}_{i,j} = x'_j .$$

Therefore for each $k \geq 0$,

$$\varphi_{k+1} \circ \Phi'_{k+1} = \tilde{\Phi}_{k+1} \circ \varphi_k . \tag{4.1}$$

Next we calculate for $(x_1, x_2, \cdots, x_{\# \tilde{V}(k)}) \in \mathbf{Z}^{\# \tilde{V}(k)}$,

$$\begin{aligned}
\tilde{\Phi}_{k+1}(x_1, x_2, \cdots, x_{\# \tilde{V}(k)}) &= (\underbrace{\hat{x}_1, \cdots, \hat{x}_1}_{\# \tilde{V}_1^{(k+1)} \text{ times}}, \underbrace{\hat{x}_2, \cdots, \hat{x}_2}_{\# \tilde{V}_2^{(k+1)} \text{ times}}, \cdots, \underbrace{\hat{x}_{\# V(k+1)}, \cdots, \hat{x}_{\# V(k+1)}}_{\# \tilde{V}_{\# V'(k+1)}^{(k+1)} \text{ times}}) \\
&= \varphi_{k+1}(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_{\# V(k+1)}),
\end{aligned}$$

where $\hat{x}_j=(x_1, x_2, \cdots, x_{\#\tilde{V}^{(k)}})\tilde{M}^{(k+1)}_{[j]}$. Thus $\mathrm{Im}\,\tilde{\Phi}_{k+1}\subset \mathrm{Im}\,\varphi_{k+1}$ holds. It is easy to see that $\varphi_k(\mathbf{Z}^{\#V^{(k)}+})\subset \mathbf{Z}^{\#\tilde{V}^{(k)}+}$ and φ_k preserves order units. Therefore $\{\varphi_n\}_{n\in\mathbf{N}}$ gives the order isomorphism between $K_0(V', E')$ and $K_0(\tilde{V}, \tilde{E})$.

Next we prove the condition (2). For each $k\geq 1$, from the construction of $\tilde{M}^{(k)}$, it is clear that for each $1\leq \zeta \leq \# V'^{(k)}$, and for all j, j' with $\sum_{l=0}^{\zeta-1}\#\tilde{V}^{(k)}_l<j, j'\leq \sum_{l=0}^{\zeta}\#\tilde{V}^{(k)}_l$ $(\#\tilde{V}^{(k)}_0=0)$,

$$\tilde{M}^{(k)}_j=\tilde{M}^{(k)}_{j'}=\tilde{M}^{(k)}_{[\zeta]}\,. \tag{4.2}$$

Here we write $\tilde{v}^{(k)}_j\in \tilde{V}^{(k)}$ instead of $j\in \tilde{V}^{(k)}$. So (4.2) is equivalent to

$$\tilde{v}^{(k)}_j\in \tilde{V}^{(k)}_\zeta \quad \text{and} \quad \tilde{v}^{(k)}_{j'}\in \tilde{V}^{(k)}_\zeta\,. \tag{4.3}$$

Then from (3.6) and (4.3),

$$\begin{aligned}\tilde{\mathbf{P}}_k&=(\#\mathscr{P}(\tilde{v}^{(k)}_1), \#\mathscr{P}(\tilde{v}^{(k)}_2), \cdots, \#\mathscr{P}(\tilde{v}^{(k)}_{\#\tilde{V}^{(k)}}))=\tilde{M}^{(1)}\tilde{M}^{(2)}\cdots\tilde{M}^{(k-1)}\tilde{M}^{(k)}\\&=(\underbrace{\#\mathscr{P}[\tilde{V}^{(k)}_1], \cdots, \#\mathscr{P}[\tilde{V}^{(k)}_1]}_{\#\tilde{V}^{(k)}_1\text{ times}}, \underbrace{\#\mathscr{P}[\tilde{V}^{(k)}_2], \cdots, \#\mathscr{P}[\tilde{V}^{(k)}_2]}_{\#\tilde{V}^{(k)}_2\text{ times}}, \cdots,\\&\qquad\qquad \underbrace{\#\mathscr{P}[\tilde{V}^{(k)}_{\#V'^{(k)}}], \cdots, \#\mathscr{P}[\tilde{V}^{(k)}_{\#V'^{(k)}}]}_{\#\tilde{V}^{(k)}_{\#V'^{(k)}}\text{ times}})\end{aligned} \tag{4.4}$$

$$=\varphi_k(\#\mathscr{P}[\tilde{V}^{(k)}_1], \#\mathscr{P}[\tilde{V}^{(k)}_2], \cdots, \#\mathscr{P}[\tilde{V}^{(k)}_{\#V'^{(k)}}])\,.$$

By (4.1), it follows that

$$\begin{aligned}\varphi_k(\mathbf{P}'_k)&=\varphi_k(M'^{(1)}M'^{(2)}\cdots M'^{(k-1)}M'^{(k)})=\varphi_k\circ\Phi'_k\circ\Phi'_{k-1}\circ\cdots\circ\Phi'_2\circ\Phi'_1\\&=\tilde{\Phi}_k\circ\tilde{\Phi}_{k-1}\circ\cdots\circ\tilde{\Phi}_2\circ\tilde{\Phi}_1\circ\varphi_0=\tilde{M}^{(1)}\tilde{M}^{(2)}\cdots\tilde{M}^{(k-1)}\tilde{M}^{(k)}=\tilde{\mathbf{P}}_k\,.\end{aligned} \tag{4.5}$$

From (4.4) and (4.5), we get the condition (2). So we are done. □

For a matrix $M^{(k)}=(m^{(k)}_{i,j})$, define $\bar{m}^{(k)}_j\equiv\sum_{i=1}^{\#V^{(k-1)}}m^{(k)}_{i,j}$ and $\bar{m}^{(k)}\equiv\max_{1\leq j\leq \#V^{(k)}}\bar{m}^{(k)}_j$. The next lemma is useful for the construction of the base diagram $\mathscr{D}'$ in the main theorem.

LEMMA 4.2 ([Su]: Lemma 5.5). *Suppose $\mathscr{D}=(V, E)$ is an unordered Bratteli diagram and $\{M^{(k)}\}_{k\geq 1}$ is the sequence of incidence matrices of $\mathscr{D}$. Then there exist $\mathscr{D}'=(V', E')$ and $\{M'^{(k)}\}_{k\geq 1}$ such that*

(1) *$K_0(V', E')\cong K_0(V, E)$ as dimension groups via an isomorphism preserving distinguished order units,*

(2) *for all $n\geq 2$, $\#\{M'^{(n)}_j \mid \bar{m}'^{(n)}_j\leq s\}<2^s$ for $\#V'^{(n-1)}\leq s\leq \bar{m}'^{(n)}$.*

PROOF. We construct $\mathscr{D}'$ by a recursive process. First, define $\#V'^{(1)}$, $M'^{(1)}$ and the map ϕ_1 by

$$\#V'^{(1)}\equiv\#V^{(1)}\,,\qquad M'^{(1)}\equiv M^{(1)}\,,$$

$$\phi_1:\{1, 2, \cdots, \#V^{(1)}\}\rightarrow(1, 2, \cdots, \#V'^{(1)}\}\qquad \phi_1(i)=i \text{ for all } 1\leq i\leq \#V^{(1)}\,.$$

Here, for $n\geq 2$, we assume $\phi_{n-1}:\{1,2,\cdots,\#V^{(n-1)}\}\rightarrow\{1,2,\cdots,\#V'^{(n-1)}\}$ are already constructed. So we shall construct $V'^{(n)}$, $M'^{(n)}$ and ϕ_n.

For $\{M_j^{(n)}\}_{j=1}^{\#V^{(n)}}$, there exists the set of vectors $\{L_i^{(n)}\}_{i\geq 1}$ such that

(1) $\{L_i^{(n)}\}_{i\geq 1}$ coincides with $\{M_j^{(n)}\}_{j=1}^{\#V^{(n)}}$ as a set,

(2) $L_i^{(n)}\neq L_j^{(n)}$ if $i\neq j$.

We write $L_j^{(n)}=(l_{1,j}^{(n)},l_{2,j}^{(n)},\cdots,l_{\#V^{(n-1)},j}^{(n)})^t$. Now we define $\#V'^{(n)}$, $M'^{(n)}$ and ϕ_n by

$$\#V'^{(n)}\equiv\#\{L_i^{(n)}\}_{i\geq 1}\,,\quad m_{i,j}'^{(n)}\equiv\sum_{s\in\phi_{n-1}^{-1}(i)}l_{s,j}^{(n)},\quad M'^{(n)}\equiv(m_{i,j}'^{(n)})\,,$$

$$\phi_n:\{1,2,\cdots,\#V^{(n)}\}\rightarrow\{1,2,\cdots,\#V'^{(n)}\}\qquad\phi_n(i)=j\underset{\text{def}}{\Longleftrightarrow}M_i^{(n)}=L_j^{(n)}\,.$$

First we shall show $K_0(V',E')\cong K_0(V,E)$. For $n\in\mathbf{N}$, define the order homomorphisms $\varphi_n:\mathbf{Z}^{\#V'^{(n)}}\rightarrow\mathbf{Z}^{\#V^{(n)}}$ by

$$\varphi_n(x_1,x_2,\cdots,x_{\#V'^{(n)}})\equiv(x_{\phi_n(1)},x_{\phi_n(2)},\cdots,x_{\phi_n(\#V^{(n)})})$$

$$\begin{array}{ccccccc}\cdots & \xrightarrow{\Phi_n'} & \mathbf{Z}^{\#V'^{(n)}} & \xrightarrow{\Phi_{n+1}'} & \mathbf{Z}^{\#V'^{(n+1)}} & \xrightarrow{\Phi_{n+2}'} & \cdots\\ & & \downarrow{\scriptstyle\varphi_n} & & \downarrow{\scriptstyle\varphi_{n+1}} & & \\ \cdots & \xrightarrow{\Phi_n} & \mathbf{Z}^{\#V^{(n)}} & \xrightarrow{\Phi_{n+1}} & \mathbf{Z}^{\#V^{(n+1)}} & \xrightarrow{\Phi_{n+2}} & \cdots\end{array}$$

Now we shall show $\Phi_{n+1}\circ\varphi_n=\varphi_{n+1}\circ\Phi_{n+1}'$, where Φ_{n+1}, Φ_{n+1}' are the order homomorphisms defined by $M^{(n+1)}$, $M'^{(n+1)}$ respectively.

$$\begin{aligned}\Phi_{n+1}\circ\varphi_n(x_1,x_2,\cdots,x_{\#V'^{(n)}})&=\Phi_{n+1}(x_{\phi_n(1)},x_{\phi_n(2)},\cdots,x_{\phi_n(\#V^{(n)})})\\&=(\tilde{x}_1,\tilde{x}_2,\cdots,\tilde{x}_{\#V^{(n+1)}})\end{aligned}$$

where

$$\tilde{x}_j=\sum_{i=1}^{\#V^{(n)}}x_{\phi_n(i)}m_{i,j}^{(n+1)}=\sum_{p=1}^{\#V'^{(n)}}\left(x_p\sum_{i\in\phi_n^{-1}(p)}m_{i,j}^{(n+1)}\right),$$

$$\begin{aligned}\varphi_{n+1}\circ\Phi_{n+1}'(x_1,x_2,\cdots,x_{\#V'^{(n)}})&=\varphi_{n+1}(x_1',x_2',\cdots,x_{\#V'^{(n+1)}}')\\&=(x_{\phi_{n+1}(1)}',x_{\phi_{n+1}(2)}',\cdots,x_{\phi_{n+1}(\#V^{(n+1)})}')\end{aligned}$$

where

$$\begin{aligned}x_{\phi_{n+1}(j)}'&=\sum_{p=1}^{\#V'^{(n)}}x_pm_{p,\phi_{n+1}(j)}'^{(n+1)}=\sum_{p=1}^{\#V'^{(n)}}\left(x_p\sum_{i\in\phi_n^{-1}(p)}l_{i,\phi_{n+1}(j)}^{(n+1)}\right)\\&=\sum_{p=1}^{\#V'^{(n)}}\left(x_p\sum_{i\in\phi_n^{-1}(p)}m_{i,j}^{(n+1)}\right).\end{aligned}$$

Therefore $\tilde{x}_j=x_{\phi_{n+1}(j)}'$ holds for all $1\leq j\leq\#V^{(n+1)}$. So we get $\Phi_{n+1}\circ\varphi_n=\varphi_{n+1}\circ\Phi_{n+1}'$.

Next we shall show $\operatorname{Im}\Phi_n\subset\operatorname{Im}\varphi_n$ for all $n\in\mathbf{N}$. For $\mathbf{x}\in\mathbf{Z}^{\#V^{(n-1)}}$, we calculate

$$\Phi_n(\mathbf{x})=(\mathbf{x}M_1^{(n)}, \mathbf{x}M_2^{(n)}, \cdots \mathbf{x}M_{\#V^{(n)}}^{(n)})=(\mathbf{x}L_{\phi_n(1)}^{(n)}, \mathbf{x}L_{\phi_n(2)}^{(n)}, \cdots \mathbf{x}L_{\phi_n(\#V^{(n)})}^{(n)}),$$

$$\varphi_n(\mathbf{x}L_1^{(n)}, \mathbf{x}L_2^{(n)}, \cdots \mathbf{x}L_{\#V'^{(n)}}^{(n)})=(\mathbf{x}L_{\phi_n(1)}^{(n)}, \mathbf{x}L_{\phi_n(2)}^{(n)}, \cdots \mathbf{x}L_{\phi_n(\#V^{(n)})}^{(n)}).$$

So we see that $\Phi_n(\mathbf{x})=\varphi_n(\mathbf{x}L_1^{(n)}, \mathbf{x}L_2^{(n)}, \cdots \mathbf{x}L_{\#V'^{(n)}}^{(n)})$. This implies $\operatorname{Im}\Phi_n\subset\operatorname{Im}\varphi_n$.

It is easy to see that $\varphi_n(\mathbf{Z}^{\#V'^{(n)}+})\subset\mathbf{Z}^{\#V^{(n)}+}$ and φ_n preserves order units. Therefore $\{\varphi_n\}_{n\in\mathbf{N}}$ gives the order isomorphism between $K_0(V', E')$ and $K_0(V, E)$. We finish the proof of the condition (1).

Next we shall show the condition (2). We see that $\bar{m}_j'^{(n)}=\sum_{i=1}^{\#V^{(n-1)}} l_{i,j}^{(n)}$ and

$$\#\{M_j'^{(n)} \mid \bar{m}_j'^{(n)}=s\}=\#\{L_j^{(n)} \mid \textstyle\sum_{i=1}^{\#V^{(n-1)}} l_{i,j}^{(n)}=s\}\leq\binom{s-1}{\#V^{(n-1)}-1},$$

where we used the following fact:

$$\#\{(n_1, n_2, \cdots n_d)\in\mathbf{N}^d \mid \textstyle\sum_{i=1}^d n_i=n\}=\binom{n-1}{d-1}. \tag{4.6}$$

And we see that

$$\#\{M_i'^{(n)} \mid \bar{m}_i'^{(n)}\leq s\}\leq\#\left\{(n_1, \cdots, n_{\#V^{(n-1)}})\in\mathbf{N}^{\#V^{(n-1)}} \,\middle|\, \#V^{(n-1)}\leq\sum_{i=1}^{\#V^{(n-1)}} n_i\leq s\right\}$$

$$=\sum_{i=\#V^{(n-1)}}^{s}\binom{i-1}{\#V^{(n-1)}-1}=\sum_{i=\#V^{(n-1)}}^{s}\left\{\binom{i}{\#V^{(n-1)}}-\binom{i-1}{\#V^{(n-1)}}\right\}=\binom{s}{\#V^{(n-1)}}<2^s,$$

where we used the formula $\binom{n}{r}=\binom{n-1}{r}+\binom{n-1}{r-1}$. So we are done. $\square$

REMARKS. (1) In the above situation, for $\#V'^{(n-1)}\leq s<\#V^{(n-1)}$, $\#\{M_j'^{(n)} \mid \bar{m}_j'^{(n)}\leq s\}=0$.

(2) We shall give the example of construction of $(V'E')$ as follows: Let

$$M^{(1)}=[10\ \ 10\ \ 20\ \ 20\ \ 20],\quad M^{(2)}=\begin{bmatrix}1&1&3&1&1&1&1&1\\1&2&4&2&2&2&2&1\\1&1&3&1&1&3&3&1\\1&2&4&2&2&4&4&1\\1&1&3&1&1&5&5&1\end{bmatrix},\quad M^{(3)}=\begin{bmatrix}1&4&1&1&1&1&1\\2&4&1&1&2&2&2\\1&3&1&1&3&3&1\\1&2&2&1&4&4&1\\1&5&1&2&5&5&1\\1&6&1&1&6&6&1\\1&7&1&1&7&7&1\\1&5&1&1&8&8&1\end{bmatrix}.$$

Then

$$M'^{(1)}=[10\ \ 10\ \ 20\ \ 20\ \ 20],\quad M'^{(2)}=\begin{bmatrix}1&1&3&1\\1&2&4&2\\1&1&3&3\\1&2&4&4\\1&1&3&5\end{bmatrix},\quad M'^{(3)}=\begin{bmatrix}2&9&2&2&9\\4&11&4&4&11\\1&3&1&1&3\\2&13&2&2&13\end{bmatrix},$$

$$(\phi_2(1), \phi_2(2), \phi_2(3), \phi_2(4), \phi_2(5), \phi_2(6), \phi_2(7), \phi_2(8)) = (1, 2, ,3, 2, 2, 4, 4, 1),$$

$$(\phi_3(1), \phi_3(2), \phi_3(3), \phi_3(4), \phi_3(5), \phi_3(6), \phi_3(7)) = (1, 2, 3, 4, 5, 5, 1).$$

We shall use the next lemma in the main theorem in order to modify a given diagram (V, E) satisfying $\{V^{(k)} \mid \# V^{(k)} \geq 2\} < \infty$.

LEMMA 4.3 ([Su]: Lemma 5.6). *Suppose $\mathscr{D} = (V, E)$ is a simple unordered Bratteli diagram satisfying $\#\{V^{(k)} \mid \# V^{(k)} \geq 2\} < \infty$. Then there exists the simple unordered Bratteli diagram $\acute{\mathscr{D}} = (\acute{V}, \acute{E})$ such that*

(1) *$K_0(V, E) \cong K_0(\acute{V}, \acute{E})$ as dimension groups via an isomorphism preserving distinguished order units,*

(2) *$\# \acute{V}^{(k)} = 2$ for all $k \geq 2$.*

PROOF. Since a dimension group associated with a contraction of diagram preserves order isomorphism, contracting $\mathscr{D}$ to $\{c_i \in Z^+ \mid \# V^{(c_i)} = 1\}_{i=0}^{\infty}$, we can assume $\# V^{(k)} = 1$ for all $k \in \mathbf{Z}^+$. Moreover we assume $m^{(k)} \geq 2$ where $(m^{(k)})$ is the k-th incidence matrix of $\mathscr{D}$ but we may identify $(m^{(k)})$ with $m^{(k)} \in \mathscr{D}$. Define the sequence of incidence matrices $\{\acute{M}^{(k)}\}_{k=1}^{\infty}$ of $\acute{\mathscr{D}}$ by

$$\acute{M}^{(k)} \equiv \begin{cases} [m^{(1)}\ m^{(1)}] & \text{if } k = 1, \\ \begin{bmatrix} \theta^{(k)} + \lambda^{(k)} & \theta^{(k)} + \lambda^{(k)} \\ \theta^{(k)} & \theta^{(k)} \end{bmatrix} & \text{if } k \geq 2, \end{cases}$$

where $\theta^{(k)} \in \mathbf{N}$ and $\lambda^{(k)} \in \{0, 1\}$ are the unique numbers satisfying $m^{(k)} = 2\theta^{(k)} + \lambda^{(k)}$ for all $k \geq 2$. It is clear that the condition (2) holds. So we shall show the condition (1). We construct sequence of maps $\{\varphi_k\}_{k=0}^{\infty}$ where, for $k \geq 0$, φ_k: $\mathbf{Z}^{\# V^{(k)}} \to \mathbf{Z}^{\# \acute{V}^{(k)}}$ will be order isomorphisms. Define φ_k by $\varphi_k(x) \equiv (x)$ if $k = 0$ and $\varphi_k(x) \equiv (x, x)$ if $k \geq 1$, where $x \in \mathbf{Z}$. It is easy to show that $\varphi_{k+1} \circ \Phi_{k+1} = \acute{\Phi}_{k+1} \circ \varphi_k$, $\operatorname{Im} \acute{\Phi}_k \subset \operatorname{Im} \varphi_k$, $\varphi_k(\mathbf{Z}^{\# V^{(k)}+}) \subset \mathbf{Z}^{\# \acute{V}^{(k)}+}$ and φ_k preserves order units, where Φ_k, $\acute{\Phi}_k$ are the order homomorphisms defined by $M^{(k)}$, $\acute{M}^{(k)}$ respectively. It is also seen easily that $\{\varphi_k\}_{k \geq 1}$ are order isomorphisms. So we are done. □

5. Preliminaries.

In this section, we pick up some lemmas in [Su] which are necessary to prove the main theorem.

LEMMA 5.1 ([Su]: Lemma 6.1). *Let $\{M^{(k)}\}_{k=1}^{\infty}$ be the sequence of positive incidence matrices of (V, E) and $N \in \mathbf{N}$ be given. Then for all $n \in \mathbf{N}$,*

$$\min_{\substack{1 \leq i \leq \# V^{(N-1)} \\ 1 \leq j \leq \# V^{(N+n)}}} m_{i,j}^{(N, N+n)} \geq \prod_{i=N}^{N+n-1} \# V^{(i)}. \tag{5.1}$$

PROOF. We prove it by induction. If $n = 1$, for all $1 \leq i \leq \# V^{(N-1)}$, $1 \leq j \leq \# V^{(N+1)}$,

$$m_{i,j}^{(N,N+1)}=\sum_{p=1}^{\#V^{(N)}} m_{i,p}^{(N)}m_{p,j}^{(N+1)}\geq\sum_{p=1}^{\#V^{(N)}} 1\cdot 1=\#V^{(N)}.$$

Therefore it holds for $n=1$. Suppose (5.1) holds for $n=k$. In the case of $n=k+1$, for all $1\leq i\leq \#V^{(N-1)}$, $1\leq j\leq \#V^{(N+k+1)}$,

$$\begin{aligned}m_{i,j}^{(N,N+k+1)}&=\sum_{p=1}^{\#V^{(N+k)}} m_{i,p}^{(N,N+k)}m_{p,j}^{(N+k+1)}\geq\sum_{p=1}^{\#V^{(N+k)}}\left\{\left(\prod_{i=N}^{N+k-1}\#V^{(i)}\right)\times 1\right\}\\&=\#V^{(N+k)}\times\prod_{i=N}^{N+k-1}\#V^{(i)}=\prod_{i=N}^{N+k}\#V^{(i)}.\end{aligned}$$

Therefore it holds for $n=k+1$, so we are done. □

LEMMA 5.2 ([Su]: Lemma 6.2). *Suppose $m, V, \theta\in\mathbf{N}$, $\lambda\in\mathbf{Z}^+$ and $1<r<2$ are numbers satisfying the following conditions:*

(1) $m=V\theta+\lambda$ *and* $0\leq\lambda<V$,

(2) $(r-1)\theta>V$ *and* $(2-r)\theta\geq 2$.

Then the following inequality holds:

$$\#\left\{(n_1,n_2,\cdots n_V)\in\mathbf{N}^V\;\middle|\;\sum_{i=1}^{V}n_i=m,\ 2\leq n_i<r\theta\ (\forall i)\right\}\geq\left(\frac{(r-1)m}{V}-r\right)^{V-1}.$$

PROOF. Let $\{l_i\}_{i=1}^{V-1}$ be a set of non-negative integers with $l_i<(r-1)\theta$. Define $\{n_i\in\mathbf{N}\}_{i=1}^{V}$ by

$$n_i\equiv\begin{cases}\theta+\lambda-l_1 & \text{if } i=1\\ \theta+l_{i-1}-l_i & \text{if } 2\leq i\leq V-1\\ \theta+l_{V-1} & \text{if } i=V.\end{cases}$$

Then we can easily verify that $\{n_i\}_{i=1}^{V}$ satisfies $\sum_{i=1}^{V}n_i=m$ and by condition (2), $2\leq n_i<r\theta$ holds for all $1\leq i\leq V$. Moreover it is easy to check that the map $(l_1,l_2,\cdots,l_{V-1})\mapsto(n_1,n_2,\cdots,n_V)$ is injective. So we get

$$\begin{aligned}&\#\left\{(n_1,n_2,\cdots n_V)\in\mathbf{N}^V\;\middle|\;\sum_{i=1}^{V}n_i=m,\ 2\leq n_i<r\theta\ (\forall i)\right\}\\&\quad\geq\#\{(l_1,l_2,\cdots,l_{V-1})\in(\mathbf{Z}^+)^{V-1}\mid 0\leq l_i<(r-1)\theta\ (\forall i)\}\\&\quad\geq([(r-1)\theta])^{V-1}=\left(\left[\frac{(r-1)(m-\lambda)}{V}\right]\right)^{V-1}\\&\quad\geq\left(\left[\frac{(r-1)(m-V)}{V}\right]\right)^{V-1}>\left(\frac{(r-1)(m-V)}{V}-1\right)^{V-1}\geq\left(\frac{(r-1)m}{V}-r\right)^{V-1},\end{aligned}$$

where [ ] is Gauss symbol, i.e. $[x]$ is the integer part of x. So we finish the proof.

□

LEMMA 5.3 ([Su]: Lemma 6.4). *For all $n \in \mathbf{N}$, the following inequality holds:*

$$\left(\frac{n}{e}\right)^n < n! < \left(\frac{n+2}{e}\right)^{n+2}.$$

PROOF. If $n=1$, the inequality holds trivially. If $n \geq 2$, then $e^n = \sum_{k=0}^{\infty} n^k/k! > n^n/n!$. Therefore the first part of the inequality holds. Next, we can calculate

$$\log(n+1)! = \sum_{k=1}^{n+1} \log k < \sum_{k=1}^{n+1} \int_k^{k+1} \log x dx = \int_1^{n+2} \log x dx$$
$$= (n+2)\log(n+2) - (n+1).$$

Since $\log(n+1) > 1$ for $n \geq 2$, we get

$$\log n! < (n+2)\log(n+2) - (n+1) - \log(n+1)$$
$$< (n+2)\{\log(n+2) - 1\} = \log((n+2)/e)^{n+2}.$$

So the second part of the inequality also holds. □

LEMMA 5.4. *Let $\mathscr{B} = \{M^{(k)}\}_{k=1}^{\infty}$ be the sequence of positive incidence matrices of simple unordered Bratteli diagram (V, E) and $N \in \mathbf{N}$, $1 \leq I \leq \# V^{(N-1)}$ be given. Suppose $\phi: \{1, 2, \cdots, \# V^{(N-1)}\} \to \{1, 2, \cdots I\}$ is an onto map. For $t \geq N$, we define $\hat{m}_{i,j}^{\prime(t)}$ by*

$$\hat{m}_{i,j}^{\prime(t)} \equiv \left(\sum_{s \in \phi^{-1}(i)} m_{s,j}^{(N,t)} \right) \Big/ \bar{m}_j^{(N,t)} \qquad \text{for} \quad 1 \leq i \leq I, \ 1 \leq j \leq \# V^{(t)}.$$

Then there exist $\{c_i\}_{i=1}^I$, $\{d_i\}_{i=1}^I$ with $0 < c_i \leq d_i < 1$ such that for all $1 \leq i \leq I$,

$$c_i \leq \inf_{\substack{1 \leq j \leq \# V^{(t)} \\ t \geq N}} \hat{m}_{i,j}^{\prime(t)}, \qquad \sup_{\substack{1 \leq j \leq \# V^{(t)} \\ t \geq N}} \hat{m}_{i,j}^{\prime(t)} \leq d_i. \tag{5.2}$$

PROOF. Define $m_{i,j}^{\prime(t)}$ and $\bar{m}_j^{\prime(t)}$ by

$$m_{i,j}^{\prime(t)} \equiv \sum_{s \in \phi^{-1}(i)} m_{s,j}^{(N,t)}, \qquad \bar{m}_j^{\prime(t)} \equiv \sum_{i=1}^{I} m_{i,j}^{\prime(t)}.$$

For any $t \geq N$, $1 \leq j \leq \# V^{(t)}$,

$$m_{i,j}^{\prime(t)} = \sum_{s \in \phi^{-1}(i)} \sum_{l=1}^{\# V^{(N)}} m_{s,l}^{(N)} m_{l,j}^{(N+1,t)} < \sum_{s \in \phi^{-1}(i)} \left(\sum_{q=1}^{\# V^{(N)}} m_{s,q}^{(N)} \times \sum_{l=1}^{\# V^{(N)}} m_{l,j}^{(N+1,t)} \right)$$
$$= \left\{ \sum_{s \in \phi^{-1}(i)} \sum_{q=1}^{\# V^{(N)}} m_{s,q}^{(N)} \right\} \times \bar{m}_j^{(N+1,t)}. \tag{5.3}$$

Thus

$$\bar{m}_j^{\prime(t)} < \bar{m}_j^{(N+1,t)} \times \|M^{(N)}\| \qquad \text{where} \quad \|M^{(N)}\| \equiv \sum_{\substack{1 \leq i \leq \# V^{(N-1)} \\ 1 \leq j \leq \# V^{(N)}}} m_{i,j}^{(N)}. \tag{5.4}$$

Also, the following inequality holds:

$$m_{i,j}'^{(t)} > \bar{m}_j^{(N+1,t)} \times \left\{ \sum_{s \in \phi^{-1}(i)} \min_{1 \le l \le \# V^{(N)}} m_{s,l}^{(N)} \right\} . \tag{5.5}$$

From (5.4) and (5.5), we get

$$\hat{m}_{i,j}'^{(t)} > c_i \qquad \text{where} \quad c_i \equiv \min_{1 \le l \le \# V^{(N)}} m_{i,l}^{(N)} \Big/ \|M^{(N)}\| . \tag{5.6}$$

It is clear that $0 < c_i < 1$ for all $1 \le i \le \# V^{(N-1)}$. Incidentally, since $\sum_{i=1}^{I} \hat{m}_{i,j}'^{(t)} = 1$ and we use (5.6), we have

$$\hat{m}_{i,j}'^{(t)} = 1 - \sum_{\substack{1 \le i' \le I \\ i' \ne i}} \hat{m}_{i',j}'^{(t)} < 1 - \sum_{\substack{1 \le i' \le I \\ i' \ne i}} c_{i'} \equiv d_i < 1 ,$$

for all $1 \le i \le I$. So we finish the proof. □

6. Proof for the infinite entropy case.

THEOREM 6.1. *Let (X, T) be a Cantor system. Then there exists a Cantor system (Y, S) such that the following conditions hold*:

(1) *(Y, S) is strongly orbit equivalent to (X, T),*

(2) *$h(S) = \infty$.*

PROOF. Let $\mathscr{D} = (V, E, \le)$ be the simple ordered Bratteli diagram which is a representation of (X, T), and let $\mathscr{B} = \{M^{(k)}\}_{k=1}^{\infty}$ be the sequence of incidence matrices $\mathscr{D}$. From the simpleness of diagram, we may assume $M^{(k)}$ is positive for any $k \ge 1$. And by Lemma 4.2, we can assume that (V, E) satisfies

$$\#\{M_i^{(k)} \mid \bar{m}_i^{(k)} \le s\} \le 2^s \tag{6.1}$$

for all $k \ge 2$ and $\# V^{(k-1)} \le s \le \bar{m}^{(k)}$. Moreover if $\#\{V^{(k)} \mid \# V^{(k)} \ge 2,\ k \ge 1\} < \infty$, as we consider within a strong orbit equivalence class, we modify $\mathscr{D}$ using Lemma 4.3 and we can also assume without the loss of generality that $\# V^{(k)} \ge 2$ for all $k \ge 1$. We shall construct $\tilde{\mathscr{D}} = (\tilde{V}, \tilde{E}, \tilde{\le})$ and $\tilde{\mathscr{B}} = \{\tilde{M}^{(k)}\}_{k=1}^{\infty}$, which is a representation of (Y, S), satisfying $h(S) = \infty$. The construction is done recursively. From Lemma 4.1 and Lemma 4.2, it suffices to construct the sequence of increasing integers $\{t_k\}_{k=0}^{\infty}$ with $t_0 = 0$ which are the depths in order to contract $\mathscr{D}$, the base diagram $\mathscr{B}' = \{M'^{(k)}\}_{k=1}^{\infty}$ and $\{\{\# \tilde{V}_i^{(k)}\}_{i=1}^{\# V'^{(k)}} \subset \mathbf{N}\}_{k=1}^{\infty}$.

The 1st step. Define $\# \tilde{V}^{(0)} \equiv 1$ and $t_0 \equiv 0$. Take any $t \ge 1$, $\alpha_0 > 1$ and fix them. We define t_1, $V'^{(1)}$, $M'^{(1)}$ and the map ϕ_1 by

$$t_1 \equiv t , \quad \# V'^{(1)} \equiv \# V^{(t_1)} , \quad M'^{(1)} \equiv M^{(t_0+1, t_1)} ,$$

$$\phi_1 : \{1, 2, \cdots, \# V^{(t_1)}\} \to \{1, 2, \cdots, \# V'^{(1)}\} \qquad \phi_1(i) = i \ \text{for all}\ 1 \le i \le \# V^{(t_1)} .$$

Here we write $V^{(t_1)}=\{v_i^{(t_1)}\}_{i=1}^{\#V^{(t_1)}}$ and $V'^{(1)}=\{v_i'^{(1)}\}_{i=1}^{\#V'^{(1)}}$. Then there exists $\{\#\tilde{V}_i^{(1)}\}_{i=1}^{\#V'^{(1)}}\subset\mathbf{N}$ such that

$$(\#\tilde{V}_i^{(1)}-2\delta(i))/(2\alpha_0)^{\#\mathscr{P}(v_i^{(t_1)})}>1 \qquad \text{for all} \quad 1\leq i\leq \#V'^{(1)}, \tag{6.2}$$

where $\delta:\mathbf{N}\to\{0,1\}$ is defined by $\delta(i)\equiv 0$ if $i\neq 1$, $\delta(i)\equiv 1$ if $i=1$. By (6.2), we see

$$(1=)\#\phi_1^{-1}(i)<\#\tilde{V}_i^{(1)}-2\delta(i) \qquad \text{for all} \quad 1\leq i\leq \#V'^{(1)}.$$

Take any $\{\#\tilde{V}_i^{(1)}\}_{i=1}^{\#V'^{(1)}}$ satisfying (6.2) and fix them. Define $\#\tilde{V}^{(1)}\equiv\sum_{i=1}^{\#V'^{(1)}}\#\tilde{V}_i^{(1)}$ and $\tilde{M}^{(1)}\in\mathbf{N}^{\#\tilde{V}^{(0)}\times\#\tilde{V}^{(1)}}$ by

$$\tilde{M}^{(1)}\equiv(\underbrace{m_1^{(1,t_1)},\cdots,m_1^{(1,t_1)}}_{\#\tilde{V}_1^{(1)}\text{ times}},\underbrace{m_2^{(1,t_1)},\cdots,m_2^{(1,t_1)}}_{\#\tilde{V}_2^{(1)}\text{ times}},\cdots,\underbrace{m_{\#V'^{(1)}}^{(1,t_1)},\cdots,m_{\#V'^{(1)}}^{(1,t_1)}}_{\#\tilde{V}_{\#V'^{(1)}}^{(1)}\text{ times}}),$$

where $(m_1^{(1,t_1)},m_2^{(1,t_1)},\cdots,m_{\#V'^{(1)}}^{(1,t_1)})=M^{(1,t_1)}$. And define $\#\mathscr{P}[\tilde{V}_i^{(1)}]$ by

$$\#\mathscr{P}[\tilde{V}_i^{(1)}]\equiv\#\mathscr{P}(v_i'^{(1)})=\#\mathscr{P}(v_i^{(t_1)}) \qquad \text{for each} \quad 1\leq i\leq\#V'^{(1)}.$$

And there exists a unique number $\alpha_1>2\alpha_0$ such that the following equality holds:

$$\sum_{i=1}^{\#V'^{(1)}}\frac{\#\tilde{V}_i^{(1)}}{(\alpha_1)^{\#\mathscr{P}[\tilde{V}_i^{(1)}]}}=1.$$

In order to satisfy the condition of induction, we define $K_i^{(1:1)}\equiv\{i\}\subset\tilde{V}^{(1)}$, $v_{\min}^{(1)}\equiv 1\in\tilde{V}^{(1)}$, $v_{\max}^{(1)}\equiv 2\in\tilde{V}^{(1)}$ (therefore $K_1^{(1:1)}=\{v_{\min}^{(1)}\}$ and $K_2^{(1:1)}=\{v_{\max}^{(1)}\})\#\{\tilde{V}^{(1)}/\approx_1\}\equiv\#\tilde{V}^{(1)}$ and $\mathrm{Dist}(\tilde{V}_j^{(1)})\equiv\#\tilde{V}_j^{(1)}$ for all $1\leq j\leq\#V'^{(1)}$.

The n-th step. For $n\geq 2$, suppose the following $(n-1)$-th data are given.

The $(n-1)$-th data:

$$t_{n-1}\in\mathbf{N},\quad \alpha_{n-1}>1,\quad \eta_{n-1}\in\mathbf{N},\quad \{\#\tilde{V}_j^{(n-1)}\}_{j=1}^{\#V'^{(n-1)}}\subset\mathbf{N},$$
$$\phi_{n-1}:\{1,2,\cdots,\#V^{(t_{n-1})}\}\to\{1,2,\cdots,\#V'^{(n-1)}\}.$$

If $n\geq 3$,

$$\{\{\#\tilde{V}_j^{(k:n-1)}\}_{j=1}^{\#\{\tilde{V}^{(n-1)}/\sim_k\}}\subset\mathbf{N}\}_{k=1}^{n-2},\qquad \{\{\#W_j^{(k:n-1)}\}_{j=1}^{\#\{\tilde{V}^{(n-1)}/\sim_k\}}\}_{k=1}^{n-2}$$
$$\{\xi_{k:n-1}:\{1,2,\cdots,\#V'^{(n-1)}\}\to\{1,2,\cdots,\#\{V^{(n-1)}/\sim_k\}\}\}_{k=1}^{n-2},$$
$$\{\kappa_{k:n-1}:\{1,2,\cdots,\#\{\tilde{V}^{(n-1)}/\approx_k\}\}\to\{1,2,\cdots,\#\{V^{(n-1)}/\sim_k\}\}\}_{k=1}^{n-2}.$$

The $(n-1)$-th assumption: For all $1\leq i\leq\#V'^{(n-1)}$,

(a.1) $(2\alpha_{n-2})^{\#\mathscr{P}[\tilde{V}_i^{(n-1)}]}<\#\tilde{V}_i^{(n-1)}-2\delta(i)$,

(a.2) $\#\phi_{n-1}^{-1}(i)<\#\tilde{V}_i^{(n-1)}-2\delta(i)$,

(a.3) $\#\mathscr{P}[\tilde{V}_i^{(n-1)}]=\#\mathscr{P}(v_i'^{(n-1)})$,

(a.4) there is the partial order $\tilde{\lesssim}$ on $\tilde{E}^{(k)}$ $(2\leq k\leq n-1)$ such that $\tilde{\lesssim}$ satisfies Assumption 2.5, 2.6, 2.7 and 2.8 with respect to $k=2,\cdots,n-1$, (a.5) $K_1^{(k:n-1)}=\{v_{\min}^{(n-1)}\}$ and $K_2^{(k:n-1)}=\{v_{\max}^{(n-1)}\}$ for all $1\le k\le n-1$.

If $n\ge 3$, for any $1\le k\le n-2$, any $1\le j\le \#\{\tilde V^{(n-1)}/\sim_k\}$, any $i\in\xi_{k:n-1}^{-1}(j)$ and any $p\in\kappa_{k:n-1}^{-1}(j)$ with $p\ne 1, 2$,

(a.6) $\eta_{n-1}\le \#(\tilde V_{i*}^{(n-1)}\cap K_p^{(k:n-1)})=\#\tilde V_{i*}^{(n-1)}/\#\{\pi_k(Con(v))\mid v\in\tilde V_{i*}^{(n-1)}\}$, $\#((\tilde V_i^{(n-1)}\setminus \tilde V_{i*}^{(n-1)})\cap K_p^{(k:n-1)})=\eta_{n-1}$ and $\#\{p'\mid K_{p'}^{(k:n-1)}\cap \tilde V_i^{(n-1)}\}>2\#\tilde V^{(n-2)}-3$,

(a.7) $\#\{\tilde V^{(n-1)}/\sim_k\}>1$,

(a.8) $\#\{\pi_k(Con(v))\mid v\in \tilde V_{i*}^{(n-1)}\}\ge \# W_j^{(k:n-1)}-\#\xi_{k:n-1}^{-1}(j)(2\#\tilde V^{(n-2)}-3)$,

(a.9) $(2\alpha_{k-1})^{\#\mathscr{P}[\tilde V_j^{(k:n-1)}]}<\# W_j^{(k:n-1)}-\#\xi_{k:n-1}^{-1}(j)(2\#\tilde V^{(n-2)}-3)$,

where $\tilde V_{i*}^{(n-1)}$ is defined by

$$\tilde V_{i*}^{(n-1)}\equiv\{v\in\tilde V_i^{(n-1)}\mid \tau(v,j)=v_{\min}^{(n-2)} \text{ for } 1\le j\le 3\,,$$
$$\tau(v,j)=v_{\max}^{(n-2)} \text{ for } \#r^{-1}(v)-2\le j\le \#r^{-1}(v)\}\,.$$

Now we shall construct the following n-th conditions using $(n-1)$-th data.

The n-th conditions: For all $1\le j\le \# V'^{(n)}$,

(n.1) $\binom{\#\tilde V^{(n-1)}}{2}<\prod_{i=1}^{\# V'^{(n-1)}}(\#\tilde V_i^{(n-1)}-2\delta(i))^{m'^{(n)}_{i,j}}<\mathrm{Dist}(\tilde V_j^{(n)})$,

(n.2) $\#\tilde V_j^{(n)}-2\delta(j)=\eta_n\,(\mathrm{Dist}(\tilde V_j^{(n)})+(2\#\tilde V^{(n-1)}-3))$,

(n.3) $\#\mathscr{P}[\tilde V_j^{(n)}]=\#\mathscr{P}(v_j'^{(n)})$,

(n.4) $(2\alpha_{n-1})^{\#\mathscr{P}[\tilde V_j^{(n)}]}<\#\tilde V_j^{(n)}-2\delta(j)$,

(n.5) $2\alpha_{n-1}<\alpha_n$, $\sum_{j=1}^{\# V'^{(n)}}\#\tilde V_j^{(n)}/\alpha_n^{\#\mathscr{P}[\tilde V_j^{(n)}]}=1$,

(n.6) $\#\phi_n^{-1}(j)<\#\tilde V_j^{(n)}-2\delta(j)$,

(n.7) there is a partial order $\tilde\le$ on $\tilde E^{(n)}$ such that $\tilde\le$ satisfies Assumption 2.5, 2.6, 2.7 and 2.8,

(n.8) $K_1^{(k:n)}=\{v_{\min}^{(n)}\}$ and $K_2^{(k:n)}=\{v_{\max}^{(n)}\}$ for all $1\le k\le n$.

If $n\ge 3$, for any $1\le k\le n$, any $1\le j\le\#\{\tilde V^{(n)}/\sim_k\}$, any $i\in\xi_{k:n}^{-1}(j)$ and any $p\in\kappa_{k:n}^{-1}(j)$ with $p\ne 1, 2$,

(n.9) $\eta_n\le\#(\tilde V_{i*}^{(n)}\cap K_p^{(k:n)})=\#\tilde V_{i*}^{(n)}/\#\{\pi_k(Con(v))\mid v\in\tilde V_{i*}^{(n)}\}$, $\#((\tilde V_i^{(n)}\setminus\tilde V_{i*}^{(n)})\cap K_p^{(k:n)})=\eta_n$ and $\#\{p'\mid K_{p'}^{(k:n)}\cap\tilde V_i^{(n)}\ne\emptyset\}>2\#\tilde V^{(n-1)}-3$,

(n.10) $\#\{\tilde V^{(n)}/\sim_k\}>1$,

(n.11) $\#\{\pi_k(Con(v))\mid v\in\tilde V_{i*}^{(n)}\}\ge\# W_j^{(k:n)}-\#\xi_{k:n}^{-1}(j)(2\#\tilde V^{(n-1)}-3)$,

(n.12) $(2\alpha_{k-1})^{\#\mathscr{P}[\tilde V_j^{(k:n)}]}<\# W_j^{(k:n)}-\#\xi_{k:n}^{-1}(j)(2\#\tilde V^{(n-1)}-3)$,

(n.13) $2\alpha_{k-1}<\alpha_{k,n}$, $\sum_{j=1}^{\#\{\tilde V^{(n)}/\sim_k\}}\# W_j^{(k:n)}/\alpha_{k,n}^{\#\mathscr{P}[\tilde V_j^{(k:n)}]}=1$,

where $\tilde V_{i*}^{(n)}$ is defined by

$$\tilde V_{i*}^{(n)}\equiv\{v\in\tilde V_i^{(n)}\mid \tau(v,j)=v_{\min}^{(n-1)} \text{ for } 1\le j\le 3,$$
$$\tau(v,j)=v_{\max}^{(n-1)} \text{ for } \#r^{-1}(v)-2\le j\le\#r^{-1}(v)\}\,.$$

Take any $t>t_{n-1}$ and fix it. First we shall construct $\# V'^{(n)}$, $M'^{(n)}$ and ϕ_n using t by the same methods in Lemma 4.2. That is to say, for $\{M_i^{(t_{n-1}+1,t)}\}_{i=1}^{\# V^{(t)}}$, there exists the set of vectors $\{L_i^{(n)}\}_{i\ge 1}$ such that $\{L_i^{(n)}\}_{i\ge 1}$ coincides with $\{M_i^{(t_{n-1}+1,t)}\}_{i=1}^{\# V^{(t)}}$ as a set and $L_i^{(n)}\ne L_j^{(n)}$ if $i\ne j$. We write $L_j^{(n)}=(l_{1,j}^{(n)}, l_{2,j}^{(n)},\cdots, l_{\# V(t_{n-1}),j}^{(n)})^t$. Now we define $\# V'^{(n)}$, $M'^{(n)}$

and the map ϕ_n by

$$\# V'^{(n)} \equiv \#\{L_i^{(n)}\}_{i \geq 1}, \quad m'^{(n)}_{i,j} \equiv \sum_{s \in \phi_{n-1}^{-1}(i)} l^{(n)}_{s,j}, \quad M'^{(n)} \equiv (m'^{(n)}_{i,j}),$$

$$\phi_n : \{1, 2, \cdots, \# V^{(t)}\} \to \{1, 2, \cdots, \# V'^{(n)}\} \qquad \phi_n(i)=j \underset{\text{def}}{\Longleftrightarrow} M_i^{(t_{n-1}+1,t)} = L_j^{(n)}.$$

By Lemma 5.4, there exist $\{c_i\}_{i=1}^{\# V'^{(n-1)}}$ and $\{d_i\}_{i=1}^{\# V'^{(n-1)}}$, which are independent of t and j, with $0 < c_i \leq d_i < 1$ such that

$$c_i \leq \hat{m}'^{(n)}_{i,j} \leq d_i. \tag{6.3}$$

Take any $r_n \in \mathbf{R}$ satisfying

$$1 < r_n < \min(3/2,\ 1/\max_{1 \leq i \leq \# V'^{(n-1)}} \{\max\{1-c_i,\ d_i\}\}) \tag{6.4}$$

and fix it. And we define $\theta^{(n)}_{i,j} \in \mathbf{N}$ and $\lambda^{(n)}_{i,j} \in \mathbf{Z}^+$ to be the unique numbers for $m'^{(n)}_{i,j}$ and $\# \tilde{V}_i^{(n-1)}$ such that, for all $1 \leq j \leq \# V'^{(n)}$,

$$m'^{(n)}_{i,j} - 6\delta(i) = (\# \tilde{V}_i^{(n-1)} - 2\delta(i))\theta^{(n)}_{i,j} + \lambda^{(n)}_{i,j},$$

$$0 \leq \lambda^{(n)}_{i,j} < \# \tilde{V}_i^{(n-1)} - 2\delta(i).$$

Moreover, for $1 \leq i \leq \# V'^{(n-1)}$, $1 \leq j$, $l \leq \# V'^{(n)}$ and $1 \leq p' \leq \#\{\tilde{V}^{(n-1)}/\approx_k\}$, define A_j, $B_{i,j}$, C_j, $D_{p',l} \in \mathbf{R}^+$, which depend on t, by

$$A_j \equiv \left\{ \left(\frac{\bar{m}'^{(n)}_j - 6}{e} \right)^6 \times \prod_{i=1}^{\# V'^{(n-1)}} \left(\frac{r_n \theta^{(n)}_{i,j} + 2}{e} \right)^{2\# \tilde{V}_i^{(n-1)}} \right\}^{-1}, \tag{6.5}$$

$$B_{i,j} \equiv \frac{1 - 6/\bar{m}'^{(n)}_j}{r_n \hat{m}'^{(n)}_{i,j} + 2\# \tilde{V}_i^{(n-1)}/\bar{m}'^{(n)}_j}, \tag{6.6}$$

$$C_j \equiv \left\{ \left(\frac{\bar{m}'^{(n)}_j - 6}{e} \right)^6 \times \left(\frac{\bar{m}'^{(n)}_j + 2}{e} \right)^{2\# \tilde{V}^{(n-1)}} \right\}^{-1}, \tag{6.7}$$

$$D_{p',l} \equiv \frac{1 - 6/\bar{m}'^{(n)}_l}{r_n \sum_{s \in \xi_{k:n-1}^{-1} \circ \kappa_{k:n-1}(p')} \hat{m}'^{(n)}_{s,l}}. \tag{6.8}$$

First we shall show that for sufficiently large t and all $1 \leq j \leq \# V'^{(n)}$, $1 \leq j' \leq \#\{\tilde{V}^{(n)}/\sim_k\}$, the following claims hold:

CLAIMS. (c.i)

$$A_j \times \prod_{i=1}^{\# V'^{(n-1)}} (B_{i,j})^{m'^{(n)}_{i,j}} > 1,$$

(c.ii) *for $n \geq 3$, for any k with $1 \leq k \leq n-2$ and any $l \in \xi_{k:n}^{-1}(j')$,*

$$C_{j'} \times \prod_{p'=1}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}} (D_{p',l})^{f_{p'}} > 1,$$

where all $f_{p'}$'s are any positive numbers satisfying $\sum_{p'=1}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}} f_{p'} = \bar{m}'^{(n)}_l$, (c.iii) *for* $n \geq 3$ *and for any* k *with* $1 \leq k \leq n-2$,

$$\alpha_{k-1}^{-\bar{m}_j'^{(n)}}(2\#\tilde{V}^{(n-1)}-3) < X_k^{\bar{m}_j'^{(n)}} - 1 ,$$

where X_k *is defined by*

$$X_k \equiv \min_{1 \leq t' \leq \#\{\tilde{V}^{(n-1)}/\sim_k\}} \frac{\#W_{t'}^{(k:n-1)} - \#\xi_{k:n-1}^{-1}(t')(2\#\tilde{V}^{(n-2)}-3)}{(2\alpha_{k-1})^{\#\mathscr{P}[\tilde{V}_{t'}^{(k:n-1)}]}} \quad (>1 \textit{ by } (a.9)) ,$$

(c.iv)

$$\#\left\{(n_k)_{k=1}^{\#\phi_{n-1}^{-1}(i)} \in \mathbf{N}^{\#\phi_{n-1}^{-1}(i)} \;\middle|\; \sum_{k=1}^{\#\phi_{n-1}^{-1}(i)} n_k = m_{i,j}'^{(n)}\right\}$$

$$< \#\left\{(n_k)_{k=1}^{\#\tilde{V}_i^{(n-1)}-2\delta(i)} \in \mathbf{N}^{\#\tilde{V}_i^{(n-1)}-2\delta(i)} \;\middle|\; \sum_{k=1}^{\#\tilde{V}_i^{(n-1)}-2\delta(i)} n_k = m_{i,j}'^{(n)} - 6\delta(i),\ 2 \leq n_k < r_n\theta_{i,j}^{(n)}\right\}$$

for all $1 \leq i \leq \#V'^{(n-1)}$,

(c.v)

$$\#\phi_n^{-1}(j) < \prod_{i=1}^{\#V'^{(n-1)}} (\#\tilde{V}_i^{(n-1)} - 2\delta(i))^{m_{i,j}'^{(n)}} ,$$

(c.vi)

$$\binom{\#\tilde{V}^{(n-1)}}{2} < \prod_{i=1}^{\#V'^{(n-1)}} (\#\tilde{V}_i^{(n-1)} - 2\delta(i))^{m_{i,j}'^{(n)}} .$$

THE PROOF OF (c.i). Let $d_i' \equiv r_n d_i$. From (6.3) and (6.4), $r_n \bar{m}_{i,j}'^{(n)} \leq d_i' < d_i/d_i = 1$ holds for all t and j. Since all $\#\tilde{V}_i^{(n-1)}$'s are constant values and $2\#\tilde{V}_i^{(n-1)}/\bar{m}_j'^{(n)} \to 0$ as $t \to \infty$, (note that by Lemma 5.1, these convergences do not depend on j,) there exists the constant d' with $0 < d' < 1$ such that, for sufficiently large t and all j,

$$\max_{1 \leq i \leq V'^{(n-1)}} \left(d_i' + \frac{2\#\tilde{V}_i^{(n-1)}}{\bar{m}_j'^{(n)}}\right) < d'$$

holds. Thus using the above inequality, we have $B_{i,j} > (1 - 6/\bar{m}_j'^{(n)})/d'$. Then we calculate

$$\left\{A_j \times \prod_{i=1}^{\#V'^{(n-1)}} (B_{i,j})^{m_{i,j}'^{(n)}}\right\}^{1/\bar{m}_j'^{(n)}} > A_j^{1/\bar{m}_j'^{(n)}} \times \prod_{i=1}^{\#V'^{(n-1)}} \left(\frac{1-6/\bar{m}_j'^{(n)}}{d'}\right)^{\bar{m}_{i,j}'^{(n)}}$$

$$= \frac{A_j^{1/\bar{m}_j'^{(n)}}(1-6/\bar{m}_j'^{(n)})}{d'} . \tag{6.9}$$

As we know the fact $\lim_{n\to\infty} n^{1/n} = 1$, it is easily seen that $A_j^{1/\bar{m}_j'^{(n)}}(1 - 6/\bar{m}_j'^{(n)}) \to 1$ as $t \to \infty$. Therefore for sufficiently large t,

$$\frac{A_j^{1/\bar{m}_j'^{(n)}}(1-6/\bar{m}_j'^{(n)})}{d'} > 1 . \tag{6.10}$$

Using (6.10) in (6.9), we get the claim (c.i).

THE PROOF OF (c.ii). The $(n-1)$-th assumption (a.7) implies that for any $1 \leq k \leq n-2$ and any $1 \leq j' \leq \#\{\tilde{V}^{(n-1)}/\sim_k\}$, $\#\xi_{k:n-1}^{-1}(j') < \#V'^{(n-1)}$. So for any p' with $1 \leq p' \leq \#\{\tilde{V}^{(n-1)}/\approx_k\}$, there exists s' such that $s' \notin \xi_{k:n-1}^{-1} \circ \kappa_{k:n-1}(p')$. In addition, from (6.3) and (6.4) there exists d'' with $0<d''<1$ such that

$$r_n \sum_{s\in \xi_{k:n-1}^{-1}\circ\kappa_{k:n-1}(p')} \hat{m}'^{(n)}_{s,l} \le r_n(1-\hat{m}'^{(n)}_{s',l})<r_n(1-c_{s'})<d''<1\,. \tag{6.11}$$

Therefore we have

$$\prod_{p'=1}^{\#\{\tilde{V}^{(n-1)}/\approx k\}} D_{p',l}^{f_{p'}} > \prod_{p'=1}^{\#\{\tilde{V}^{(n-1)}/\approx k\}} \left(\frac{1-6/\bar{m}'^{(n)}_l}{d''}\right)^{f_{p'}} = \left(\frac{1-6/\bar{m}'^{(n)}_l}{d''}\right)^{\bar{m}'^{(n)}_l}. \tag{6.12}$$

Incidentally, using the fact $\lim_{n\to\infty} n^{1/n}=1$, we see that $C_j^{1/\bar{m}'^{(n)}_l}(1-6/\bar{m}'^{(n)}_l)\to 1$ as $t\to\infty$. So for sufficiently large t, $C_j^{1/\bar{m}'^{(n)}_l}(1-6/\bar{m}'^{(n)}_l)>d''$. From (6.12) and the above inequality, we have for sufficiently large t

$$\left\{C_j\times \prod_{p'=1}^{\#\{\tilde{V}^{(n-1)}/\approx k\}} (D_{p',l})^{f_{p'}}\right\}^{1/\bar{m}'^{(n)}_l} > C_j^{1/\bar{m}'^{(n)}_l}\times \frac{1-6/\bar{m}'^{(n)}_l}{d''}>1\,.$$

The above inequality implies (c.ii).

THE PROOF OF (c.iii). Since $\alpha_{k-1}>1$ and $X_k>1$, we see that $\alpha_{k-1}^{-\bar{m}'^{(n)}_j}\to 0$, $X_k^{\bar{m}'^{(n)}_j}\to\infty$ as $t\to\infty$. And since $(2\#\tilde{V}^{(n-1)}-3)$ is constant, the claim (c.iii) hold for sufficiently large t.

THE PROOF OF (c.iv). Using (4.6), it is easily seen that

$$\#\left\{(n_1, n_2, \cdots, n_{\#\phi_{n-1}^{-1}(i)})\in \mathbf{N}^{\#\phi_{n-1}^{-1}(i)} \;\middle|\; \sum_{k=1}^{\#\phi_{n-1}^{-1}(i)} n_k = m'^{(n)}_{i,j}\right\}$$

$$=\binom{m'^{(n)}_{i,j}-1}{\#\phi_{n-1}^{-1}(i)-1}<(m'^{(n)}_{i,j})^{\#\phi_{n-1}^{-1}(i)-1}\,.$$

By the way, as $\theta^{(n)}_{i,j}$ is monotone increasing with respect to t, $\#\tilde{V}^{(n-1)}_i$ and r_n are constant, we may assume

$$(r_n-1)\theta^{(n)}_{i,j}>\#\tilde{V}^{(n-1)}_i\,,\qquad (2-r_n)\theta^{(n)}_{i,j}\ge 2\,.$$

Therefore by Lemma 5.2, we can get

$$\#\left\{(n_k)_{k=1}^{\#\tilde{V}^{(n-1)}_i-2\delta(i)}\in \mathbf{N}^{\#\tilde{V}^{(n-1)}_i-2\delta(i)} \;\middle|\; \sum_{k=1}^{\#\tilde{V}^{(n-1)}_i-2\delta(i)} n_k = m'^{(n)}_{i,j}-6\delta(i),\ 2\le n_k<r_n\theta^{(n)}_{i,j}\ (\forall k)\right\}$$

$$\ge\left(\frac{(r_n-1)(m'^{(n)}_{i,j}-6\delta(i))}{\#\tilde{V}^{(n-1)}_i-2\delta(i)}-r_n\right)^{\#\tilde{V}^{(n-1)}_i-2\delta(i)-1}.$$

Recalling that $m'^{(n)}_{i,j}$ is also monotone increasing with respect to t, and using the $(n-1)$-th assumption (a.2), for sufficiently large t we have

$$(m'^{(n)}_{i,j})^{\#\phi_{n-1}^{-1}(i)-1}<\left(\frac{(r_n-1)(m'^{(n)}_{i,j}-6\delta(i))}{\#\tilde{V}^{(n-1)}_i-2\delta(i)}-r_n\right)^{\#\tilde{V}^{(n-1)}_i-2\delta(i)-1}.$$

By the above inequality, for all $1 \leq j \leq \# V'^{(n)}$, we have

$$\#\left\{(n_1, n_2, \cdots, n_{\#\phi_n^{-1}(i)}) \in \mathbf{N}^{\#\phi_n^{-1}(i)} \;\middle|\; \sum_{k=1}^{\#\phi_n^{-1}(i)} n_k = m'^{(n)}_{i,j}\right\}$$

$$< \#\left\{(n_k)^{\#\tilde{V}_i^{(n-1)}-2\delta(i)} \in \mathbf{N}^{\#\tilde{V}_i^{(n-1)}-2\delta(i)} \;\middle|\; \sum_{k=1}^{\#\tilde{V}_i^{(n-1)}-2\delta(i)} n_k = m'^{(n)}_{i,j} - 6\delta(i),\ 2 \leq n_k < r_n \theta^{(n)}_{i,j}\ (\forall k)\right\}.$$

THE PROOF OF (c.v). For $L_j^{(n)} = (l_{1,j}^{(n)}, l_{2,j}^{(n)}, \cdots, l^{(n)}_{\#V^{(t_{n-1})},j})^t$, define $\bar{l}_j^{(n)} \equiv \sum_{i=1}^{\#V^{(t_{n-1})}} l_{i,j}^{(n)}$. By the construction of $M'^{(n)}$, we see that $\bar{l}_j^{(n)} = \bar{m}'^{(n)}_j$. Then we can calculate by (6.1),

$$\#\phi_n^{-1}(j) = \#\{M_s^{(t_{n-1}+1,t)} \mid M_s^{(t_{n-1}+1,t)} = L_j^{(n)}\} \leq \#\{M_s^{(t_{n-1}+1,t)} \mid \bar{m}_s^{(t_{n-1}+1,t)} = \bar{m}'^{(n)}_j\}$$
$$\leq \#\{M_s^{(t)} \mid \bar{m}_s^{(t)} \leq \bar{m}'^{(n)}_j\} \leq 2^{\bar{m}'^{(n)}_j}.$$

Define $p_{\min} \equiv \min_{1 \leq i \leq \# V'^{(n-1)}} \#\mathscr{P}(v_i'^{(n-1)})$. As we know

$$\#\mathscr{P}(v_i'^{(n-1)}) = m'^{(1,n-1)}_{1,i} = \sum_{k=1}^{\#V'^{(1)}} m'^{(1)}_{1,k} m'^{(2,n-1)}_{k,i} = \sum_{k=1}^{\#V'^{(1)}} \#\mathscr{P}(v_k'^{(1)}) m'^{(2,n-1)}_{k,i} > \sum_{k=1}^{\#V'^{(1)}} \#\mathscr{P}(v_k'^{(1)}),$$

we get $p_{\min} > \sum_{k=1}^{\#V'^{(1)}} \#\mathscr{P}(v_k'^{(1)})$. Using $(n-1)$-th assumption (a.1) and (a.3), we calculate

$$\prod_{i=1}^{\#V'^{(n-1)}} (\#\tilde{V}_i^{(n-1)} - 2\delta(i))^{m'^{(n)}_{i,j}} > \prod_{i=1}^{\#V'^{(n-1)}} (2\alpha_{n-2})^{\#\mathscr{P}[\tilde{V}_i^{(n-1)}]} > (2\alpha_{n-2})^{p_{\min}\bar{m}'^{(n)}_j}$$
$$> \left(\prod_{k=1}^{\#V'^{(1)}} (2\alpha_{n-2})^{\#\mathscr{P}(v_k'^{(1)})}\right)^{\bar{m}'^{(n)}_j} > 2^{\bar{m}'^{(n)}_j}.$$

Therefore we get the inequality in (c.v).

THE PROOF OF (c.vi). Recalling that $\#\tilde{V}^{(n-1)}$ is constant and $\#\mathscr{P}(v_j'^{(n)})$'s are monotone increasing with respect to t. So we can easily verify that for sufficiently large t, the following inequality holds:

$$\binom{\#\tilde{V}^{(n-1)}}{2} < \prod_{i=1}^{\#V'^{(n-1)}} (\#\tilde{V}_i^{(n-1)} - 2\delta(i))^{m'^{(n)}_{i,j}}.$$

So we are done.

Next we shall decide the values of t_n, η_n, $\#\tilde{V}_j^{(n)} \in \mathbf{N}$. If $n=2$, take any $t > t_1$ satisfying the claims (c.i), (c.iv), (c.v) and (c.vi), and fix it. If $n \geq 3$, take any $t > t_{n-1}$ satisfying all claims and fix it. First define t_n by $t_n \equiv t$. Next, take any $\eta_n \in \mathbf{N}$ satisfying the following inequalities and fix it:

$$\eta_n \times \frac{\prod_{i=1}^{\#V'^{(n-1)}} (\#\tilde{V}_i^{(n-1)} - 2\delta(i))^{m'^{(n)}_{i,j}}}{(2\alpha_{n-1})^{\#\mathscr{P}[V_j'^{(n)}]}} > 1 \qquad \text{for all} \quad 1 \leq j \leq \# V'^{(n)}. \tag{6.13}$$

Define $\{\tilde{M}_{[j]}^{(n)} \in \mathbf{N}^{\#\tilde{V}^{(n-1)}}\}_{j=1}^{\#V'^{(n)}}$ and $\{\{\tilde{M}_{i,j}^{(n)} \in \mathbf{N}^{\#\tilde{V}_i^{(n-1)}}\}_{i=1}^{\#V'^{(n-1)}}\}_{j=1}^{\#V'^{(n)}}$ by $\tilde{M}_{[j]}^{(n)} \equiv (\tilde{M}_{1,j}^{(n)}, \tilde{M}_{2,j}^{(n)}, \cdots, \tilde{M}^{(n)}_{\#V'^{(n-1)},j})^t$, and they satisfy the following conditions:

(c.1) $\tilde{M}_{[i]}^{(n)} \neq \tilde{M}_{[j]}^{(n)}$ if $i \neq j$,

(c.2) $\tilde{M}_{i,j}^{(n)} \in \{(n_k)_{k=1}^{\#\tilde{V}_i^{(n-1)}} \in \mathbf{N}^{\#\tilde{V}_i^{(n-1)}} \mid \sum_{k=1}^{\#\tilde{V}_i^{(n-1)}} n_k = m_{i,j}^{\prime(n)},\ 2 \leq n_k < r_n \theta_{i,j}^{(n)} (\forall k)\}$, where r_n is defined by (6.4),

(c.3) if we write $\tilde{M}_{[j]}^{(n)} = (\tilde{m}_{1,[j]}^{(n)}, \tilde{m}_{2,[j]}^{(n)}, \cdots, \tilde{m}_{\#\tilde{V}^{(n-1)},[j]}^{(n)})$, then $\tilde{m}_{1,[j]}^{(n)} = \tilde{m}_{2,[j]}^{(n)} = 3$ holds,

(c.4) there exist i, j and p with $1 \leq p \leq \#\{\tilde{V}^{(n-1)}/\approx_1\}$ such that

$$\sum_{l \in K_p^{(1:n-1)}} \tilde{m}_{l,[i]}^{(n)} \neq \sum_{l \in K_p^{(1:n-1)}} \tilde{m}_{l,[j]}^{(n)} . \tag{6.14}$$

Lastly define $\#\tilde{V}_j^{(n)}$ by

$$\#\tilde{V}_j^{(n)} \equiv \eta_n \times (\mathrm{Dist}(\tilde{V}_j^{(n)}) + (2\#\tilde{V}^{(n-1)} - 3)) + 2\delta(j) , \tag{6.15}$$

where $\mathrm{Dist}(\tilde{V}_j^{(n)}) \in \mathbf{N}$ is defined by

$$\mathrm{Dist}(\tilde{V}_j^{(n)}) \equiv \frac{(\sum_{i=3}^{\#\tilde{V}^{(n-1)}} \tilde{m}_{i,[j]}^{(n)})!}{\prod_{i=3}^{\#\tilde{V}^{(n-1)}} \tilde{m}_{i,[j]}^{(n)}!} = \frac{(\tilde{m}_j^{\prime(n)} - 6)!}{\prod_{i=3}^{\#\tilde{V}^{(n-1)}} \tilde{m}_{i,[j]}^{(n)}!} \tag{6.16}$$

(cf. (2.1)). Let $\#\tilde{V}^{(n)} \equiv \sum_{j=1}^{\#V^{\prime(n)}} \#\tilde{V}_j^{(n)}$. We define $\tilde{M}^{(n)} \in \mathbf{N}^{\#\tilde{V}^{(n-1)} \times \#\tilde{V}^{(n)}}$ by

$$\tilde{M}^{(n)} \equiv (\underbrace{\tilde{M}_{[1]}^{(n)}, \cdots, \tilde{M}_{[1]}^{(n)}}_{\#\tilde{V}_1^{(n)} \text{ times}}, \underbrace{\tilde{M}_{[2]}^{(n)}, \cdots, \tilde{M}_{[2]}^{(n)}}_{\#\tilde{V}_2^{(n)} \text{ times}}, \cdots, \underbrace{\tilde{M}_{[\#V^{\prime(n)}]}^{(n)}, \cdots, \tilde{M}_{[\#V^{\prime(n)}]}^{(n)}}_{\#\tilde{V}_{\#V^{\prime(n)}}^{(n)} \text{ times}}) .$$

Here what we have to show is as follows:

CLAIMS. (c.vii) *To see that the above condition* (c.1) *is guaranteed,*

(c.viii) *To see that the above condition* (c.4) *is guaranteed,*

(c.ix) $\prod_{i=1}^{\#V^{\prime(n-1)}} (\#\tilde{V}_i^{(n-1)} - 2\delta(i))^{m_{i,j}^{\prime(n)}} < \mathrm{Dist}(\tilde{V}_j^{(n)})$ *for all* $1 \leq j \leq \#V^{\prime(n)}$,

(c.x) *The quantities defined above satisfy the n-th conditions.*

THE PROOF OF (c.vii). It is clear that if $M_i^{\prime(n)} \neq M_j^{\prime(n)}$, then $\tilde{M}_{[i]}^{(n)} \neq \tilde{M}_{[j]}^{(n)}$. So what we have to show is that for any fixed j and for any $k, k' \in \{s \mid M_s^{\prime(n)} = M_j^{\prime(n)}\}$ with $k \neq k'$, we can construct $\tilde{M}_{[k]}^{(n)}$, $\tilde{M}_{[k']}^{(n)}$ satisfying $\tilde{M}_{[k]}^{(n)} \neq \tilde{M}_{[k']}^{(n)}$. By the construction of $M^{\prime(n)}$, we see

$$\#\{s \mid M_s^{\prime(n)} = M_j^{\prime(n)}\} \leq \prod_{i=1}^{\#V^{\prime(n-1)}} \#\left\{(n_k)_{k=1}^{\#\phi_{n-1}^{-1}(i)} \in \mathbf{N}^{\#\phi_{n-1}^{-1}(i)} \,\middle|\, \sum_{k=1}^{\#\phi_{n-1}^{-1}(i)} n_k = m_{i,j}^{\prime(n)} (\forall k)\right\} .$$

$\tilde{M}_{[k]}^{(n)}$, $\tilde{M}_{[k']}^{(n)}$ satisfies (c.2) respectively and by the claim (c.iv) we see that

$$\prod_{i=1}^{\#V^{\prime(n-1)}} \#\left\{(n_k)_{k=1}^{\#\phi_{n-1}^{-1}(i)} \in \mathbf{N}^{\#\phi_{n-1}^{-1}(i)} \,\middle|\, \sum_{k=1}^{\#\phi_{n-1}^{-1}(i)} n_k = m_{i,j}^{\prime(n)}\right\} <$$

$$\prod_{i=1}^{\#V^{\prime(n-1)}} \#\left\{(n_k)_{k=1}^{\#\tilde{V}_i^{(n-1)} - 2\delta(i)} \in \mathbf{N}^{\#\tilde{V}_i^{(n-1)} - 2\delta(i)} \,\middle|\, \sum_{k=1}^{\#\tilde{V}_i^{(n-1)} - 2\delta(i)} n_k = m_{i,j}^{\prime(n)} - 6\delta(i),\ 2 \leq \forall n_k < r_n \theta_{i,j}^{(n)}\right\} .$$

The last part of the above inequality implies the maximum possible value for incidence vectors in $\mathbf{N}^{\#\tilde{V}^{(n-1)}}$ satisfying (c.2). Therefore we can choose incidence vectors satisfying $\tilde{M}^{(n)}_{[k]} \neq \tilde{M}^{(n)}_{[k']}$. So we finish the proof.

THE PROOF OF (c.viii). Suppose $\{\tilde{M}^{(n)}_{[j]} \in \mathbf{N}^{\#\tilde{V}^{(n-1)}}\}_{j=1}^{\#V'^{(n)}}$ satisfies the condition (c.1), (c.2) and (c.3) but not (c.4), i.e. for any i, j and all $1 \leq p \leq \#\{\tilde{V}^{(n-1)}/\approx_1\}$,

$$\sum_{l \in K_p^{(1:n-1)}} \tilde{m}^{(n)}_{l,[i]} = \sum_{l \in K_p^{(1:n-1)}} \tilde{m}^{(n)}_{l,[j]} \,. \tag{6.17}$$

Then we shall show that we can modify $\{\tilde{M}^{(n)}_{[j]} \in \mathbf{N}^{\#\tilde{V}^{(n-1)}}\}_{j=1}^{\#V'^{(n)}}$ so as to satisfy the condition (c.4).

Take any i with $1 \leq i \leq \#V'^{(n)}$ and any t with $1 \leq t \leq \#V'^{(n-1)}$ and fix them. Since the $(n-1)$-th assumption (a.6) holds, there exists $p, p' \in \{q \mid K_q^{(k:n-1)} \cap \tilde{V}_t^{(n-1)} \neq \varnothing\}$ with $p \neq p', p \neq 1, 2, p' \neq 1, 2$ and s, s' with $s \in K_p^{(k:n-1)} \cap \tilde{V}_t^{(n-1)}$ and $s' \in K_{p'}^{(k:n-1)} \cap \tilde{V}_t^{(n-1)}$ such that $\tilde{m}^{(n)}_{s,[i]} + 1 < r_n \theta^{(n)}_{t,i}$ and $\tilde{m}^{(n)}_{s',[i]} - 1 \geq 2$.

Now we define new incidence vector $\tilde{M}^{(n)}_{[i]} = (h_1, h_2, \cdots h_{\#\tilde{V}^{(n-1)}})^t$ by

$$h_k \equiv \begin{cases} \tilde{m}^{(n)}_{s,[i]} + 1 & \text{if } k = s \\ \tilde{m}^{(n)}_{s',[i]} - 1 & \text{if } k = s' \\ \tilde{m}^{(n)}_{k,[i]} & \text{otherwise} \,. \end{cases}$$

It is easy to check that new $\tilde{M}^{(n)}_{[i]}$ satisfies (c.2) and (c.3). Then by (6.17), for j with $j \neq i$ we see

$$\sum_{l \in K_p^{(1:n-1)}} h_l \neq \sum_{l \in K_p^{(1:n-1)}} \tilde{m}^{(n)}_{l,[j]} \,, \qquad \sum_{l \in K_{p'}^{(1:n-1)}} h_l \neq \sum_{l \in K_{p'}^{(1:n-1)}} \tilde{m}^{(n)}_{l,[j]} \,. \tag{6.18}$$

Therefore (c.4) holds. Lastly we check (c.1) holds. We assume there exists $j' \neq i$ such that

$$\tilde{M}^{(n)}_{[j']} = (h_1, h_2, \cdots h_{\#\tilde{V}^{(n-1)}})^t \,. \tag{6.19}$$

However, it is clear that (6.19) does not imply (6.18). This is a contradiction. So the condition (c.1) holds.

THE PROOF OF (c.ix). Using Lemma 5.3, we can calculate

$$\begin{aligned} Dist(\tilde{V}_j^{(n)}) &= \frac{(\bar{m}_j'^{(n)} - 6)!}{\prod_{i=3}^{\#\tilde{V}^{(n-1)}} \tilde{m}^{(n)}_{i,[j]}!} \\ &> \frac{((\bar{m}_j'^{(n)} - 6)/e)^{\bar{m}_j'^{(n)} - 6}}{\prod_{i=3}^{\#\tilde{V}^{(n-1)}} ((\tilde{m}^{(n)}_{i,j} + 2)/e)^{(\tilde{m}^{(n)}_{i,j} + 2)}} > \frac{((\bar{m}_j'^{(n)} - 6)/e)^{\bar{m}_j'^{(n)} - 6}}{\prod_{i=1}^{\#V'^{(n-1)}} ((r_n \theta^{(n)}_{i,j} + 2)/e)^{m'^{(n)}_{i,j} + 2\#\tilde{V}_i^{(n-1)}}} \\ &> \left\{ \left(\frac{\bar{m}_j'^{(n)} - 6}{e} \right)^6 \prod_{i=1}^{\#V'^{(n-1)}} \left(\frac{r_n \theta^{(n)}_{i,j} + 2}{e} \right)^{2\#\tilde{V}_i^{(n-1)}} \right\}^{-1} \times \prod_{i=1}^{\#V'^{(n-1)}} \left(\frac{\bar{m}_j'^{(n)} - 6}{r_n \theta^{(n)}_{i,j} + 2} \right)^{m'^{(n)}_{i,j}} \\ &= A_j \times \prod_{i=1}^{\#V'^{(n-1)}} \left(\frac{\bar{m}_j'^{(n)} - 6}{r_n \theta^{(n)}_{i,j} + 2} \right)^{m'^{(n)}_{i,j}} \qquad \text{(by (6.5))} \,. \end{aligned}$$

By (6.7), we see that

$$\frac{\bar{m}_j'^{(n)}-6}{r_n\theta_{i,j}^{(n)}+2}=\frac{\bar{m}_j'^{(n)}-6}{(\#\tilde{V}_i^{(n-1)}-2\delta(i))(r_n\theta_{i,j}^{(n)}+2)}(\#\tilde{V}_i^{(n-1)}-2\delta(i))$$
$$>B_{i,j}(\#\tilde{V}_i^{(n-1)}-2\delta(i))\,.$$

From the claim (c.i), we have

$$\mathrm{Dist}(\tilde{V}_j^{(n)})>A_j\times\prod_{i=1}^{\#V'(n-1)}(B_{i,j}(\#\tilde{V}_i^{(n-1)}-2\delta(i)))^{m_{i,j}'^{(n)}}$$
$$>\prod_{i=1}^{\#V'(n-1)}(\#\tilde{V}_i^{(n-1)}-2\delta(i))^{m_{i,j}'^{(n)}}\,.$$

The proof of (c.x). (n.1) $\Leftrightarrow$ (c.vi) and (c.ix).

(n.2) $\Leftrightarrow$ (6.15).

(n.3) Since the construction of $\tilde{M}^{(n)}$ satisfies the assumption in Lemma 4.1, (n.3) holds by Lemma 4.1.

(n.4) By (6.15), (c.ix) and (6.13), we have

$$\frac{\#\tilde{V}_j^{(n)}-2\delta(j)}{(2\alpha_{n-1})^{\#\mathscr{P}[\tilde{V}_j^{(n)}]}}>\frac{\eta_n\times\mathrm{Dist}(\tilde{V}_j^{(n)})}{(2\alpha_{n-1})^{\#\mathscr{P}[\tilde{V}_j^{(n)}]}}>\eta_n\times\frac{\prod_{i=1}^{\#V'(n-1)}(\#\tilde{V}_i^{(n-1)}-2\delta(i))^{m_{i,j}'^{(n)}}}{(2\alpha_{n-1})^{\#\mathscr{P}[V_j'^{(n)}]}}>1\,.$$

Therefore (n.4) holds.

(n.5) Let α_n be the positive number satisfying the following equality:

$$\sum_{j=1}^{\#V'(n)}\frac{\#\tilde{V}_j^{(n)}}{\alpha_n^{\#\mathscr{P}[\tilde{V}_j^{(n)}]}}=1\,.$$

The above equality implies that for all $2\le j\le\#V'^{(n)}$, $\#\tilde{V}_j^{(n)}<\alpha_n^{\#\mathscr{P}[\tilde{V}_j^{(n)}]}$. By (n.4) we see that $2\alpha_{n-1}<\alpha_n$. Therefore (n.5) holds.

(n.6) $\Leftrightarrow$ (c.v), (n.1) and (n.2).

(n.7) $\Leftrightarrow$ (n.1), (n.2) and Lemma 2.9.

(n.8) $\Leftrightarrow$ (3.17) in Lemma 2.9.

(n.9) From the definition of $\tilde{V}_{i*}^{(n)}$, Assumption 2.8 (1) and Lemma 2.9,

$$\#\tilde{V}_{i*}^{(n)}=\eta_n\times\mathrm{Dist}(\tilde{V}_i^{(n)})\,,\tag{6.20}$$
$$\#(\tilde{V}_i^{(n)}\setminus\tilde{V}_{i*}^{(n)})=\eta_n(2\#\tilde{V}^{(n-1)}-3)+2\delta(i)\,.\tag{6.21}$$

Now we observe the value of $\#\{\pi_k(Con(v))\mid v\in\tilde{V}_{i*}^{(n)}\}$. For $v\in\tilde{V}^{(n)}$, define the ordered vector of v for $\approx_k$, $O.vec_k(v)$ by

$$O.vec_k(v)=([K_{p_1}^{(k:n-1)}],[K_{p_2}^{(k:n-1)}],\cdots,[K_{p_q}^{(k:n-1)}])\underset{\text{def}}{\Longleftrightarrow}\tau(v,t)\in K_{p_t}^{(k:n-1)}$$

for $1\le t\le q\equiv\#r^{-1}(v)$. Since $K_1^{(k:n-1)}=\{v_{\min}^{(n-1)}\}$, $K_2^{(k:n-1)}=\{v_{\max}^{(n-1)}\}$ by the $(n-1)$-th assumption (a.5) and for any $v \in \tilde{V}_i^{(n)}$, and p' with $1 \le p' \le \#\{\tilde{V}^{(n-1)}/\approx_k\}$,

$$\#\{t \in \{1, 2, \cdots \#r^{-1}(v)\} \mid \tau(v, t) \in K_{p'}^{(k:n-1)}\} = \sum_{l \in K_{p'}^{(k;n-1)}} \tilde{m}_{l,[i]}^{(n)}, \tag{6.22}$$

we have

$$\begin{aligned}\#\{\boldsymbol{\pi}_k(Con(v)) \mid v \in \tilde{V}_{i*}^{(n)}\} &= \#\{O.vec_k(v) \mid v \in \tilde{V}_{i*}^{(n)}\} \\ &\le \frac{(\bar{m}_i'^{(n)} - 6)!}{\prod_{p'=3}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}} (\sum_{l \in K_{p'}^{(k;n-1)}} \tilde{m}_{l,[i]}^{(n)})!} \equiv \mathrm{Max}_k(\tilde{V}_{i*}^{(n)}).\end{aligned} \tag{6.23}$$

Let $O_k(\tilde{V}_{i*}^{(n)})$ be the set of vectors arranging all $\sum_{l \in K_{p'}^{(k;n-1)}} \tilde{m}_{l,[i]}^{(n)}$ vertices of $[K_{p'}^{(k:n-1)}]$ for $3 \le p' \le \#\{\tilde{V}^{(n-1)}/\approx_k\}$. It is easy to see that $\mathrm{Max}_k(\tilde{V}_{i*}^{(n)}) = \#O_k(\tilde{V}_{i*}^{(n)})$. Here, we shall show that the inequality (6.23) is in fact an equality, i.e.

$$\#\{\boldsymbol{\pi}_k(Con(v)) \mid v \in \tilde{V}_{i*}^{(n)}\} = \frac{(\bar{m}_i'^{(n)} - 6)!}{\prod_{p'=3}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}} (\sum_{l \in K_{p'}^{(k;n-1)}} \tilde{m}_{l,[i]}^{(n)})!}. \tag{6.24}$$

Suppose $w_k = ([K_{p_1}^{(k:n-1)}], [K_{p_2}^{(k:n-1)}], \cdots, [K_{p_q}^{(k:n-1)}])$ is any ordered vector for $\approx_k$ satisfying that

(1) $q = \bar{m}_i'^{(n)}$ $(= \#r^{-1}([\tilde{V}_i^{(n)}]))$,
(2) $([K_{p_4}^{(k:n-1)}], [K_{p_5}^{(k:n-1)}], \cdots, [K_{p_{q-4}}^{(k:n-1)}], [K_{p_{q-3}}^{(k:n-1)}]) \in O_k(\tilde{V}_{i*}^{(n)})$,
(3) $p_1 = p_2 = p_3 = 1$ and $p_{q-2} = p_{q-1} = p_q = 2$.

And for w_k, take any ordered vector $w = (v_1, v_2, \cdots, v_q)$ satisfying that

(4) $\#\{t \mid v_t = l \in \tilde{V}^{(n-1)}\} = \tilde{m}_{l,[i]}^{(n)}$ for all $1 \le l \le \#\tilde{V}^{(n-1)}$,
(5) $v_t \in K_{p_t}^{(k:n-1)}$ for all $1 \le t \le q$,
(6) $v_1 = v_2 = v_3 = v_{\min}^{(n-1)}$ and $v_{q-2} = v_{q-1} = v_q = v_{\max}^{(n-1)}$.

Since $\tilde{V}_{i*}^{(n)}$ satisfies (6.20) and from Assumption 2.8 (2), there exists $v \in \tilde{V}_{i*}^{(n)}$ such that $O.vec(v) = w$ holds. And from the above conditions (1), (2), $\cdots$, and (6), $O.vec_k(v) = w_k$ holds. This implies (6.24) holds. Therefore the inequality (6.23) is in fact an equality.

Next we observe the value of $\#(\tilde{V}_{i*}^{(n)} \cap K_p^{(k:n)})$. Let $\{\tilde{V}_{i,l}^{(n)}\}_{l=1}^{\eta_n}$ be disjoint subsets of $\tilde{V}_i^{(n)}$ satisfying Assumption 2.8. Then the following equality holds:

$$\#(\tilde{V}_{i*}^{(n)} \cap K_p^{(k:n)}) = \eta_n \times \#(\tilde{V}_{i,l}^{(n)} \cap \tilde{V}_{i*}^{(n)} \cap K_p^{(k:n)}). \tag{6.25}$$

Take any $v \in \tilde{V}_{i,l}^{(n)} \cap \tilde{V}_{i*}^{(n)} \cap K_p^{(k:n)}$ and fix it. And there exist unique $\{p_t\}_{t=1}^{q}$ $(q = \#r^{-1}(v))$ such that

$$O.vec_k(v) = ([K_{p_1}^{(k:n-1)}], [K_{p_2}^{(k:n-1)}], \cdots, [K_{p_q}^{(k:n-1)}]).$$

Each $\tilde{V}_{i,l}^{(n)}$ has partial distinct ordering and $\#(\tilde{V}_{i,l}^{(n)} \cap \tilde{V}_{i*}^{(n)}) = \mathrm{Dist}(\tilde{V}_i^{(n)})$. In addition, v satisfies (6.22) and

$$\prod_{p'=3}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}} \prod_{l \in K_{p'}^{(k;n-1)}} \tilde{m}_{l,[i]}^{(n)}! = \prod_{t=3}^{\#\tilde{V}^{(n-1)}} \tilde{m}_{t,[i]}^{(n)}!.$$

It follows that

$$\begin{aligned}\#(\tilde{V}^{(n)}_{i,l} \cap \tilde{V}^{(n)}_{i*} \cap K^{(k:n)}_p) &= \#\{u \in \tilde{V}^{(n)}_{i,l} \cap \tilde{V}^{(n)}_{i*} \mid O.vec_k(u) = O.vec_k(v)\} \\ &= \#\{u \in \tilde{V}^{(n)}_{i,l} \cap \tilde{V}^{(n)}_{i*} \mid \tau(u, t) \in K^{(k:n-1)}_{p_t}\} \\ &= \prod_{p'=3}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}} \frac{(\sum_{l \in K^{(k;n-1)}_{p'}} \tilde{m}^{(n)}_{l,[i]})!}{\prod_{l \in K^{(k;n-1)}_{p'}} \tilde{m}^{(n)}_{l,[i]}!} \\ &= \frac{\prod_{p'=3}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}} (\sum_{l \in K^{(k;n-1)}_{p'}} \tilde{m}^{(n)}_{l,[i]})!}{\prod_{t=3}^{\#\tilde{V}^{(n-1)}} \tilde{m}^{(n)}_{t,[i]}!} .\end{aligned}$$

Therefore from (6.25) and the above equality, we get

$$\#(\tilde{V}^{(n)}_{i*} \cap K^{(k:n)}_p) = \eta_n \times \frac{\prod_{p'=3}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}} (\sum_{l \in K^{(k;n-1)}_{p'}} \tilde{m}^{(n)}_{l,[i]})!}{\prod_{t=3}^{\#\tilde{V}^{(n-1)}} \tilde{m}^{(n)}_{t,[i]}!} . \tag{6.26}$$

Incidentally by (6.24) and (6.16), we calculate

$$\begin{aligned}\frac{\#\tilde{V}^{(n)}_{i*}}{\#\{\pi_k(Con(v)) \mid v \in \tilde{V}^{(n)}_{i*}\}} &= \frac{\eta_n \times \mathrm{Dist}(\tilde{V}^{(n)}_i)}{\{\pi_k(Con(v)) \mid v \in \tilde{V}^{(n)}_{i*}\}} \\ &= \frac{\eta_n \times \{(\tilde{m}'^{(n)}_i - 6)!\}/(\prod_{t=3}^{\#\tilde{V}^{(n-1)}} \tilde{m}^{(n)}_{t,[i]}!)}{\{(\tilde{m}'^{(n)}_i - 6)!\}/\{\prod_{p'=3}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}} (\sum_{l \in K^{(k;n-1)}_{p'}} \tilde{m}^{(n)}_{l,[i]})!\}} \\ &= \eta_n \times \frac{\prod_{p'=3}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}} (\sum_{l \in K^{(k;n-1)}_{p'}} \tilde{m}^{(n)}_{l,[i]})!}{\prod_{t=3}^{\#\tilde{V}^{(n-1)}} \tilde{m}^{(n)}_{t,[i]}!} = \#(\tilde{V}^{(n)}_{i*} \cap K^{(k:n)}_p) .\end{aligned}$$

This implies that, for any $i \in \xi^{-1}_{k:n}(j)$ and any $p \in \kappa^{-1}_{k:n}(j)$ with $p \neq 1, 2$,

$$\eta_n \leq \#(\tilde{V}^{(n)}_{i*} \cap K^{(k:n)}_p) = \#\tilde{V}^{(n)}_{i*}/\#\{\pi_k(Con(v)) \mid v \in \tilde{V}^{(n)}_{i*}\} .$$

Lastly we shall show

$$\#((\tilde{V}^{(n)}_i \backslash \tilde{V}^{(n)}_{i*}) \cap K^{(k:n)}_p) = \eta_n . \tag{6.27}$$

From (6.21), for any $1 \leq l \leq \eta$,

$$\#(\tilde{V}^{(n)}_{i,l} \backslash (\tilde{V}^{(n)}_{i*} \cap \tilde{V}^{(n)}_{i,l})) = 2\#\tilde{V}^{(n-1)} - 3 .$$

Take any $u, v \in \tilde{V}^{(n)}_{i,l} \backslash (\tilde{V}^{(n)}_{i*} \cap \tilde{V}^{(n)}_{i,l})$ with $u \neq v$ and fix them. By (2.3) in Lemma 2.9, it is easy to check

$$\{t \mid \tau(u, t) = v^{(n-1)}_{\min} \text{ or } v^{(n-1)}_{\max}\} \neq \{t \mid \tau(v, t) = v^{(n-1)}_{\min} \text{ or } v^{(n-1)}_{\max}\} . \tag{6.28}$$

Since $K^{(k:n-1)}_1 = \{v^{(n-1)}_{\min}\}$, $K^{(k:n-1)}_2 = \{v^{(n-1)}_{\max}\}$, (6.28) implies

$$O.vec_k(u) \neq O.vec_k(v) . \tag{6.29}$$

So it is easy observation that

$$\#(K^{(k:n)}_p \cap \tilde{V}^{(n)}_{i,l} \backslash (\tilde{V}^{(n)}_{i*} \cap \tilde{V}^{(n)}_{i,l})) = 1$$

for all p with $p \in \kappa_{k:n}^{-1} \circ \xi_{k:n}(i)$ and $p \neq 1, 2$. This implies (6.27) holds. Moreover by (6.29),

$$\#\{p \mid K_p^{(k:n)} \cap \tilde{V}_i^{(n)} \neq \emptyset\} = \#\{p \mid K_p^{(k:n)} \cap \tilde{V}_{i*}^{(n)} \neq \emptyset\} + \#\{p \mid K_p^{(k:n)} \cap (\tilde{V}_i^{(n)} \setminus \tilde{V}_{i*}^{(n)}) \neq \emptyset\}$$
$$\geq \#\{p \mid K_p^{(k:n)} \cap \tilde{V}_{i*}^{(n)} \neq \emptyset\} + (2\#\tilde{V}^{(n-1)} - 3) > 2\#\tilde{V}^{(n-1)} - 3\,.$$

So we finish the proof of (n.9).

(n.10) (3.14) implies that $u, v \in \tilde{V}^{(n)}$ and for any $2 \leq k \leq n-1$, $u \not\sim_1 v$ implies $u \not\sim_k v$. Therefore immediately we see that for any $2 \leq k \leq n-1$, $\#\{\tilde{V}^{(n)}/\sim_1\} > 1$ implies $\#\{\tilde{V}^{(n)}/\sim_k\} > 1$. So it is sufficient to show $\#\{\tilde{V}^{(n)}/\sim_1\} > 1$. By the construction of $\tilde{M}^{(n)}$ and the definition of $\sim_1$, (6.14) in (c.4) implies that $\xi_{1:n}(i) \neq \xi_{1:n}(j)$. So $\#\{\tilde{V}^{(n)}/\sim_1\} > 1$ and we finish the proof.

(n.11) From the definition of $\sim_k$, it is easily seen that for any $i, i' \in \xi_{k:n}^{-1}(j)$ with $i \neq i'$,

$$\{\pi_k(Con(v)) \mid v \in \tilde{V}_{i*}^{(n)}\} = \{\pi_k(Con(v)) \mid v \in \tilde{V}_{i'*}^{(n)}\}\,.$$

And we see that

$$\tilde{V}_j^{(k:n)} = \bigcup_{i \in \xi_{k:n}^{-1}(j)} \tilde{V}_i^{(n)} = \bigcup_{i \in \xi_{k:n}^{-1}(j)} (\tilde{V}_{i*}^{(n)} \cup \tilde{V}_{ic}^{(n)})\,,$$

where $\tilde{V}_{ic}^{(n)} \equiv \tilde{V}_i^{(n)} \setminus \tilde{V}_{i*}^{(n)}$. By the same arguments of (n.9), for any $i, i' \in \xi_{k:n}^{-1}(j)$ we see

$$\{\pi_k(Con(v)) \mid v \in \tilde{V}_{i*}^{(n)}\} \cap \{\pi_k(Con(v)) \mid v \in \tilde{V}_{i'c}^{(n)}\} = \emptyset\,.$$

So we calculate

$$\{\pi_k(Con(v)) \mid v \in \tilde{V}_{i*}^{(n)}\} = \left\{\pi_k(Con(v)) \mid v \in \bigcup_{i' \in \xi_{k:n}^{-1}(j)} \tilde{V}_{i'*}^{(n)}\right\}$$
$$= \left\{\pi_k(Con(v)) \mid v \in \tilde{V}_j^{(k:n)} \setminus \bigcup_{i' \in \xi_{k:n}^{-1}(j)} \tilde{V}_{i'c}^{(n)}\right\}$$
$$= W_j^{(k:n)} \setminus \left\{\pi_k(Con(v)) \mid v \in \bigcup_{i' \in \xi_{k:n}^{-1}(j)} \tilde{V}_{i'c}^{(n)}\right\}.$$

Therefore we get

$$\#\{\pi_k(Con(v)) \mid v \in \tilde{V}_{i*}^{(n)}\} = \#W_j^{(k:n)} - \#\left\{\pi_k(Con(v)) \mid v \in \bigcup_{i' \in \xi_{k:n}^{-1}(j)} \tilde{V}_{i'c}^{(n)}\right\}$$
$$\geq \#W_j^{(k:n)} - \sum_{i' \in \xi_{k:n}^{-1}(j)} \#\{\pi_k(Con(v)) \mid v \in \tilde{V}_{i'c}^{(n)}\}$$
$$= \#W_j^{(k:n)} - \#\xi_{k:n}^{-1}(j)(2\#\tilde{V}^{(n-1)} - 3)\,.$$

So we are done.

(n.12) Take any $i \in \xi_{k:n}^{-1}(j)$ and fix it. Then using Lemma 5.3

$$\# W_j^{(k:n)} = \#\{\pi_k(Con(v)) \mid v \in \tilde{V}_j^{(k:n)}\} \geq \#\{\pi_k(Con(v)) \mid v \in \tilde{V}_{i*}^{(n)}\}$$

$$= \frac{(\bar{\tilde{m}}_{[i]}^{(n)} - 6)!}{\prod_{p'=3}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}} \left(\sum_{l \in K_{p'}^{(k;n-1)}} \tilde{m}_{l,[i]}^{(n)} \right)!} \qquad \text{(by (6.24))}$$

$$> \frac{(\bar{\tilde{m}}_{[i]}^{(n)} - 6)!}{\prod_{p'=1}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}} \left(\sum_{l \in K_{p'}^{(k;n-1)}} \tilde{m}_{l,[i]}^{(n)} \right)!}$$

$$> \frac{((\bar{\tilde{m}}_{[i]}^{(n)} - 6)/e)^{\bar{\tilde{m}}_{[i]}^{(n)} - 6}}{\prod_{p'=1}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}} \left(\left(\sum_{l \in K_{p'}^{(k;n-1)}} \tilde{m}_{l,[i]}^{(n)} + 2 \right) \Big/ e \right)^{\sum_{l \in K_{p'}^{(k;n-1)}} \tilde{m}_{l,[i]}^{(n)} + 2}} \equiv (*) . \tag{6.30}$$

Since $\bar{\tilde{m}}_{[i]}^{(n)} = \sum_{p'=1}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}} \sum_{l \in K_{p'}^{(k;n-1)}} \tilde{m}_{l,[i]}^{(n)}$, we calculate

$$(*) > \prod_{p'=1}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}} \left(\frac{\bar{\tilde{m}}_{[i]}^{(n)} - 6}{\sum_{l \in K_{p'}^{(k;n-1)}} \tilde{m}_{l,[i]}^{(n)} + 2} \right)^{f_{p'}} \times E \equiv (**) , \tag{6.31}$$

where $f_{p'} \equiv \sum_{l \in K_{p'}^{(k;n-1)}} \tilde{m}_{l,[i]}^{(n)}$ and

$$E \equiv \left\{ ((\bar{\tilde{m}}_{[i]}^{(n)} - 6)/e)^6 \times \prod_{p'=1}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}} \left(\left(\sum_{l \in K_{p'}^{(k;n-1)}} \tilde{m}_{l,[i]}^{(n)} + 2 \right) \Big/ e \right)^2 \right\}^{-1} .$$

Incidentally we see that $\#\{\tilde{V}^{(n-1)}/\approx_k\} \leq \#\tilde{V}^{(n-1)}$, $\sum_{l \in K_{p'}^{(k;n-1)}} \tilde{m}_{l,[i]}^{(n)} < \bar{\tilde{m}}_{[i]}^{(n)}$ and $\bar{\tilde{m}}_{[i]}^{(n)} = \bar{m}_i'^{(n)}$. So we see that

$$\prod_{p'=1}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}} \left(\left(\sum_{l \in K_{p'}^{(k;n-1)}} \tilde{m}_{l,[i]}^{(n)} + 2 \right) \Big/ e \right)^2 < \left(\frac{\bar{m}_i'^{(n)} + 2}{e} \right)^{2\#\tilde{V}^{(n-1)}}$$

Using above inequality, we get

$$E > C_i \tag{6.32}$$

where C_i is defined by (6.7). Moreover, for $i \in \xi_{k:n-1}^{-1}(j)$, we can calculate

$$\sum_{l \in K_{p'}^{(k;n-1)}} \tilde{m}_{l,[i]}^{(n)} + 2 = \sum_{j'=1}^{\#V'^{(n-1)}} \sum_{l \in K_{p'}^{(k;n-1)} \cap \tilde{V}_{j'}^{(n-1)}} \tilde{m}_{l,[i]}^{(n)} + 2$$

$$< \sum_{j' \in \xi_{k:n-1}^{-1} \circ \kappa_{k:n-1}(p')} \#(K_{p'}^{(k:n-1)} \cap \tilde{V}_{j'}^{(n-1)}) r_n \theta_{j',i}^{(n)} ,$$

$$\frac{\bar{\tilde{m}}_{[i]}^{(n)} - 6}{\sum_{l \in K_{p'}^{(k;n-1)}} \tilde{m}_{l,[i]}^{(n)} + 2} > \frac{\bar{m}_i'^{(n)} - 6}{\sum_{j' \in \xi_{k:n-1}^{-1} \circ \kappa_{k:n-1}(p')} \#(K_{p'}^{(k:n-1)} \cap \tilde{V}_{j'}^{(n-1)}) r_n \theta_{j',i}^{(n)}}$$

$$> \frac{\bar{m}_i'^{(n)} - 6}{r_n \sum_{j' \in \xi_{k:n-1}^{-1} \circ \kappa_{k:n-1}(p')} \dfrac{\#\tilde{V}_{j'}^{(n-1)} \theta_{j',i}^{(n)}}{\#\{\pi_k(Con(v)) \mid v \in \tilde{V}_{j'*}^{(n-1)}\}}}$$

$$>\frac{\bar{m}_i'^{(n)}-6}{r_n \sum_{j'\in\xi_{k:n-1}^{-1}\circ\kappa_{k:n-1}(p')} m_{j',i}'^{(n)}}\times(\#W^{(k:n-1)}_{\kappa_{k:n-1}(p')}-\#\xi_{k:n-1}^{-1}\circ\kappa_{k:n-1}(p')(2\#\tilde{V}^{(n-2)}-3)).$$

Therefore

$$\prod_{p'=1}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}}\left((\bar{\tilde{m}}_{[i]}^{(n)}-6)/\left(\sum_{l\in K_{p'}^{(k;n-1)}}\tilde{m}_{l,[i]}^{(n)}+2\right)\right)^{f_{p'}}$$
$$>\prod_{p'=1}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}}\{D_{p',i}\times(\#W^{(k:n-1)}_{\kappa_{k:n-1}(p')}-\#\xi_{k:n-1}^{-1}\circ\kappa_{k:n-1}(p')(2\#\tilde{V}^{(n-2)}-3))\}^{f_{p'}},\quad(6.33)$$

where $D_{p',i}$ is defined by (6.8). From (6.31), (6.32) and (6.33), we see that

$$(**)>C_i\times\prod_{p'=1}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}}\{D_{p',i}\times(\#W^{(k:n-1)}_{\kappa_{k:n-1}(p')}-\#\xi_{k:n-1}^{-1}\circ\kappa_{k:n-1}(p')(2\#\tilde{V}^{(n-2)}-3))\}^{f_{p'}}.\quad(6.34)$$

Using the claim (c.ii), we get from (6.30), (6.31) and (6.34),

$$\#W_j^{(k:n)}>\prod_{p'=1}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}}(\#W^{(k:n-1)}_{\kappa_{k:n-1}(p')}-\#\xi_{k:n-1}^{-1}\circ\kappa_{k:n-1}(p')(2\#\tilde{V}^{(n-2)}-3))^{f_{p'}}.$$

Since

$$\#\mathscr{P}[\tilde{V}_j^{(k:n)}]=\sum_{p'=1}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}}\left(\#\mathscr{P}[\tilde{V}^{(k:n-1)}_{\kappa_{k:n-1}(p')}]\sum_{l\in K_{p'}^{(k;n-1)}}\tilde{m}_{l,[i]}^{(n)}\right)=\prod_{p'=1}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}}\#\mathscr{P}[\tilde{V}^{(k:n-1)}_{\kappa_{k:n-1}(p')}]f_{p'},\quad(6.35)$$

we can calculate as follows:

$$\frac{\#W_j^{(k:n)}}{(2\alpha_{k-1})^{\#\mathscr{P}[\tilde{V}_j^{(k:n)}]}}>\prod_{p'=1}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}}\left(\frac{\#W^{(k:n-1)}_{\kappa_{k:n-1}(p')}-\#\xi_{k:n-1}^{-1}\circ\kappa_{k:n-1}(p')(2\#\tilde{V}^{(n-2)}-3)}{(2\alpha_{k-1})^{\#\mathscr{P}[\tilde{V}^{(k:n)}_{\kappa_{k:n-1}(p')}]}}\right)^{f_{p'}}$$
$$>X_k^{\bar{m}_i'^{(n)}}\qquad\text{(by (c.iii))}.$$

Incidentally we see that

$$\#\xi_{k:n}^{-1}(j)\le\#\left\{\tilde{M}_{[t]}^{(n)}\;\middle|\;\sum_{l\in K_{p'}^{(k;n-1)}}\tilde{M}_{l,[t]}^{(n)}=\sum_{l\in K_{p'}^{(k;n-1)}}\tilde{M}_{l,[i]}^{(n)}\text{ for all }1\le p'\le\#\{\tilde{V}^{(n-1)}/\approx_k\}\right\}$$
$$\le\#\{\tilde{M}_{[t]}^{(n)}\mid\bar{\tilde{m}}_{[t]}^{(n)}=\bar{\tilde{m}}_{[i]}^{(n)}\}=\#\{M_t'^{(n)}\mid\bar{m}_t'^{(n)}=\bar{m}_i'^{(n)}\}$$
$$\le\#\{M_s^{(t_{n-1}+1,t_n)}\mid\bar{m}_s^{(t_{n-1}+1,t_n)}=\bar{m}_i'^{(n)}\}\le\#\{M_s^{(t_n)}\mid\bar{m}_s^{(t_n)}\le\bar{m}_i'^{(n)}\}$$
$$\le2^{\bar{m}_i'^{(n)}}\qquad\text{(by (6.1))}.\quad(6.36)$$

So we have

$$
\begin{aligned}
\frac{\#\xi_{k:n}^{-1}(j)}{(2\alpha_{k-1})^{\#\mathscr{P}[\tilde{V}_j^{(k:n)}]}} &\leq \frac{2^{\bar{m}_i'^{(n)}}}{(2\alpha_{k-1})^{\#\mathscr{P}[\tilde{V}_j^{(k:n)}]}} \qquad \text{(by (6.36))} \\
&= \prod_{p'=1}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}} 2^{f_{p'}} \Bigg/ \prod_{p'=1}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}} (2\alpha_{k-1})^{\#\mathscr{P}[\tilde{V}_{\kappa_{k:n-1}(p')}^{(k:n-1)}]f_{p'}} \qquad \text{(by (6.35))} \\
&< \prod_{p'=1}^{\#\{\tilde{V}^{(n-1)}/\approx_k\}} (\alpha_{k-1}{}^{\#\mathscr{P}[\tilde{V}_{\kappa_{k:n-1}(p')}^{(k:n-1)}]})^{f_{p'}} = \alpha_{k-1}^{-\bar{m}_i'^{(n)}} . \qquad (6.37)
\end{aligned}
$$

In addition we use the claim (c.iii) and (6.37). Then we get

$$
\begin{aligned}
\frac{\# W_j^{(k:n)} - \#\xi_{k:n}^{-1}(j)(2\#\tilde{V}^{(n-1)}-3)}{(2\alpha_{k-1})^{\#\mathscr{P}[\tilde{V}_j^{(k:n)}]}} &> \frac{\# W_j^{(k:n)}}{(2\alpha_{k-1})^{\#\mathscr{P}[\tilde{V}_j^{(k:n)}]}} - \alpha_{k-1}^{-\bar{m}_i'^{(n)}}(2\#\tilde{V}^{(n-1)}-3) \\
&> X_k^{\bar{m}_i'^{(n)}} - (X_k^{\bar{m}_i'^{(n)}} - 1) = 1 .
\end{aligned}
$$

Therefore we finish the proof.

(n.13) Let $\alpha_{k,n}$ be the positive number satisfying the following equality:

$$
\sum_{j=1}^{\#\{\tilde{V}^{(n)}/\sim_k\}} \frac{\# W_j^{(k:n)}}{\alpha_{k,n}^{\#\mathscr{P}[\tilde{V}_j^{(k:n)}]}} = 1 .
$$

The above equality implies that for all $1 \leq j \leq \#\{\tilde{V}^{(n)}/\sim_k\}$, $\# W_j^{(k:n)} < \alpha_{k,n}^{\#\mathscr{P}[\tilde{V}_j^{(k:n)}]}$. By (n.12) we see $2\alpha_{k-1} < \alpha_{k,n}$. Therefore (n.13) holds.

So we finish the proof of (c.ix).

Therefore we can construct recursively the simple unordered Bratteli diagram $\mathscr{D}' = (V', E')$ and the simple ordered Bratteli diagram $\tilde{\mathscr{D}} = (\tilde{V}, \tilde{E}, \tilde{\leq})$ satisfying, for all $n \geq 2$, the following conditions:

(A) $K_0(V, E) \cong K_0(V', E')$ as dimension groups via an isomorphism preserving distinguished order units,

(B) $K_0(V', E') \cong K_0(\tilde{V}, \tilde{E})$ as dimension groups via an isomorphism preserving distinguished order units,

(C) $(\tilde{V}, \tilde{E})$ satisfies Assumption 2.4,

(D) there is the partial order $\tilde{\leq}$ on $\tilde{E}$ such that $\tilde{\leq}$ satisfies Assumption 2.5, 2.6, 2.7 and 2.8,

(E) $2\alpha_{n-1} < \alpha_n$, $\sum_{j=1}^{\#\{\tilde{V}^{(n)}/\approx\}} \#\tilde{V}_j^{(n)} / \alpha_n^{\#\mathscr{P}[\tilde{V}_j^{(n)}]} = 1$,

(F) for any k with $1 \leq k < n$, $2\alpha_{k-1} < \alpha_{k,n}$ and $\sum_{j=1}^{\#\{\tilde{V}^{(n)}/\sim_k\}} \# W_j^{(k:n)} / \alpha_{k,n}^{\#\mathscr{P}[\tilde{V}_j^{(k:n)}]} = 1$.

Let (Y, S) be the Cantor system defined by $\tilde{\mathscr{D}}$. By (A) and (B), (Y, S) is strongly orbit equivalent to (X, T). Since $\tilde{\mathscr{D}}$ satisfies (C) and (D), we see by Lemma 3.9 and (F)

$$
h(S_k(t)) = \log \alpha_{k,k+t} > \log \alpha_{k-1} + \log 2 . \qquad (6.38)
$$

And from Lemma 3.7 and (6.38), we obtain

$$
h(S_k) = \lim_{t\to\infty} h(S_k(t)) > \log \alpha_{k-1} + \log 2 . \qquad (6.39)
$$

By (E), we see $\lim_{k\to\infty}\alpha_k=\infty$. Therefore by Lemma 3.6 and (6.39), we can get

$$h(S)=\lim_{k\to\infty}h(S_k)=\infty\ .$$

So we finish the proof of the theorem. □

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