

Another Type of Instanton Bundles on $Gr_2(\mathbf{C}^{n+2})$

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1. Introduction.

The purpose of the present paper is to show the existence of an anti-self-dual connection on 2-plane complex Grassmannian $Gr_2(\mathbf{C}^{n+2})$, to classify instantons and to describe the moduli space. The reason why we use the terminology “instanton” is that our anti-self-dual connections are nothing but 1-instantons in the case $n=1$ (CP^2). However we also have proved that there exists another generalization of instantons on CP^2 to $Gr_2(\mathbf{C}^{n+2})$ [N-N2]. The structure group reflects the main difference between them. In the present paper, $SU(r)$ -bundles are taken into account, while in [N-N2], $Sp(r)$ -bundles are considered. By the isomorphism $SU(2) \cong Sp(1)$, our two series of generalizations coincide with instantons on CP^2 in the case $n=1$. On HP^n , which is another typical example of quaternion-Kähler manifolds, there exists a generalization of instantons on 4-dimensional sphere $S^4 \cong HP^1$. This instanton bundle also has $Sp(r)$ as a structure group and so, odd Chern classes of this bundle vanish. Since the cohomology groups $H^{4i}(HP^n, \mathbf{Z}) \cong \mathbf{Z}$ for $i=0, 1, \dots, n$ and the others vanish, odd Chern classes of an arbitrary bundle on HP^n necessarily vanish. On the contrary, our examples have the non-vanishing third Chern classes. In higher dimensional case, these are the first examples such that higher degree odd Chern classes do not vanish.

As for the existence of anti-self-dual connections, Mamone-Capria and Salamon first give the above examples of instanton bundles on HP^n and prove that a well-known Horrocks bundle on CP^5 can be obtained as the pull-back of an anti-self-dual bundle on HP^2 [M-S]. Applying the monad given by Donaldson [D] to higher dimensional case, $Sp(r)$ -instanton bundles on $Gr_2(\mathbf{C}^{n+2})$ are exhibited in [N-N2]. In both cases, the typical examples of 1-instantons are homogeneous bundles with canonical connections. The author determines all irreducible homogeneous bundles with anti-self-dual canonical connections over compact quaternion symmetric spaces and give a deformation of canonical connections [Na-3]. Adapting this point of view, we will deform the canonical connection on a direct sum of a line bundle and a homogeneous bundle on $Gr_2(\mathbf{C}^{n+2})$.

To classify anti-self-dual bundles, we make use of the theory of monads on the

Salamon twistor space.

MAIN THEOREM 1. *Let E be a vector bundle on $Gr_2(\mathbf{C}^{n+2})$ ($n \geq 2$) which has*

- (1) *an anti-self-dual connection with the structure group $SU(r)$, where $r \geq 3$, and*
- (2) *$c_2(E) = xy$, $c_3(E) = xy(x - y)$ and $c_4(E) = x^3y - x^2y^2 + xy^3$.*

We denote by \tilde{E} the pull back bundle of E on the twistor space F .

Then, \tilde{E} is the cohomology bundle of the following monad,

$$(M) \quad \mathcal{O}(-1, 0) \rightarrow \underline{V} \oplus \mathcal{O}(-1, 1) \rightarrow \mathcal{O}(0, 1),$$

where \underline{V} is a trivial bundle $F \times V$ of rank $r + 1$.

In [M-S], Mamone-Capria and Salamon derive a monad using Beilinson's spectral sequence of CP^{2n+1} . Then they need some vanishing theorems. First a vanishing theorem about an anti-self-dual bundle on higher dimensional positive quaternion-Kähler manifold is obtained by the author [Na-2]. This vanishing theorem and an inductive argument give a complete classification of $Sp(r)$ -instantons on HP^n [K-N]. This is a direct generalization of ADHM-construction [A]. Next it is showed that there exists a spectral sequence for holomorphic bundles on the generalized flag manifold F^{2n+1} which is the twistor space of $Gr_2(\mathbf{C}^{n+2})$ [N-N2] (see §2). (In the case $n=1$, our spectral sequence coincides with Buchdahl's spectral sequence [Bu].) This spectral sequence, combined with extended vanishing theorems [N-N1], implies a classification of $Sp(r)$ -instantons on $Gr_2(\mathbf{C}^{n+2})$ [N-N2]. In both cases, since $Sp(r)$ are structure groups, the dual bundles are isomorphic to the original bundles. However, a similar isomorphism can not be carried in our case, because the third Chern class does not vanish. Hence, we need to take a pair of a bundle and its dual into account. (Note that our monad (M) is not self-dual in the sense of [O-S-S, p. 282].) We use a spectral sequence in the case $n=2$ ($Gr_2(\mathbf{C}^4)$) and an induction in higher dimensional cases. In the latter case, a slightly stronger theorem (Main Theorem 1') will be proved. This argument also makes a proof in [N-N2] be complete. Main Theorem 1 can be regarded as a generalization of a classification by Buchdahl [Bu].

Finally the moduli space will be described.

MAIN THEOREM 2. *The moduli space of anti-self-dual connections on E satisfying the hypothesis in Main Theorem 1 is identified with an open cone over $\mathbf{P}(\mathbf{C}^{n+2})$, where \mathbf{C}^{n+2} is the standard representation space of $SU(n+2)$.*

In [N-N3], following Donaldson [D], we give a description of the moduli of $Sp(n)$ -instantons on $Gr_2(\mathbf{C}^{n+2})$ in a coordinate-based fashion by using the embedding of F^{2n+1} into $CP^{n+1} \times CP^{n+1*}$. On the other hand, the author shows that this moduli can be described by the representation theory via the Bott-Borel-Weil theorem and this moduli is identified with an open cone over $\mathbf{P}(\wedge^2 \mathbf{C}^{n+2})$ where $\wedge^2 \mathbf{C}^{n+2}$ is one of the irreducible representation spaces of $SU(n+2)$ [Na-3]. Now, we also make use of the representation of $SU(n+2)$ in a slightly different manner from [Na-3]. For example,

this method makes us enable to observe the degeneration of anti-self-dual connections easily. In this case, the set of singular points is only a quaternion hypersurface $Gr_2(\mathbf{C}^{n+1})$. In the case of $Sp(n)$ -instantons on $Gr_2(\mathbf{C}^{n+2})$, one of \mathbf{HP}^i 's where $i=0, 1, \dots, [n/2]$ appears as the singular set [N-N3]. The structure of $SU(n+2)$ -orbits in \mathbf{C}^{n+2} and $\wedge^2 \mathbf{C}^{n+2}$ causes these phenomena.

2. Preliminaries.

Let M be a connected quaternion-Kähler manifold with non-zero scalar curvature and Z be the Salamon twistor space of M [S].

From the definition, the vector bundle $\wedge^2 T^*M$ has the following holonomy invariant decomposition:

$$\wedge^2 T^*M = S^2\mathbf{H} \oplus S^2\mathbf{E} \oplus (S^2\mathbf{H} \oplus S^2\mathbf{E})^\perp,$$

where \mathbf{H} and \mathbf{E} are vector bundles associated with the standard representations of $Sp(1)$ and $Sp(n)$, respectively. For example, \mathbf{H} is a tautological quaternionic line bundle when the base space is a quaternionic projective space \mathbf{HP}^n .

DEFINITION 2.1. An $\omega \in \Omega^2 T^*M$ is called a *self-dual* (resp. *anti-self-dual*) form if $\omega \in \Gamma(S^2\mathbf{H})$ (resp. $\Gamma(S^2\mathbf{E})$).

This definition reduces to the usual one on a 4-dimensional oriented Riemannian manifold in the case $n=1$. We shall investigate metric connections on a complex vector bundle F equipped with a hermitian metric h .

DEFINITION 2.2 ([G-P], [M-S] and [Ni]). A connection ∇ is called (resp. *anti*-)self-dual if its curvature 2-form R^∇ is a (resp. anti-)self-dual form.

THEOREM 2.3 ([G-P], [M-S] and [Ni]). *Self-dual and anti-self-dual connections are Yang-Mills connections.*

REMARK. If M is compact, self-dual and anti-self-dual connections actually minimize the Yang-Mills functional ([G-P] and [M-S]). Moreover, it is known that there is an essentially unique non-flat self-dual connection over a simply connected quaternion-Kähler manifold whose dimension is greater than or equal to 8 [Na-1].

Let E be an anti-self-dual bundle on M and \tilde{E} be the pull-back bundle of E on Z . Then, it is known that \tilde{E} is a holomorphic bundle with the induced structure ([M-S] and [Ni]). The author showed a vanishing theorem about \tilde{E} at first and this vanishing theorem was extended as follows.

THEOREM 2.4 ([Na-2] and [N-N]). *Let M be a $4n$ -dimensional compact quaternion-Kähler manifold with positive scalar curvature and Z be the twistor space of M . If \tilde{E} is the pull back bundle of E on M which has a unitary structure and an anti-self-dual*

connection, then we have

$$\begin{aligned} H^i(Z, \tilde{E}(k)) &= 0 \quad \text{for } 1 \leq i \leq n \text{ and } i+k+1 < 0, \\ H^1(Z, \tilde{E}(-2)) &= 0, \\ H^2(Z, \tilde{E}(-3)) &= 0 \quad \text{for } n \geq 2, \end{aligned}$$

and

$$\begin{aligned} H^i(Z, \tilde{E}(k)) &= 0 \quad \text{for } n+1 \leq i \leq 2n \text{ and } i+k > 0, \\ H^{2n-1}(Z, \tilde{E}(-2n+1)) &= 0 \quad \text{for } n \geq 2, \\ H^{2n}(Z, \tilde{E}(-2n)) &= 0. \end{aligned}$$

REMARK. A line bundle $\mathcal{O}(1)$ on the twistor space corresponds to L in [S].

From now on, we focus our attention on a complex Grassmanian manifold of 2-planes:

$$Gr_2(\mathbf{C}^{n+2}) = SU(n+2)/S(U(2) \times U(n)).$$

The twistor space of this manifold is a generalized flag manifold F^{2n+1} :

$$F^{2n+1} = SU(n+2)/S(U(1) \times U(n) \times U(1)).$$

In other words, F^{2n+1} is represented as follows:

$$F^{2n+1} = \{(l, V) \mid 0 \in l \subset V \subset \mathbf{C}^{n+2}, \dim l = 1 \text{ and } \dim V = n+1\},$$

where l and V are complex vector subspaces. Then the twistor fibration $\pi: F \rightarrow Gr_2(\mathbf{C}^{n+2})$ is

$$\pi((l, V)) = l \oplus V^\perp.$$

So, note that this is *not* a holomorphic fibration [S]. (When no confusion can arise, the dimension $2n+1$ will be omitted.)

We give a quick review of the geometry of this generalized flag manifold F^{2n+1} and refer to [N-N2] for more details.

First, we describe the ring structure of the cohomology of F . The twistor space F^{2n+1} of $Gr_2(\mathbf{C}^{n+2})$ has double holomorphic fibrations to \mathbf{P}^{n+1} and \mathbf{P}^{n+1*} such that

$$\begin{aligned} p_1: (l, V) &\rightarrow [l], \\ p_2: (l, V) &\rightarrow [V]. \end{aligned}$$

We denote by x and y pull-back elements of $H^2(F, \mathbf{Q})$ of the standard positive generators of $H^2(\mathbf{P}^{n+1}, \mathbf{Z})$ and $H^2(\mathbf{P}^{n+1*}, \mathbf{Z})$ respectively. Then, by Leray-Hirsch theorem, the cohomology ring $H^*(F, \mathbf{Q})$ is generated by x and y . In our proof of Main Theorem 1, we need to know the ring structure of cohomology in the case F^5 ($n=2$).

Bernstein-Gelfand-Gelfand theorem [B-G-G] gives that there is a relation on F^5 such that

$$(2.1) \quad \begin{aligned} x^3 - x^2y + xy^2 - y^3 &= 0, \\ x^4 &= 0, \quad x^3y - x^2y^2 + xy^3 = 0, \quad y^4 = 0, \\ x^3y^2 - x^2y^3 &= 0. \end{aligned}$$

The fundamental class of F^5 is $x^3y^2 = x^2y^3$.

On the other hand, using the twistor fibration, we have $H^*(Gr_2(\mathbf{C}^{n+2}), \mathbf{Q})$ is regarded as a subring of $H^*(F, \mathbf{Q})$ [B-G-G]. Hence, an element of $H^*(Gr_2(\mathbf{C}^{n+2}), \mathbf{Q})$ may be written with x and y .

Next, we introduce a spectral sequence of Beilinson type for a holomorphic vector bundle on the flag manifold. To represent this spectral sequence, we define vector bundles on F .

DEFINITION 2.5. A vector bundle Q_1 denotes the quotient bundle of $P_1^* \Omega_{\mathbf{P}^{n+1}}$ by $\mathcal{O}(-1, -1)$, where $\Omega_{\mathbf{P}^{n+1}}$ is the holomorphic cotangent bundle on \mathbf{P}^{n+1} and in general, $\mathcal{O}(p, q)$ is the line bundle $p_1^* \mathcal{O}(p) \otimes p_2^* \mathcal{O}(q)$. The quotient bundle Q_2 is defined in a similar way:

$$\begin{aligned} 0 &\rightarrow \mathcal{O}(-1, -1) \rightarrow p_1^* \Omega_{\mathbf{P}^{n+1}} \rightarrow Q_1 \rightarrow 0, \\ 0 &\rightarrow \mathcal{O}(-1, -1) \rightarrow p_2^* \Omega_{\mathbf{P}^{n+1}} \rightarrow Q_2 \rightarrow 0. \end{aligned}$$

Using vector bundles Q_1 and Q_2 , we can show an analogue of theorem of Beilinson on \mathbf{P}^m and refer to [N-N2] for a proof of the next proposition.

PROPOSITION 2.6. For an arbitrary holomorphic vector bundle S on F , there exists a spectral sequence $E_r^{p,q}$ converging to

$$E_\infty^{p,q} = \begin{cases} \sum_{p=0}^{2n+1} E_\infty^{-p,p} = S & \text{if } p+q=0 \\ 0 & \text{otherwise.} \end{cases}$$

The E_1 -terms satisfy exact sequences

$$\begin{aligned} \cdots \rightarrow \sum_{r=0}^{p-1} H^q(F, \wedge^r Q_1^* \otimes S(-p, 0)) \otimes \wedge^{p-1-r} Q_2^*(0, -p) &\rightarrow E_1^{-p,q} \\ &\rightarrow \sum_{r=0}^p H^q(F, \wedge^r Q_1^* \otimes S(-p, 0)) \otimes \wedge^{p-r} Q_2^*(0, -p) \rightarrow \cdots \end{aligned}$$

where, for example, $S(p, q)$ means $S \otimes \mathcal{O}(p, q)$ and \sum denotes the direct sum.

Since the above spectral sequence is used in the case $n=2$, we give vanishing theorems for anti-self-dual bundles on $Gr_2(\mathbf{C}^4)$. These vanishing theorems can be obtained from Theorem 2.4 and an induction argument ([N-N2; Theorem 4.10]).

THEOREM 2.7. *Let E be an anti-self-dual bundle with a hermitian structure on $Gr_2(\mathbb{C}^4)$ and \tilde{E} be the pull back bundle on the twistor space F^5 . Then, we have*

$$\begin{aligned} H^0(F^5, \tilde{E}(p, q)) &= 0 \quad \text{if } p+q \leq -1, & H^1(F^5, \tilde{E}(p, q)) &= 0 \quad \text{if } p+q \leq -2, \\ H^2(F^5, \tilde{E}(p, q)) &= 0 \quad \text{if } p+q \leq -4, & H^3(F^5, \tilde{E}(p, q)) &= 0 \quad \text{if } p+q \geq -2, \\ H^4(F^5, \tilde{E}(p, q)) &= 0 \quad \text{if } p+q \geq -4, & H^5(F^5, \tilde{E}(p, q)) &= 0 \quad \text{if } p+q \geq -5. \end{aligned}$$

The twistor space F^5 is a homogeneous Kähler manifold and line bundles $\mathcal{O}(p, q)$ are homogeneous bundles on F^5 . Hence, by the Bott-Borel-Weil theorem, we can know the dimension of the cohomology groups for $\mathcal{O}(p, q)$ (see, for example, [K]).

THEOREM 2.8. *There exist the following formulae:*

$$\dim H^i(F^5, \mathcal{O}(p, q)) = (-1)^i \frac{1}{12} (p+1)(p+2)(q+1)(q+2)(p+q+3)$$

if

$$\begin{aligned} i=0 & \quad \text{for } p \geq 0 \text{ and } q \geq 0, \\ i=2 & \quad \text{for } p \leq -3 \text{ and } p+q \geq -2 \text{ or } q \leq -3 \text{ and } p+q \geq -2, \\ i=3 & \quad \text{for } p \geq 0 \text{ and } p+q \leq -4 \text{ or } q \geq 0 \text{ and } p+q \leq -4, \\ i=5 & \quad \text{for } p \leq -3 \text{ and } q \leq -3, \end{aligned}$$

and the other cohomology groups vanish.

Finally, we introduce the Ward correspondence. To do so, we make use of the real structure σ on the twistor space which is induced by the quaternionic structure [S].

WARD CORRESPONDENCE. *There is a one-to-one correspondence between anti-self-dual bundles with unitary structures on a quaternion-Kähler manifold M and holomorphic vector bundles E on the twistor space such that*

- (1) *the restricted bundles $E|_{\mathbb{P}_x^1}$ to the fibre \mathbb{P}_x^1 are trivial for all $x \in M$, and*
- (2) *there is an isomorphism $\tau: E \rightarrow \sigma^* \bar{E}^*$ with $(\sigma^* \bar{\tau})^* = \tau$ which induces a positive definite hermitian form on sections of $E|_{\mathbb{P}_x^1}$ for all $x \in M$.*

3. Classification.

In this section, we give a proof of Main Theorem 1. We employ an induction argument about the dimension of the base manifold $Gr_2(\mathbb{C}^{n+2})$. Therefore, we classify anti-self-dual bundles E satisfying the hypothesis of Main Theorem 1 on $Gr_2(\mathbb{C}^4)$ ($n=2$) at first.

REMARK. Throughout this section, we do not distinguish between an anti-self-dual bundle on $Gr_2(\mathbb{C}^{n+2})$ and its pull-back on F^{2n+1} , and we use the same symbol E for both.

LEMMA 3.1. Let $\chi(E(p, q))$ be the holomorphic Euler characteristics for $E(p, q)$ on F^5 :

$$\chi(E(p, q)) = \sum_{i=0}^5 (-1)^i \dim H^i(F^5, E(p, q)).$$

Then we have

$$\begin{aligned} \chi(E(p, q)) = r & \left\{ 1 + \frac{11}{6}(p+q) + (p+q)^2 + \frac{5}{4}pq + \frac{1}{6}(p+q)^3 + \frac{7}{6}pq(p+q) \right. \\ & \left. + \frac{1}{4}pq(p+q)^2 + \frac{1}{4}p^2q^2 + \frac{1}{12}p^2q^2(p+q) \right\} \\ & - \left\{ 3 + \frac{5}{2}(p+q) + \frac{3}{2}p + \frac{1}{2}(p+q)^2 + 2pq + \frac{1}{2}p^2 + \frac{1}{2}pq(p+q) \right\}. \end{aligned}$$

PROOF. Note that $c_4(E) = 0$, because of the relation (2.1). A direct computation shows that

$$\begin{aligned} ch(E) &= r - xy + \frac{1}{2}xy(x-y) + \frac{1}{12}x^2y^2 - \frac{1}{24}x^2y^2(x-y), \\ ch(\mathcal{O}(p, q)) &= 1 + (px + qy) + \frac{1}{2}(px + qy)^2 + \frac{1}{6}(px + qy)^3 + \frac{1}{24}(px + qy)^4 \\ & \quad + \frac{1}{120}(px + qy)^5, \\ td(F^5) &= 1 + \frac{3}{2}(x+y) + \left\{ (x+y)^2 + \frac{1}{3}xy \right\} + \left\{ \frac{3}{8}(x+y)^3 + \frac{1}{2}xy(x+y) \right\} \\ & \quad + \frac{11}{6}x^2y^2 + \frac{1}{2}x^2y^2(x+y), \end{aligned}$$

where ch means the Chern character and td means the Todd class. Then the Hirzebruch-Riemann-Roch theorem and our relations (2.1) yield our desired result. \square

COROLLARY 3.2. Under the same notation as in Lemma 3.1, we have

$$\chi(E(p, q)) = \begin{cases} 0 & \text{if } p+q = -3, \\ \frac{1}{12}(q+1)(q+2)\{rq(q+1)+6\} & \text{if } p+q = -2, \\ q(q+2)\{\frac{r}{6}(q+1)(q-1)+1\} & \text{if } p+q = -1, \\ r-3+\frac{q}{4}\{rq(q^2-5)+6(q+1)\} & \text{if } p+q = 0, \end{cases}$$

and so

$$\chi(E(-1, -1)) = \chi(E(0, -2)) = \chi(E(-1, 0)) = \chi(E(1, -2)) = 0$$

and $\chi(E) = r - 3$.

We also consider the dual bundle E^* . Note that we can obtain a similar formulae for the bundle E^* , if we interchange the roles of p and q .

Next, we introduce two divisors using the holomorphic fibrations $p_1 : F^5 \rightarrow \mathbf{P}^3$ and $p_2 : F^5 \rightarrow \mathbf{P}^{3*}$. We fix linear subspaces \mathbf{P}^2 and \mathbf{P}^{2*} in \mathbf{P}^3 and \mathbf{P}^{3*} respectively, in such a way that the intersection $p_1^{-1}(\mathbf{P}^2) \cap p_2^{-1}(\mathbf{P}^{2*})$ is the twistor space F^3 of $Gr_2(\mathbf{C}^3)$. A divisor $p_1^{-1}(\mathbf{P}^2)$ is denoted by Y_1 and $p_2^{-1}(\mathbf{P}^{2*})$ is denoted by Y_2 . From our definition, we get exact sequences of sheaves:

$$(3.1) \quad 0 \rightarrow \mathcal{O}(-1, 0) \rightarrow \mathcal{O} \rightarrow \mathcal{O}_{Y_1} \rightarrow 0,$$

$$(3.2) \quad 0 \rightarrow \mathcal{O}_{Y_1}(0, -1) \rightarrow \mathcal{O}_{Y_1} \rightarrow \mathcal{O}_{F^3} \rightarrow 0,$$

$$(3.3) \quad 0 \rightarrow \mathcal{O}(0, -1) \rightarrow \mathcal{O} \rightarrow \mathcal{O}_{Y_2} \rightarrow 0,$$

$$(3.4) \quad 0 \rightarrow \mathcal{O}_{Y_2}(-1, 0) \rightarrow \mathcal{O}_{Y_2} \rightarrow \mathcal{O}_{F^3} \rightarrow 0,$$

where $\mathcal{O}_{Y_i}(p, q)$ denotes a restriction of $\mathcal{O}(p, q)$ to Y_i for $i = 1, 2$.

The next lemma has been shown implicitly in [N-N2]. Buchdahl's vanishing theorems [Bu] and Theorem 2.7 and the exact sequences (3.1–3.4) imply the desired result. (For more details, see the paragraph before Theorem 4.10 in [N-N2].)

LEMMA 3.3. *Let E be an arbitrary anti-self-dual bundle on $Gr_2(\mathbf{C}^4)$. Then we have the following vanishing theorems:*

$$H^0(Y_i, E(p, q)) = 0 \quad \text{if } p + q \leq -1, \quad H^1(Y_i, E(p, q)) = 0 \quad \text{if } p + q \leq -2,$$

$$H^3(Y_i, E(p, q)) = 0 \quad \text{if } p + q \geq -3, \quad H^4(Y_i, E(p, q)) = 0 \quad \text{if } p + q \geq -4,$$

where $i = 1$ or 2 .

We denote by $h^i(E(p, q))$ the dimension of $H^i(F^5, E(p, q))$.

PROPOSITION 3.4. *Let E be an anti-self-dual bundle on $Gr_2(\mathbf{C}^4)$ satisfying the hypothesis of Main Theorem 1. Then we have*

$$h^2(E(p, q)) = 0 \quad \text{if } p = 0, -1 \text{ or } q = -1, -2 \text{ or } p \geq 1 \text{ and } q \geq 0,$$

$$h^2(E^*(p, q)) = 0 \quad \text{if } p = -1, -2 \text{ or } q = 0, -1 \text{ or } p \geq 0 \text{ and } q \geq 1,$$

$$h^3(E(p, q)) = 0 \quad \text{if } p = -1, -2 \text{ or } q = -2, -3 \text{ or } p \leq -3 \text{ and } q \leq -4,$$

$$h^3(E^*(p, q)) = 0 \quad \text{if } p = -2, -3 \text{ or } q = -1, -2 \text{ or } p \leq -4 \text{ and } q \leq -3.$$

PROOF. When we make use of Serre duality and the isomorphism $K_{F^5} \cong \mathcal{O}(-3, -3)$ (K_{F^5} is the canonical bundle on F^5), it suffices to prove vanishing theorems about h^2 .

Step 1. From Theorem 2.7 and Corollary 3.2, we obtain

$$h^2(E(-1, -1)) = h^2(E(0, -2)) = h^2(E^*(-1, -1)) = h^2(E^*(-2, 0)) = 0.$$

We tensor the sequence (3.1) with $E(p, q)$ and take the long exact sequence of cohomology groups

$$\cdots \rightarrow H^1(Y_1, E(p, q)) \rightarrow H^2(F^5, E(p-1, q)) \rightarrow H^2(F^5, E(p, q)) \rightarrow \cdots$$

This, together with Lemma 3.3 and $h^2(E(-1, -1)) = h^2(E(0, -2)) = 0$, implies that $h^2(E(-2, -1)) = h^2(E(-1, -2)) = 0$. In a similar way, we get $h^2(E^*(-2, -1)) = h^2(E^*(-3, 0)) = 0$. Applying Serre duality, we obtain $h^3(E(0, -3)) = h^3(E^*(-1, -2)) = 0$. These vanishing theorems, Theorem 2.7 and Corollary 3.2 yield that $h^2(E(0, -3)) = h^2(E^*(-1, -2)) = 0$.

Step 2. If we use Serre duality, the results in Step 1 imply

$$h^3(E(-2, -2)) = h^3(E(-1, -3)) = h^3(E(-1, -2)) = h^3(E(0, -3)) = 0,$$

$$h^3(E^*(-2, -2)) = h^3(E^*(-3, -1)) = h^3(E^*(-1, -2)) = h^3(E^*(-2, -1)) = 0.$$

The same argument as in the last part of Step 1 gives that $h^3(E(-2, -1)) = h^3(E^*(-3, 0)) = 0$. Next, we take the same long exact sequence as in Step 1:

$$\cdots \rightarrow H^2(F^5, E(p, q)) \rightarrow H^2(Y_1, E(p, q)) \rightarrow H^3(F^5, E(p-1, q)) \rightarrow \cdots$$

When we substitute $(-1, -1)$, $(-1, -2)$, $(0, -2)$ and $(0, -3)$ into (p, q) , our vanishing theorems in Steps 1 and 2 yield

$$h^2(Y_1, E(-1, -1)) = h^2(Y_1, E(-1, -2)) = h^2(Y_1, E(0, -2)) = h^2(Y_1, E(0, -3)) = 0,$$

where $h^i(Y_1, E(p, q)) = \dim H^i(Y_1, E(p, q))$. By a similar method, we have

$$\begin{aligned} h^2(Y_1, E^*(-2, 0)) &= h^2(Y_1, E^*(-2, -1)) = h^2(Y_1, E^*(-1, -1)) \\ &= h^2(Y_1, E^*(-1, -2)) = 0. \end{aligned}$$

Buchdahl's vanishing theorems $H^1(F^3, E(p, q)) = 0$ for $p + q \leq -2$ and the exact sequence (3.2) imply the injectivity of $H^2(Y_1, E(p, q-1)) \rightarrow H^2(Y_1, E(p, q))$, if $p + q \leq -2$. Consequently, we obtain

$$h^2(Y_1, E(0, q)) = 0 \quad \text{if } q \leq -1, \quad h^2(Y_1, E(-1, q)) = 0 \quad \text{if } q \leq -2,$$

$$h^2(Y_1, E^*(-1, q)) = 0 \quad \text{if } q \leq -1, \quad h^2(Y_1, E^*(-2, q)) = 0 \quad \text{if } q \leq 0.$$

By definition, Y_1 is smooth and the adjunction formula yields the isomorphism between the canonical bundle K_{Y_1} and $\mathcal{O}_{Y_1}(-2, -3)$. Combined with the above vanishing, Serre duality implies

$$h^2(Y_1, E(0, q)) = h^2(Y_1, E(-1, q)) = 0 \quad \text{if } q \in \mathbf{Z},$$

$$h^2(Y_1, E^*(-1, q)) = h^2(Y_1, E^*(-2, q)) = 0 \quad \text{if } q \in \mathbf{Z}.$$

Using vanishing theorems $h^2(E(-1, 0)) = h^2(E^*(-2, 0)) = 0$ and $h^2(Y_1, E) = h^2(Y_1, E^*(-1, 0)) = 0$, we obtain from the exact sequence (3.1) that $h^2(E(0, -1)) =$

$$h^2(E^*(-1, 0))=0.$$

Step 3. Since Y_1 is $p_1^{-1}(\mathbf{P}^2)$, the dual Euler sequence on \mathbf{P}^2 gives

$$0 \rightarrow p_1^* \Omega_{\mathbf{P}^2}^1 \rightarrow \mathcal{O}_Y(-1, 0)^{\oplus 3} \rightarrow \mathcal{O}_Y \rightarrow 0.$$

This, together with Lemma 3.3, implies that $H^1(Y_1, p_1^* \Omega_{\mathbf{P}^2}^1 \otimes E(p, q))=0$ if $p+q \leq -1$. Dualizing this sequence and using $TP^2 \cong \Omega_{\mathbf{P}^2}^1(3)$, we get from the above vanishing theorem $H^2(Y_1, E(p, q)) \rightarrow H^2(Y_1, E(p+1, q))^{\oplus 3}$ are injective if $p+q \leq -4$. Hence, vanishing theorems $h^2(Y_1, E(-1, q))=h^2(Y_1, E^*(-2, q))=0$ ($q \in \mathbf{Z}$) yield that

$$h^2(Y_1, E(-2, q))=0 \quad \text{if } q \leq -2, \quad h^2(Y_1, E^*(-3, q))=0 \quad \text{if } q \leq -1.$$

By induction with respect to p and Serre duality, we have

$$\begin{aligned} h^2(Y_1, E(p, q))=0 & \quad \text{if } p \leq -2 \text{ and } q \leq -2 \text{ or } p \geq 1 \text{ and } q \geq -2, \\ h^2(Y_1, E^*(p, q))=0 & \quad \text{if } p \leq -3 \text{ and } q \leq -1 \text{ or } p \geq 0 \text{ and } q \geq -1. \end{aligned}$$

These vanishing and the long exact sequence associated with the sequence (3.1) show inductively that

$$\begin{aligned} h^2(E(p, -1))=h^2(E(p, -2))=0 & \quad \text{if } q \in \mathbf{Z}, \\ h^2(E^*(p, 0))=h^2(E^*(p, -1))=0 & \quad \text{if } q \in \mathbf{Z}. \end{aligned}$$

Using Serre duality, we have

$$\begin{aligned} h^3(E(p, -2))=h^3(E(p, -3))=0 & \quad \text{if } q \in \mathbf{Z}, \\ h^3(E^*(p, -1))=h^3(E^*(p, -2))=0 & \quad \text{if } q \in \mathbf{Z}. \end{aligned}$$

Step 4. In this final step, we make use of the other divisor Y_2 of F^5 . Vanishing theorems in Step 3 and the sequence (3.3) imply that

$$\begin{aligned} h^2(Y_2, E(p, -1))=h^2(Y_2, E(p, -2))=0 & \quad \text{if } p \in \mathbf{Z}, \\ h^2(Y_2, E^*(p, 0))=h^2(Y_2, E^*(p, -1))=0 & \quad \text{if } p \in \mathbf{Z}. \end{aligned}$$

It is shown in a similar way as in Step 3 that $H^2(Y_2, E(p, q)) \rightarrow H^2(Y_2, E(p, q+1))^{\oplus 3}$ are injective, if $p+q \leq -4$. Consequently, we have

$$\begin{aligned} h^2(Y_2, E(p, q))=0 & \quad \text{if } p \leq -1 \text{ and } q \leq -3 \text{ or } p \geq -1 \text{ and } q \geq 0, \\ h^2(Y_2, E^*(p, q))=0 & \quad \text{if } p \leq -2 \text{ and } q \leq -2 \text{ or } p \geq -2 \text{ and } q \geq 1. \end{aligned}$$

Using again the sequence (3.3), we obtain inductively that

$$\begin{aligned} h^2(E(p, q))=0 & \quad \text{if } p \geq -2 \text{ and } q \geq 1, \\ h^2(E^*(p, q))=0 & \quad \text{if } p \geq -2 \text{ and } q \geq 1, \end{aligned}$$

completing the proof. \square

PROPOSITION 3.5. *Let E be an anti-self-dual bundle on $Gr_2(\mathbb{C}^4)$ satisfying the hypothesis of Main Theorem 1. Then we have*

$$\begin{aligned} h^1(E(0, -1)) &= 1, & h^1(E(-1, 0)) &= h^1(E(1, -2)) = 0, \\ h^1(E^*(-1, 0)) &= 1, & h^1(E^*(0, -1)) &= h^1(E^*(-2, 1)) = 0, \\ h^4(E(-2, -3)) &= 1, & h^4(E(-3, -2)) &= h^4(E(-1, -4)) = 0, \\ h^4(E^*(-3, -2)) &= 1, & h^4(E^*(-2, -3)) &= h^4(E^*(-4, -1)) = 0. \end{aligned}$$

PROOF. Theorem 2.7 and Corollary 3.2 imply

$$0 = \chi(E(-1, 0)) = -h^1(E(-1, 0)) + h^2(E(-1, 0)).$$

By Proposition 3.4, we know $h^2(E(-1, 0)) = 0$. Hence, $h^1(E(-1, 0)) = 0$. In a similar way, we get desired results about h^1 and applying Serre duality, we also obtain results about h^4 . \square

THEOREM 3.6. *Let E be an anti-self-dual bundle on $Gr_2(\mathbb{C}^4)$ satisfying the hypothesis of Main Theorem 1. Then we have $h^0(E) = r - 3$ and $h^1(E) = 0$ on the twistor space F^5 . Moreover, E decomposes into a direct sum $E = E' \oplus T$, where E' is an $SU(3)$ anti-self-dual bundle and T is a flat bundle of rank $r - 3$.*

PROOF. Theorem 2.7, Corollary 3.2 and Proposition 3.4 yield $h^0(E) - h^1(E) = r - 3$ and so $h^0(E) \geq r - 3$.

In general, a holomorphic section $s \in H^0(Z, E)$ corresponds to a covariant constant section of E over M , where Z is the twistor space of a quaternion-Kähler manifold M and E is an anti-self-dual bundle on M . (See, for example, [W-W: p. 422] or [Na-2]. A direct computation in [Na-2] shows this fact.) Consequently, there exists a holomorphically trivial bundle T such that $E \cong T \oplus E'$ and the rank of T is greater than or equal to $r - 3$, where E' is a subbundle of E . Then we have the rank of E' is less than or equal to 3. However, if the rank of E' is less than 3, $c_3(E) = c_3(E')$ vanishes. This is a contradiction and we have $rk(T) = r - 3$ and $rk(E') = 3$, where $rk(T)$ means the rank of T and so on. The same argument implies that E' has no section. Hence we get $h^0(E) = h^0(T) = rk(T) = r - 3$ and so $h^1(E) = 0$. \square

Due to Theorem 3.6, from now on, we assume that the rank of E is 3 and so, $h^0(E) = h^1(E) = 0$. Then we can apply the same method for the dual bundle E^* and we have $h^0(E^*) = h^1(E^*) = 0$.

LEMMA 3.7. *Let E be an $SU(3)$ anti-self-dual bundle on $Gr_2(\mathbb{C}^4)$ satisfying the hypothesis of Main Theorem 1. Then we have $h^0(E(p, -p)) = 0$, $h^0(E^*(p, -p)) = 0$, $h^5(E(-p-3, p-3)) = 0$ and $h^5(E^*(-p-3, p-3)) = 0$ where $p \neq \pm 1$.*

PROOF. The homogeneous bundle $\mathcal{O}(p, -p)$ on F^5 is the pull-back bundle of an anti-self-dual bundle on $Gr_2(\mathbb{C}^4)$ from [Na-3; Theorem 3.4]. Consequently, $E(p, -p)$

is also the pull-back of an anti-self-dual bundle. A direct computation shows that the third Chern class $c_3(E(p, -p)) = -(p-1)(p+1)^2xy(x-y)$ under the relation (2.1). The same argument as in Theorem 3.6 implies that if $h^0(E(p, -p))$ does not vanish, $E(p, -p)$ has a trivial summand and $c_3(E(p, -p))$ vanishes. This is a contradiction when $p \neq \pm 1$. A similar way gives vanishing about the dual bundle E^* and Serre duality gives vanishing about h^5 . \square

LEMMA 3.8. *Under the same assumption as in Lemma 3.7, we have $h^1(E(2, -2)) = h^1(E^*(-2, 2)) = h^4(E(-1, -5)) = h^4(E^*(-5, -1)) = 0$.*

PROOF. From Corollary 3.2, we have $\chi(E(2, -2)) = 0$. Theorem 2.7, Proposition 3.4 and Lemma 3.7 give the desired result. \square

We define

$$W_1 = H^1(F^5, E(0, -1))^* \quad \text{and} \quad W_2 = H^1(F^5, E^*(-1, 0))^* .$$

By Proposition 3.5, we have $\dim W_1 = \dim W_2 = 1$. On the other hand, we obtain from our definition that $H^1(W_1 \otimes E(0, -1)) \cong W_1 \otimes H^1(E(0, -1)) \cong \text{End}(H^1(E(0, -1)))$ and $H^1(W_2 \otimes E^*(-1, 0)) \cong \text{End}(H^1(E^*(-1, 0)))$, where we regard W_i as trivial bundles on F^5 with fibres W_i for $i = 1, 2$. (For brevity, we will omit F^5 in cohomology groups.) Consequently, using the identity elements of $\text{End}(H^1(E(0, -1)))$ and $\text{End}(H^1(E^*(-1, 0)))$ respectively, we have the extensions S_1 and S_2 such that

$$(3.5) \quad 0 \rightarrow E \rightarrow S_1 \rightarrow W_1^*(0, 1) \rightarrow 0 ,$$

$$(3.6) \quad 0 \rightarrow E^* \rightarrow S_2 \rightarrow W_2^*(1, 0) \rightarrow 0 .$$

Dualizing (3.6), tensoring with $W_1(0, -1)$ and taking the associated cohomology sequence, we get from Theorem 2.8

$$H^1(S_2^* \otimes W_1(0, -1)) \cong H^1(E \otimes W_1(0, -1)) \cong \text{End}(H^1(E(0, -1))) .$$

Hence, there is the compatible extension V such that

$$\begin{array}{ccccccc} 0 & \longrightarrow & S_2^* & \longrightarrow & V & \longrightarrow & W_1^*(0, 1) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & E & \longrightarrow & S_1 & \longrightarrow & W_1^*(0, 1) \longrightarrow 0 . \end{array}$$

Therefore, we have the display of a monad

$$W_2(-1, 0) \rightarrow V \rightarrow W_1^*(0, 1)$$

such that

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & 0 & \longrightarrow & W_2(-1, 0) & \longrightarrow & S_2^* & \longrightarrow & E & \longrightarrow & 0 \\
 & & & \parallel & & \downarrow & & \downarrow & & \\
 (3.7) & 0 & \longrightarrow & W_2(-1, 0) & \longrightarrow & V & \longrightarrow & S_1 & \longrightarrow & 0 \\
 & & & & & \downarrow & & \downarrow & & \\
 & & & & & W_1^*(0, 1) & = & W_1^*(0, 1) & & \\
 & & & & & \downarrow & & \downarrow & & \\
 & & & & & 0 & & 0 & &
 \end{array}$$

To prove Main Theorem 1, we must determine the bundle V . We will apply Proposition 2.6 for the bundle $V(0, -1)$ and consequently, we consider cohomology groups $H^i(V(p, q))$ and $H^1(V \otimes Q_1^*(p, q))$.

LEMMA 3.9. *Let S_1 be the bundle defined as above. Then we have $h^i(S_1(p, q))=0$ if*

- $i=0$ for $p+q \leq 0$ and $(p, q) \neq (-1, 1), (0, 0), (0, -1), (1, -1)$,
- $i=1$ for $p+q \leq -2$ or $(p, q) = (-1, 0), (0, 0), (1, -2), (2, -2)$,
- $i=2$ for $p+q \leq -4$ or $p \geq -1$ and $q \geq -2$ or $(p, q) = (-2, -1), (0, -3)$,
- $i=3$ for $p+q \geq -2$ or $p \leq -1$ and $q \leq -2$ or $(p, q) = (-2, -1), (0, -3)$,
- $i=4$ for $p+q \geq -4$ or $(p, q) = (-3, -2), (-3, -3), (-1, -4), (-1, -5)$,
- $i=5$ for $p+q \geq -6$ and $(p, q) \neq (-4, -2), (-2, -4)$.

PROOF. From the exact sequence (3.5), we get a long exact sequence:

$$\cdots \rightarrow H^i(E(p, q)) \rightarrow H^i(S_1(p, q)) \rightarrow W_1^* \otimes H^i(\mathcal{O}(p, q+1)) \rightarrow \cdots$$

Our vanishing theorems for $H^i(E(p, q))$ (Theorems 2.7 and 3.6, Propositions 3.4 and 3.5, Lemmas 3.7 and 3.8) and the Bott-Borel-Weil theorem for $H^i(\mathcal{O}(p, q+1))$ (Theorem 2.8) yield our results. \square

LEMMA 3.10. *Let V be the bundle defined as above. Then we have $h^i(V(p, q))=0$ if*

- $i=0$ for $p+q \leq 0$ and $(p, q) \neq (-1, 1), (0, 0), (0, -1), (1, -1)$,
- $i=1$ for $p+q \leq -2$ or $(p, q) = (-1, 0), (0, 0), (1, -2), (2, -2)$,
- $i=2$ for $p+q \leq -4$ or $p \geq -1$ and $q \geq -2$ or $(p, q) = (-2, -1), (0, -3)$,

- $i=3$ for $p+q \geq -2$ or $p \leq -1$ and $q \leq -2$ or $(p, q) = (-2, -1), (0, -3)$,
 $i=4$ for $p+q \geq -4$ or $(p, q) = (-3, -2), (-3, -3), (-1, -4), (-1, -5)$,
 $i=5$ for $p+q \geq -6$ and $(p, q) \neq (-4, -2), (-3, -3), (-2, -3), (-2, -4)$.

PROOF. From the second row of (3.7), we get a long exact sequence:

$$\cdots \rightarrow W_2 \otimes H^i(\mathcal{O}(p-1, q)) \rightarrow H^i(V(p, q)) \rightarrow H^i(S_1(p, q)) \rightarrow \cdots$$

Our vanishing theorems for $H^i(S_1(p, q))$ (Lemma 3.9) and the Bott-Borel-Weil theorem for $H^i(\mathcal{O}(p-1, q))$ (Theorem 2.8) yield our results. \square

LEMMA 3.11. *Let V be the bundle defined as above. For brevity, we denote by T the pull-back bundle $p_1^*TP^3$, where TP^3 is the holomorphic tangent bundle of P^3 . Then we have $h^i(V \otimes T(p, q)) = 0$ if*

- $i=0$ for $p+q \leq -2$ and $(p, q) \neq (-1, -1)$ or $(p, q) = (1, -2)$,
 $i=1$ for $p+q \leq -4$ or $(p, q) = (-2, -1), (-1, 0), (0, -2), (0, -3), (1, -2)$,
 $i=2$ for $p \leq -1$ and $q \leq -2$ or $p \geq -2$ and $q \geq -2$,
 $i=3$ for $p+q \geq -3$ or $(p, q) = (-3, -1), (-3, -2), (-3, -3), (-1, -3)$,
 $i=4$ for $p+q \geq -5$ and $(p, q) \neq (-2, -3)$,
 $i=5$ for $p+q \geq -7$ and $(p, q) \neq (-5, -2), (-4, -3), (-3, -3), (-3, -4)$.

PROOF. Using the Euler sequence on P^3 and the holomorphic fibration p_1 , we obtain

$$(3.8) \quad 0 \rightarrow \mathcal{O}_{F^5} \rightarrow \mathcal{O}(1, 0)^{\oplus 4} \rightarrow T \rightarrow 0.$$

Taking the associated long exact sequence, we have

$$\cdots \rightarrow H^i(V(p+1, q))^{\oplus 4} \rightarrow H^i(V \otimes T(p, q)) \rightarrow H^{i+1}(V(p, q)) \rightarrow \cdots$$

Our vanishing theorems for $H^i(V(p, q))$ (Lemma 3.10) yield our results. \square

LEMMA 3.12. *Under the same assumption and the notation as in Lemma 3.11, we have $h^i(V \otimes Q_1^*(p, q)) = 0$ if*

- $i=0$ for $p+q \leq -2$ and $(p, q) \neq (-1, -1)$ or $(p, q) = (1, -2)$,
 $i=1$ for $p+q \leq -4$ or $(p, q) = (-2, -1), (0, -3)$,
 $i=2$ for $p \leq -1$ and $q \leq -1$ and $(p, q) \neq (-1, -2)$,
 $i=3$ for $p \geq -2$ and $q \geq -3$ and $(p, q) \neq (-2, -2), (-2, -3)$
 and $(p, q) = (-3, -2), (-3, -3)$,
 $i=4$ for $p+q \geq -4$ or $(p, q) = (-3, -2), (-1, -4)$,

$i=5$ for $p+q \geq -6$ and $(p, q) \neq (-3, -3)$ or $(p, q) = (-2, -5)$.

PROOF. Dualizing the exact sequence in Definition 2.5, we use Lemmas 3.10 and 3.11. \square

LEMMA 3.13. *Under the same hypothesis as in Lemma 3.10, we have $h^0(V(0, -1)) = h^1(V(0, -1)) = 0$.*

PROOF. From the second row of (3.7) and the Bott-Borel-Weil theorem (Theorem 2.8), we obtain $H^i(V(0, -1)) \cong H^i(S_1(0, -1))$ for $i=0, 1, \dots, 5$. Making use of (3.5) and vanishing theorems $h^0(E(0, -1))=0$ (Theorem 2.7) and $h^1(\mathcal{O})=0$ (Theorem 2.8), we get an exact sequence:

$$0 \rightarrow H^0(S_1(0, -1)) \rightarrow W_1^* \otimes H^0(\mathcal{O}) \rightarrow H^1(E(0, -1)) \rightarrow H^1(S_1(0, -1)) \rightarrow 0.$$

From our definition of the extension of E by $W_1^*(0, 1)$, the Bockstein operator $W_1^* \otimes H^0(\mathcal{O}) \rightarrow H^1(E(0, -1)) = W_1^*$ is the identity. Hence, $h^0(S_1(0, -1)) = h^1(S_1(0, -1)) = 0$. \square

LEMMA 3.14. *Under the same hypothesis as in Lemma 3.10, $h^3(V(-3, -1)) = 1$.*

PROOF. By Lemma 3.10, we get $h^i(V(-3, -1)) = 0$ for $i \neq 3$. A direct computation and our definition of E and V give $ch(V) = ch(\mathcal{O}^{\oplus 4} \oplus \mathcal{O}(-1, 1))$. Consequently, the Hirzebruch-Riemann-Roch theorem implies $\chi(V(-3, -1)) = 4\chi(\mathcal{O}(-3, -1)) + \chi(\mathcal{O}(-4, 0))$. The Bott-Borel-Weil theorem (Theorem 2.8) yields $\chi(\mathcal{O}(-3, -1)) = 0$ and $\chi(\mathcal{O}(-4, 0)) = -1$ and so, $h^3(V(-3, -1)) = 1$. \square

LEMMA 3.15. *Under the same hypothesis as in Lemma 3.10, we have $h^0(V \otimes Q_1^*(-1, -1)) = 0$ and the identification $H^1(V \otimes Q_1^*(-1, -1)) \cong W_1^* \otimes \mathbb{C}^4$, where \mathbb{C}^4 is the standard representation space of $SU(4)$.*

PROOF. The exact sequence (3.8), Lemmas 3.10 and 3.13 yield $h^0(V \otimes T(-1, -1)) = h^1(V \otimes T(-1, -1)) = 0$. These vanishing, combined with the dualized exact sequence in Definition 2.5 and Lemma 3.10, imply that $h^0(V \otimes Q_1^*(-1, -1)) = 0$ and $H^0(V) \cong H^1(V \otimes Q_1^*(-1, -1))$. From the second row of (3.7) and Theorem 2.8, we get $H^0(V) \cong H^0(S_1)$. Theorem 3.6 and (3.5) yield that $H^0(S_1) \cong W_1^* \otimes H^0(\mathcal{O}(0, 1))$. The Bott-Borel-Weil theorem implies the identification $H^0(\mathcal{O}(0, 1)) \cong \mathbb{C}^4$ as the representation space of $SU(4)$. \square

LEMMA 3.16. *Under the same hypothesis as in Lemma 3.10, $h^3(V \otimes Q_1^*(-3, -1)) = 0$.*

PROOF. In the same way as in Lemma 3.14, we obtain $\chi(V \otimes Q_1^*(-3, -1)) = 4\chi(Q_1^*(-3, -1)) + \chi(Q_1^*(-4, 0))$. Since the bundle Q_1^* is homogeneous, the Bott-Borel-Weil theorem implies that $\chi(Q_1^*(-3, -1)) = 0$ and $\chi(Q_1^*(-4, 0)) = 0$. Hence, Lemma 3.12 yields $h^3(V \otimes Q_1^*(-3, -1)) = 0$. \square

LEMMA 3.17. *Under the same hypothesis as in Lemma 3.10, $h^3(V \otimes Q_1^*(-4, -1)) = h^4(V \otimes Q_1^*(-4, -1)) = 0$.*

PROOF. Using the homogeneity of bundles Q_1 and Q_1^* , we get an isomorphism between Q_1^* and $Q_1(3, -1)$. Consequently, $h^i(V \otimes Q_1^*(-4, -1)) = h^i(V \otimes Q_1(-1, -2))$ for $i=0, 1, \dots, 5$.

Serre duality implies that $h^4(V(-2, -3)) = h^1(V^*(-1, 0))$ and $h^5(V(-2, -3)) = h^0(V^*(-1, 0))$. Dualizing the first column of (3.7), we obtain from Theorem 2.8 that $H^i(V^*(-1, 0)) \cong H^i(S_2(-1, 0))$, for $i=0, 1, \dots, 5$. Making use of (3.6) and vanishing theorems $h^0(E^*(-1, 0)) = 0$ (Theorem 2.7) and $h^1(\mathcal{O}) = 0$ (Theorem 2.8), we get an exact sequence:

$$0 \rightarrow H^0(S_2(-1, 0)) \rightarrow W_2^* \otimes H^0(\mathcal{O}) \rightarrow H^1(E^*(-1, 0)) \rightarrow H^1(S_2(-1, 0)) \rightarrow 0.$$

From our definition of the extension of E^* by $W_2^*(1, 0)$, the Bockstein operator $W_2^* \otimes H^0(\mathcal{O}) \rightarrow H^1(E^*(-1, 0)) = W_2^*$ is the identity. Hence, $h^0(S_2(-1, 0)) = h^1(S_2(-1, 0)) = 0$ and so, $h^4(V(-2, -3)) = h^5(V(-2, -3)) = 0$.

The exact sequence in Definition 2.5, Lemma 3.10 and the above vanishing theorem yield that $H^i(V \otimes T^*(-1, -2)) \cong H^i(V \otimes Q_1(-1, -2))$ for $i=3, 4$. Next, dualizing (3.8), we obtain from Lemma 3.10 that $H^i(V \otimes T^*(-1, -2)) = 0$ for $i=3, 4$. \square

THEOREM 3.18. *Under the same hypothesis as in Lemma 3.10, we have an isomorphism between V and $W_1^* \otimes \mathbb{C}^4 \oplus \mathcal{O}(-1, 1)$, where $W_1 = H^1(E(0, -1))^*$ and \mathbb{C}^4 is the standard representation space of $SU(4)$.*

PROOF. We apply our spectral sequence (Proposition 2.6) to the vector bundle $V(0, -1)$. Our vanishing theorems (Lemmas 3.10, 3.12–3.17) imply that

$$E_1^{-1,1} \cong W_1^* \otimes \mathbb{C}^4 \otimes \mathcal{O}(0, -1), \quad E_1^{-3,3} \cong \mathcal{O}(-1, 0),$$

and the other E_1 -terms vanish. Hence, by Proposition 2.6, we have

$$0 \rightarrow W_1^* \otimes \mathbb{C}^4 \otimes \mathcal{O}(0, -1) \rightarrow V(0, -1) \rightarrow \mathcal{O}(-1, 0) \rightarrow 0.$$

However, the Bott-Borel-Weil theorem (Theorem 2.8) yields that $H^1(\mathcal{O}(1, 0) \otimes \mathcal{O}(0, -1)) \cong H^1(\mathcal{O}(1, -1)) = 0$ and so, the above exact sequence splits. Consequently, we obtain $V(0, -1) \cong W_1^* \otimes \mathbb{C}^4 \otimes \mathcal{O}(0, -1) \oplus \mathcal{O}(-1, 0)$. \square

Theorem 3.18, together with Theorem 3.6, yields Main Theorem 1 in the case $n=2$.

4. Classification II.

In this section, we also use the same symbol E for an anti-self-dual bundle on $Gr_2(\mathbb{C}^{n+2})$ and its pull-back on F^{2n+1} .

We apply Theorem 2.4 to an anti-self-dual bundle on $Gr_2(\mathbb{C}^{n+2})$.

PROPOSITION 4.1. *Let E be an anti-self-dual bundle with a hermitian structure on $Gr_2(\mathbf{C}^{n+2})$. We assume that $p+q$ is an even number. Then we have for $2 \leq i \leq n$*

$$\begin{aligned} H^1(F^{2n+1}, E(p, q)) &= 0 & \text{if } p+q \leq -2, \\ H^i(F^{2n+1}, E(p, q)) &= 0 & \text{if } \begin{cases} i: \text{odd and } p+q \leq -i-3, \\ i: \text{even and } p+q \leq -i-2. \end{cases} \end{aligned}$$

REMARK. The assumption that $p+q$ is even is caused by the non-existence of the line bundle $L(\mathcal{O}(1))$ in [S]. Under the notation in Theorem 2.4, we have an identification $\mathcal{O}(2) \cong \mathcal{O}(1, 1)$. However, this assumption is not needed from the viewpoint of the Penrose transform ([Ba] and [M-S]). We will give an elementary proof for this fact in Proposition 4.3.

For $H^0(F^{2n+1}, E(p, q))$, we have the next lemma (see also [N-N2; Lemma 3.3]).

LEMMA 4.2. *Let E be an anti-self-dual bundle with a hermitian structure on $Gr_2(\mathbf{C}^{n+2})$. Then we have $H^0(F^{2n+1}, E(p, q)) = 0$ if $p+q \leq -1$.*

PROOF. Let s be a section of $E(p, q)$. We denote by \mathbf{P}_x the twistor fibre ($x \in Gr_2(\mathbf{C}^{n+2})$). From [Na-3; Lemma 3.3], we have $E(p, q)|_{\mathbf{P}_x} \cong E_x \otimes \mathcal{O}_{\mathbf{P}^1}(p+q)$. If $p+q \leq -1$, $H^0(\mathbf{P}^1, \mathcal{O}(p+q)) = 0$. Hence, the restricted section $s|_{\mathbf{P}_x}$ vanishes. \square

PROPOSITION 4.3. *The assumption in Proposition 4.1 that $p+q$ is an even number is unnecessary.*

PROOF. We employ an induction. When $n=1$, this result is obtained in [Bu] and when $n=2$, this is nothing but Theorem 2.7.

Using the holomorphic fibration $p_1: F^{2n+1} \rightarrow \mathbf{P}^{n+1}$, we denote by Y_1 the divisor $p_1^{-1}(\mathbf{P}^n)$ in a similar way as in §3. Hence we have exact sequences of sheaves:

$$(4.1) \quad 0 \rightarrow \mathcal{O}(-1, 0) \rightarrow \mathcal{O} \rightarrow \mathcal{O}_{Y_1} \rightarrow 0,$$

$$(4.2) \quad 0 \rightarrow \mathcal{O}_{Y_1}(0, -1) \rightarrow \mathcal{O}_{Y_1} \rightarrow \mathcal{O}_{F^{2n-1}} \rightarrow 0.$$

From Lemma 4.2, the sequence (4.2) and the hypothesis of induction, we obtain that for $2 \leq i \leq n-1$,

$$(4.3) \quad \begin{aligned} H^0(Y_1, E(p, q-1)) &\cong H^0(Y_1, E(p, q)) & \text{if } p+q \leq -1, \\ H^1(Y_1, E(p, q-1)) &\cong H^1(Y_1, E(p, q)) & \text{if } p+q \leq -2, \\ H^i(Y_1, E(p, q-1)) &\cong H^i(Y_1, E(p, q)) & \text{if } \begin{cases} i: \text{odd and } p+q \leq -i-3, \\ i: \text{even and } p+q \leq -i-2. \end{cases} \end{aligned}$$

The sequence (4.1), Proposition 4.1 and Lemma 4.2 imply that

$$H^0(Y_1, E(p, q)) = 0 \quad \text{if } p+q \text{ is odd and } p+q \leq -1.$$

This, together with (4.3), yields $H^0(Y_1, E(p, q))=0$ if $p+q \leq -1$. Using again (4.1) and Proposition 4.1, we get $H^1(F^{2n+1}, E(p-1, q))=0$ if $p+q$ is even and is less than or equal to -2 . Consequently, we have $H^1(F^{2n+1}, E(p, q))=0$ if $p+q \leq -2$.

The similar way implies the desired result. \square

REMARK. In this proof, we also obtain for $2 \leq i \leq n-1$

$$(4.4) \quad \begin{aligned} H^0(Y_1, E(p, q)) &= 0 && \text{if } p+q \leq -1, \\ H^1(Y_1, E(p, q)) &= 0 && \text{if } p+q \leq -2, \\ H^i(Y_1, E(p, q)) &= 0 && \text{if } \begin{cases} i : \text{odd and } p+q \leq -i-3, \\ i : \text{even and } p+q \leq -i-2. \end{cases} \end{aligned}$$

From now on, we give a proof of Main Theorem 1 in the case $n \geq 3$. To do so, we will prove a slightly stronger theorem.

MAIN THEOREM 1'. Let E be a holomorphic vector bundle of rank r on F^{2n+1} ($n \geq 2$) which satisfies the following condition.

- (1) The bundle E (resp. E^*) and the restricted bundle $E|_{F^{2m+1}}$ (resp. $E^*|_{F^{2m+1}}$) to an arbitrary $F^{2m+1} \subset F^{2n+1}$ satisfies the same type of vanishing theorems as in Lemma 4.2 and Proposition 4.3, where $1 \leq m \leq n$.
- (2) $c_1(E)=0, c_2(E)=xy, c_3(E)=xy(x-y)$ and $c_4(E)=x^3y-x^2y^2+xy^3$.

Then, E is the cohomology bundle of the following monad,

$$(M) \quad \mathcal{O}(-1, 0) \rightarrow \underline{V} \oplus \mathcal{O}(-1, 1) \rightarrow \mathcal{O}(0, 1),$$

where \underline{V} is a trivial bundle $F \times V$ of rank $r+1$.

REMARK 1. In §3, we mainly use vanishing theorems and a spectral sequence for holomorphic vector bundles. Moreover, F^{2m+1} can be regarded as the twistor space of $Gr_2(\mathbb{C}^{m+2})$. Consequently, we also have proved Main Theorem 1' in the case $n=2$. Hence, we employ an induction with respect to the dimension of the base space F^{2n+1} . We assume that Main Theorem 1' is true on F^{2n-1} ($n \geq 3$).

REMARK 2. For brevity, we refer to the above condition (1) as Lemma 4.2 and Proposition 4.3. Note that the condition (1) implies (4.4) from a similar argument as in the proof of Proposition 4.3.

LEMMA 4.4. Let E be a holomorphic vector bundle on F^{2n+1} satisfying the hypothesis of Main Theorem 1'. Then we have

$$\dim H^1(F^{2n+1}, E(0, -1)) = 1 \quad \text{and} \quad \dim H^1(F^{2n+1}, E^*(-1, 0)) = 1.$$

PROOF. Fix $F^{2n-1} \subset Y_1 \subset F^{2n+1}$ and we restrict the bundle E to F^{2n+1} . The restricted bundle satisfies the conditions in Main Theorem 1', because $n \geq 3$. (The third Chern class $c_3(E)$ does not vanish.) From the hypothesis of induction, the restricted

bundle $E|_{F^{2n-1}}$ is the cohomology bundle of the monad (M). Now the Bott-Borel-Weil theorem implies that $H^i(F^{2n-1}, \mathcal{O}(-1, -1)), H^i(F^{2n-1}, \mathcal{O}(0, -1))$ and $H^i(F^{2n-1}, \mathcal{O}(-1, 0))$ vanish for $i=0, 1$. Combined with the first row and the first column of the display of (M), these vanishing yields that $H^0(\mathcal{O}) \cong H^1(F^{2n-1}, E(0, -1))$. Consequently, we have $\dim H^1(F^{2n-1}, E(0, -1)) = 1$ and $\dim H^1(F^{2n-1}, E^*(-1, 0)) = 1$ in a similar way.

If $n \geq 3$, the Bott-Borel-Weil theorem yields that

$$\begin{aligned}
 H^2(F^{2n-1}, \mathcal{O}(p, q)) = 0 & \quad \text{if} \quad \begin{cases} p \geq -2 \text{ and } q \geq -2, \\ \text{or } p+q \leq -3 \end{cases} \\
 H^3(F^{2n-1}, \mathcal{O}(p, q)) = 0 & \quad \text{if} \quad \begin{cases} p \geq -3 \text{ and } q \geq -3, \\ \text{or } p \leq -1 \text{ and } q \leq -1. \end{cases}
 \end{aligned}$$

This, together with the display of the monad (M), implies that $H^2(F^{2n-1}, E(p, q)) = 0$ if $p \leq 0$ and $q \leq -1$. Hence, Proposition 4.3 and the exact sequence (4.2) yield that $H^2(Y_1, E(p, q-1)) \cong H^2(Y_1, E(p, q))$ if $p \leq 0, q \leq -1$ and $(p, q) \neq (0, -1)$. From (4.4), we obtain

$$(4.5) \quad H^2(Y_1, E(p, q)) = 0 \quad \text{if } p \leq 0, q \leq -1 \text{ and } (p, q) \neq (0, -1).$$

Since $H^1(Y_1, E(0, -2)) = 0$ from (4.4) and $H^2(Y_1, E(0, -2)) = 0$ from (4.5), (4.2) implies that $H^1(Y, E(0, -1)) \cong H^1(F^{2n-1}, E(0, -1))$.

On the other hand, (4.1), (4.4) and (4.5) yield $H^2(F^{2n+1}, E(p-1, q)) \cong H^2(F^{2n+1}, E(p, q))$ if $p \leq 0, q \leq -1$ and $(p, q) \neq (0, -1)$. From Proposition 4.3, we get

$$(4.6) \quad H^2(F^{2n+1}, E(p, q)) = 0 \quad \text{if } p \leq 0, q \leq -1 \text{ and } (p, q) \neq (0, -1).$$

Since $H^1(F^{2n+1}, E(-1, -1)) = 0$ from Proposition 4.3 and $H^2(F^{2n+1}, E(-1, -1)) = 0$ from (4.6), (4.1) implies that $H^1(F^{2n+1}, E(0, -1)) \cong H^1(Y_1, E(0, -1))$. Consequently, we have $\dim H^1(F^{2n+1}, E(0, -1)) = \dim H^1(F^{2n-1}, E(0, -1)) = 1$.

In a similar way, we obtain $\dim H^1(F^{2n+1}, E^*(-1, 0)) = \dim H^1(F^{2n-1}, E^*(-1, 0)) = 1$. \square

We define

$$W_1 = H^1(F^{2n+1}, E(0, -1))^* \text{ and } W_2 = H^1(F^{2n+1}, E^*(-1, 0))^*.$$

By Lemma 4.4, we have $\dim W_1 = \dim W_2 = 1$. Using the identity elements of $\text{End}(H^1(E(0, -1)))$ and $\text{End}(H^1(E^*(-1, 0)))$ respectively, we have the extensions S_1 and S_2 such that

$$(4.7) \quad 0 \rightarrow E \rightarrow S_1 \rightarrow W_1^*(0, 1) \rightarrow 0,$$

$$(4.8) \quad 0 \rightarrow E^* \rightarrow S_2 \rightarrow W_2^*(1, 0) \rightarrow 0,$$

in a similar way as in §3. Since $n \geq 3$, the Bott-Borel-Weil theorem implies that $H^1(F^{2n+1}, \mathcal{O}(p, q)) = H^2(F^{2n+1}, \mathcal{O}(p, q)) = 0$. Dualizing (4.8), tensoring with $W_1(0, -1)$

and taking the associated cohomology sequence, we get

$$H^1(F^{2n+1}, S_2^* \otimes W_1(0, -1)) \cong H^1(F^{2n+1}, E \otimes W_1(0, -1)) \\ \cong \text{End}(H^1(F^{2n+1}, E(0, -1))).$$

Hence, there is the compatible extension V_1 such that

$$\begin{array}{ccccccc} 0 & \longrightarrow & S_2^* & \longrightarrow & V_1 & \longrightarrow & W_1^*(0, 1) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & E & \longrightarrow & S_1 & \longrightarrow & W_1^*(0, 1) \longrightarrow 0. \end{array}$$

Therefore, we have the display of a monad

$$(M1) \quad W_2(-1, 0) \rightarrow V_1 \rightarrow W_1^*(0, 1)$$

such that

$$(4.9) \quad \begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & W_2(-1, 0) & \longrightarrow & S_2^* & \longrightarrow & E \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & W_2(-1, 0) & \longrightarrow & V_1 & \longrightarrow & S_1 \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & W_1^*(0, 1) = W_1^*(0, 1) & & \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

To prove Main Theorem 1', we must determine the bundle V_1 .
Now we introduce another monad on F^{2n+1} :

$$(M2) \quad W_2(-1, 0) \rightarrow \underline{V} \oplus \mathcal{O}(-1, 1) \rightarrow W_1^*(0, 1),$$

where \underline{V} is a trivial bundle of rank $r + 1$. The cohomology bundle of (M2) satisfies the condition (1) and the condition (2) in Main Theorem 1' by the Bott-Borel-Weil theorem and a direct computation respectively.

PROPOSITION 4.5. *We have an isomorphism on F^{2n-1} that*

$$a : \underline{V} \oplus \mathcal{O}(-1, 1)|_{F^{2n-1}} \cong V_1|_{F^{2n-1}}.$$

PROOF. By the hypothesis of induction, there are isomorphisms between the cohomology bundles of the restricted monad (M1) to F^{2n-1} and the cohomology bundles of the restricted monad (M2) to F^{2n-1} . From a theorem of Okonek-Schneider-

Spindler ([O-S-S; Corollary 1, p. 279]), if $H^i(F^{2n-1}, \underline{V}^*(-1, 0) \oplus \mathcal{O}(0, -1))$, $H^i(F^{2n-1}, V_1(0, -1))$, $H^j(F^{2n-1}, \mathcal{O}(-1, -1))$, $H^i(F^{2n-1}, V_1^*(-1, 0))$ and $H^i(F^{2n-1}, \underline{V}(0, -1) \oplus \mathcal{O}(-1, 0))$ vanish for $i=0, 1$ and $j=1, 2$, we obtain the desired isomorphism.

Since \underline{V} is a trivial bundle, the Bott-Borel-Weil theorem implies that $H^i(F^{2n-1}, \underline{V}^*(-1, 0) \oplus \mathcal{O}(0, -1))=0$, $H^j(F^{2n-1}, \mathcal{O}(-1, -1))=0$ and $H^i(F^{2n-1}, \underline{V}(0, -1) \oplus \mathcal{O}(-1, 0))=0$ for $i=0, 1$ and $j=1, 2$.

Next, since $H^i(F^{2n-1}, \mathcal{O}(-1, -1))=0$ ($i=0, \dots, 2n-1$) by the Bott-Borel-Weil theorem, if we restrict the second row of the display (4.9) to F^{2n-1} , we have $H^i(F^{2n-1}, V_1(0, -1)) \cong H^i(F^{2n-1}, S_1(0, -1))$ for $i=0, 1$.

Making use of the second column of (4.9) and Lemma 4.2, we obtain an exact sequence:

$$\begin{aligned} 0 \rightarrow H^0(F^{2n+1}, S_1(0, -1)) \rightarrow W_1^* \\ \rightarrow H^1(F^{2n+1}, E(0, -1)) \rightarrow H^1(F^{2n+1}, S_1(0, -1)) \rightarrow 0, \end{aligned}$$

where $H^1(F^{2n+1}, \mathcal{O})=0$ by the Bott-Borel-Weil theorem. The definition of the extension yields that $W_1^* \rightarrow H^1(F^{2n+1}, S_1(0, -1))$ is the identity. Hence $H^i(F^{2n+1}, S_1(0, -1))=0$ for $i=0, 1$. Moreover, from the second column of the display (4.9), Lemma 4.2 and Proposition 4.3, the Bott-Borel-Weil theorem implies that

$$(4.10) \quad \begin{aligned} H^0(F^{2n+1}, S_1(p, q)) &= 0 & \text{if } p+q \leq -1, \\ H^1(F^{2n+1}, S_1(p, q)) &= 0 & \text{if } p+q \leq -2, \\ H^2(F^{2n+1}, S_1(p, q)) &= 0 & \text{if } p+q \leq -4. \end{aligned}$$

Then the exact sequence (4.1) and (4.10) yield that

$$(4.11) \quad H^0(Y_1, S_1(p, q))=0 \quad \text{if } p+q \leq -1,$$

$$(4.12) \quad H^1(Y_1, S_1(p, q))=0 \quad \text{if } p+q \leq -3.$$

Using again the second column of the restricted display (4.9) to F^{2n-1} , we obtain from Lemma 4.2 and Proposition 4.3 that $H^0(F^{2n-1}, S_1(p, q))=0$ if $p+q \leq -1$ and $(p, q) \neq (0, -1)$ and $H^1(F^{2n-1}, S_1(p, q))=0$ if $p+q \leq -2$. These vanishing, combined with (4.2), shows that

$$(4.13) \quad H^1(Y_1, S_1(p, q-1)) \cong H^1(Y_1, S_1(p, q)) \quad \text{if } p+q \leq -2.$$

By (4.12) and (4.13), we have $H^1(Y_1, S_1(p, q))=0$ if $p+q \leq -2$. These, together with (4.2) and (4.10), yield that $H^0(F^{2n-1}, S_1(p, q))=0$ if $p+q \leq -1$, in particular, $H^0(F^{2n-1}, S_1(0, -1))=0$. Then, from the second column of the restricted display (4.9), we have

$$0 \rightarrow W_1^* \rightarrow H^1(F^{2n-1}, E(0, -1)) \rightarrow H^1(F^{2n-1}, S_1(0, -1)) \rightarrow 0.$$

Lemma 4.4 and its proof implies that $\dim W_1^* = \dim H^1(F^{2n-1}, E(0, -1)) = 1$ and so,

$$H^1(F^{2n-1}, S_1(0, -1))=0.$$

As for $H^i(F^{2n-1}, V_1^*(-1, 0))$ ($i=0, 1$), we may apply a similar method to the dual monad of (M1). \square

PROPOSITION 4.6. *There exists a unique element $A \in H^0(F^{2n+1}, \text{End}(\underline{V} \oplus \mathcal{O}(-1, 1), V_1))$ such that the restriction A to F^{2n-1} corresponds to a in Proposition 4.5.*

PROOF. Since $\text{End}(\underline{V} \oplus \mathcal{O}(-1, 1), V_1) \cong \underline{V}^* \otimes V_1 \oplus V_1(1, -1)$, if $H^i(Y_1, V_1(0, -1))$, $H^i(Y_1, V_1(-1, 0))$, $H^i(F^{2n+1}, V_1(-1, 1))$ and $H^i(F^{2n+1}, V_1(-2, 1))$ vanish for $i=0, 1$, from (4.1) and (4.2), we obtain that $H^0(F^{2n-1}, \text{End}(V \oplus \mathcal{O}(-1, 1), V_1)) \cong H^0(F^{2n+1}, \text{End}(V \oplus \mathcal{O}(-1, 1), V_1))$ and so, we have the desired A .

First Proposition 4.5 and the Bott-Borel-Weil theorem imply that

$$(4.14) \quad \begin{aligned} H^0(F^{2n-1}, V_1(p, q)) &= 0 && \text{if } p \leq -1 \text{ or } q \leq -2 \text{ or } (p, q) = (0, -1), \\ H^1(F^{2n-1}, V_1(p, q)) &= 0. \end{aligned}$$

By (4.2) and (4.14), we have for $i=0, 1$.

$$(4.15) \quad \begin{aligned} H^i(Y_1, V_1(p, q-1)) &\cong H^i(Y_1, V_1(p, q)) \\ &\text{if } p \leq -1 \text{ or } q \leq -2 \text{ or } (p, q) = (0, -1). \end{aligned}$$

The second row of (4.9), (4.10) and the Bott-Borel-Weil theorem yields that

$$(4.16) \quad \begin{aligned} H^0(F^{2n+1}, V_1(p, q)) &= 0 && \text{if } p+q \leq -1, \\ H^1(F^{2n+1}, V_1(p, q)) &= 0 && \text{if } p+q \leq -2, \\ H^2(F^{2n+1}, V_1(p, q)) &= 0 && \text{if } p+q \leq -4. \end{aligned}$$

From (4.1) and (4.16), we obtain that

$$\begin{aligned} H^0(Y_1, V_1(p, q)) &= 0 && \text{if } p+q \leq -1, \\ H^1(Y_1, V_1(p, q)) &= 0 && \text{if } p+q \leq -3. \end{aligned}$$

These vanishing, together with (4.15), imply that for $i=0, 1$,

$$(4.17) \quad H^i(Y_1, V_1(p, q))=0 \quad \text{if } p \leq -1 \text{ or } q \leq -2 \text{ or } (p, q) = (0, -1).$$

In a similar way, (4.1), (4.16), (4.17) shows that for $i=0, 1$,

$$H^i(F^{2n+1}, V_1(p, q))=0 \quad \text{if } p \leq -1 \text{ or } q \leq -2 \text{ or } (p, q) = (0, -1). \quad \square$$

THEOREM 4.7. *The vector bundle V_1 in the monad (M1) is identified with $\underline{V} \oplus \mathcal{O}(-1, 1)$, where \underline{V} is a trivial bundle of rank $r+1$.*

PROOF. By Proposition 4.6, we obtain a homomorphism $A : \underline{V} \oplus \mathcal{O}(-1, 1) \rightarrow V_1$ such that the restriction $A|_{F^{2n-1}}$ is an isomorphism. Hence we also have $\det A : \wedge^{r+2}(\underline{V} \oplus \mathcal{O}(-1, 1)) \rightarrow \wedge^{r+2} V_1$. Since \underline{V} is trivial, $\wedge^{r+2}(\underline{V} \oplus \mathcal{O}(-1, 1)) \cong \mathcal{O}(-1, 1)$. On

the other hand, from the monad (M1), we get $c(V_1) = c(E)c(\mathcal{O}(-1, 0))c(\mathcal{O}(0, 1))$ and so, $c_1(\bigwedge^{r+2} V_1) = c_1(\mathcal{O}(-1, 1))$. Consequently, we regard $\det A$ as an element of $H^0(F^{2n+1}, \mathcal{O}(-1, 1)^* \otimes \mathcal{O}(-1, 1)) \cong H^0(F^{2n+1}, \mathcal{O}) \cong \mathbb{C}$. Since $A|_{F^{2n-1}}$ is an isomorphism, $\det A = \det A|_{F^{2n-1}} \neq 0$, and so A is also an isomorphism. \square

5. Moduli spaces.

To describe homomorphisms in the monad (M), we make use of the expression of F^{2n+1} as a homogeneous space. For brevity, $SU(n+2)$ is denoted by G and $S(U(1) \times U(n) \times U(1))$ is denoted by K_Z . Let \mathbb{C}^{n+2} be the standard representation space of G with a G -invariant hermitian inner product h . Now we denote by e (resp. f) the highest (resp. lowest) weight vector with the norm 1 in \mathbb{C}^{n+2} . Then, by the restriction of the action of G to K_Z , we have two irreducible representation spaces $\mathbb{C}e$ and $\mathbb{C}f$ of K_Z . We also obtain an irreducible representation $\mathbb{C}e \otimes f$ of K_Z by the tensor product. Under this notation, we have

$$\mathcal{O}(-1, 0) = G \times_{K_Z} \mathbb{C}e, \quad \mathcal{O}(0, 1) = G \times_{K_Z} \mathbb{C}f \quad \text{and} \quad \mathcal{O}(-1, 1) = G \times_{K_Z} \mathbb{C}e \otimes f.$$

Hence, for example, an element of $\mathcal{O}(-1, 0)$ is denoted by $[g, ce]$, where c is a complex number and $[g, ce]$ is the coset represented by $(g, ce) \in G \times \mathbb{C}e$.

PROPOSITION 5.1. *Let α and β be homomorphisms in the monad (M):*

$$(M) \quad \mathcal{O}(-1, 0) \xrightarrow{\alpha} V \oplus \mathcal{O}(-1, 1) \xrightarrow{\beta} \mathcal{O}(0, 1).$$

Then, there exist $A \in \text{Hom}(\mathbb{C}^{n+2}, V)$, $B \in \text{Hom}(V, \mathbb{C}^{n+2})$, $z \in \mathbb{C}^{n+2}$ and $w \in \mathbb{C}^{n+2}$ such that*

$$(5.1) \quad \alpha([g, ce]) = ([g], cAe), [g, ch(z, gf)e \otimes f],$$

$$(5.2) \quad \beta([g, v], [g, c'e \otimes f]) = [g, \{h(Bv, gf) + c'w(ge)\}f],$$

where $g \in G$, $c, c' \in \mathbb{C}$, and $v \in V$.

PROOF. A homomorphism α is regarded as an element of $H^0(\text{Hom}(\mathcal{O}(-1, 0), V \oplus \mathcal{O}(-1, 1))) \cong V \otimes H^0(\mathcal{O}(1, 0)) \oplus H^0(\mathcal{O}(0, 1))$. (For brevity, we omit F^{2n+1} in cohomology groups.) The Bott-Borel-Weil theorem implies that $H^0(\mathcal{O}(1, 0)) \cong \mathbb{C}^{n+2*}$ and $H^0(\mathcal{O}(0, 1)) \cong \mathbb{C}^{n+2}$. Consequently, α is identified with an element of $\text{Hom}(\mathbb{C}^{n+2}, V) \oplus \mathbb{C}^{n+2}$. In a similar way, β belongs to $\text{Hom}(V, \mathbb{C}^{n+2}) \oplus \mathbb{C}^{n+2*}$. For example, the method of Kostant [K] yields the explicit expressions of α and β . \square

Since (M) is a monad, α is an injection, β is a surjection and $\beta \circ \alpha = 0$.

LEMMA 5.2. *A homomorphism α in (M) is injective if and only if A is injective. A homomorphism β in (M) is surjective if and only if B is surjective.*

PROOF. If A is injective, (5.1) implies that α is injective. We assume that A is not injective. Then, $\text{Ker } A (\neq \{0\})$ is a subspace in \mathbf{C}^{n+2} . Let u be an element in $\text{Ker } A$ with the norm 1. Since $n \geq 2$, there exists $v \in \mathbf{C}^{n+2}$ such that $h(v, v) = 1$, $h(u, v) = 0$ and $h(z, v) = 0$. Considering the standard representation of $SU(n+2)$, we obtain $g \in G$ such that $u = ge$ and $v = gf$. Hence, from our choice of u, v and g , (5.1) yields that $\alpha([g, e]) = 0$. This is a contradiction with the injectivity of α .

As for β and B , the surjectivity of B implies the surjectivity of β by (5.2). If B is not surjective, there exists $g \in G$ such that $h(Bv, gf) = 0$ for an arbitrary $v \in V$ and $w(ge) = 0$. Consequently, β is not surjective. \square

COROLLARY 5.3. *That the rank r of E is greater than or equal to $n+1$ is a necessary condition for the existence of E satisfying the conditions in Main Theorem 1 (or 1').*

PROOF. The vector bundle E is the cohomology bundle of (M) by Main Theorem 1. From Proposition 5.2, the dimension of the vector space V is greater than or equal to $n+2$. \square

LEMMA 5.4. *A composition homomorphism $\beta \circ \alpha$ is a 0-map if and only if there exists a constant $c \in \mathbf{C}$ such that $BA + z \otimes w = c \text{Id}_{\mathbf{C}^{n+2}}$, where we regard $z \otimes w$ as an element of $\mathbf{C}^{n+2} \otimes \mathbf{C}^{n+2*} \cong \text{End}(\mathbf{C}^{n+2})$ and $\text{Id}_{\mathbf{C}^{n+2}}$ is the identity on \mathbf{C}^{n+2} .*

PROOF. This proof is a slight modification of [Na-3, Proposition 5.1.2].
From (5.1) and (5.2), $\beta \circ \alpha = 0$ if and only if

$$h(g^{-1}BAge, f) + w(ge)h(g^{-1}z, f) = h(\{g^{-1}(BA + z \otimes w)g\}e, f) = 0,$$

for an arbitrary $g \in G$.

As a representation space, $\text{End}(\mathbf{C}^{n+2})$ is decomposed into $\mathfrak{sl}(n+2) \oplus \mathbf{C} \text{Id}_{\mathbf{C}^{n+2}}$. According to this decomposition, $BA + z \otimes w$ is assumed to be expressed as $X + c \text{Id}_{\mathbf{C}^{n+2}}$, where c is a constant. Then, we have $h(\{g^{-1}(BA + z \otimes w)g\}e, f) = h(g^{-1}Xge, f)$. Combined with the irreducibility of the adjoint representation of G , $X \neq 0$ if and only if there exists $g \in G$ such that $h(g^{-1}Xge, f) \neq 0$. \square

THEOREM 5.5. *Let E be a vector bundle satisfying the conditions in Main Theorem 1. If E has an irreducible $SU(r)$ anti-self-dual connection, then we have $r = n+1$ and an identification between V and \mathbf{C}^{n+2} .*

PROOF. First, we assume that $H^0(E) \neq 0$. The same argument as in the proof of Theorem 3.6 implies that E has a trivial subbundle with a flat connection. This is a contradiction with the irreducibility and so $H^0(E) = 0$.

The first row of the display of (M) and the Bott-Borel-Weil theorem yield that $H^i(\text{Ker } \beta) \cong H^i(E)$ for $i = 0, \dots, 2n+1$. Consequently, $H^0(\text{Ker } \beta) = 0$. Since $n \geq 2$, the Bott-Borel-Weil theorem implies that $H^1(\mathcal{O}(p, q)) = 0$ for arbitrary p, q . By the first column of the display of (M) , we obtain

$$(5.3) \quad 0 \rightarrow V \rightarrow H^0(\mathcal{O}(0, 1)) \rightarrow H^1(\text{Ker } \beta) \rightarrow 0.$$

Using again the Bott-Borel-Weil theorem, we have $H^0(\mathcal{O}(0, 1)) \cong \mathbf{C}^{n+2}$ as the representation space of G . Hence, $\dim H^1(E) = \dim H^1(\text{Ker } \beta) = n+2 - \dim V$ and so, $\dim V \leq n+2$. Since $\text{rank } E = \dim V - 1$, we obtain $\text{rank } E \leq n+1$. However, Corollary 5.3 asserts that $\text{rank } E \geq n+1$. Consequently we have $\text{rank } E = n+1$ and $H^1(E) = 0$. Then, (5.3) yields that $V \cong H^0(\mathcal{O}(0, 1)) \cong \mathbf{C}^{n+2}$. \square

By Theorem 5.5, we assume that the trivial bundle \underline{V} in the monad (M) is $\underline{\mathbf{C}^{n+2}} = F^{2n+1} \times \mathbf{C}^{n+2}$ and the monad (M) is described as

$$(MI) \quad \mathcal{O}(-1, 0) \xrightarrow{\alpha} \underline{\mathbf{C}^{n+2}} \oplus \mathcal{O}(-1, 1) \xrightarrow{\beta} \mathcal{O}(0, 1).$$

Then, note that A and B are automorphisms on \mathbf{C}^{n+2} by Lemma 5.2.

PROPOSITION 5.6. *Monads (MI) and (MI') are isomorphic to each other, in other words, the following diagram is commutative;*

$$\begin{array}{ccccc} (MI) : \mathcal{O}(-1, 0) & \xrightarrow{\alpha} & \underline{\mathbf{C}^{n+2}} \oplus \mathcal{O}(-1, 1) & \xrightarrow{\beta} & \mathcal{O}(0, 1) \\ & p \downarrow & \downarrow F & & \downarrow q \\ (MI') : \mathcal{O}(-1, 0) & \xrightarrow{\alpha'} & \underline{\mathbf{C}^{n+2}} \oplus \mathcal{O}(-1, 1) & \xrightarrow{\beta'} & \mathcal{O}(0, 1) \end{array}$$

where F is an automorphism of $\underline{\mathbf{C}^{n+2}} \oplus \mathcal{O}(-1, 1)$, p and q are automorphisms of $\mathcal{O}(-1, 0)$ and $\mathcal{O}(0, 1)$, respectively, if and only if there exists a non-zero constant a, b and c such that $aB'A' = bBA$, $cz = az'$ and $bw = cw'$ under the notations in (5.1) and (5.2).

PROOF. The Bott-Borel-Weil theorem implies that $H^0(\text{End}(\mathcal{O}(-1, 0))) \cong H^0(\text{End}(\mathcal{O}(0, 1))) \cong \mathbf{C}$ and $H^0(\text{End}(\underline{\mathbf{C}^{n+2}} \oplus \mathcal{O}(-1, 1))) \cong H^0(\text{End}(\underline{\mathbf{C}^{n+2}})) \oplus H^0(\mathcal{O}) \cong \text{End}(\underline{\mathbf{C}^{n+2}}) \oplus \mathbf{C}$. Consequently p and q can be regarded as non-zero constants and the automorphism F is expressed as $(C, r) \in \text{Aut}(\underline{\mathbf{C}^{n+2}}) \oplus \mathbf{C}^*$. Then the commutative diagram, (5.1) and (5.2) yield that

$$(5.4) \quad (([g], CAge), r[g, h(z, gf)e \otimes f]) = p(([g], A'ge), [g, h(z', gf)e \otimes f]),$$

$$(5.5) \quad q\{h(Bv, gf) + w(ge)\} = h(B' Cv, gf) + rw'(ge)$$

for arbitrary $g \in G$ and $v \in \mathbf{C}^{n+2}$. Consequently, (5.4) implies that $CAge = pA'ge$ and $rh(z, gf) = ph(z', gf)$ for an arbitrary $g \in G$. From the irreducibility of the standard representation of G , we have $CA = pA'$ and $rz = pz'$. If we put $v=0$ in (5.5), the irreducibility yields $qw = rw'$. Then we obtain $qB = B'C$, using again the irreducibility. Now A and B are automorphisms and so, $C = pA'A^{-1} = qB'^{-1}B$.

Conversely, if $aB'A' = bBA$, $cz = az'$ and $bw = cw'$, we may put $C = aA'A^{-1} = bB'^{-1}B$, $p = a$, $q = b$ and $r = c$. \square

Let \mathcal{M}^C be the set consisting of the isomorphism classes of the cohomology bundles

of the monad (MI). We call \mathcal{M}^c the complex moduli space.

THEOREM 5.7. *The complex moduli space \mathcal{M}^c is identified with*

$$\{(z, w) \in \mathbb{C}^{n+2} \times \mathbb{C}^{n+2*} \mid w(z) \neq 1\} / \mathbb{C}^*,$$

where \mathbb{C}^* -action is defined as $p \cdot (z, w) = (pz, \frac{1}{p}w)$ for $p \in \mathbb{C}^*$.

PROOF. Using a theorem of Okonek-Schneider-Spindler ([O-S-S; Corollary 1, p. 279]) and the Bott-Borel-Weil theorem, we have a bijection between the isomorphism classes of the monads (MI) and the isomorphism classes of the cohomology bundles. From Propositions 5.1 and 5.6 and Lemmas 5.2 and 5.4, we obtain

$$\mathcal{M}^c = \{(A, B, z, w, c) \in \text{Aut}(\mathbb{C}^{n+2}) \times \text{Aut}(\mathbb{C}^{n+2}) \times \mathbb{C}^{n+2} \times \mathbb{C}^{n+2*} \times \mathbb{C} \mid BA + z \otimes w = c \text{Id}_{\mathbb{C}^{n+2}}\} / \sim,$$

where $(A, B, z, w, c) \sim (A', B', z', w', c')$ means that there exist non-zero constants p, q and r such that $pB'A' = qBA$, $pz' = rz$, $rw' = qw$ and so, $pc' = qc$.

If $c=0$, then $BA = -z \otimes w$. Since $z \otimes w$ is not an automorphism, there is a contradiction and so, $c \neq 0$. Using our \mathbb{C}^* -action, we may put $c=1$. Then, if we fix (z, w) , BA is uniquely determined and automorphisms A and B are uniquely determined up to the equivalence relation. However, we must consider the condition that $\text{Id}_{\mathbb{C}^{n+2}} - z \otimes w$ is an automorphism, because BA is an automorphism. It is easy to show that $\text{Id}_{\mathbb{C}^{n+2}} - z \otimes w$ is invertible if and only if $w(z) \neq 1$. \square

REMARK. Making use of the proof of Theorem 5.7, we obtain another description of \mathcal{M}^c :

$$(5.6) \quad \mathcal{M}^c = \{(z, w, c) \in \mathbb{C}^{n+2} \times \mathbb{C}^{n+2*} \times \mathbb{C}^* \mid w(z) \neq c\} / \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*,$$

where the $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ -action is defined as $(p, q, r) \cdot (z, w, c) = (\frac{r}{p}z, \frac{q}{r}w, \frac{q}{p}c)$ for $p, q, r \in \mathbb{C}^*$.

To obtain the moduli of anti-self-dual connections, the reality condition (the Ward correspondence in §2) must be taken into account. First, we describe the real structure σ on F^{2n+1} . We define $j \in G$ as $je = f$, $je = -f$ and $ju = u$ for an arbitrary $u \in \mathbb{C}^{n+2}$ which is orthogonal to e and f . Then we have $\sigma([g]) = [gj]$ (for example, [Na-3]). Let $\mathbf{P}_x = \pi^{-1}(x)$ be a twistor fibre, where $x \in \text{Gr}_2(\mathbb{C}^{n+2})$.

PROPOSITION 5.8. *Let E be the cohomology bundle of the monad (MI). The restricted bundle $E|_{\mathbf{P}_x}$ to the twistor fibre is trivial for each $x \in \text{Gr}_2(\mathbb{C}^{n+2})$ if and only if (z, w, c) in (5.6) satisfies $w(u)h(z, u) + w(v)h(z, v) \neq c$ for arbitrary $u, v \in \mathbb{C}^{n+2}$ such that $|u|=|v|=1$ and $h(u, v)=0$.*

PROOF. This proof is a slight modification of [O-S-S, Lemma 4.2.3, p. 325].

From the theorem of Grothendieck ([O-S-S, Theorem 2.1.1, p. 22]) and $c_1(E)=0$, $E|_{\mathbf{P}_x}$ is trivial if and only if for an arbitrary non-zero section s of $E|_{\mathbf{P}_x}$ we have $s(z) \neq 0$ for all z in \mathbf{P}_x .

Since $\mathcal{O}(-1, 0)|_{\mathbf{P}_x} \cong \mathcal{O}(-1)$, $\mathcal{O}(0, 1)|_{\mathbf{P}_x} \cong \mathcal{O}(1)$ and $\mathcal{O}(-1, 1)|_{\mathbf{P}_x} \cong \mathcal{O}$ by [Na-3, Lemma 3.3], the display of the restricted monad (MI) to the twistor fibre \mathbf{P}_x implies that $I: H^0(\mathbf{P}_x, E|_{\mathbf{P}_x}) \cong H^0(\mathbf{P}_x, \text{Ker } \beta|_{\mathbf{P}_x}) \rightarrow \mathbf{C}^{n+2} \oplus \mathbf{C}$ is injective.

If $E|_{\mathbf{P}_x}$ is trivial, the injectivity of $I: H^0(\mathbf{P}_x, E|_{\mathbf{P}_x}) \rightarrow \mathbf{C}^{n+2} \oplus \mathbf{C}$ yields that there exists a subspace $E_x \subset \mathbf{C}^{n+2} \oplus \mathbf{C}$ such that

$$(5.7) \quad \bigcap_{[g] \in \mathbf{P}_x} \text{Ker } \beta_{[g]} = E_x \quad \text{and} \quad \bigcup_{[g] \in \mathbf{P}_x} \text{Im } \alpha_{[g]} \cap E_x = \{0\},$$

where we denote $\alpha([g], \cdot)$ by $\alpha_{[g]}: \mathbf{C}e \rightarrow \mathbf{C}^{n+2} \oplus \mathbf{C}$ and $\beta([g], (\cdot, \cdot))$ by $\beta_{[g]}: \mathbf{C}^{n+2} \oplus \mathbf{C} \rightarrow \mathbf{C}f$, using (5.1) and (5.2). We claim that if $[g_1]$ and $[g_2]$ are different points in \mathbf{P}_x , then $\text{Im } \alpha_{[g_1]} \cap \text{Im } \alpha_{[g_2]} = \{0\}$. Let $Sp(1)$ be the subgroup in G of which the corresponding Lie algebra is generated by the highest root and $U(1)$ be the standard subgroup of $Sp(1)$. Note that $j \in Sp(1) \setminus U(1)$. Then by definition of the twistor space $[S]$, there exists $s \in Sp(1) \setminus U(1)$ such that $g_2 = g_1 s$. We assume that there exists a non-zero constant c such that $\alpha([g_1, ce]) = \alpha([g_2, e])$. By (5.1), we have $cAg_1 e = Ag_1 se$. Lemma 5.2, yields that $se = ce$ and so, $s \in U(1)$. This is a contradiction. Hence, $\mathbf{C}^{n+2} \oplus \mathbf{C}$ is decomposed into $\text{Im } \alpha_{[g_1]} \oplus \text{Im } \alpha_{[g_2]} \oplus E_x$. Then $\beta_{[g_2]} \circ \alpha_{[g_1]}: \mathbf{C}e \rightarrow \mathbf{C}f$ is an isomorphism by (5.7).

Next, we assume that the restricted bundle $E|_{\mathbf{P}_x}$ is not trivial and so, there exist a non-zero section $s \in H^0(\mathbf{P}_x, E|_{\mathbf{P}_x})$ and $g_1 \in G$ such that $s([g_1]) = 0$. By the injectivity of I , there exists a unique $(u, c) \in \mathbf{C}^{n+2} \oplus \mathbf{C}$ such that $I(s([g])) = ([g], (u, c))$, where $[g] \in \mathbf{P}_x$. From the definition of I , we obtain $\beta([g], (u, c)) = 0$ for an arbitrary $[g] \in \mathbf{P}_x$. Since $s([g_1]) = 0$, there exists a constant c' such that $\alpha([g_1, c'e]) = ([g_1], (u, c))$. Consequently, we have $\beta_{[g]} \circ \alpha_{[g_1]}(c'e) = \beta([g], (u, c)) = 0$. Thereby, $E|_{\mathbf{P}_x}$ is trivial if and only if $\beta_{[g_2]} \circ \alpha_{[g_1]}: \mathbf{C}e \rightarrow \mathbf{C}f$ is an isomorphism for arbitrary $[g_1] \neq [g_2] \in \mathbf{P}_x$. However, using $j \in Sp(1) \subset G$, we can substitute $g_1 j$ for g_2 .

On the other hand, a direct computation and the definition of $j \in G$ imply that $\beta_{[g_1 j]} \circ \alpha_{[g_1]}(e) = \{-h(BAge, ge) + w(gf)h(z, gf)\}f$. Hence, $E|_{\mathbf{P}_x}$ is trivial if and only if

$$(5.8) \quad -h(BAge, ge) + w(gf)h(z, gf) \neq 0,$$

for an arbitrary $[g] \in \mathbf{P}_x$. Since $BA + z \otimes w = c \text{Id}_{\mathbf{C}^{n+2}}$ by Lemma 5.4, (5.8) is equivalent to $w(ge)h(z, ge) + w(gf)h(z, gf) \neq c$. Taking account of the standard representation of G , we obtain the desired result. \square

By [Na-3, Lemma 5.1.7], we have isomorphisms $s_1: \mathcal{O}(-1, 0) \cong \sigma^* \overline{\mathcal{O}(0, 1)^*}$ and $s_2: \mathcal{O}(0, 1) \cong \sigma^* \overline{\mathcal{O}(-1, 0)^*}$. Since $\mathcal{O}(-1, 1)$ is the pull-back of an anti-self-dual bundle from [Na-3, Theorem 3.4], the Ward correspondence implies that $s: \mathcal{O}(-1, 1) \cong \sigma^* \overline{\mathcal{O}(-1, 1)}$. We call these isomorphisms the standard isomorphisms. More explicitly, the standard isomorphisms are expressed as:

$$\begin{aligned} s_1([g, e]) &= ([g], [gj, -h(f, \cdot)]), & s_2([g, f]) &= ([g], [gj, h(e, \cdot)]), \\ s([g, e \otimes f]) &= ([g], [gj, h(e \otimes f, \cdot)]), \end{aligned}$$

where the last h means the induced metric from $h|_{C_e}$ and $h|_{C_f}$.

PROPOSITION 5.9. *Let E be the cohomology bundle of the monad (MI). Moreover, the restricted bundle $E|_{\mathbb{P}_x^1}$ is assumed to be trivial for every x in $Gr_2(\mathbb{C}^{n+2})$. Then, there is an isomorphism $\tau : E \rightarrow \sigma^* \bar{E}^*$ with $(\sigma^* \bar{\tau})^* = \tau$ which induces a positive definite hermitian form on sections of $E|_{\mathbb{P}_x^1}$ for every $x \in Gr_2(\mathbb{C}^{n+2})$ if and only if there exist a hermitian metric on $\underline{\mathbb{C}^{n+2}}$, a hermitian metric on $\mathcal{O}(-1, 1)$ and isomorphisms $\mathcal{O}(-1, 0) \cong \sigma^* \overline{\mathcal{O}(0, 1)^*}$ and $\mathcal{O}(0, 1) \cong \sigma^* \overline{\mathcal{O}(-1, 0)^*}$.*

PROOF. From the hypothesis, $\sigma^* \bar{E}^*$ is the cohomology bundle of the monad;

$$\sigma^* \overline{\mathcal{O}(0, 1)^*} \xrightarrow{\sigma^* \bar{\beta}^*} \sigma^* \overline{\mathbb{C}^{n+2} \oplus \mathcal{O}(-1, 1)^*} \xrightarrow{\sigma^* \bar{\alpha}^*} \sigma^* \overline{\mathcal{O}(-1, 0)^*}.$$

We can check the conditions in a theorem of Okonek-Schneider-Spindler ([O-S-S, Lemma 4.1.3, p. 276]) using the Bott-Borel-Weil theorem and standard isomorphisms $\mathcal{O}(-1, 0) \cong \sigma^* \overline{\mathcal{O}(0, 1)^*}$, $\mathcal{O}(0, 1) \cong \sigma^* \overline{\mathcal{O}(-1, 0)^*}$ and $\mathcal{O}(-1, 1) \cong \sigma^* \overline{\mathcal{O}(-1, 1)^*}$. Then we obtain that there is an isomorphism $\tau : E \rightarrow \sigma^* \bar{E}^*$ if and only if there exist isomorphisms $\underline{\mathbb{C}^{n+2}} \oplus \mathcal{O}(-1, 1) \cong \sigma^* \overline{\mathbb{C}^{n+2}^*} \oplus \sigma^* \overline{\mathcal{O}(-1, 1)^*}$, $\mathcal{O}(-1, 0) \cong \sigma^* \overline{\mathcal{O}(0, 1)^*}$ and $\mathcal{O}(0, 1) \cong \sigma^* \overline{\mathcal{O}(-1, 0)^*}$. Since $H^0(\mathcal{O}(-1, 1)) = H^0(\mathcal{O}(1, -1)) = 0$, we have $\underline{\mathbb{C}^{n+2}} \cong \sigma^* \overline{\mathbb{C}^{n+2}^*}$ and $\mathcal{O}(-1, 1) \cong \sigma^* \overline{\mathcal{O}(-1, 1)^*}$. The restricted bundle $\underline{\mathbb{C}^{n+2}}|_{\mathbb{P}_x}$ and $\mathcal{O}(-1, 1)|_{\mathbb{P}_x}$ are trivial by [Na-3, Lemma 3.3]. Hence, these isomorphisms induce non-degenerate hermitian forms on $\underline{\mathbb{C}^{n+2}}$ and $\mathcal{O}(-1, 1)$ respectively.

Next we take the condition imposed upon τ into account. Since $E|_{\mathbb{P}_x^1}$ is trivial, under the notation in the proof of Proposition 5.8, this condition yields E_x has a positive hermitian inner product. If we make use of an identification $\mathcal{O}(-1, 1)|_{\mathbb{P}_x} \cong \mathbb{P}_x \times \mathbb{C}_x$, the proof of Proposition 5.8 yields that

$$(5.9) \quad \text{Im } \alpha_{[g]} \oplus \text{Im } \alpha_{[gj]} \oplus E_x = \mathbb{C}^{n+2} \oplus \mathbb{C}_x,$$

where $[g] \in \mathbb{P}_x$. Since there exists $g \in G$ such that $h(z, gf) = h(z, gje) = 0$, we have $\mathbb{C}_x \subset E_x$ for this $g \in G$ by (5.1). Then the induced hermitian form on $\mathcal{O}(-1, 1)|_{\mathbb{P}_x}$ is positive, because of the positivity of the hermitian inner product on E_x . The non-degeneracy of the hermitian form implies the positivity of this hermitian form and so, $\mathcal{O}(-1, 1)$ has a hermitian metric. On the other hand, the property of the standard representation of G , Lemma 5.2 and (5.9) yield that the vector space spanned by $\bigcup_{x \in Gr_2(\mathbb{C}^{n+2})} E_x$ has \mathbb{C}^{n+2} as a subspace. Consequently, the induced hermitian form on $\underline{\mathbb{C}^{n+2}}$ is also positive.

Conversely, a hermitian metric on $\underline{\mathbb{C}^{n+2}}$ induces an isomorphism $\underline{\mathbb{C}^{n+2}} \cong \sigma^* \overline{\mathbb{C}^{n+2}^*}$ and a hermitian metric on $\mathcal{O}(-1, 1)$ induces an isomorphism $\mathcal{O}(-1, 1) \cong \sigma^* \overline{\mathcal{O}(-1, 1)^*}$. Combined with isomorphisms $\mathcal{O}(-1, 0) \cong \sigma^* \overline{\mathcal{O}(0, 1)^*}$ and $\mathcal{O}(0, 1) \cong \sigma^* \overline{\mathcal{O}(-1, 0)^*}$, these induce the desired $\tau : E \rightarrow \sigma^* \bar{E}^*$ under the hypothesis that $E|_{\mathbb{P}_x}$ is trivial. \square

Therefore, to describe the moduli space, we fix the G -invariant hermitian inner product h on \mathbb{C}^{n+2} and the standard isomorphisms s, s_1, s_2 .

PROPOSITION 5.10. *Under this fixed isomorphisms $\mathbf{C}^{n+2} \cong \sigma^* \overline{\mathbf{C}^{n+2}}$, $s : \mathcal{O}(-1, 1) \cong \sigma^* \overline{\mathcal{O}(-1, 1)}$, $s_1 : \mathcal{O}(-1, 0) \cong \sigma^* \overline{\mathcal{O}(0, 1)}$ and $s_2 : \mathcal{O}(0, 1) \cong \sigma^* \overline{\mathcal{O}(-1, 0)}$, the following two conditions are equivalent:*

(1) *There exists a commutative diagram*

$$\begin{array}{ccccc}
 \mathcal{O}(-1, 0) & \xrightarrow{\alpha} & \mathbf{C}^{n+2} \oplus \mathcal{O}(-1, 1) & \xrightarrow{\beta} & \mathcal{O}(0, 1) \\
 \downarrow & & \downarrow & & \downarrow \\
 \sigma^* \overline{\mathcal{O}(0, 1)} & \xrightarrow{\sigma^* \beta^*} & \sigma^* \overline{\mathbf{C}^{n+2} \oplus \mathcal{O}(-1, 1)} & \xrightarrow{\sigma^* \alpha^*} & \sigma^* \overline{\mathcal{O}(-1, 0)}.
 \end{array}$$

(2) *For all u and v in \mathbf{C}^{n+2} , $h(Au, v) = h(u, Bv)$ and $w = -h(\cdot, z)$.*

PROOF. If the diagram is commutative, a direct computation shows that for all u in \mathbf{C}^{n+2} and all g in G , $h(Age, u) = \overline{h(Bu, ge)}$ and $h(z, gf) = \overline{w(gf)}$. The irreducibility of the standard action of G implies the condition (2). Now it is clear that (2) yields (1). \square

We denote by A^* the adjoint operator of A with respect to h . Let \mathcal{M} be the moduli space of anti-self-dual connections on E satisfying the hypothesis in Main Theorem 1.

PROOF OF MAIN THEOREM 2. From Lemma 5.4 and Proposition 5.10, we have $A^*Au - h(u, z)z = cu$ for an arbitrary $u \in \mathbf{C}^{n+2}$. In particular, $A^*Az = (|z|^2 + c)z$, and if u is orthogonal to z , then $A^*Au = cu$. Hence A is an automorphism on \mathbf{C}^{n+2} if and only if c is a positive real number. On the other hand, Propositions 5.8 and 5.10 imply that $|z|^2 \neq -c$. However, this condition is satisfied automatically because of the positivity of c .

By Proposition 5.6, (z, c) and (z', c') induce the isomorphic monads if and only if there exists a non-zero constant $p \in \mathbf{C}^*$ such that $(z', c') = (pz, |p|^2c)$. Consequently, we obtain

$$\mathcal{M} = \{(z, c) \in \mathbf{C}^{n+2} \times \mathbf{R}^+\} / \mathbf{C}^*,$$

where \mathbf{C}^* -action is defined as $p \cdot (z, c) = (pz, |p|^2c)$.

Next, we focus our attention on $|z|^2 + c$ which is an eigenvalue of A^*A . Since $|pz|^2 + |p|^2c = |p|^2(|z|^2 + c)$, we can normalize in such a way that $|z|^2 + c = 1$ using the \mathbf{C}^* -action. Then the \mathbf{C}^* -action is reduced to S^1 -action such that $p \cdot (z, c) = (pz, c)$ where $p \in S^1 \subset \mathbf{C}^*$. Therefore, taking account of the positivity of c , we obtain that

$$\mathcal{M} = \{z \in \mathbf{C}^{n+2} \mid |z|^2 < 1\} / S^1. \quad \square$$

REMARK. Under the assumption that $|z|^2 + c = 1$, we may put

$$\begin{cases} Az = z \\ Au = \sqrt{1 - |z|^2}u \end{cases} \quad \text{if } u \in \mathbf{C}^{n+2} \text{ is orthogonal to } z.$$

First we suppose that $z = 0$. Then $A = \text{Id}_{\mathbf{C}^{n+2}}$. From Propositions 5.1 and 5.10, the cohomology bundle (MI) is decomposed into $\mathcal{O}(-1, 1)$ and the cohomology bundle of

the monad:

$$\mathcal{O}(-1, 0) \rightarrow \underline{\mathbf{C}^{n+2}} \rightarrow \mathcal{O}(0, 1).$$

This monad is the standard monad induced by ϖ_{n+1} . (This terminology is defined in [Na-3, Definition 4.4].) Moreover, the cohomology bundle of the standard monad is homogeneous by [Na-3, Theorem 4.5]. In fact, the cohomology bundle of (MI) is isomorphic to a direct sum $Q_2(0, 1) \oplus \mathcal{O}(-1, 1)$. Consequently, the “vertex” of \mathcal{M} corresponds to a reducible connection. (A $SU(n+1)$ -anti-self-dual connection reduces to a $U(n) \times U(1)$ connection.) The centralizer of $U(n) \times U(1)$ in $SU(n+1)$ is $U(1)$. We denote by Z_{n+1} the center of $SU(n+1)$. The group $U(1)/Z_{n+1}$ is nothing but S^1 in the description of the moduli in Main Theorem 2.

Finally, we put $|z|^2 = 1$. Then $\text{Ker } A = \{u \in \mathbf{C}^{n+2} \mid u \perp z\}$. For brevity, $\text{Ker } A$ is expressed as z^\perp . The proof of Lemma 5.2 implies that $\alpha([g, e]) = 0$ if and only if $ge \in z^\perp$ and $gf \in z^\perp$. Combined with Proposition 5.10, the monad (MI) does not define a vector bundle on $Gr_2(z^\perp)$. In the case $n=1$, this is a well-known fact, because $Gr_2(z^\perp)$ is one point.

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