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The Involutions of Compact Symmetric Spaces, IV

Tadashi NAGANO and Makiko Sumi TANAKA

Sophia University and International Christian University

Introduction.

In this part, we will discuss and determine the signature $\tau(M)$ and the g-signature g-$\tau(M)$ (See [AS]) of every compact oriented symmetric space M, (making the description of $\tau(M)$ in [N88] intelligible, in particular) as well as self-intersections $SI(N;M)$ of subspaces N in M which are beautifully related to g-$\tau(M)$ by the g-signature theorem or the generalized Lefschetz fixed point theorem of Atiyah-Singer [AS] in case g is an orientation-preserving involution of M. An example of geometric applications will be added.

Our method is basically to apply the Atiyah-Singer theory [AS], especially the g-signature theorem (1.3) below to our geometric results ([N88] and others). We have determined those invariants for each space in a few ways. The value of $\tau(M)$ was stated in [N88] with a very brief explanation (involving a careless mistake), but we will give a more detailed proof. Informations on g-$\tau(M)$ and on $SI(N;M)$ are reciprocal to some extent. The self-intersection $SI(N;M)=[B]$ is realized by a (symmetric) subspace B of M (in the cases discussed in this paper).

0.1 THEOREM ([N88], 10.1). *If the signature* $\tau(M)$ $(\geqq 0)$ *is positive for a simple 1-connected* M, *then* $\tau(M)$ *equals the indicated value below*: $\frac{1}{2}\tau(G_{2p}^{o}(\mathbf{R}^{2n}))=\tau(G_p(\mathbf{C}^n))=\tau(G_p(\mathbf{H}^n))=\chi(G_p(\mathbf{R}^n))$, *the Euler number of* $G_p(\mathbf{R}^n)$, $\tau(\mathrm{EII})=4$, $\tau(\mathrm{EIII})=3$, $\tau(\mathrm{EVI})=7=\tau(\mathrm{EVIII})$, $\tau(\mathrm{EIX})=8$, *and* $\tau(\mathrm{FII})=1=\tau$ (GI). Here the symbols for the symmetric spaces are Cartan's [H] with a few exceptions such as $G_p(V)$ meaning the Grassmann manifold of the p dimensional subspaces of a vector space V and $G_p^o(\mathbf{R}^n)$ which means that of the oriented p-subspaces of $\mathbf{R}^n$; $G_p^o(\mathbf{R}^n)$ is 1-connected except that it consists of two points for $p=0$ or n and $G_1^o(\mathbf{R}^2)$ is a circle. The known Euler number $\chi(G_p(\mathbf{R}^n))$ equals the binomial coefficient ${}_{[n/2]}C_{[p/2]}$ *if* $p(n-p)=\dim G_p(\mathbf{R}^n)$ is even and 0 otherwise.

0.2 COROLLARY. *One has*

$$(0\leqq)3\tau(M)\leqq\chi(M) \qquad \textit{if } \dim M>0\,,$$

and this is sharp. In particular, the equality

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$$\tau(M)=\chi(M) \text{ implies } \chi(M)=0=\tau(M).$$

0.2a REMARK. In case $\dim M=4$, one might recall the Hitachin-Thorpe-Itoh inequality $(3/2)\tau(M)\leqq\chi(M)$ for every Einstein-Weyl manifold M. And a good proof is wanted for 0.2.

0.2b REMARK. Let us add another sharp estimate $\chi(M)\leqq\#_2(M)$, where $\#_2(M)$ is the maximal cardinality of finite trivial subspaces B (which are, by definition, finite sets and symmetric spaces with all the point symmetries s_x equal to the identity 1_B) of M (See [N88] for the reference). In case M is a group, $\#_2(M)$ is essentially the same as the 2-rank of Borel-Serre. It has something to do with the homological torsion. The equality obtains if M is Kaehlerian. M. Takeuchi proved $\#_2(M)=\dim H^*(M, \mathbf{Z}_2)$ for every R-space (3.1).

The concept of the signature or index has a long history; we will quote some results in Section 1. A few years after we determined $\tau(M)$ for every symmetric space M ([N88]), Hirzebruch et al. [HS] established a unified and (theoretically) practical formula for the symmetric spaces and determined $\tau(M)$ for most M; for some spaces, Bliss et al. [BMP] "had to resort to machine computation" and for the spaces the machine could not work out (and left with question marks in [HS]). Y. Shimizu used combinatorics to determine their $\tau(M)$ by their theorem in his Master Thesis (1991). Their results agree with ours, naturally.

Our method is geometric, appealing to the g-signature theorem of Atiyah and Singer (1.3), an analog of the Lefschetz fixed point theorem, and other theorems such as (1.4). For determining the g-signature g-$\tau(M)$ and even the signature $\tau(M)$, we will always use an involution g, because we know the fixed point set $F(g, M)$ of g acting on M for every pair (g, M) (described in [N88]). One will observe many interesting geometric facts. For an easy example, if M is Kaehlerian (e.g. $M=G_p(\mathbf{C}^n)$) and g is "a complex conjugation" (e.g. $F(g, M)=G_p(\mathbf{R}^n)$), then we will prove g-$\tau(M)=\chi(F(g, M))$ (Proposition 3.2). Since the g-signature theorem involves the self-intersection $SI(N;M)$ of a submanifold N, we will determine it in several cases. One of the methods is to use an appropriate double fibration as in integral geometry or the Penrose transform (or the Klein correspondence for it) (See 4.11a and Theorem 4.11b). Since we choose g to be an involutive automorphism, the related self-intersection is represented by a symmetric space (a subspace in the sense of 1.8a). We will not have to use precise description of the cohomology ring which is known for almost but not every M or a general formula for the Euler class of a homogeneous vector bundle (to be applied to the normal bundle to N).

Based on determined values of $SI(N;M)$, we will explain certain geometric applications (to cohomology rings and some analogs or variants of projective geometry) for a sequence of exceptional spaces, FII, EIII, EVI and EVIII (Propositions 5.1 and 5.4) in Section 5.

1. Signature and preliminaries.

We will briefly explain general facts about $\tau(M)$ and g-$\tau(M)$ which we will or could use later; see [AS] for better explanations. Also a minimum number of basic definitions and symbols concerning the geometric theory of symmetric spaces will be recalled in this section (See [N88] for more details).

The signature $\tau(M)$ is the index of the quadratic form on $H^{2k}(M)=H^{2k}(M;\mathbf{R})$ defined with the cup product and the orientation. It is reasonable to define $\tau(M)$ to be zero if $\dim M$ is not a multiple of 4. $\tau(M)$ and g-$\tau(M)$ are also defined as the indices of certain elliptic complexes.

1.1. *The g-signature g-$\tau(M)$ equals the signature $\tau(M)$ if g is homotopic with 1_M, the identity map, or, more generally, if g acts on $H^{2k}(M)$ as the identity* [AS], $4k:=\dim M$.

1.2. *$\tau(M)$ is a certain Pontryagin number of M (the Hirzebruch index theorem).* Hence (i) an m-fold covering space of M has the signature $=m\tau(M)$. (ii) $\tau(M)=0$ if M is a group or M is a covering space of a non-orientable manifold. (iii) $\tau(M\times N)=\tau(M)\times\tau(N)$; actually this product formula is valid in a much more general setting ([AS], p. 580) including the case where the dimension of M or N is not divisible by 4.

1.3 Proposition (The g-signature theorem [AS] p. 583). *Let X be a compact oriented manifold of dimension $4k$, and let g be an orientation preserving involution of X. Then one has*

$$g\text{-}\tau(X)=\tau(SI(F(g,X)\,;\,X))\,,$$

where $F(g,X)$ is the fixed point set of g acting on X and $SI(F(g,X);X)$ means the self-intersection (See [AS] p. 583) *of $F(g,X)$ in X.*

We will use the following theorem of Hattori-Taniguchi [HT] also.

1.4 Proposition ([HT] p. 722. Cf. [AS] p. 594). *If the circle group $U(1)$ acts on a compact oriented manifold X, then the signature $\tau(X)$ equals the sum of $\tau(F_j)$ for the connected components F_j of the fixed point set $F(U(1),X)$ such that* $\operatorname{codim} F_j\equiv 0 \bmod 4$; *$F_j$ is oriented in a certain way by means of the action of $U(1)$ as in* [AS]. *(The sum of $\tau(F_k)$ for F_k with* $\operatorname{codim} F_k\equiv 2 \bmod 4$ *equals* 0).

The twistor space was used to prove the next proposition.

1.5 Proposition (Thm 2.1, [NT87]). *The signature of a quaternionic Kaehler manifold X of dimension $4n$ is equal to b_{2n}, the Betti number of X at the half dimension $2n$, up to sign.*

The Hodge index theorem ([GH] p. 126) reads, for symmetric spaces:

1.6 PROPOSITION. *The signature of a symmetric Kaehler manifold M is equal to $P(i)$, the Poincaré polynomial $P(t)$ with the imaginary unit i substituted.*

1.7. We will work on the compact connected simple symmetric spaces unless otherwise mentioned. Such a space M is a homogeneous space G/K of a compact connected Lie group G which acts on M as isometries; the action is locally effective usually. M admits the point symmetry s_o for every point o of M, which is an isometry and acts on the tangent space T_oM as -1 times the identity. One knows that

1.7a. s_o is homotopic to the identity 1_M if and only if (1) the Euler number $\chi(M)$ is positive, or equivalently, (2) the rank $r(K)$ equals $r(G)$ for $M=G/K$.

1.8. A map f of a symmetric space M into another is called *a homomorphism* if f commutes with every point symmetry; $f\circ s_x = s_{f(x)}\circ f$. Somewhat similar definition of a homomorphism is found in [KN] at p. 227. If M is connected, every s_x is an automorphism. Since M is connected (in this paper), a homomorphism is nothing but a totally geodesic map. Hence the automorphism group $\mathrm{Aut}(M)$ coincides with the affine transformation group $A(M)$. $A(M)$ equals the isometry group $I(M)$ if M is semi-simple and connected, since an affine map preserves the Ricci tensor. A homomorphism $f: M\to N$ gives rise to a homomorphism $I(f): I(M)\hat{}\to I(N)$ of a covering group of the $I(M)$ with respect to which f is equivariant.

1.8a. The involutions we will consider will be automorphisms; hence their fixed point sets are *subspaces* (i.e. symmetric spaces whose inclusion maps are homomorphisms).

1.9 COROLLARY TO 1.4. *Let $s\in G$ have a finite order. Then one has the signature $\tau(M)=\sum\tau(F_j)$ for the connected components F_j of $F(s, M)$ if F_j is oriented in an appropriate way.*

PROOF. s is then a member of some subgroup $T\cong U(1)$ *of* G. Since T acts on $F(s, M)$ and $F(T, M)$ is its subspace, one obtains the conclusion by using 1.4 twice; $\tau(M)$ equals the sum of the signatures of the components of $F(T, M)$ as well as $\tau(F(s, M))$. □

We call each component of $F(s_o, M)$ *a polar of o in M*; the polars in M are concretely described in [N88].

1.10 COROLLARY. (i) *The signature $\tau(M)$ equals the sum of those of the polars of a point, if $\chi(M)>0$.* (ii) *One has $\tau(M)=0$ otherwise.*

PROOF. (i) is immediate from 1.9. In the case of (ii), one has $r(K)<r(G)$ and hence G contains a circle group T which fixes no point as was proved by Hopf-Samelson [HS] (easy to see if M is symmetric); and 1.4 gives $\tau(M)=0$. □

1.10a REMARK. The polars in M are thus closely related to M in terms of topology. For another example, M is orientable if and only if every polar has an even dimension

([NT95] 5.1). Also the Lefschetz number Lef(s_o) equals the sum of their Euler numbers ([AS] p. 574); hence, in case $\chi(M)$ is positive, the sum equals $\chi(M)$. (In case M is a group, Lef(s_o) equals $2^{r(M)}$, $r(M)$ denoting the rank of M.)

1.10b. The conclusion of 1.10 (i) is true even if M is a compact 1-connected group and the polars are appropriately oriented. It is not true, however, if M is not.

1.11. g-$\tau(X)$ is determined by the action of g on the real cohomology of X; hence it depends on the homotopy class of g only ([AS] p. 578).

1.12 PROPOSITION. *One has* s_o-$\tau(M)=\tau(M)$ *for the point symmetry* s_o, *if* $\dim M \equiv 0 \bmod 4$.

PROOF. This equality obtains by 1.1, since s_o acts trivially on $H^p(M)$ which consists of the G-invariant (or, equivalently, harmonic) p-forms on M, provided p is even. □

1.13. We will specify involutions, say s, of M by means of components of $F(s, M)$, since just one of them uniquely determines s. The correspondence with the conventional definitions were given in [N88].

For a point p of $F(s, M)$, the connected component $F(s \circ s_p, M)_{(p)}$ through p of $F(s \circ s_p, M)$ meets $F(s, M)$ at p in such a way that the tangent space $T_pF(s \circ s_p, M)$ is the orthogonal complement of $T_pF(s, M)$ in T_pM. We say that $F(s \circ s_p, M)$ is *c-orthogonal to* $F(s, M)$ *at* p.

1.14. We denote by $M^{\%}$ *the bottom space* (or the adjoint space [H]) of a simple M; that is, every connected space which is locally isomorphic with M is a convering space of $M^{\%}$ (if not identical). $M^{\%}$ is the orbit space $M/C(I(M))$ by the center of $I(M)$; and $C(I(M))$ is bijective with the inverse image $\pi^{-1}(o^{\%})$ of a point $o^{\%}$ of $M^{\%}$ under the covering morphism $\pi: M \to M^{\%}$. In case M is a group, $M^{\%}$ is thus the adjoint group ad(M).

Every space $M=G/K$ is a subspace of G (which can be appropriately chosen among the covering groups of the identity component, $I(M)_{(1)}$ of $I(M)$) (the Cartan embedding). The monomorphism: $M \to G$ projects onto the one: $M^{\%} \to G^{\%}$. Theoretically, its existence compensates for the unfortunate fact that the projection $G \to G/K$ is not a homomorphism.

1.15. *A pole* p of a point o is a polar which is a singleton. There exists a double covering morphism $\pi: M \to M''$ satisfying $\pi(o)=\pi(p)$. *A centriole* means a connected component of the set of the midpoints of geodesic arcs joining o with p, which is known to be a subspace; in fact the projection π carries every centriole onto a polar of $\pi(o)$. In other words, a centriole is a connected component of the fixed point set $F(s_o \circ \gamma, M)$ where γ is the deck transformation for π. We used to call $F(s_o \circ \gamma, M)$ the centrosome $C(o, p)$ for the pair (o, p).

2. Determination of the signature.

We will determine the signature $\tau(M)$ for every compact symmetric space M, which we may assume is simple and 1-connected with $\tau(M)\geqq 0$ to prove the theorem 0.1.

2.0. $\tau(M)=0$ if M is a group (1.2ii).

2.1. $\tau(\mathrm{AI})=0=\tau(\mathrm{AII})$ by (1.10ii) and $\mathrm{AI}(2)=S^2$.

2.2. $\tau(G_p(\mathbf{C}^n))=\chi(G_p(\mathbf{R}^n))$.

PROOF 1. This obtains by the Hodge formula (1.6), applied to the Poincaré polynomial $\prod_{1\leqq k\leqq p}(1-t^{2(n-p+k)})/(1-t^{2k})$ of $G_p(\mathbf{C}^n)$.

PROOF 2. Alternatively, one may use the involution $s:=I_1$, $I_1\in \mathrm{U}(1)\subset \mathrm{U}(n)$, such that $F(s, G_p(\mathbf{C}^n))=G_p(\mathbf{C}^{n-1})\coprod G_{p-1}(\mathbf{C}^{n-1})$. The self-intersection of $F(s, G_p(\mathbf{C}^n))$ in $G_p(\mathbf{C}^n)$ is $G_p(\mathbf{C}^{n-2})\coprod G_{p-2}(\mathbf{C}^{n-2})$ (See 4.6). Hence $\tau(G_p(\mathbf{C}^n))=I_1\text{-}\tau(G_p(\mathbf{C}^n))=\tau(G_p(\mathbf{C}^{n-2}))+\tau(G_{p-2}(\mathbf{C}^{n-2}))$. By induction, we obtain $\tau(G_p(\mathbf{C}^n))=\chi(G_p(\mathbf{R}^n))$, knowing $\tau(\text{a point})=1$, $\tau(G_1(\mathbf{C}^2))=\tau(S^2)=0$ and the values of $\chi(G_p(\mathbf{R}^n))$ (See 0.1). □

2.3. (i) $\tau(G_r(\mathbf{R}^m))=0$ if m or $r(m-r)$ is odd. (ii) If $r=2p$ and $m=2n$ are even, then $\tau(G_{2p}(\mathbf{R}^{2n}))=\chi(G_p(\mathbf{R}^n))$.

PROOF. (i) is taken care of by 1.2 (ii) and 1.10. For (ii), one has $F(J_n, G_{2p}(\mathbf{R}^{2n}))=G_p(\mathbf{C}^n)$ for the involution J_n defined with a complex structure of $\mathbf{R}^{2n}$. Thus 2.2 gives $\tau(G_{2p}(\mathbf{R}^{2n}))=J_n\text{-}\tau(G_{2p}(\mathbf{R}^{2n}))=\tau(G_p(\mathbf{C}^n))=\chi(G_p(\mathbf{R}^n))$ by 1.3. □

2.4. $\tau(\mathrm{DIII})=0$.

PROOF. $\mathrm{DIII}(n):=\mathrm{SO}(2n)/\mathrm{U}(n)$, $n>1$, may be thought of as a connected component of the set of all the complex structures $J:\mathbf{R}^{2n}\to\mathbf{R}^{2n}$ which lie in $\mathrm{SO}(2n)$; the set has two components by the orientations J defines. Hence $\mathrm{DIII}(n)$ admits an involution s such that $F(s, \mathrm{DIII}(n))$ is 2 copies of $\mathrm{DIII}(2)\times\mathrm{DIII}(n-2)$, and $\mathrm{DIII}(2)\cong S^2$. Thus 2.4 follows from 1.9, 1.2 (iii) and $\tau(S^2)=0$. □

2.5. $\tau(G_p(\mathbf{H}^n))=\chi(G_p(\mathbf{R}^n))$.

PROOF. The quaternion Grassmann manifold $G_p(\mathbf{H}^n)$ admits an involution whose fixed point set is exactly $G_p(\mathbf{C}^n)$ and which lies in some $\mathrm{U}(1)\subset\mathrm{Sp}(n)$. Thus one has $\tau(G_p(\mathbf{H}^n))=\tau(G_p(\mathbf{C}^n))=\chi(G_p(\mathbf{R}^n))$ by 1.9 (or 1.10 (i)) and 2.2. □

2.6. $\tau(\mathrm{CI})=0$.

PROOF. Similarly to DIII, $\mathrm{CI}(n)=\mathrm{Sp}(n)/\mathrm{U}(n)$ admits an involution in $\mathrm{U}(1)$ whose fixed point set is $\mathrm{CI}(1)\times\mathrm{CI}(n-1)$, $\mathrm{CI}(1)\cong S^2$. □

2.6a REMARK. 2.6 and 2.4 are immediate from 1.6 if one notes that $1+t^2$ divides the Poincaré polynomials of $\mathrm{CI}(n)$ and of $\mathrm{DIII}(n)$.

2.7. $\tau(\mathrm{GI})=1$.

PROOF. The polars of $\mathrm{GI}:=G_2/\mathrm{SO}(4)$ are $\{o\}\amalg S^2\cdot S^2$ and $\tau(S^2\cdot S^2)=0$. And 1.10 (i) applies. □

2.8. $\tau(\mathrm{FI})=0$.

PROOF. One has $F(s_{\mathrm{II}},\mathrm{FI})=G_4^o(\mathbf{R}^9)$ and 2.3 applies along with 1.10 and 1.3. □

2.9. $\tau(\mathrm{FII})=1$.

PROOF. The polars of FII are $\{o\}\amalg S^8$. □

2.10. $\tau(\mathrm{EI})=0$ by (1.10 ii).

2.10a. Thus $\tau(M)=\chi(M)=0$ if the root system $R(M)$ is E_6 or A_r, $r>1$.

2.11. $\tau(\mathrm{EII})=4$.

PROOF. One has $F(s_{\mathrm{III}},\mathrm{EII})=G_4^o(\mathbf{R}^{10})\amalg\mathrm{DIII}(5)$ and $\tau(G_4^o(\mathbf{R}^{10}))\pm\tau(\mathrm{DIII}(5))=4+0$. One obtains the result with 1.9, 2.3 and 2.4. □

2.12. $\tau(\mathrm{EIII})=3$.

PROOF. $F(s_{\mathrm{II}},\mathrm{EIII})=G_2(\mathbf{C}^6)\amalg S^2\times G_1(\mathbf{C}^6)$ and $\tau(G_2(\mathbf{C}^6))+0=3$.

2.13. $\tau(\mathrm{EIV})=0$, since $\chi(\mathrm{EIV})=0$ (1.10 ii).

2.14. $\tau(\mathrm{EV})=0$.

PROOF. $\mathrm{EV}^{\%}$ is not orientable (1.10a), since $\mathrm{EV}^{\%}$ has odd dimensional polars such as $\mathrm{AII}(4)/\mathbf{Z}_4$. Hence 1.2(ii) applies. For other proofs, one knows $\frac{1}{2}\dim\mathrm{EV}$ is odd and that all the polars have lower dimensions than $\frac{1}{2}\dim\mathrm{EV}$. □

2.14a. Thus $\tau(M)=0$ if the root system $R(M)$ is A_r, E_6 or E_7.

2.15. $\tau(\mathrm{EVI})=7$.

PROOF 1. Since EVI is **H**-Kählerian, $\tau(\mathrm{EVI})$ equals the Betti number $b_{32}=7$, $\dim\mathrm{EVI}=64$, by 1.5.

PROOF 2. One knows $F(s_o,\mathrm{EVI})=\{o\}\amalg G_4^o(\mathbf{R}^{12})\amalg S^2\cdot\mathrm{DIII}(6)$. Ignoring the orientations of the subspaces, we obtain $\tau(\mathrm{EVI})=1\pm6+0$ by 1.9, 2.3 and 2.4. One also has $F(s_{\mathrm{VII}},\mathrm{EVI})=\mathrm{EII}\amalg\mathrm{EIII}$ and hence $\tau(\mathrm{EVI})=\pm4\pm3$ by 2.11 and 2.12. From these two we conclude $\tau(\mathrm{EVI})=1+6=7$. □

2.16. $\tau(\mathrm{EVII})=0$.

PROOF. $\mathrm{EVII}^{\%}$ is not orientable by 1.9a. Also $\dim\mathrm{EVII}=54$. □

2.17. $\tau(\mathrm{EVIII})=7$.

PROOF. $F(s_{\mathrm{IX}}, \mathrm{EVIII}) = S^2 \cdot \mathrm{EV} \amalg \mathrm{EVI}$ and $\tau(S^2 \cdot \mathrm{EV}) + \tau(\mathrm{EVI}) = 0 \pm 7$. $\square$

2.18. $\tau(\mathrm{EIX}) = 8$.

PROOF. EIX is also **H**-Kählerian. For another proof, one uses $F(s_{\mathrm{VIII}}, \mathrm{EIX}) = G_4^o(\mathbf{R}^{16}) \amalg \mathrm{DIII}(8)^{\%}$ and $\tau(G_4^0(\mathbf{R}^{16})) + \tau(\mathrm{DIII}(8)^{\%}) = \pm 8 + 0$ by 2.4 and 1.9. $\square$

We have thus determined the signature of every compact symmetric space, completing the proof of 0.1.

3. The g-signatures of symmetric spaces.

We will determine g-$\tau(M)$, the g-signature, of each compact connected simple symmetric space M where g is involutive and preserves the orientation. We may assume g is not homotopic with a point symmetry s_o (1.12), the identity 1_M, (1.1), or a covering transformation. Although we do not describe such a g in details, we like to remark that even if G in $M = G/K$ does not admit outer involutions, it is possible that, for a member b of the connected G, the map $\mathrm{ad}(b) : aK \mapsto \mathrm{ad}(b)(aK)$ gives such an involution of M as g in the above; since b must satisfy an obvious condition to let $\mathrm{ad}(b)$ act on M, the transformation $\mathrm{ad}(b)$ is not necessarily homotopic to 1_M.

Since our method is, mainly, to apply the g-signature theorem (1.3), we need self-intersections which will be determined in Section 4.

We begin with an important class of involutions g such that the fixed point sets $F(g; M)$ are what are called R-spaces (3.1 below).

3.1 DEFINITION. *An R-space N* in a Kählerian symmetric space M is the fixed point set $F(t, M)$ of an anti-holomorphic involution t. Thus $\dim N$ is half the $\dim M$; M may be thought of as "a complexification" of N.

3.2 PROPOSITION. *Let N be an R-space in a Kählerian symmetric space M, $F(t, M) = N$. Then one has* (i) *the normal bundle $T^{\perp}N$ is isomorphic with TN, and hence $T^{\perp}N$ has the Euler number χN*; (ii) *N is connected* ([N88] 9.1); (iii) *$SI(N; M) = \chi N$; and* (iv) *t-$\tau(M) = \chi N$.*

PROOF. (i) The complex structure J of the T_oM carries T_oN onto its orthogonal complement for every point o of N, since $T_o^{\perp}N$ is the eigenspace for the eigenvalue -1 of t acting on T_oM and this t anticommutes with J. The linear isomorphism $J : T_oN \to T_o^{\perp}N$ clearly extends to an isomorphism $TN \to T^{\perp}N$. (ii) Since $M = G/K$ is a Kählerian space, $K(o) = o$, K contains a circle group U(1) in its center; U(1) acting on T_oM contains J. First we will show that U(1) acts trivially on every polar M^+ of o in M. Let p be a point of M^+. The corresponding meridian $M^-_{(p)} := F(s_p \circ s_o, M)_{(p)}$ is a complex submanifold. Hence J stabilizes the tangent space $T_oM^-_{(p)}$. Thus U(1) stabilizes the subspace $M^-_{(p)}$. Therefore U(1) fixes the pole p of o in $M^-_{(p)}$, as asserted. Now let J_{U} denote a corresponding member of U(1) to J; its differential is J if restricted to T_oM.

The subspace $J_U(N_{(o)})$ has the tangent space $T_o^{\perp}N$ at o, where $N_{(o)}$ denotes the component of N throu o; that is, $J_U(N_{(o)})$ is c-orthogonal to $N_{(o)}$ at o. Since U(1) fixes the points of every polar M^+, so it stabilizes every polar $N_{(o)}^+ \subset M^+$ of o in $N_{(o)}$. Hence $N_{(o)}$ shares all the polars of o with $J_U(N_{(o)})$. Therefore N is connected by (5.2) in [N88]. (An alternative proof can be obtained by means of Wolf's theorem (3.3 in [W]). (iii) is obvious from (i) and (ii). (iv) follows from (iii) and 1.3, the Atiyah-Singer theorem. □

3.3 REMARK. In most cases, the equality $t\text{-}\tau(M)=\tau(M)$ obtains. Precisely, this is valid for $(N, M)=(G_r(\mathbf{R}^{2r}), G_r(\mathbf{C}^{2r}))$, $(G_r(\mathbf{H}^n), G_{2r}(\mathbf{C}^{2n}))$, $(\mathrm{SO}(n), \mathrm{DIII}(n))$, $(T\cdot \mathrm{AII}(n), \mathrm{DIII}(2n))$, $(T\cdot \mathrm{AI}(n), \mathrm{CI}(n))$, $(\mathrm{Sp}(2n), \mathrm{CI}(2n))$, $(G_2(\mathbf{H}^4)^{\%}, \mathrm{EIII})$, (FII, EIII), $(T\cdot \mathrm{EIV}, \mathrm{EVII})$, and $(\mathrm{AII}(4)/\mathbf{Z}_2, \mathrm{EVII})$. The other cases are $(\mathrm{U}(r), G_r(\mathbf{C}^{2r}))$, $(S^p\cdot S^q, G_2^o(\mathbf{R}^{p+q+2}))$, $(N, N\times N^-)$ where N is Kählerian and the diagonal of the product of N and its "complex conjugate" N^- in $N\times N^-$. In every case, one has $t\text{-}\tau(M)=\mathrm{Lef}(t)$, the Lefschetz number.

3.4 ASSERTION. *One has* $s_{\mathrm{I}}\text{-}\tau(M)=\tau(M)$ *for every* 1*-connected space* $M=\mathrm{E}_6/K$, *that is,* M *is* EI, EII, EIII *or* EIV *where* s_{I} *is an outer involution of* E_6 *(which can be) induced on the spaces* EI *through* EIV.

PROOF. We may assume that s_{I} is the point symmetry at a point of EI, since the outer involutions of E_6 are homotopic with each other. Then the equality is trivial for EI and EIV. (For $M=$ EII, one has the result by 4.14 obviously, but we will give another proof here.) The fixed point set $F(s_{\mathrm{I}}, \mathrm{EII})$ is the disjoint union of $\mathrm{CI}(4)^{\%}$ and a lower dimensional subspace (4.1); $2\dim \mathrm{CI}(4)^{\%}=\dim \mathrm{EII}$. The c-orthogonal space to $\mathrm{CI}(4)^{\%}$ at o is isomorphic with $\mathrm{CI}(4)^{\%}$ and these two subspaces meet in o, $G_2(\mathbf{C}^4)^{\%}\cong G_2(\mathbf{R}^6)$ and $T\cdot \mathrm{AI}(4)/\mathbf{Z}_2$, while $G_2(\mathbf{C}^4)^{\%}$ is contained in the polar $G_2(\mathbf{C}^6)$ of o in EII and $T\cdot \mathrm{AI}(4)/\mathbf{Z}_2$ in the polar $S^2\cdot G_3(\mathbf{C}^6)$. One knows $SI(G_2(\mathbf{R}^6); G_2(\mathbf{C}^6))=\chi G_2(\mathbf{R}^6)=3$ by 3.2. $T=S^1$ is homologous to zero in S^2. (See 4.19 for another proof.) Finally one has $F(s_{\mathrm{I}}, \mathrm{EIII})$ is $G_2(\mathbf{H}^4)^{\%}$, which is an R-space. By (3.2), one has $s_{\mathrm{I}}\text{-}\tau(\mathrm{EIII})=\chi(G_2(\mathbf{H}^4)^{\%})=3$. □

3.4a REMARK. One notes $s_{\mathrm{I}}\text{-}\tau(M)\leqq \mathrm{Lef}(s_{\mathrm{I}})$ for the above spaces by $\mathrm{Lef}(s_{\mathrm{I}})=12$, 12, 3 and 4 for $M=$ EI, EII, EIII and EIV respectively.

3.5 ASSERTION. *Let* g *denote* $\gamma\circ s_o$, *where* γ *is the deck transformation for a double covering morphism*: $M\to M''$. *Then one has* $g\text{-}\tau(M)=0$ *provided there is only one centriole* (1.15) *for the pair* (o, p) *of* o *and its pole* $p:=\gamma(o)$ *(that is, the centrosome* $F(g; M)$ *is connected).*

PROOF. $F(g; M)$ is homologous to zero in this case by 4.3. □

3.5a REMARK. Examples are given in 4.3a.

3.6. $g\text{-}\tau(G_p^o(\mathbf{R}^n))=\tau(G_p^o(\mathbf{R}^{n-2}))$, where $F(g; G_p^o(\mathbf{R}^n))=G_p^o(\mathbf{R}^{n-1})$. ($g$ is defined with the reflection I_1 in a hyperplane of $\mathbf{R}^n$. g preserves the orientation if both p and $n-p$ are even. Of course $p\leqq n-2$.)

PROOF. One has $SI(G_p^o(\mathbf{R}^{n-1}); G_p^o(\mathbf{R}^n)) = G_p^o(\mathbf{R}^{n-2})$ by linear algebra (4.5). $\square$

3.7. g-$\tau(G_r(\mathbf{C}^{2r}))=0$ for g satisfying $F(g; G_r(\mathbf{C}^{2r}))=\mathrm{CI}(r)$. ($g$ has $\mathrm{Lef}(g)=2^r \neq \chi G_r(\mathbf{C}^{2r})$.)

PROOF. One knows $F(g \circ s_o; G_r(\mathbf{C}^{2r}))=\mathrm{OIII}(r)$ $(=\mathrm{DIII}(r) \amalg \mathrm{DIII}(r))$ for $o \in \mathrm{CI}(r)$. s_o is homotopic with the identity and $\mathrm{OIII}(r)$ has dimension $<\frac{1}{2}\dim G_r(\mathbf{C}^{2r})$. $\square$

4. Self-intersections of some subspaces.

We will compute the self-intersection $SI(N; M)$ of N in M (See [AS] p. 583) in several cases, where M is a compact symmetric space and N is one of the connected components of the fixed point set $F(t, M)$ of an involutive automorphism t of M.

We have various methods to find SI, especially 4.11b and 4.11c, which we find useful. SI is obvious in some cases (4.1). The g-signature theorem may work if $2\dim N = \dim M$.

4.1. Obvious facts. (i) One has $SI(N; M)=0$ if $2\dim N < \dim M$. (ii) One has $SI(N; M)=g$-$\tau(M)$ if N is a component of $F(t, M)$ such that $2\dim N = \dim M$ and the other components have dimensions $<\dim N$, where t and g are orientation preserving involutions of M which are homotopic ($t \simeq g$). One finds SI by g-τ or τ. (iii) If the homology class $[N]=0$ in $H_*(M; \mathbf{Z})$, then $SI(N; M)=0$. (iv) In case N is an R-space in a Kaehlerian space M, one can use 3.2; see 3.3.

4.1a EXAMPLES. (1) $SI(G_1(\mathbf{H}^3); \mathrm{FI})=0$ by (i). Hence $SI(S^2 \cdot \mathrm{CI}(3); \mathrm{FI})=0$ by (ii) and 2.8. (2) $SI(S^8; \mathrm{FII})=1$ by (ii); in this case $t=s_o$ and $F(s_o \cdot \mathrm{FII})=\{o\} \amalg S^8$ and $\tau(\mathrm{FII})=1$ (2.9). The fact $SI=1$ is obvious from the plane projective geometry in which the line is S^8. (3) $SI(G_1(\mathbf{H}^3); \mathrm{FII})=1$ by (2) and (ii). (4) $SI(\mathrm{FI}; \mathrm{EI})=0$ by (iii); indeed FI is disjoint from certain congruent subspaces $\cong$ FI (as proved in 4.11d). (5) Similarly one obtains $SI(N; M)=0$ by (iii) for the pairs $(N; M)=SI(\mathrm{FI}; \mathrm{EI})$, $(\mathrm{FII}; \mathrm{EIV})$, $(\mathrm{EII}; \mathrm{EV})$, $(G_4(\mathbf{C}^8)^{\%}; \mathrm{EV})$, $(\mathrm{EIII}, \mathrm{EVII})$ and $(\mathrm{DIII}(p) \times \mathrm{DIII}(q); \mathrm{DIII}(p+q))$ (See 4.10 for details). (5) $SI(G_2(\mathbf{H}^4)^{\%}; \mathrm{EIII})=3$ and $SI(\mathrm{AII}(4)/\mathbf{Z}_2; \mathrm{EVII})=0$ both by (iv).

4.1b. Other examples of 4.1 (ii). (1) $SI(G_p(\mathbf{C}^n); G_p(\mathbf{H}^n))=\tau(G_p(\mathbf{H}^n))$, a fact used to prove 2.5. (2) $SI(G_2(\mathbf{C}^6); \mathrm{EIII})=\tau(\mathrm{EIII})$ $(=3)$. (3) $SI(G_4(\mathbf{C}^8)^{\%}; \mathrm{EVI})=\tau(\mathrm{EVI})$ $(=7)$. (4) $SI(G_8(\mathbf{R}^{16})^{\#}; \mathrm{EVIII})=\tau(\mathrm{EVIII})$. (5) $SI(G_p(\mathbf{C}^n); G_{2p}(\mathbf{R}^{2n}))=\tau(G_p(\mathbf{C}^n))=\chi(G_p(\mathbf{R}^n))$. One obtains similar formulas by replacing $G_{2p}(\mathbf{R}^{2n})$ with $G_{2p}^o(\mathbf{R}^{2n})$ and $G_{2p}(\mathbf{R}^{2n})^{\#}$. (6) $SI(S^2 \cdot G_3(\mathbf{C}^6); \mathrm{EII})=4$. (7) $SI(G_2(\mathbf{C}^4) \cdot G_2(\mathbf{C}^4); G_8^o(R^{12}))=6$.

4.2. If a subspace N of M is isomorphic with a 1-connected group, then its normal bundle $T^{\perp}(N, M)$ is trivial and hence one obtains $SI(N; M)=0$.

4.3 PROPOSITION. *One has $SI(N; M)=0$ if N is a centriole* (1.15) *which meets minimal geodesic arcs from a point o to one of its poles, p, at their midpoints.*

PROOF. We will show that N is actually homologous to zero in M. We know ([N92]) that N is an orbit of (the identity component of) the isotropy subgroup K of the connected isometry group G at o; K also fixes p. Hence every point q of N is the midpoint of a minimal geodesic arc c joining o to p. By the minimality, q is the unique point at which c meets N and c is the unique minimal one from o to p that contains q. Therefore one can deform N to o smoothly by moving every point q of N along the above unique geodesic c toward o; thus N is contractible to the point o. $\square$

4.3a. Examples include the spaces which appear in the Bott periodicity. They are $\mathrm{U}(m)\to G_m(\mathbf{C}^{2m})\to\mathrm{U}(2m)$ and $\mathrm{O}(m)\to G_m(\mathbf{R}^{2m})\to T\cdot\mathrm{AI}(2m)\to\mathrm{CI}(2m)\to\mathrm{Sp}(2m)\to G_{2m}(\mathbf{H}^{4m})\to T\cdot\mathrm{AII}(4m)\to\mathrm{DIII}(8m)\to\mathrm{O}(16m)$ joined with monomorphisms, where N is a centriole satisfying the condition above in M for each $N\to M$.

The self-intersections can be determined easily with linear algebra in such cases as the following 4.5 through 4.9. Also one knows $SI(S^8,\mathrm{FII})=1$ by plane projective geometry.

4.4. $SI(G_r(\mathbf{R}^n);G_r(\mathbf{R}^{n+q}))=G_r(\mathbf{R}^{n-q})$ if $r+q\leqq n$ and $=0$ if not.

PROOF. If $r+q>n$, (4.1) applies by $\dim G_r(\mathbf{R}^n)=r(n-r)$. Assume $r+q\leqq n$. Then $\mathbf{R}^n$ is the direct sum of $\mathbf{R}^{n-q}$ and $\mathbf{R}^q$. $\mathbf{R}^{n+q}$ contains a q-dimensional subspace V such that the sum $V+\mathbf{R}^{n-q}$ is n-dimensional and meets $\mathbf{R}^n$ in $\mathbf{R}^{n-q}$. Then $G_r(\mathbf{R}^n)$ meets $G_r(V+\mathbf{R}^{n-q})$ transversely in $G_r(\mathbf{R}^{n-q})$. $\square$

Similarly, one obtains the following 4.5 through 4.7.

4.5. $SI(G_r^o(\mathbf{R}^n);G_r^o(\mathbf{R}^{n+q}))=G_r^o(\mathbf{R}^{n-q})$ if $r+q\leqq n$ and $=0$ if not.

4.6. $SI(G_r(\mathbf{C}^n);G_r(\mathbf{C}^{n+q}))=G_r(\mathbf{C}^{n-q})$ if $r+q\leqq n$ and $=0$ if not.

4.7. $SI(G_r(\mathbf{H}^n);G_r(\mathbf{H}^{n+q}))=G_r(\mathbf{H}^{n-q})$ if $r+q\leqq n$ and $=0$ if not.

4.8. $SI(G_1(\mathbf{C}^p)\times G_1(\mathbf{C}^q);G_2(\mathbf{C}^{p+q}))=\min\{p,q\}$.

PROOF. We write $[e_1\wedge e_2\wedge\cdots\wedge e_p]$ for the space with basis $(e_1,\cdots,e_p)$. Then $N:=G_1(\mathbf{C}^p)\times G_1(\mathbf{C}^q)$ in $G_2(\mathbf{C}^{p+q})$ is the space of the 2-dim. subspaces spanned by a nonzero member of $[e_1\wedge\cdots\wedge e_p]$ and that of $[f_1\wedge f_2\wedge\cdots\wedge f_q]$. We may assume $p\leqq q$. Let $b\in\mathrm{SU}(p+q)$ carry e_i into $c_ie_i+s_if_i$, f_i into $-s_ie_i+c_if_i$ and f_j into f_j, where $i\leqq p<j$, $c_i^2+s_i^2=1$ and $(c_1/s_1)^2,\cdots,(c_p/s_p)^2$ are all distinct. Then $b(N)$ meets N only at $[e_1\wedge f_1]$, $[e_2\wedge f_2]$, $\cdots$, and $[e_p\wedge f_p]$, as is easily seen. Thus the desired $SI\leqq p$. Moreover one has $F(\mathrm{ad}(I_p);G_2(C^{p+q}))=G_2(\mathbf{C}^p)\amalg N\amalg G_2(\mathbf{C}^q)$ for an involutive member I_p of $\mathrm{U}(p+q)$. By 4.6, $\tau SI(G_2(\mathbf{C}^q);G_2(\mathbf{C}^{p+q}))=\tau G_2(\mathbf{C}^{q-p})$ if $q-p\geqq 2$ and $=0$ if not. Hence $\tau G_2(\mathbf{C}^{q+p})-\tau G_2(\mathbf{C}^{q-p})=[(p+q)/2]-[(q-p)/2]=p$ (whether or not $q-p\geqq 2$). And 1.3 applies. $\square$

4.9. $SI(G_2^o(\mathbf{R}^p)\cdot G_2^o(\mathbf{R}^q);G_4^o(\mathbf{R}^{p+q}))=\min\{p,q\}$ if p and q are even.

PROOF. We assume $p\leqq q=:p+m$. Similarly to the above proof, we use $b\in$

$SO(p+q)$ which carries e_{2i-1} into $c_i e_{2i-1}+s_i f_{2i}$, e_{2i} into $c_i e_{2i}+s_i f_{2i-1}$, f_{2i-1} into $-s_i e_{2i-1}+c_i f_{2i}$, f_{2i} into $-s_i e_{2i}+c_i f_{2i-1}$, and f_j into f_j for $1 \leqq i \leqq p/2$ and $j>p$, where $c_i^2+s_i^2=1$, c_i being all distinct. Then $N:=G_2^o(\mathbf{R}^p)\cdot G_2^o(\mathbf{R}^q)$ meets $b(N)$ in $[\pm e_{2i-1}\wedge e_{2i}\wedge f_{2i-1}\wedge f_{2i}]$ only. $\square$

4.10. $SI(\mathrm{DIII}(p)\times \mathrm{DIII}(q); \mathrm{DIII}(p+q))=0$.

PROOF. $\mathrm{DIII}(p)$ is the totality of the complex structures of $\mathbf{R}^{2p}$ which define a given orientation of $\mathbf{R}^{2p}$. An involution $I_{2p}\in SO(2p+2q)$ gives the eigenspace decomposition $\mathbf{R}^{2p+2q}=\mathbf{R}^{2p}\oplus\mathbf{R}^{2q}$. Correspondingly one has 2 copies of a subspace $\mathrm{DIII}(p)\times\mathrm{DIII}(q)$. These copies are isotopic with each other, since there is a member of $SO(2p+2q)$ which stabilizes both $\mathbf{R}^{2p}$ and $\mathbf{R}^{2q}$ and reverses their orientations. $\square$

4.11. $SI(\mathrm{CI}(p)+\mathrm{CI}(q); \mathrm{CI}(p+q))=\mathrm{CI}(1)\times\cdots\times\mathrm{CI}(1)\times\mathrm{CI}(q-p)=\mathrm{CI}(1)^p\times \mathrm{CI}(q-p)$ if $p\leqq q$. ($\mathrm{CI}(1)\cong S^2$.)

4.11a. We need preliminaries before the proof, which will be used in other cases as well. Let s and t be commutative involutive members of a compact Lie group G. Write K for $F(\mathrm{ad}(s), G)$ and H for $F(\mathrm{ad}(t), G)$. Then one has two fibrations $\pi_K: G/K\cap H\to G/K$ and $\pi_H: G/K\cap H\to G/H$ (π_k or π_H is not a homomorphism). To a point x of G/K, there corresponds the subspace $x^{\check{}}:=\pi_H\circ\pi_K^{-1}(x)$ of G/H which is isomorphic with $K/K\cap H\cong K(o)$, $H(o)=\{o\}$. This is not only bijective (by $K=F(\mathrm{ad}(s), G)$), but also G-equivariant diffeomorphism of G/K onto those subspaces $\{x^{\check{}}\mid x\in G/K\}$ on which G acts transitively. Similarly, G/H is G-equivariantly diffeomorphic with $\{y^{\hat{}}:=\pi_K\circ\pi_H^{-1}(y)\cong H/K\cap H\mid y\in G/H\}$. Moreover one has a kind of duality: the point x lies on the subspace $y^{\hat{}}$ for a point $y\in G/H$ if and only if $x^{\check{}}$ contains y. (Cf. [N88] 5.8, 11.1).

Now 4.11 follows from the next theorem for the case in which $G=Sp(p+q)$, $K=U(p+q)$, and $H=Sp(p)\times Sp(q)$ so that $H/K\cap H=\mathrm{CI}(p)\times\mathrm{CI}(q)$.

4.11b THEOREM. *Let $M=G/K$ and $N=G/H$ be compact symmetric spaces such that $K=F(\sigma, G)$ and $H=F(\tau, G)$ for commuting involutions σ and τ of G. Assume that* (1) *H and K are connected,* (2) *G/H has an equal rank to the subspace $K(o)\cong K/K\cap H$, $N\ni o=H(o)$, and* (3) *the Weyl group of $K(o)$ coincides with that of N. Then one has*

$$SI(H/K\cap H; M)=[H_o/H_o\cap K]$$

and this is connected, where H_0 is the centralizer $C(A, H)$ of a maximal torus $A\subset G/H$ in H, $A\ni o$.

PROOF. In the notations of (4.11a), $o^{\hat{}}$ is an H-orbit which may be identified with $H/K\cap H$. The maps $x\mapsto x^{\check{}}$ and $y\mapsto y^{\hat{}}$ are injective, since H and K are maximal subgroups of G. A point x of M lies in the intersection $o^{\hat{}}\cap y^{\hat{}}$ if and only if $x^{\check{}}$ contains both o and y by (4.11a).

We want to place $y^{\hat{}}$ close to $o^{\hat{}}$ and in a generic position; then $y^{\hat{}}$ will meet $o^{\hat{}}$ transversely. As the first step, we choose y in a maximal torus $A\subset N$ so the geodesics in A which pass through y and o are dense in A. We may assume that y lies in a neighborhood U of o which satisfies these conditions: (U1) U is open and convex, (U2) U does not meet the cut locus of o (This is the case if U is "small") and (U3) $U \cap x^{\sim}$ is also convex for every $x^{\sim} \ni o$. We can make U satisfy (U3), since $x^{\sim}$ is compact and totally geodesic; and (U3) will follow easily if one notes that $\tilde{x}$ is obtained by rotating the orbit $K(o)$ with some $h \in H$; $hK(o) = x^{\sim}$.

Now we choose y in $U \cap A$ and, by (2), A in $K(o)$. If $x^{\sim} = hK(o)$ contains y, then $x^{\sim}$ contains the whole A. The isotropy subgroup $h(K \cap H)h^{-1}$ contains some h_1 which carries $h(A)$ back to A. Moreover the same $h(K \cap H)h^{-1}$ contains some h_2 such that $h_2 \circ h_1 \circ h$ restricted to A is the identity map 1_A by (3). Therefore the centralizer H_0 of A contains some h_0 which carries $K(o)$ onto $x^{\sim}$. That is, the totality of the subspaces $x^{\sim}$ which contain both o and y is H_o-equivariantly diffeomorphic with the space $H_0/H_0 \cap K$.

We will show that this is connected by (1). Let A' denote a subgroup of G which is a covering group of A. Then H_0A' is the union of all the maximal tori of G which contain A'; hence H_0A' is connected. Thus $H_0/H_o \cap K = H_0A'/(H_0 \cap K)A'$ is also connected ($A' \subset K$). Therefore $y^{\wedge}$ meets $o^{\wedge}$ in a connected subspace $\cong H_0/H_0 \cap K$.

Besides, $y^{\wedge}$ does transversely; that is, $\dim H_0/H_0 \cap K = 2\dim o^{\wedge} - \dim M$. The proof is done with easy calculations in view of $\dim H_0 = \dim H - \dim N + \dim A$, the assumption (2) and the fact to the effect that $H_0 \cap K$ is the $K(o)$ counterpart of H_0. □

4.11c REMARK. Although the assumption (3) may look too strong, the theorem is useful in determining $SI(N; M)$ in case $2\dim N > \dim M$. One might get a better theorem by replacing H_0 with the normalizer $N(A, H)$, that is, by adding the Weyl group $W(N)$ of N to H_0. We will also write $W(R)$ for $W(N)$, since $W(N)$ depends on the root system $R = R(N)$ only. *If* $2\dim o^{\wedge} = \dim M$ (*for simplisity*), *then*, *without assumption* (3) *in* 4.11b, *one can find* $y \in N$ *such that* $o^{\wedge} \cap y^{\wedge}$ *consists of points which are* $\# W(N)/\# W(K(o))$, *the quotient of the orders*, *in number*; *hence one has*

$$(0 \leqq) SI(H/K \cap H\ ;\ M) \leqq \# W(N)/\# W(K(o))\,.$$

The equality obtains in many cases. For example, one has $SI(\mathrm{FII}; \mathrm{EIII}) = \chi(\mathrm{FII}) = 3$, since FII is an R-space (3.2 iii), while $W(N) = W(\mathrm{EIV}) = W(\mathrm{A}_2)$ and $W(K(o)) = W(T \cdot S^9) = W(\mathrm{A}_1)$ give $\# W(N)/\# W(K(o)) = \chi(G_1(\mathbf{C}^3)) = 3$. For another example (4.8) for $p = q$, one recalls $N = \mathrm{G}_p(\mathbf{C}^{p+q})$ and $K(o) = S^2 \times G_{p-1}(\mathbf{C}^{p+q-2})$ to have $\# W(N)/\# W(K(o)) = \# W(\mathrm{Sp}(p))/\# W(\mathrm{Sp}(1) \times \mathrm{Sp}(p-1)) = \chi(G_1(\mathbf{H}^p)) = p$. 4.20 and 4.21 are other examples.

There are, however, cases in which the equality fails such as 4.19, 4.22 (ii), 4.23, 4.24 and 4.25. In these cases $H/K \cap H$ are C-spaces (which are not globally Kählerian) in **H**-Kähler spaces M, where a C-space means a locally Kählerian subspace of an **H**-Kähler space whose complex structure is related to the quaternion structure of M in a certain way; $G_1(\mathbf{C}^n)$ in $G_1(\mathbf{H}^n)$ and $\mathrm{CI}(4)^{\%}$ in EII are examples.

4.11d REMARK. The "duality" (4.11a) may be used to prove $SI(\mathrm{FI}; \mathrm{EI})=0$ (See 4.1a (4)) more directly. Thus consider the double fibration $N:=\mathrm{EIV} \leftarrow \mathrm{E}_6/H \cap K \rightarrow M:=$ EI. One has $K/H \cap K \cong G_1(\mathbf{H}^3)$ and $H/H \cap K \cong \mathrm{FI}$. One knows the root system $R(G_1(\mathbf{H}^3))$ is BC_1 and that of EIV is A_2; hence the rank $r(G_1(\mathbf{H}^3))$ is less than $r(N)$. Thus N contains a point y such that no geodesic passing through y and o, $H(o)=o$, is congruent with any geodesic in $G_1(\mathbf{H}^3)$; hence no subspace $x^{\sim} \cong G_1(\mathbf{H}^3)$ contains $\{y, o\}$. By the duality, $o^{\wedge} \cong \mathrm{FI}$ does not meet $y^{\wedge}$. (The non-existence of the geodesic is also seen by observing that the dimension of the principal orbit $\approx H/H_0$ of H acting on N, which admits a fibration $S^8 \rightarrow H/H_0 \rightarrow \mathrm{FII}$, equals $8+16<25<\dim N-1$.)

4.11e ACKNOWLEDGEMENT. We owe the computations of H_0 in all the cases to Hiroshi Tamaru, for which we thank him.

We will give a few applications of 4.11b: 4.12 through 4.18 (as well as 4.4, 4.5, etc.).

4.12. $SI(G_4^o(\mathbf{R}^9); \mathrm{FI})=G_4^o(\mathbf{R}^7)$.

PROOF. The subspace $x^{\wedge} \cong G_1(\mathbf{H}^3)$ has the same root system BC_1 as $N=\mathrm{FII}$. Hence the subspace $y^{\wedge}=G_4^o(\mathbf{R}^9)$ of FI meets $o^{\wedge}$ in a subspace $\cong H_0/H_0 \cap K \cong \mathrm{SO}(7)^{\sim}/\mathrm{SO}(3)^{\sim} \cdot \mathrm{SO}(4)^{\sim}=G_4^o(\mathbf{R}^7)$, as claimed. □

4.12a. $SI(S^8; \mathrm{FII})=1$.

4.13. $SI(G_4^o(\mathbf{R}^{10}); \mathrm{EII})=G_4^o(\mathbf{R}^6)$.

4.13a. $SI(T \cdot G_5^o(\mathbf{R}^{10}); \mathrm{EI})=\mathrm{UI}(4):=\mathrm{U}(4)/\mathrm{O}(4)$.

$T \cdot G_5^o(\mathbf{R}^{10})=F(s_{\mathrm{III}}; \mathrm{EI})$ (misprinted in [N88]). Since EI is 1-connected, UI(4) is homologous to zero, which reaffirms $\tau(\mathrm{EI})=0$.

4.14. $SI(\mathrm{FI}; \mathrm{EII})=SI(\mathrm{EII}; \mathrm{EVI})=SI(\mathrm{EVI}; \mathrm{EIX})=G_4^o(\mathbf{R}^8)$.

PROOF. H_0 is $\mathrm{SO}(8)^{\sim}$ in these three cases. □

4.14a REMARK. The spaces in 4.14 and 4.13 are all **H**-Kählerian.

4.15. (i) $SI(\mathrm{DIII}(5); \mathrm{EIII})=G_2(\mathbf{C}^4)$. (ii) $SI(G_2^o(\mathbf{R}^{10}); \mathrm{EIII})=1$.

PROOF. $M \cong \mathrm{EIII} \cong N$. $H_0 \cong \mathrm{U}(4) \cong \mathrm{U}(1) \cdot \mathrm{SO}(6)^{\sim}$. (i) $K \cap H \cong \mathrm{U}(1) \cdot \mathrm{U}(5)$. $H_0/H_0 \cap K \cong G_2(\mathbf{C}^4)$. (ii) $K \cap H \cong \mathrm{U}(1) \cdot \mathrm{U}(1) \cdot \mathrm{SO}(8)^{\sim}$. $H_0/H_0 \cap K=1$. □

4.16. $SI(\mathrm{CI}(n); G_n(\mathbf{C}^{2n}))=[S^2 \times \cdots \times S^2]=[(S^2)^n]$.

4.17. Let M be a compact, 1-connected and simple Lie group and B be its subspace with the same root system as M. Then one has

$$SI(B\,;\,M)=[\text{the maximal torus}]=0\,.$$

PROOF. Let G denote the group $M \times M$. We write H for its diagonal subgroup $\{(x, x) \in G\}$. Let τ be the involution of M for the space B; thus $B \cong M/F(\tau, M)$. Let ι be the involution: $(x, y) \mapsto (y, x)$ of G. And we write K for $F(\iota \circ (\tau \times \tau), G)$. Noting that G/K

is a symmetric space which is isomorphic with M, we identify M with G/K. Now we have the setting in Theorem 4.11b. Hence $SI(B;M)=H_0/H_0\cap K$. H_0 is a maximal torus in H obviously. Therefore $SI(B;M)$ is a maximal torus in M. □

4.18. $SI(G_8^o(\mathbf{R}^{10});\mathrm{EIII})=1$.

PROOF. The root systems of $G_8^o(\mathbf{R}^{10})$ and $N\cong\mathrm{EIII}$ are B_2 and BC_2 respectively, q.e.d. A more geometric observation makes the result obvious. $G_8^o(\mathbf{R}^{10})$ is isomorphic with a polar in EIII and the corresponding meridian, which meets the polar at a single point (transversely). □

4.19. $SI(\mathrm{CI}(4)^{\%};\mathrm{EII})=4$. (Hence s_{I}-$\tau(\mathrm{EII})=s_{\mathrm{IV}}$-$\tau(\mathrm{EII})=4$.)

PROOF. One knows $F(s_{\mathrm{I}},\mathrm{FII})=\mathrm{CI}(4)^{\%}\amalg G_1(\mathbf{H}^4)$. Thus s_{I}-$\tau(\mathrm{EII})=\tau SI(\mathrm{CI}(4)^{\%};\mathrm{EII})$ by 1.3 and 4.1 (ii). On the other hand, one has s_{I}-$\tau(\mathrm{EII})=s_{\mathrm{IV}}$-$\tau(\mathrm{EII})=\tau SI(\mathrm{FI};\mathrm{EII})=\tau(G_4^o(\mathbf{R}^8))=4$ by 4.14. □

4.20. $SI(\mathrm{DIII}(5);\mathrm{EII})=2$.

PROOF. Applying 1.3 to $F(s_{\mathrm{III}},\mathrm{EII})=\mathrm{DIII}(5)\amalg G_4^o(\mathbf{R}^{10})$, one obtains $4=\tau(\mathrm{EII})=\tau(SI(\mathrm{DIII}(5);\mathrm{EII}))+\tau(SI(G_4^o(\mathbf{R}^{10});\mathrm{EII}))$, for which we know $\tau(SI(G_4^o(\mathbf{R}^{10});\mathrm{EII}))=\tau(G_4^o(\mathbf{R}^6))=2$ by 4.13 and 0.1. □

4.20a. Incidentally, $\# W(N)/\# W(K(o))=2$ in this case (4.11c), N being EIII, $K(o)=S^2\cdot\mathbf{C}P^5$.

4.21. $3=SI(\mathrm{FII};\mathrm{EIII})=SI(\mathrm{EIII};\mathrm{EVI})=SI(\mathrm{EVI};\mathrm{EVIII})=3$.

PROOF. The first equality obtains by 4.1a (iv). One sees $\# W(N)/\# W(K(o))=3$ in all the three cases (4.11c). As to EIII in EVI, the g-τ theorem 1.3 applied to $F(s_{\mathrm{VII}},\mathrm{EVI})=\mathrm{EII}\amalg\mathrm{EIII}$ gives $7=\tau(\mathrm{EVI})=4+SI(\mathrm{EIII};\mathrm{EVI})$ by 4.14. For EVI in EVIII, we use the similar fact: $F(s_{\mathrm{IX}},\mathrm{EVIII})=S^2\cdot\mathrm{EV}\amalg\mathrm{EVI}$. For a point $o\in\mathrm{EVI}$, the connected component $F_o:=F(s_o\circ s_{\mathrm{IX}},\mathrm{EVIII})_{(o)}$, the c-orthogonal space to EVI at o, happens to be isomorphic with EVI. F_o meets EVI in their common polar $G_4^o(\mathbf{R}^{12})$ and $\{o\}$. $G_4^o(\mathbf{R}^{12})\cong G_8^o(\mathbf{R}^{12})$ is a subspace of a polar $G_8(\mathbf{R}^{16})^{\#}$ in EVIII and hence has $SI(G_8^o(\mathbf{R}^{12});G_8(\mathbf{R}^{16})^{\#})=\pm 2$ by 4.5. We thus infer that $SI(\mathrm{EVI};\mathrm{EVIII})=1\pm 2$. But EVI cannot meet a congruent EVI at a single point, say o. Indeed if it did, those two subspaces $\cong$ EVI would meet the polar $G_8(\mathbf{R}^{16})^{\#}$ of o in disjoint subspaces $\cong G_8^o(\mathbf{R}^{12})$, contrary to the above fact $SI=2$. □

4.21a COROLLARY. (i) $SI(S^2\cdot\mathrm{EV};\mathrm{EVIII})=G_4^o(\mathbf{R}^8)$, *whose* τ *equals* 4. (ii) $SI(G_2^o(\mathbf{R}^4)\cdot G_6^o(\mathbf{R}^{12}))/\mathbf{Z}_2;G_8(\mathbf{R}^{16})^{\#})=G_4^o(\mathbf{R}^8)$.

PROOF. By the above proof, we have $7=\tau(\mathrm{EVIII})=3+\tau SI(S^2\cdot\mathrm{EV};\mathrm{EVIII})$; hence this $\tau SI=4$. Trying to find this SI, we use the double fibration and the notation in 4.11b and 4.11c for $N=\mathrm{EIX}$ and $K/H\cap K=\mathrm{DIII}(8)^{\%}$. We obtain $H_0/H_0\cap K=G_4^o(\mathbf{R}^8)$

and $\# W(N)/\# W(K(o)) = \# W(\mathrm{F}_4)/\# W(\mathrm{B}_4) = \chi(\mathrm{FII}) = 3$. Since we know $\tau(G_4^o(\mathbf{R}^8)) = 4$, we conclude (i): $SI = G_4^o(\mathbf{R}^8)$. For (ii), one notes that the components $\cong S^2 \cdot \mathrm{EV}$ of $F(s_{\mathrm{IX}}, \mathrm{EVIII})$ and $F(s_o \circ s_{\mathrm{IX}}, \mathrm{EVIII})$ meet in the connected subspace $(G_2^o(\mathbf{R}^4) \cdot G_6^o(\mathbf{R}^{12}))/\mathbf{Z}_2$. Thus (ii) follows from (i).

4.21b Remark. Let us add a sort of reaffirmation of (ii). By definition, $G_8(\mathbf{R}^{16})^\#$ consists of the oriented 8-dimensional linear subspaces of $\mathbf{R}^{16}$, identified with their orthogonal complements. We work on the covering space $G_8^o(\mathbf{R}^{16})$, in which $B := G_2^o(\mathbf{R}^4) \cdot G_6^o(\mathbf{R}^{12})$ is a component of $F(I_4, G_8^o(\mathbf{R}^{16}))$ together with $G_a^o(\mathbf{R}^4) \cdot G_b^o(\mathbf{R}^{12})$ for $(a, b) = (0, 8)$ and $(4, 4)$. Both of these are $G_8^o(\mathbf{R}^{12}) \cong G_4^o(\mathbf{R}^{12})$ and have $SI = 2$ by 4.5, while we know $\tau G_8^o(\mathbf{R}^{16}) = 12$. Hence the SI in (ii) has the signature $\tau(SI) = \frac{1}{2}(12 - 2 - 2) = 4 = \tau(G_4^o(\mathbf{R}^8))$.

4.22. (i) $SI(G_8^o(\mathbf{R}^{12}); \mathrm{EVI}) = 3$. (ii) $SI(S^2 \cdot \mathrm{DIII}(6); \mathrm{EVI}) = 4$.

Proof. These subspaces are the polars $M^+(p)$ and $M^+(q)$ as in the proof 2 of 2.15. They are also congruent with the meridians $M^-(p)$ and $M^-(q)$. The intersection $M^+(p) \cap M^-(p)$ is $\{p\} \amalg G_4^o(\mathbf{R}^8)$, the polars of $M^+(p)$, while $M^+(q) \cap M^-(q)$ is $\{q\} \amalg G_2(\mathbf{C}^6) \amalg T \cdot T \cdot \mathrm{AII}(3)$. Hence 1.3 gives $SI(M^+(p); \mathrm{EVI}) = 1 + SI(G_4^o(\mathbf{R}^8); M^+(p)) = 1 \pm 2$ by 4.5 (actually, 4.11c applies to show this $SI = 3$) and $SI(M^+(q); \mathrm{EVI}) = 1 + SI(G_2(\mathbf{C}^6); M^+(q)) + SI(T \cdot T \cdot \mathrm{AII}(3); M^+(q)) = 1 + SI(G_2(\mathbf{C}^6); M^+(q)) + 0$ by 4.1 (iii). The subspace $G_2(\mathbf{C}^6)$ lies in the polar $G_8^o(\mathbf{R}^{12})$ of q. Hence $SI(G_2(\mathbf{C}^6); M^+(q)) = \pm 3$ by 4.1b (5). Therefore $7 = \tau(\mathrm{EVI}) = (1 \pm 2) + (1 \pm 3)$; the question of the signs $\pm$ (or the orientations) is resolved. (Let us add that, concerning S^2 in $S^2 \cdot \mathrm{DIII}(6)$, its homology class $[S^2] = 0$ in $H_2(\mathrm{EVI}, \mathbf{R})$ but $[S^2] \neq 0$ in $H_2(\mathrm{EVI}, \mathbf{Z}_2)$.) □

4.23. $SI(G_4(\mathbf{C}^8)^\%; \mathrm{EVI}) = 7$.

Proof. Since $F(s_{\mathrm{V}}, \mathrm{EVI}) = G_2(\mathbf{C}^8) \amalg G_4(\mathbf{C}^8)^\%$, we have the self-intersection $= \tau(\mathrm{EVI})$ by 1.3 and 4.1b. □

4.24. $SI(S^2 \cdot \mathrm{EVII}; \mathrm{EIX}) = 1 + 3 = 4$.

Proof. $S^2 \cdot \mathrm{EVII}$ is a polar together with EVI in EIX. Hence $8 = \tau(\mathrm{EIX}) = \tau G_4^o(\mathbf{R}^8) + \tau SI(S^2 \cdot \mathrm{EVII}; \mathrm{EIX})$ by 4.14. One knows $\tau G_4^o(\mathbf{R}^8) = 4$. □

Note $[S^2] \neq 0$ in $\mathrm{H}_2(\mathrm{EIX}, \mathbf{Z}_2)$.

4.25. $SI(\mathrm{DIII}(8)^\%; \mathrm{EIX}) = 8$.

Proof. $\mathrm{DIII}(8)^\% \amalg G_4^o(\mathbf{R}^{16})$ is the fixed point set of an inner involution of EIX. Since $G_4^o(\mathbf{R}^{16})$ has a lower dimension, one has $8 = \tau(\mathrm{EIX}) = \tau SI(\mathrm{DIII}(8)^\%; \mathrm{EIX})$; also $2 \dim \mathrm{DIII}(8)^\% = \dim \mathrm{EIX}$. □

In the diagrams below, the arrows mean monomorphisms. The lower indices in parentheses as in $M_{(s)}$ indicate the signature τM. The number attached to each monomorphism $N \to M$ denotes $\tau(SI(N; M))$.

$$\begin{tikzcd}
S^8 = G_8^o(\mathbf{R}^9) \arrow[r, "2"'] \arrow[d, "1"] & G_8^o(\mathbf{R}^{10}) \arrow[r, "2"'] \arrow[d, "1"] & G_8^o(\mathbf{R}^{12}) \arrow[r, "2"'] \arrow[d, "3"] & G_8(\mathbf{R}^{16})^{\#} \arrow[d, "7"] \\
\mathrm{FII}_{(1)} \arrow[r, "3"'] & \mathrm{EIII}_{(3)} \arrow[r, "3"'] & \mathrm{EVI}_{(7)} \arrow[r, "3"'] & \mathrm{EVIII}_{(7)} \\
 & \mathrm{DIII}(5)_{(0)} \arrow[r, "0"'] \arrow[u, "2"'] & S^2\cdot\mathrm{DIII}(6)_{(0)} \arrow[r, "0"'] \arrow[u, "4"'] & \mathrm{DIII}(8)^{o}/_{(0)} \arrow[u, "0"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{FII}_{(1)} \arrow[r, "3"'] & \mathrm{EIII}_{(3)} \arrow[r, "3"'] & \mathrm{EVI}_{(7)} \arrow[r, "3"'] & \mathrm{EVIII}_{(7)} & \\
G_1(\mathbf{H}^3)_{(1)} \arrow[r, "3"'] \arrow[u, "1"'] & G_2(\mathbf{C}^6)_{3)} \arrow[r, "3"'] \arrow[u, "3"'] & G_4^o(\mathbf{R}^{12})_{(6)} \arrow[r, "3"'] \arrow[u, "3"'] & \mathrm{EVI}_{(7)} \arrow[u, "3"'] & \textbf{H-Kähler}
\end{tikzcd}$$

$$\begin{tikzcd}
G_1(\mathbf{H}^3)_{(1)} \arrow[r, "3"'] \arrow[d, "0"] & G_2(\mathbf{C}^6)_{(3)} \arrow[r, "3"'] \arrow[d, "0"] & G_4^o(\mathbf{R}^{12})_{(6)} \arrow[r, "3"'] \arrow[d, "3"] & \mathrm{EVI}_{(7)} \arrow[d, "4"] & \textbf{H-Kähler} \\
\mathrm{FI}_{(0)} \arrow[r, "4"'] & \mathrm{EII}_{(4)} \arrow[r, "4"'] & \mathrm{EVI}_{(7)} \arrow[r, "4"'] & \mathrm{EIX}_{(8)} & \textbf{H-Kähler} \\
S^2\cdot\mathrm{CI}(3)_{(0)} \arrow[r, "0"'] \arrow[u, "0"'] & S^2\cdot G_3(\mathbf{C}^6)_{(0)} \arrow[r, "0"'] \arrow[u, "4"'] & S^2\cdot\mathrm{DIII}(6)_{(0)} \arrow[r, "0"'] \arrow[u, "4"'] & S^2\cdot\mathrm{EVII}_{(0)} \arrow[u, "4"'] & \textbf{C-spaces}
\end{tikzcd}$$

REMARK. Let (N, M) be the pair (FI, EII), (EII, EVI) or (EVI, EIX). Then $\tau(SI(N;M))=\sum_p \tau(SI(N^+(p);M^+(p)))$ summed up for the polars of o (including $\{0\}$).

$$\begin{tikzcd}
G_4^o(\mathbf{R}^9)_{(0)} \arrow[r, "2"'] \arrow[d, "0"] & G_4^o(\mathbf{R}^{10})_{(4)} \arrow[r, "2"'] \arrow[d, "2"] & G_4^o(\mathbf{R}^{12})_{(6)} \arrow[r, "2"'] \arrow[d, "3"] & G_4^o(\mathbf{R}^{16})_{(8)} \arrow[d, "0"] & \textbf{H-Kähler} \\
\mathrm{FI}_{(0)} \arrow[r, "4"'] & \mathrm{EII}_{(4)} \arrow[r, "4"'] & \mathrm{EVI}_{(7)} \arrow[r, "4"'] & \mathrm{EIX}_{(8)} & \textbf{H-Kähler} \\
 & \mathrm{DIII}(5)_{(0)} \arrow[r, "0"'] \arrow[u, "2"'] & S^2\cdot\mathrm{DIII}(6)_{(0)} \arrow[r, "0"'] \arrow[u, "4"'] & \mathrm{DIII}(8)^{o}/_{(0)} \arrow[u, "8"'] & \textbf{C-spaces}
\end{tikzcd}$$

5. An application.

5.1 PROPOSITION. *There exists a harmonic* 8*-form* $\omega \in H^8(\mathrm{EVIII})$ *on* EVIII *such that its pullbacks by the monomorphisms* $S^8 \to \mathrm{FII} \to \mathrm{EIII} \to \mathrm{EVI} \to \mathrm{EVIII}$ *induce* ω *to a harmonic form (denoted by* ω*) on each of these spaces,* M*, whose exterior power* $\bigwedge^m \omega = \omega^m$, $8m = \dim M$*, does not vanish. Naturally, every power* $\bigwedge^k \omega$ *induced on* M *is harmonic (hence invariant under* $G = I(M)_{(1)}$*).*

PROOF. Each space M' in the sequence $\cdots \to M' \to M \to$ has nonzero self- intersection in the next ambient space M, $\dim M = 2\dim M'$; $SI(S^8;\mathrm{FII}) = 1$, and $SI(M';M) = 3$ for the other M' by 4.21. Hence there is a harmonic 8-form $\omega_1 \in H^8(\mathrm{FII})$ such that $\omega_1 \wedge \omega_1 \neq 0$ and its induced form is not zero on S^8. ω_1 extends to a harmonic form

ω_2 on EIII with $\omega_2 \wedge \omega_2 \neq 0$, and so forth. Eventually ω_1 extends to the desired form ω on EVIII (with $\bigwedge^{16} \omega \neq 0$). $\square$

5.2 PROBLEM. M' probably has the smallest volume among the submanifolds in the homology class $[M']$, as M. Berger proved in the case of $(M', M) = (S^8, \mathrm{FII})$.

5.3 DEFINITION. *A regular triplet* in M is a set $\{o, x, y\}$ of three points of the connected symmetric space which satisfies $s_o \circ s_x \circ s_y = 1_M$.

5.3a REMARK. One sees s_o, s_x and s_y act on $\{o, x, y\}$ as the identity, since the point symmetries $s_o, \cdots$ are automorphisms, 1.8. If M is one of the four projective planes, then a regular triplet exists and is unique up to congruence by $\#_2 M = 3$ (See 0.2b). A regular triplet is a sort of regular orthogonal triangle.

5.4 PROPOSITION. *Let $M' \to M$ be a part of the monomorphisms* FII$\to$EIII$\to$EVI$\to$EVIII. *Then the intersection $M' \cap bM'$ of M' with an arbitrary congruent space, $b \in G$, contains a regular triplet.*

5.4a REMARK. This is an easy consequence of 4.21; a detailed proof will be given in a forthcoming paper. Let us add that the linear isotropy representation of those spaces S^8, FII, $\cdots$, EVIII are spinor (or half-spinor). We want to understand "a Clifford-Kähler" structures generalizing the complex and the quaternionic Kähler structures; for instance, some constant multiple of ω in 5.1 might deserve the name of the fundamental form.

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Present Addresses:
TADASHI NAGANO
DEPARTMENT OF MATHEMATICS, SOPHIA UNIVERSITY,
KIOICHO, CHIYODA-KU, TOKYO, 102–8554 JAPAN.

MAKIKO SUMI TANAKA
DEPARTMENT OF MATHEMATICS, SCIENCE UNIVERSITY OF TOKYO,
CHIBA, 278–0022 JAPAN.
e-mail: makiko@ma.noda.sut.ac.jp