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Strong Ergodic Theorems for Non-Lipschitzian Mappings of Asymptotically Nonexpansive Type in Uniformly Convex Banach Spaces

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Abstract. In this paper we establish strong ergodic theorems for non-Lipschitzian mappings of asymptotically nonexpansive type in uniformly convex Banach spaces.

Introduction.

Throughout this paper X denotes a uniformly convex Banach space, C a nonempty bounded closed convex subset of X, and T a mapping from C into itself.

The asymptotic behavior of asymptotically nonexpansive mappings has been studied by many authors. There appear in the literature the following three definitions of an asymptotically nonexpansive mapping:

(c_1) (Goebel and Kirk [3]) There exists a sequence $\{a_k\}$ with $\lim_{k\to\infty} a_k=1$ such that $\|T^k u-T^k v\|\leqslant a_k\|u-v\|$ for $u, v\in C$ and integers $k\geqslant 0$. In this case we say that T is *asymptotically nonexpansive in the strong sense.*

(c_2) (Kirk [4]) T^K is continuous for some positive integer K and

$$\overline{\lim}_{k\to\infty} \sup_{v\in C}(\|T^k u-T^k v\|-\|u-v\|)\leqslant 0 \qquad \text{for} \quad u\in C\,. \tag{0.1}$$

In this case we say that T is *asymptotically nonexpansive in the weak sense.*

(c_3) (Bruck, Kuczumow and Reich [2]) T is called *asymptotically nonexpansive in the intermediate sense* if T^K is continuous for some positive integer K and

$$\overline{\lim_{k\to\infty}}\ \sup_{u,v\in C}(\|T^k u-T^k v\|-\|u-v\|)\leqslant 0\,. \tag{0.2}$$

DEFINITION 0.1. A sequence $\{x_n\}_{n\geqslant 0}$ in X is said to be *strongly almost convergent* to an element x in X if (the strong limit) $\lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1} x_{i+k}=x$ uniformly in $k=0, 1, 2, \cdots$.

The purpose of this paper is to prove the following strong ergodic theorems.

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THEOREM 0.1. *Let T be asymptotically nonexpansive in the weak sense, and let x be an element in C. The following* (a) *and* (b) *are equivalent*:

(a) $\{T^n x\}$ *is strongly almost convergent to a fixed point of T.*

(b) $$\lim_{l,m,n\to\infty} \|\tfrac{1}{2}(S_n T^{l+n}x + S_m T^{l+m}x) - T^l(\tfrac{1}{2}S_n T^n x + \tfrac{1}{2}S_m T^m x)\| = 0\,,$$

where $S_n = (1/n)\sum_{i=0}^{n-1} T^i$ for $n = 1, 2, \cdots$.

THEOREM 0.2. *Let T be asymptotically nonexpansive in the intermediate sense, and let x be an element in C. If*

(0.3) $$\lim_{n\to\infty} \|T^{n+i}x - T^n x\| \text{ exists uniformly in } i = 1, 2, \cdots,$$

then $\{T^n x\}$ is strongly almost convergent to a fixed point of T.

REMARKS 0.1. 1) Since $\lim_{n\to\infty} \|T^{n+i}x - T^n x\|$ exists for every $i = 1, 2, \cdots$, in (0.3) we require the uniformity of the limit in i. We can prove that if there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

(0.4) $$\lim_{k\to\infty} \|T^{n_k+i}x - T^{n_k}x\| \text{ exists uniformly in } i = 1, 2, \cdots$$

then (0.3) is satisfied. We see also that if $\{T^{n_k}x\}$ is strongly convergent for some subsequence $\{n_k\}$ of $\{n\}$ then (0.4) and hence (0.3) is satisfied. So, in particular, if T is compact then (0.3) is satisfied. 2) Clearly if T is asymptotically nonexpansive in the strong sense then it is asymptotically nonexpansive in the intermediate sense. But the converse does not hold (see Example 0.1 below). So Theorem 0.2 is a generalization of the result by Oka [7] and Krüppel and Górnicki [6]. 3) Theorems 0.1 and 0.2 can be generalized to the case of almost-orbits of T (see §2).

EXAMPLE 0.1. Let $C = [0, 1]$ and φ be the Cantor ternary function. We define $T: C \to C$ by

$$\begin{aligned} Tu &= u/2 && \text{if } 0 \leqslant u \leqslant 1/2\,, \\ &= \varphi(u)/2 && \text{if } 1/2 < u \leqslant 1\,. \end{aligned}$$

Then for $n = 1, 2, \cdots$, $T^n u = u/2^n$ $(0 \leqslant u \leqslant 1/2)$ and $T^n u = \varphi(u)/2^n$ $(1/2 < u \leqslant 1)$. Therefore, for each $n \geqslant 1$, T^n is continuous, but it is not Lipschitz continuous and a fortiori T is not asymptotically nonexpansive in the strong sense. Since $\sup_{u,v\in[0,1]}(|T^n u - T^n v| - |u - v|) \leqslant 1/2^{n-1} \to 0$ as $n \to \infty$, T is asymptotically nonexpansive in the intermediate sense.

1. Proofs of theorems.

We see that for every sequence $\{x_n\}_{n\geqslant 0}$ in X the following equality holds true: For any l, $p \geqslant 1$ and $k \geqslant 0$

$$(1.1)\qquad \frac{1}{l}\sum_{i=0}^{l-1} x_{i+k}=\frac{1}{l}\sum_{i=0}^{l-1}\left(\frac{1}{p}\sum_{j=0}^{p-1} x_{i+j+k}\right)+\frac{1}{lp}\sum_{i=1}^{p-1}(p-i)(x_{i+k-1}-x_{i+k+l-1})\,.$$

Similarly as in the proof of [5, Lemma 2] we have

LEMMA 1.1. *Let T be asymptotically nonexpansive in the weak sense and let x be an element in C. If* (b) *in Theorem* 0.1 *is satisfied, then $\{\|S_nT^nx-f\|\}$ is convergent for every $f\in F(T)$, where $F(T)$ denotes the set of fixed points of T.*

REMARK 1.1. We note that $F(T)$ is not empty if T is asymptotically nonexpansive in the weak sense. See [4].

LEMMA 1.2. *Let T be asymptotically nonexpansive in the weak sense and let x be an element in C. If* (b) *in Theorem* 0.1 *is satisfied, then $\{S_nT^nx\}$ is strongly convergent to a fixed point of T.*

PROOF. Take an $f\in F(T)$ and set $u_n=S_nT^nx-f$ for $n\geqslant 1$. By Lemma 1.1, $\{\|u_n\|\}$ is convergent. Set $d=\lim_{n\to\infty}\|u_n\|$. By $\lim_{n\to\infty}\|u_{n+1}-u_n\|=0$ we have

$$(1.2)\qquad \lim_{n\to\infty}\|u_{n+i}+u_n\|=2d\qquad \text{for every } i\geqslant 0\,.$$

Since $S_{n+k}T^{n+k}x=(n+k)^{-1}\sum_{i=0}^{n+k-1}S_nT^{n+k+i}x+v(n,k)$ by (1.1) and $\|v(n,k)\|\leqslant(n-1)\operatorname{diam}C/2(n+k)$, where $v(n,k)=[n(n+k)]^{-1}\sum_{i=1}^{n-1}(n-i)[T^{n+k+i-1}x-T^{2(n+k)+i-1}x]$ and $\operatorname{diam}C$ denotes the diameter of C, we get

$$\begin{aligned}(1.3)\qquad \|u_{n+k}+u_{m+k}\| &= \Bigg\|\frac{1}{n+k}\sum_{i=0}^{n+k+1}(S_nT^{n+k+i}x+S_mT^{m+k+i}x-2f)\\ &\quad+\frac{n-m}{(m+k)(n+k)}\sum_{i=0}^{n+k-1}(S_mT^{m+k+i}x-f)+v(n,k)+v(m,k)\\ &\quad+\frac{1}{m+k}\sum_{i=n+k}^{m+k-1}(S_mT^{m+k+i}x-f)\Bigg\|\\ &\leq\frac{2}{n+k}\sum_{i=0}^{n+k-1}\|\tfrac{1}{2}(S_nT^{n+k+i}x+S_mT^{m+k+i}x)-f\|\\ &\quad+\left[\frac{2(m-n)}{m+k}+\frac{n-1}{2(n+k)}+\frac{m-1}{2(m+k)}\right]\operatorname{diam}C\end{aligned}$$

for $m\geqslant n\geqslant 1$ and $k\geqslant 0$. By (b) and (0.1), for any $\varepsilon>0$ there exists an integer $N(\varepsilon)>0$ such that

$$\|\tfrac{1}{2}(S_nT^{l+n}x+S_mT^{l+m}x)-T^l(S_nT^nx/2+S_mT^mx/2)\|<\varepsilon/2\,,$$

$$\|T^l(S_nT^nx/2+S_mT^mx/2)-f\|<\varepsilon/2+\|S_nT^nx/2+S_mT^mx/2-f\|$$

for $l,m,n\geqslant N(\varepsilon)$. Therefore, if $m\geqslant n\geqslant N(\varepsilon)$ and $k\geqslant N(\varepsilon)$ then

$$\begin{aligned}&\|(S_nT^{n+k+i}x+S_mT^{m+k+i}x)/2-f\|\\&\quad\leqslant\|(S_nT^{n+k+i}x+S_mT^{m+k+i}x)/2-T^{k+i}(S_nT^nx/2+S_mT^mx/2)\|\\&\qquad+\|T^{k+i}(S_nT^nx/2+S_mT^mx/2)-f\|<\varepsilon/2+\varepsilon/2+\|S_nT^nx/2+S_mT^mx/2-f\|\\&\quad\leqslant\varepsilon+\|u_n+u_m\|/2\end{aligned}$$

for every $i\geqslant 0$. Combining this with (1.3) we obtain

$$\|u_{n+k}+u_{m+k}\|\leqslant 2\varepsilon+\|u_n+u_m\|+\left[\frac{2(m-n)}{m+k}+\frac{n-1}{2(n+k)}+\frac{m-1}{2(m+k)}\right]\operatorname{diam}C$$

for $m\geqslant n\geqslant N(\varepsilon)$ and $k\geqslant N(\varepsilon)$. Letting $k\to\infty$, we see from (1.2) that

$$2d\leqslant 2\varepsilon+\|u_n+u_m\|\leqslant 2\varepsilon+\|u_n\|+\|u_m\| \qquad \text{for every } m,\ n\geqslant N(\varepsilon)\,.$$

This shows that $\lim_{n,m\to\infty}\|u_n+u_m\|=2d$. Since $\|u_n\|\to d$ as $n\to\infty$, the uniform convexity of X implies $\lim_{n,m\to\infty}\|S_nT^nx-S_mT^mx\|=\lim_{n,m\to\infty}\|u_n-u_m\|=0$, whence $\{S_nT^nx\}$ converges strongly. Put $y=\lim_{n\to\infty}S_nT^nx$.

We want to show $y\in F(T)$. Since for $l\geqslant 0$, $\|S_nT^{l+n}x-S_nT^nx\|\to 0$ as $n\to\infty$, we have

$$\lim_{n\to\infty}S_nT^{l+n}x=y \qquad \text{for every } l\geqslant 0\,. \tag{1.4}$$

Let $\varepsilon>0$ be arbitrarily given. By (b) with $m=n$ and (0.1) there exists an integer $N_1(\varepsilon)>0$ such that $\|T^lS_nT^nx-S_nT^{l+n}x\|<\varepsilon$ and $\|T^lS_nT^nx-T^ly\|<\varepsilon+\|S_nT^nx-y\|$ for $l,n\geqslant N_1(\varepsilon)$, which implies

$$\begin{aligned}\|T^ly-y\|&\leqslant\|T^ly-T^lS_nT^nx\|+\|T^lS_nT^nx-S_nT^{l+n}x\|+\|S_nT^{l+n}x-y\|\\&<2\varepsilon+\|S_nT^nx-y\|+\|S_nT^{l+n}x-y\| \qquad \text{for } l,\ n\geqslant N_1(\varepsilon)\,.\end{aligned}$$

Letting $n\to\infty$, it follows from (1.4) that $\|T^ly-y\|\leqslant 2\varepsilon$ for $l\geqslant N_1(\varepsilon)$. Therefore we obtain $\lim_{l\to\infty}T^ly=y$. Combining this with the continuity of T^K we have $y\in F(T)$. Q.E.D.

PROOF OF THEOREM 0.1. Suppose that there exists an element $y\in F(T)$ such that $\lim_{n\to\infty}S_nT^kx=y$ uniformly in $k\geqslant 0$. By this assumption and (0.1), for any $\varepsilon>0$ there exists an integer $N(\varepsilon)>0$ such that if n, $l\geqslant N(\varepsilon)$ then $\|S_nT^kx-y\|<\varepsilon/3$ for $k\geqslant 0$ and $\|T^lu-y\|<\|u-y\|+\varepsilon/3$ for $u\in C$. Therefore, if l, m, $n\geqslant N(\varepsilon)$ then we have

$$\begin{aligned}&\|\tfrac12(S_nT^{l+n}x+S_mT^{l+m}x)-T^l(\tfrac12 S_nT^nx+\tfrac12 S_mT^mx)\|\\&\leqslant\|\tfrac12(S_nT^{l+n}x+S_mT^{l+m}x)-y\|+\|T^l(\tfrac12 S_nT^nx+\tfrac12 S_mT^mx)-y\|\\&<\tfrac12(\|S_nT^{l+n}x-y\|+\|S_mT^{l+m}x-y\|)+\|\tfrac12(S_nT^nx+S_mT^mx)-y\|+\varepsilon/3<\varepsilon\,.\end{aligned}$$

So, (b) holds good.

Conversely, suppose that (b) is satisfied. By virtue of Lemma 1.2 there exists an element $y\in F(T)$ such that $\lim_{n\to\infty}S_nT^nx=y$. This implies

$$(1.5)\qquad \lim_{n,l\to\infty} S_nT^{n+l+k}x=y \qquad \text{uniformly in } k\geqslant 0\,.$$

In fact, let $\varepsilon>0$ be arbitrarily given. By (b) with $m=n$, there exists an integer $N_1(\varepsilon)>0$ such that $\|S_nT^{n+j}x-T^jS_nT^nx\|<\varepsilon/3$ for $j, n\geqslant N_1(\varepsilon)$. So, if $l, n\geqslant N_1(\varepsilon)$ then $\|S_nT^{n+l+k}x-T^{l+k}S_nT^nx\|<\varepsilon/3$ for every $k\geqslant 0$. By (0.1) and $\lim_{n\to\infty} S_nT^nx=y$, there exists an integer $N_2(\varepsilon)>0$ such that if l, $n\geqslant N_2(\varepsilon)$ then $\|T^{l+k}S_nT^nx-y\|<\varepsilon/3+\|S_nT^nx-y\|<2\varepsilon/3$ for every $k\geqslant 0$. Consequently, if n, $l\geqslant\max\{N_1(\varepsilon), N_2(\varepsilon)\}$ then $\|S_nT^{n+l+k}x-y\|\leqslant\|S_nT^{n+l+k}x-T^{l+k}S_nT^nx\|+\|T^{l+k}S_nT^nx-y\|<\varepsilon$ for every $k\geqslant 0$. So we have (1.5).

By (1.5), for any $\varepsilon>0$ there exists an integer $N(=N(\varepsilon))>0$ such that

$$(1.6)\qquad \|S_NT^{2N+k}x-y\|<\varepsilon \qquad \text{for every } k\geqslant 0\,.$$

Since $S_nT^kx=(1/n)\sum_{i=0}^{2N-1} S_NT^{k+i}x+(1/n)\sum_{i=0}^{n-2N-1} S_NT^{2N+k+i}x+(1/nN)\sum_{i=1}^{N-1}(N-i)\times(T^{k+i-1}x-T^{k+i+n-1}x)$ by (1.1), we see from (1.6) that if $n>2N$ then

$$\begin{aligned}\|S_nT^kx-y\|&\leqslant\left\|\frac{1}{n}\sum_{i=0}^{2N-1}(S_NT^{k+i}x-y)\right\|+\left\|\frac{1}{n}\sum_{i=0}^{n-2N-1}(S_NT^{2N+k+i}x-y)\right\|\\&\quad+\frac{N-1}{2n}\operatorname{diam}C\leq\frac{2N}{n}\operatorname{diam}C+\frac{1}{n}\sum_{i=0}^{n-2N-1}\|S_NT^{2N+k+i}x-y\|\\&\quad+\frac{N-1}{2n}\operatorname{diam}C\\&<\left(\frac{2N}{n}+\frac{N-1}{2n}\right)\operatorname{diam}C+\varepsilon \qquad \text{for every } k\geqslant 0\,.\end{aligned}$$

Therefore $\overline{\lim}_{n\to\infty}\sup_{k\geqslant 0}\|S_nT^kx-y\|\leqslant\varepsilon$, i.e., $\lim_{n\to\infty}\sup_{k\geqslant 0}\|S_nT^kx-y\|=0$. Q.E.D.

PROOF OF THEOREM 0.2. Suppose that (0.3) is satisfied. It suffices to show that (b) in Theorem 0.1 is satisfied. Let $\varepsilon>0$ be arbitrarily given. By virtue of [8, Lemma 2.3] there exist an integer $N_1(\varepsilon)>0$ and $\delta_\varepsilon>0$ such that if $l\geqslant N_1(\varepsilon)$, $k\geqslant 2$, $x_i\in C$ $(i=1, 2, \cdots, k)$ and if $\|x_i-x_j\|-\|T^lx_i-T^lx_j\|<\delta_\varepsilon$ for $1\leqslant i, j\leqslant k$, then $\|T^l(\sum_{i=1}^k r_ix_i)-\sum_{i=1}^k r_iT^lx_i\|<\varepsilon$ for every $r=(r_1, r_2, \cdots, r_k)\in\Delta^{k-1}$, where $\Delta^{k-1}=\{r=(r_1, r_2, \cdots, r_k);\ r_i\geqslant 0\ (i=1, 2, \cdots, k) \text{ and } \sum_{i=1}^k r_i=1\}$. Consequently, if $l\geqslant N_1(\varepsilon)$, $n, m\geqslant 1$, x_i, $y_i\in C$ and if $\max\{\|x_i-x_j\|-\|T^lx_i-T^lx_j\|, \|x_i-y_p\|-\|T^lx_i-T^ly_p\|, \|y_p-y_q\|-\|T^ly_p-T^ly_q\|; 0\leqslant i, j\leqslant n-1, 0\leqslant p, q\leqslant m-1\}<\delta_\varepsilon$, then

$$(1.7)\qquad \left\|T^l\left(\sum_{i=0}^{n-1} r_ix_i+\sum_{i=0}^{m-1} t_iy_i\right)-\left(\sum_{i=0}^{n-1} r_iT^lx_i+\sum_{i=0}^{m-1} t_iT^ly_i\right)\right\|<\varepsilon$$

for any r_i, $t_i\geqslant 0$ with $\sum_{i=0}^{n-1} r_i+\sum_{i=0}^{m-1} t_i=1$. By (0.3) there exists an integer $N_2(\varepsilon)>0$ such that $\beta(i)-\delta_\varepsilon/2<\|T^nx-T^{n+i}x\|<\beta(i)+\delta_\varepsilon/2$ for $n\geqslant N_2(\varepsilon)$ and $i\geqslant 0$, where $\beta(i)=\lim_{n\to\infty}\|T^nx-T^{n+i}x\|$. Hence, if n, $m\geqslant N_2(\varepsilon)$ and $l\geqslant 0$ then $\|T^{i+n}x-T^{j+m}x\|-\|T^{l+i+n}x-T^{l+j+m}x\|<\delta_\varepsilon/2+\beta(|j+m-i-n|)-(\beta(|j+m-i-n|)-\delta_\varepsilon/2)=\delta_\varepsilon$ for i, $j\geqslant 0$.

So, using (1.7) with $r_i=1/2n$, $x_i=T^{i+n}x$ for $0\leqslant i\leqslant n-1$ and $t_i=1/2m$, $y_i=T^{i+m}x$ for $0\leqslant i\leqslant m-1$, we obtain that if l, m, $n\geqslant\max\{N_1(\varepsilon), N_2(\varepsilon)\}$ then $\|T^l((1/2)S_nT^nx+(1/2)S_mT^mx)-(1/2)(S_nT^{l+n}x+S_mT^{l+m}x)\|<\varepsilon$, i.e., (b) in Theorem 0.1 is satisfied. So, by virtue of Theorem 0.1, $\{T^nx\}$ is strongly almost convergent to a fixed point of T. Q.E.D.

REMARK 1.2. To prove Theorem 0.2 we have used [8, Lemma 2.3]. As shown in [8], the proof of that lemma is based on [8, Lemma 2.1] which is stated as follows:

LEMMA. *Suppose that T is asymptotically nonexpansive in the intermediate sense. Then, for $\varepsilon>0$ there exist an integer $N_\varepsilon>0$ and $\delta_{2,\varepsilon}>0$ such that if $k\geqslant N_\varepsilon$, x_1, $x_2\in C$ and if $\|x_1-x_2\|-\|T^kx_1-T^kx_2\|\leqslant\delta_{2,\varepsilon}$, then*

$$\|T^k(r_1x_1+r_2x_2)-r_1T^kx_1-r_2T^kx_2\|<\varepsilon$$

for all $r_1\geqslant 0$ and $r_2\geqslant 0$ with $r_1+r_2=1$.

Since the proof in [8] of this lemma is incomplete, we give a proof of the lemma here.

PROOF. Let δ be the modulus of uniform convexity of X and define a function $d:[0,\infty)\rightarrow[0,\infty)$ by

$$d(t)=\begin{cases}\frac{1}{2}\int_0^t\delta(s)ds & \text{if } 0\leqslant t\leqslant 2\,,\\ d(2)+\frac{1}{2}\delta(2)(t-2) & \text{if } t>2\,.\end{cases}$$

Then d is strictly increasing, continuous and convex, and satisfies

$$2r_1r_2d(\|u-v\|)\leqslant 1-\|r_1u+r_2v\| \tag{1.8}$$

for r_1, $r_2\geqslant 0$ with $r_1+r_2=1$, $\|u\|\leqslant 1$ and $\|v\|\leqslant 1$.

Let $\varepsilon>0$ be arbitrarily given. Choose an $\eta_\varepsilon>0$ such that $\eta_\varepsilon<\varepsilon/3$ and $(D/4)(1+9D^2/\varepsilon^2)d^{-1}(\eta_\varepsilon/D)<\varepsilon$, and put $\delta_{2,\varepsilon}=\min\{\eta_\varepsilon/2, D\}$ where D (>0) is the diameter of C. By (0.2) there exists an integer $N_\varepsilon>0$ such that if $k\geqslant N_\varepsilon$ then

$$\|T^kp-T^kq\|<\|p-q\|+\delta_{2,\varepsilon} \qquad \text{for every } p, q\in C\,. \tag{1.9}$$

Let $k\geqslant N_\varepsilon$ and let x_1, $x_2\in C$ with $\|x_1-x_2\|-\|T^kx_1-T^kx_2\|\leqslant\delta_{2,\varepsilon}$, and let r_1, $r_2\geqslant 0$ with $r_1+r_2=1$.

We first consider the case when $\|x_2-x_1\|\geqslant\varepsilon/3$ and both r_1, $r_2\geqslant\varepsilon/3D$. Put $u=(T^kx_2-T^k(r_1x_1+r_2x_2))/[r_1(1+9D\delta_{2,\varepsilon}/\varepsilon^2)\|x_2-x_1\|]$ and $v=(T^k(r_1x_1+r_2x_2)-T^kx_1)/[r_2(1+9D\delta_{2,\varepsilon}/\varepsilon^2)\|x_2-x_1\|]$. Then $\|u\|\leqslant 1$ and $\|v\|\leqslant 1$ by (1.9), because $\|x_2-(r_1x_1+r_2x_2)\geqslant\varepsilon^2/9D$ and $\|r_1x_1+r_2x_2-x_1\|\geqslant\varepsilon^2/9D$. Since $\|r_1T^kx_1+r_2T^kx_2-T^k(r_1x_1+r_2x_2)\|\leqslant\alpha D\|u-v\|/2$, where $\alpha=(1/2)(1+9D^2/\varepsilon^2)$, by (1.8) and $r_1u+r_2v=(T^kx_2-T^kx_1)/[(1+9D\delta_{2,\varepsilon}/\varepsilon^2)\|x_2-x_1\|]$ we have

$$\begin{aligned}
&d((2/\alpha D)\|r_1T^kx_1+r_2T^kx_2-T^k(r_1x_1+r_2x_2)\|)\\
&\quad =d((2/\alpha D)r_1r_2(1+9D\delta_{2,\varepsilon}/\varepsilon^2)\|x_2-x_1\|\,\|u-v\|)\\
&\quad \leqslant(2/\alpha D)r_1r_2(1+9D\delta_{2,\varepsilon}/\varepsilon^2)\|x_2-x_1\|d(\|u-v\|)\\
&\quad \leqslant(1/\alpha D)(\|x_2-x_1\|-\|T^kx_2-T^kx_1\|+(9D\delta_{2,\varepsilon}/\varepsilon^2)\|x_2-x_1\|)\\
&\quad \leqslant(\delta_{2,\varepsilon}/\alpha D)(1+9D^2/\varepsilon^2)\leqslant\eta_\varepsilon/D\,.
\end{aligned}$$

Here we have used the convexity of d and $d(0)=0$. Therefore we obtain from the choice of η_ε

$$\|r_1T^kx_1+r_2T^kx_2-T^k(r_1x_1+r_2x_2)\|\leqslant(D/4)(1+9D^2/\varepsilon^2)d^{-1}(\eta_\varepsilon/D)<\varepsilon\,.$$

We next consider the case when $\|x_2-x_1\|\geqslant\varepsilon/3$ and $r_i<\varepsilon/3D$, where $i=1$ or 2. By (1.9), $\|T^k(r_1x_1+r_2x_2)-T^kx_{3-i}\|<r_i\|x_2-x_1\|+\delta_{2,\varepsilon}<\varepsilon/3+\delta_{2,\varepsilon}$ and $\|r_1T^kx_1+r_2T^kx_2-T^kx_{3-i}\|=r_i\|T^kx_2-T^kx_1\|<r_i\|x_2-x_1\|+\delta_{2,\varepsilon}<\varepsilon/3+\delta_{2,\varepsilon}$, which implies

$$\begin{aligned}
&\|T^k(r_1x_1+r_2x_2)-(r_1T^kx_1+r_2T^kx_2)\|\\
&\quad\leqslant\|T^k(r_1x_1+r_2x_2)-T^kx_{3-i}\|+\|T^kx_{3-i}-(r_1T^kx_1+r_2T^kx_2)\|\\
&\quad<2\varepsilon/3+2\delta_{2,\varepsilon}<2\varepsilon/3+\eta_\varepsilon<\varepsilon\,.
\end{aligned}$$

Finally, in the case when $\|x_2-x_1\|<\varepsilon/3$ we see from (1.9)

$$\begin{aligned}
&\|T^k(r_1x_1+r_2x_2)-(r_1T^kx_1+r_2T^kx_2)\|\\
&\quad\leqslant\|T^k(r_1x_1+r_2x_2)-T^kx_1\|+\|T^kx_1-(r_1T^kx_1+r_2T^kx_2)\|\\
&\quad<r_2\|x_2-x_1\|+\delta_{2,\varepsilon}+r_2\|T^kx_2-T^kx_1\|<2r_2\|x_2-x_1\|+2\delta_{2,\varepsilon}<2\varepsilon/3+\eta_\varepsilon<\varepsilon\,.
\end{aligned}$$

Q.E.D.

2. Concluding remarks.

DEFINITION 2.1 (Bruck [1]). A sequence $\{x_n\}_{n\geqslant0}$ in C is called an *almost-orbit* of T if

$$\lim_{n\to\infty}\left[\sup_{m\geqslant0}\|x_{n+m}-T^mx_n\|\right]=0\,. \tag{2.1}$$

The same argument as in §1 yields the following Theorems 2.1 and 2.2 which are extensions of Theorems 0.1 and 0.2 respectively.

THEOREM 2.1. *Let T be asymptotically nonexpansive in the weak sense, and let $\{x_n\}_{n\geqslant0}$ in C be an almost-orbit of T. The following* (a) *and* (b) *are equivalent:*

(a) *$\{x_n\}_{n\geqslant0}$ is strongly almost convergent to a fixed point of T.*

(b)

$$\lim_{l,m,n\to\infty}\left\|T^l\left(\frac{1}{2n}\sum_{i=0}^{n-1}x_{i+n}+\frac{1}{2m}\sum_{i=0}^{m-1}x_{i+m}\right)-\left(\frac{1}{2n}\sum_{i=0}^{n-1}T^lx_{i+n}+\frac{1}{2m}\sum_{i=0}^{m-1}T^lx_{i+m}\right)\right\|=0\,.$$

THEOREM 2.2. *Let T be asymptotically nonexpansive in the intermediate sense, and let $\{x_n\}_{n\geq 0}$ in C be an almost-orbit of T. If*

$$\lim_{n\to\infty} \|x_{n+i}-x_n\| \text{ exists uniformly in } i=1,2,\cdots, \tag{2.2}$$

then $\{x_n\}_{n\geq 0}$ is strongly almost convergent to a fixed point of T.

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