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# Birational Geometry of Plane Curves

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## 1. Introduction.

The purpose of this paper is to study curves on rational surfaces from the viewpoint of birational geometry. We begin by recalling basic notions and elementary results of birational geometry of plane curves (see [4], [5] and [8]).

Let $C$ be a curve on a surface $S$. Here by curves and surfaces we mean projective irreducible varieties of dimension 1 and 2, respectively, which are defined over an algebraically closed field of characteristic zero. We shall study such pairs $(S, C)$. Two pairs $(S, C)$ and $(S_1, C_1)$ are said to be *birationally equivalent* if there exists a birational map $f: S \to S_1$ such that the proper image $f[C]$ of $C$ by $f$ coincides with $C_1$. Here the proper image $f[C]$ is by definition the closure of the image $f(x)$ of the generic point $x$ of $C$. When there is no danger of confusion, we say that $C$ is birationally equivalent to $C_1$ as imbedded curves if two pairs $(S, C)$ and $(S_1, C_1)$ are birationally equivalent. A pair $(W, D)$ is said to be *non-singular*, if both $W$ and $D$ are non-singular. In this case, we have complete linear systems $|m(D+K_W)|$ for any $m>0$, where $K_W$ indicates a canonical divisor of $W$. The dimension $\dim|m(D+K_W)|+1$ depends on both $D$ and $W$. But to simplify the notation, we use the symbol $P_m[D]$ to denote $\dim|m(D+K_W)|+1$. From this we define the *Kodaira dimension* $\kappa[D]$ of $(W, D)$ to be the degree of $P_m[D]$ as a function in $m$. It is easy to see that $P_m[D]$ and $\kappa[D]$ are birational invariants of $(W, D)$ in the above sense. In general, for $n \geqq m$, the dimensions $\dim|mD+nK_W|$ are also birational invariants. To verify this, let $h: V \to W$ be a birational morphism where both $V$ and $W$ are non-singular. We assume that $D$ is non-singular and the proper inverse image of $D$ by $h$, denoted by $D_1$, is also non-singular. Then we have

$$mD_1 + nK_V \sim h^*(mD + nK_W) + mR'_h + (n-m)R_h\,,$$

where $R'_h$ is the logarithmic ramification divisor and $R_h$ is the ramification divisor (see [4]). Here the symbol $\sim$ denotes the linear equivalence between divisors. These divisors are effective and the images of these by $h$ are finite sets of points. Hence,

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$$|mD_1+nK_V|=h^*|mD+nK_W|+mR'_h+(n-m)R_h\,.$$

From this, it follows that

$$\dim|mD_1+nK_V|=\dim|mD+nK_W|\,.$$

Given a pair $(S, C)$, one has a non-singular model which is by definition a non-singular pair $(W, D)$ being birationally equivalent to $(S, C)$. Define $P_m[C]$ to be $P_m[D]$. In the same way one can define Kodaira dimension $\kappa[C]$ to be $\kappa[D]$. When $S$ is a rational surface, $P_1[C]$ is equal to $g(C)$, which is the geometric genus of $C$. Hence, $P_1[C]$ vanishes if and only if $C$ is a rational curve. It was proved by Coolidge [1] that $P_2[C]$ vanishes if and only if $(S, C)$ is birationally equivalent to $(\mathbf{P}^2$, line). In the sections 6 and 8, generalizing this we shall prove the next result.

THEOREM 1. *Given a pair $(S, C)$, let $P_m$ denote $P_m[C]$. Curves $C$ with $P_2 \leqq 1$ are classified as follows.*

1. *If $P_2=0$, then $C$ is birationally equivalent to a line on $\mathbf{P}^2$ as imbedded curves.*
2. *If $P_1=P_2=1$, then $C$ is birationally equivalent to a non-singular cubic on $\mathbf{P}^2$ as imbedded curves.*
3. *If $P_2=1$ and $P_3=0$, then $C$ is birationally equivalent to a rational sextic curve with ten double points on $\mathbf{P}^2$ as imbedded curves.*
4. *In the above statement, the condition $P_2=1$ and $P_3=0$ can be replaced by $P_1=0$, $P_6=1$.*
5. *If $P_1=0$ and $P_2=1$, then $C$ is birationally equivalent to a rational curve of degree $3m$ which has nine $m-$ple points and one double point on $\mathbf{P}^2$ as imbedded curves.*

Moreover, plurigenera of such curves with $m>2$ in the case 5) are as follows: $P_3=1$, $P_4=[2-4/m]+1$, $P_5=[2-5/m]+1$, $P_6=[3-6/m]+1$, where the symbol $[X]$ denotes the integral part of a number $X$. In particular $P_6 \geqq 2$, if $m>2$.

The next purpose is the study of minimal models of pairs. A non-singular pair $(S, D)$ is said to be *relatively minimal*, whenever the intersection number $D \cdot E \geqq 2$ for any exceptional curve (of the first kind) $E$ on $S$ such that $E \neq D$. In this case every birational morphism from $(S, D)$ into another non-singular pair $(S_1, D_1)$ turns out to be isomorphic. Moreover, the pair $(S, D)$ is said to be *minimal*, if every birational map from any non-singular pair $(S_1, D_1)$ into $(S, D)$ turns out to be regular. It was shown in the proof of Proposition 3 in [5] that any relatively minimal pair $(S, D)$ is minimal if $\kappa[D]=2$. In this case, the self-intersection number $D^2$ is a birational invariant. Moreover, if $\kappa[D] \geqq 0$, $D^2$ is also a birational invariant except for the case in which $\kappa[D]=0$ and $P_1[D]=1$. Assuming that the non-singular pair $(S, D)$ is relatively minimal and $g=P_1[D]>0$, we have the following result.

1) *If $\kappa[D]=0$, then $g=1$ and $D^2=8$ or $9$.*

2) *If $\kappa[D]=1$, then* A) *$g=1$ and $D^2=4g-4=0$ or* B) *$g \geqq 2$ and $D^2=4g+4 \geqq 12$.*

In the case A), such pairs $(S, D)$ are obtained from plane curves of degree $3m \geqq 6$

with nine $m-$ ple points. In the case B), those pairs $(S, D)$ are obtained as non-singular models of $(\mathbf{P}^2, C)$ where $C$ is a plane curve of degree $d \geqq 4$ with only one $(d-2)$ ple point.

A curve $C$ on $\mathbf{P}^2$ is said to be a *curve of type* $[d; m_0, \cdots, m_r]$, where $d$ is the degree of $C$ and the multiplicities of all the singular points (including infinitely near singular points) are $m_0, \cdots, m_r$. Here we usually assume $m_0 \geqq \cdots \geqq m_r \geqq 2$. Whenever $m_0 = m_1 = \cdots = m_{f-1}$, the symbol $[d; m_0^f, m_f, \cdots, m_r]$ may be employed. If $C$ is a non-singular curve of degree $d$, $C$ is said to be a curve of type $[d; 1]$. Given a pair $(S, D)$, we say *the plane type of the curve $D$ is $T$* whenever there exists a birational equivalence between $(S, D)$ and $(\mathbf{P}^2, C)$ such that the type of $C$ is $T$.

One of the main problems of birational geometry of plane curves is to give some birational characterizations of plane curves in terms of birational invariants such as $D^2$, genus $g$ and plurigenera. In the section 5, we shall prove the following result.

THEOREM 2 (Cf. Corollary of Theorem 2 in [5]). *Suppose that $\kappa[D]=2$, $g=P_1[D] \geqq 1$ and that $(S, D)$ is relatively minimal. In this case, $D^2 \leqq 4g+4$.*

1. *If $D^2=4g+4$, then $g=3$ and the plane type of the curve is $[4; 1]$. The case in which $D^2=4g+3$ does not occur.*
2. *If $D^2=4g+2$, then $g=4$ and the plane type of the curve is $[5; 2^2]$.*
3. *If $D^2=4g+1$, then* A) *$g=5$ and the plane type of the curve is $[5; 2]$ or* B) *$g=6$ and the plane type of the curve is $[5; 1]$.*
4. *If $D^2=4g$, then $g=6$ and the plane type of the curve is $[6; 3, 2]$.*
5. *If $D^2=4g-1$, then $g=7$ and* A) *the plane type of the curve is $[6; 3]$ or* B) *the plane type is $[7; 4, 2^2]$.*
6. *If $D^2=4g-2$, then $g=8$ and the plane type of the curve is $[7; 4, 2]$.*
7. *If $D^2=4g-3$, then $g=9$ and* A) *the type of the curve is $[8; 5, 2^2]$ or* B) *the type is $[7; 4]$.*
8. *If $D^2=4g-4$, then* A) *$g=10$ and the plane type of the curve is $[8; 5, 2]$,* B) the plane type of the curve is $[9; 6, 2^3]$ *or* C) *$g=9$ and the plane type is $[7; 3^2]$ or* D) *$2 \leqq g = 10-\delta \leqq 10$ and the plane type is $[6; 2^\delta]$ where $0 \leqq \delta \leqq 8$.*

REMARK. Recently, O. Matsuda [8] has succeeded in determining all the possible types of #-minimal models of minimal pairs $(S, D)$ with $4g-5 \geqq D^2 \geqq 4g-10$.

In order to state Theorem 3, we use the notation of type of curves on some rational surfaces, which will be introduced in the next section.

THEOREM 3. *Under the same hypothesis as in Theorem* 2,

1. *$D^2 \leqq 3g+7$.*
2. *If $D^2=3g+7$, then* A) *$g=3$ and the plane type of the curve is $[4; 1]$ or* B) *$g=6$ and the plane type is $[5; 1]$.*
3. *If $D^2=3g+6$, then* A) *$g=10$ and the plane type of the curve is $[6; 1]$ or* B) *the curve is birationally equivalent to a curve of type $[3*e, b; 1]$ as imbedded curves where $e \geqq 3b$ and $b \geqq 2$ or $e \geqq 4$ and $b=1$ or $e \geqq 3$ and $b=0$. In this case*

*$g=2e-2-3b$ holds.*

4. *If $D^2=3g+5$, then $g=9$ and* A) *the plane type of the curve is* [6; 2] *or* B) *the plane type of the curve is* $[7; 3^2]$.
5. *if $D^2=3g+4$, then* A) $g=15$ *and the plane type of the curve is* [7; 1] *or* B) $g=12$ *and the plane type of the curve is* [7; 3] *or* C) *the plane type of the curve is* [8; 4, 3] *or* D) $g=8$ *and the plane type of the curve is* $[6; 2^2]$.

REMARK. If $\kappa[D]=1$ and $g \geqq 2$, then $D^2=4g+4$. Therefore, under this assumption, the equality $D^2=3g+7$ implies that the plane type of the curve is [5; 3] with $g=3$. Further, the equality $D^2=3g+6$ implies that the plane type is [5; 3, 2] or [4; 2] with $g=2$.

In birational geometry of plane curves, the following problem may be interesting.

*Given a plane curve C, take all plane curves D which are birationally equivalent to C as imbedded curves: Find all the curves D that have the minimal degree among such curves.*

Since the existence of the curve with minimal degree is obvious, the problem is to find and characterize curves with minimal degree among curves birationally equivalent to the given curve $C$ as imbedded curves.

The following notion is a kind of minimality introduced in p. 62 of [5].

Let $L$ be a line and $C$ a curve on the plane $\mathbf{P}^2$. We say that $C$ is *L-relatively minimal*, if for any birational map $h$ from $\mathbf{P}^2$ into iself, the degree increases, i.e. $h[C]\cdot L \geqq C\cdot L$. Furthermore, if the equality $h[C]\cdot L=C\cdot L$ implies that $h$ is linear, then we say that $C$ is *L-minimal.* Non-singular plane curves of degree $>3$ are $L$-minimal.

REMARKS. 1. *A straight line is L-relatively minimal, but not L-minimal.* Indeed, the quadratic transformation defined by $x=x_1$, $y=x_1y_1$ transforms the line $y=0$ into the line $y_1=0$.

2. *Conics are not L-relatively minimal.*

3. *Non-singular cubics C are L-relatively minimal, but not L-minimal,* since a Cremona plane transformation with center P, Q, R on $C$ transforms $C$ into another non-singular cubic $C_1$.

As a corollary to Theorem 4, we shall show that

*a plane curve of type $[d; m_0, m_1, \cdots, m_r]$ is L-minimal, if $d>m_0+2m_1$.*

Moreover, we shall generalize this result in the case of curves on $\Sigma_b$.

As a corollary to Theorem 5, we shall show that

*If $d>m_0+m_1+m_2$ is satisfied, then a birational map of $\mathbf{P}^2$ preserving C is linear.*

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## 2. Minimal models of rational surfaces.

We start with recalling basic notions concerning relatively minimal models of rational surfaces.

It is well known that given a rational surface $S$, after contracting all exceptional curves on $S$ successively, we have relatively minimal models of $S$. Relatively minimal models of rational surfaces are the projective plane $\mathbf{P}^2$ or $\mathbf{P}^1\times\mathbf{P}^1$ or a $\mathbf{P}^1-$ bundle over $\mathbf{P}^1$ with a section $\Delta_\infty$ of negative self intersection number. The last surface is denoted by a symbol $\Sigma_b$ where $-b$ denotes the self intersection number of the section $\Delta_\infty$. For simplicity, we let $\Sigma_0$ denote the product surface $\mathbf{P}^1\times\mathbf{P}^1$. The Picard group of $\Sigma_b$ ($b\geqq 0$) is generated by a section $\Delta_\infty$ and a fiber $F_u=\rho^{-1}(u)$ of the $\mathbf{P}^1-$ bundle, where $\rho:\Sigma_b\to\mathbf{P}^1$ is the projection.

Let $C$ be an irreducible curve on $\Sigma_b$. Then there exist integers $\sigma$ and $e$ such that

$$C\sim\sigma\Delta_\infty+eF_u\,.$$

We have $C\cdot F_u=\sigma$ and $C\cdot\Delta_\infty=e-b\cdot\sigma$. Hereafter, suppose that $C\neq\Delta_\infty$. Thus $C\cdot\Delta_\infty\geqq 0$ and hence, $e\geqq\sigma\cdot b$. If $b>0$ then $\Delta_\infty^2=-b<0$ and such a section $\Delta_\infty$ is uniquely determined. For a surface $\mathbf{P}^1\times\mathbf{P}^1$, we have $F_u\sim\mathbf{P}^1\times$ point and $\Delta_\infty\sim$ point $\times\mathbf{P}^1$. We may assume that $e\geqq\sigma$. Thus $\sigma$ and $e$ are uniquely determined for a given curve $C$ on $\Sigma_b$. Letting $\pi$ be the virtual genus of $C$, we have

$$\begin{aligned}2\pi-2=C^2+K\cdot C&=(\sigma\Delta_\infty+eF_u)\cdot((\sigma-2)\Delta_\infty+(e-b-2)F_u)\\&=b(1-\sigma)\sigma+2(e\sigma-e-\sigma)\,.\end{aligned}$$

Hence,

$$\pi=(e-1)(\sigma-1)-b\sigma(\sigma-1)/2\,,$$

$$C^2=2e\sigma-\sigma^2b\,.$$

We assume $C$ to be singular. Let $m(C)$ denote the highest multiplicity of the singular points of $C$. We take a singular point $p_1$ on $C$ with $\mathrm{mult}_{p_1}(C)=m(C)$, that is denoted by $m_1$. Blowing up at center $p_1$, we obtain a surface $S_1$ and a proper birational morphism $\mu_1: S_1\to S_0=\Sigma_b$ which satisfies

$$\mu_1^*(C)\sim C_1+m_1E_1\,,$$

where $E_1$ is the exceptional curve $\mu_1^{-1}(p_1)$ and $C_1$ is the proper transform of $C$ by $\mu_1^{-1}$. Letting $K_0$ and $K_1$ denote canonical divisors of $S_0=\Sigma_b$ and $S_1$, respectively, we have

$$K_1\sim\mu_1^*(K_0)+E_1\,.$$

Letting $m_2$ denote $m(C_1)$ and taking $p_2$ on $C_1$ such that $\mathrm{mult}_{p_2}(C_1)=m_2$, we have a surface $S_2$ and a birational morphism $\mu_2: S_2\to S_1$ which is obtained by blowing up at

$p_2$. Continuing this process, we obtain a sequence of birational morphisms $\mu_1, \mu_2, \cdots, \mu_r$ such that the composition $\mu$ of these morphisms gives rise to a minimal resolution of the singularities of the imbedded curve $C$:

$$W=S_r \xrightarrow{\mu_r} S_{r-1} \xrightarrow{\mu_{r-1}} \cdots \xrightarrow{\mu_2} S_1 \xrightarrow{\mu_1} S_0=\Sigma_b .$$

Thus letting $m_j=\mathrm{mult}_{p_j}(C_{j-1})$, we have a sequence of integers $m_1, m_2, \cdots, m_r$ such that $m_1 \geqq m_2 \geqq \cdots \geqq m_r \geqq 2$. Here, $C_0$ stands for $C$. In this case, the curve $C$ of a pair $(\Sigma_b, C)$ is said to be a *curve of type* $[\sigma * e, b; m_1, m_2, \cdots, m_r]$. For simplicity, $[\sigma * e, 0; m_1, m_2, \cdots, m_r]$ is rewritten as $[\sigma * e; m_1, m_2, \cdots, m_r]$. In the case where $C$ is itself non-singular, we put $r=0$ or $r=1$ and $m_1=1$ by convention. Moreover, if $C$ is non-singular, we say that $C$ is the curve of type $[\sigma * e, b; 1]$.

## 3. Elementary transformations.

We shall introduce special kinds of birational transformations, called elementary transformations. Take a point $p$ on $\Sigma_b$. Blowing up at $p$, we have a birational morphism $\mu: S_1 \to S_0=\Sigma_b$. Then letting $F$ be a fiber $\rho^{-1}(\rho(p))$ of $\Sigma_b$ and letting $E$ be the exceptional curve $\mu^{-1}(p)$, we have

$$\mu^*(\sigma\Delta_\infty + eF_u) \sim \mu^*(C) \sim C' + mE ,$$

$$\mu^*(F_u) \sim \mu^*(F) \sim F' + E .$$

Here $F'$ and $C'$ denote the proper inverse images of $F$ and $C$, respectively. If $p \in \Delta_\infty$ then denoting by $\Delta'_\infty$ the image of $\Delta_\infty$ we have $(\Delta'_\infty)^2 = -b-1$. Moreover, $\mu^*(\Delta_\infty) \sim \Delta'_\infty + E$, and

$$C' \sim \sigma(\Delta'_\infty + E) + e(F' + E) - mE .$$

Since $F'^2 = -1$, $F'$ becomes an exceptional curve. Contracting $F'$ into a non-singular point $p'$ we get a non-singular surface $S'$ and a proper birational morphism $\mu': S_1 \to S'$. By $\Delta'_\infty \cdot F' = \Delta_\infty \cdot F - 1 = 1 - 1 = 0$, $\mu'$ is isomorphic around $\Delta'_\infty$. Thus, the image $\Delta''_\infty$ of $\Delta'_\infty$ by $\mu'$ is isomorphic to $\Delta'_\infty$. Hence,

$$(\Delta''_\infty)^2 = \Delta'^2_\infty = \Delta^2_\infty - 1 = -b-1 .$$

This implies that $S'$ is isomorphic to $\Sigma_{b+1}$. The image of $C'$ by $\mu'$ is denoted by $C_0$, that satisfies

$$C_0 \sim \sigma' \Delta''_\infty + e' F_v ,$$

for some integers $\sigma'$ and $e'$, where $F_v$ is a fiber of the $\mathbf{P}^1$-bundle $\Sigma_{b+1}$. The inverse image of $F_v$ by $\mu'$ satisfies

$$\mu'^*(F_v) \sim F' + E .$$

Let $m'$ denote the multiplicity of $C_0$ at $p'$. By the same argument as before, we obtain

$$C' \sim \sigma'\Delta''_\infty + e'(F' + E) - m'F' .$$

Since $E$, $F'$ and $\Delta''_\infty$ are linearly independent, it follows that

$$\sigma' = \sigma, \sigma + e - m = e', e = e' - m' .$$

Hence,

$$m' = \sigma - m, e' = e + m' = e + \sigma - m .$$

Also in the case when $p \notin \Delta_\infty$, we get the similar result and finally we obtain the following proposition.

PROPOSITION 1. 1. *If $p \in \Delta_\infty$, then $S' = \Sigma_{b+1}$ and $m' = \sigma - m$, $e' = e + m' = e + \sigma - m$.*
2. *If $p \notin \Delta_\infty$, then $b > 0$ and $S' = \Sigma_{b-1}$, $m' = \sigma - m$, $e' = e - m$.*

The birational map $\mu \cdot \mu'^{-1}$ is called *elementary transformation of type* I with center $p$.

Let $D$ be a non-singular curve on $S$. We may suppose that the pair $(S, D)$ is relatively minimal. First we suppose that $D$ cannot be transformed into an exceptional curve by any birational map : $S \to W$ where $W$ is a non-singular surface.

If $S = \mathbf{P}^2$, then the degree of $D > 2$.

If $S \neq \mathbf{P}^2$, then after successive blowing downs of exceptional curves, we have a birational morphism $\lambda$: $S \to \Sigma_b$, and the image $\lambda(D) = C$ is a curve on $\Sigma_b$. The type of $C$ is denoted by $[\sigma * e, b; m_1, m_2, \cdots, m_r]$. Suppose that $\sigma \neq 0$ or 1, in other words, $C$ is neither a fiber of $\Sigma_b$ nor a section. If $\sigma < 2m_1$, then perform an elementary transformation of type I with center $p_1$, where $\mathrm{mult}_{p_1}(C) = m_1$. The transformed curve has the type $[\sigma * e', b'; m', m_2, \cdots, m_r]$, where $b' = b \pm 1$, $m' = \sigma - m_1$ and $m' < m_1$. It should be noted that $m' < m_2$ may occur.

After a finite number of elementary transformations of type I, we can assume that $C$ is transformed into a fiber or a section of $\Sigma_b$ or a curve $C$ which satisfies $\sigma \geqq 2m_1$. During this process, $\sigma$ is invariant. But $e$ may increase or decrease.

If $b = 0$, then we have an isomorphism $\varepsilon : \Sigma_0 \to \Sigma_0$ defined by $\varepsilon(x, y) = (y, x)$. The isomorphism $\varepsilon$ exchanges $\Delta_\infty$ and $F_u$. The isomorphism defined by the map $\varepsilon$ is called an *elementary transformation of type* II.

After a finite succession of elementary transformations of type I and II, we can assume $\sigma = 0$ or $\sigma = 1$ or $\sigma \geqq 2m_1$ and moreover if $b = 0$, then we assume that $\sigma \geqq 2m_1$ and $\sigma \leqq e$.

In the case $b = 1$, we have $\Delta_\infty^2 = -1$; hence $\Delta_\infty$ is also an exceptional curve. Take a point $p$ from $S - \Delta_\infty$ and blow up at $p$. Then we have a non-singular surface $U$ and a proper birational morphism $\mu : U \to \Sigma_1$. The inverse image of $p$ is an exceptional curve $E$, that satisfies $\Delta_\infty \cap E = \varnothing$. Letting $C'$ denote the proper inverse image of $C$, we have

$$C' \sim \sigma\Delta_\infty + e(F' + E) - m_1 E .$$

Contracting $\Delta_\infty$ into a non-singular point $q$, we have a non-singular surface $W$ and a proper birational morphism $\lambda : U \to W$. $W$ is isomorphic to $\Sigma_1$, which has a $\mathbf{P}^1$ — fibering. The image of $E$ is a section of the fibering, which we denote by $\Delta$. The image $C_0$ of $C'$ by $\lambda$ is written as follows for some $\sigma'$ and $e'$ in the space of linear equivalence classes:

$$C_0 \sim \sigma'\Delta + e'F_v .$$

Here $F_v$ denotes a general fiber of the $\mathbf{P}^1$— bundle of $W$. By the same argument as before, we have

$$\sigma' = e - m_1,\ e' = e,\ m' = e - \sigma ,$$

where $m'$ indicates the multiplicity of $C_0$ at $q$. The birational map $\varphi : W \to \Sigma_1$ obtained from composing $\mu$ and $\lambda^{-1}$ is called an *elementary transformation of type* III with center $p$.

Now we take a point $p_1$ where $m_1 = \mathrm{mult}_{p_1}(C) = m(C)$. If $e - \sigma < m_1$, then $\Delta_\infty$ does not pass through $p_1$, since $e - \sigma = \Delta_\infty \cdot C < \mathrm{mult}_{p_1}(C) = m_1$. Thus we can apply elementary transformation of type III with center $p_1$ and then the transformed curve $C_0$ has the type $[\sigma' * e', 1; m', m_2, \cdots, m_r]$, where $m' = e - \sigma < m_1$ and $\sigma' = e - m_1 < \sigma$. $m'$ may be smaller than $m_2$.

Finally we consider the case when $C$ is itself non-singular. If $b = 1$ and $e - \sigma = m_1 = 1$, then $\Delta_\infty$ is an exceptional curve with $\Delta_\infty \cdot C = 1$. This implies that $(\Sigma_1, C)$ is not relatively minimal. If $\sigma < 2$, then $\sigma = 1$ or 0. In each case, it is easy to see that $(S, C)$ is birationally equivalent to $(S_0, E)$ where $E$ is an exceptional curve on a non-singular surface $S_0$. Therefore, observing the invariants $\sigma$, $e$ and the highest multiplicities $m_1$ and the number of singular points $p_i$ with $m_i = m_1$ under elementary transformations of type I, II, III, we obtain the following result.

PROPOSITION 2. *Let $(S, D)$ be a relatively minimal pair. Suppose that $D$ is not transformed into an exceptional curve by any birational map $S \to W$. Then either* A) *$S$ is a projective plane and $D$ is a non-singular plane curve with degree $\geqq 3$ or* B) *$(S, D)$ is birationally equivalent to $(\Sigma_b, C)$ that satisfies the condition: $\sigma \geqq 2m_1$. Moreover, if $b = 0$, then $e \geqq \sigma$ and if $b = 1$, then $e - \sigma \geqq m_1$. Furthermore, if $b = m_1 = 1$, then $e - \sigma \geqq 2$.*

DEFINITION. *When the condition in the statement* B) *is satisfied, the pair $(\Sigma_b, C)$ or just $C$ is said to be #-minimal.*

We shall give some examples of types of curves of #-minimal pairs and examine types which vary under certain types of birational transformations.

A curve $C_1$ of type $[\sigma * e, 1; m_1, \cdots, m_r]$ is birationally equivalent to a plane curve of type $[e; e - \sigma, m_1, \cdots, m_r]$. If $C_1$ is #-minimal, then $e - \sigma \geqq m_1$ and $\sigma \geqq 2m_1$; hence $e \geqq e - \sigma + 2m_1$. Writing $d = e$, $m_0 = e - \sigma$, the above equation is rewritten as $d \geqq m_0 + 2m_1$.

In general, for a plane curve of type $[d; m_0, m_1, m_2, \cdots, m_r]$, an equality $d < m_0 + m_1 + m_2$ is said to be *the Noether inequality*. Hence the inequality $d \geqq m_0 + 2m_1$

derived from the condition of ♯-minimality is stronger than the converse of the Noether inequality. It is my understanding that inequalities defining ♯-minimality are closely related to the converse of the Noether inequality.

Let $C$ be a curve on $\Sigma_0 = \mathbf{P}^1 \times \mathbf{P}^1$ of type $[\sigma * e; m_1, \cdots, m_r]$ as imbedded curves. Blowing up at $p_1$, one sees that $C$ is birationally equivalent to a plane curve of type $[e+\sigma-m_1; e-m_1, \sigma-m_1, m_2, \cdots, m_r]$. If $C$ is ♯-minimal, then $e-m_1 \geqq \sigma-m_1 \geqq m_1 \geqq m_2$. Conversely, we let $C_0$ be a plane curve of type $[d; m_0, m_1, \cdots, m_r]$. If $p_0$ and $p_1$ are distinct points on $\mathbf{P}^2$, then $C_0$ is birationally equivalent to a curve $C_1$ of type $[(d-m_0)*(d-m_1); d-m_0-m_1, m_2, \cdots, m_r]$.

Note that the condition $d-m_0-m_1 \geqq m_2$ is satisfied if the converse of the Noether inequality $d \geqq m_0+m_1+m_2$ holds. However, the condition of the ♯-minimality for the curve $C_1$ implies that $d-m_0 \geqq 2(d-m_0-m_1)$, i.e. $m_0+2m_1 \geqq d$. If $p_1$ is infinitely near to $p_0$, then $C_0$ is birationally equivalent to a curve of type $([d-m_0)*(2d-m_0-m_1), 2; d-m_0-m_1, m_2, \cdots, m_r]$.

EXAMPLES. 1) Curves of the type $[3*e, b; 1]$ are birationally equivalent to plane curves of type $[e-b+1; e-b-2, 2^{b-1}]$ as imbedded curves. Here the $b-1$ double points are infinitely near singular points.

2) Curves of type $[\sigma*e; 1]$ are birationally equivalent to plane curves of type $[\sigma+e-1; e-1, \sigma-1]$ and curves of type $[\sigma*(e+\sigma), 2; 1]$ are birationally equivalent to plane curves of type $[\sigma+e-1; e-1, \sigma-1]$. However, the singular points of curves of the former type are distinct points on $\mathbf{P}^2$ and the second one of the singular points of curves of the latter type is an infinitely near singular point.

## 4. ♯-minimal models.

Let $(\Sigma_b, C)$ be a ♯-minimal pair and suppose that $C$ has type $[\sigma*e, b; m_1, m_2, \cdots, m_r]$. Then by applying a finite sequence of blowing ups, we have a minimal resolution $\mu: S \to \Sigma_b$ of singularities of $C$ and the relations among canonical divisors and the inverse images of the curves are as follows:

$$K \sim \mu^*(K_0) + \sum_{i=1}^{r} E_i ,$$

$$D - \mu^*(C) - \sum_{i=1}^{r} m_i E_i .$$

Here, $K_0$ denotes a canonical divisor of $S_0 = \Sigma_b$ and the total inverse images $\mu^* E_i$ of $E_i$ are denoted by the same symbols. Moreover, for simplicity, total inverse images of divisors by $\mu$ shall be denoted by the same symbols. Hence, we can write as follows:

$$D+m_1K\sim C+m_1K_0+\sum_{i=1}^{r}(m_1-m_i)E_i\,,$$

$$C+m_1K_0\sim(\sigma-2m_1)\Delta_\infty+(e-m_1(b+2))F_u\,.$$

By hypothesis of ♯-minimality, $\sigma-2m_1\geqq 0$ and $e-m_1(b+2)\geqq 0$; thus $C+m_1K_0$ is a divisor linearly equivalent to an effective divisor and hence $|D+m_1K|$ is not empty. Thus $|m_1D+m_1K|\neq\varnothing$, which implies that $\kappa[D]=\kappa(D+K,S)\geqq 0$. Suppose that there exits an irreducible curve $\Gamma\neq D$ satisfying $(D+K)\cdot\Gamma<0$. Then since $\kappa(D+K,S)\geqq 0$, it follows that $\Gamma^2<0$ and $K\cdot\Gamma<-D\cdot\Gamma\leqq 0$. Hence, $\Gamma$ is an exceptional curve such that $D\cdot\Gamma=0$. However $\Gamma$ is not one of the $E_i$, because $D\cdot E_i=m_i\geqq 2$. Thus

$$-m_1=D\cdot\Gamma+m_1K\cdot\Gamma=(D+m_1K)\cdot\Gamma\geqq(C+m_1K_0)\cdot\Gamma\geqq 0\,.$$

This is a contradiction.

When $g=P_1[D]>0$, we have $(D+K)\cdot D=2g-2\geqq 0$, which establishes that $D+K$ is nef.

When $g=0$, putting $\nu=-D^2$, we have $\nu>2$. Define a $\mathbf{Q}-$ divisor $Z_\nu$ to be $D+\nu/(\nu-2)\cdot K$, which satisfies $Z_\nu\cdot D=0$. We shall verify $\nu>3$. Indeed, we assume that $D^2=-\nu=-3$ and $D\cdot K=1$. Claim that $m_1\geqq 3$. Actually, suppose that $m_1=2$. Then we have $|D+2K|=|D+m_1K|\neq\varnothing$. However, since $(D,D+2K)=D^2+2D\cdot K=-3+2=-1$, we have $|D+2K|=D+|2K|$, which is void since $S$ is a rational surface.

In resolving the singularities of the pair $(\Sigma_b, C)$, we have a pair $(W, B)$ obtained from $(\Sigma_b, C)$ by blowing up all the singular points with multiplicities $m_j\geqq 3$. Then letting $K_1$ denote a canonical divisor on $W$, we have $\kappa(B+3K_1, W)=\kappa(D+3K, S)\geqq 0$. Denoting by $\delta$ the number of double points (including infinitely near singular points) on $C$, we have

$$B^2=4\delta-3,\ B\cdot K_1=1-2\delta\,.$$

Supposing that $\delta>0$, we have $B^2=4\delta-3>0$. First, we claim that $B+3K_1$ is nef. Actually if an irreducible curve $\Gamma_1$ satisfies $(B+3K_1,\Gamma_1)<0$, then $\Gamma_1^2<0$; hence $B\neq\Gamma_1$. Therefore, $\Gamma_1$ turns out to be an exceptional curve on $W$. Hence, $B\cdot\Gamma_1<-3K_1\cdot\Gamma_1=3$. Since $m_j\geqq 3$ and $(B+3K_1, E_j)=m_j-3$, $\Gamma_1$ cannot coincide with $E_j$. Therefore, $B\cdot\Gamma_1-m_1=(B+m_1K_1,\Gamma_1)\geqq(C+m_1K_0,\Gamma_1)\geqq 0$; thus $B\cdot\Gamma_1\geqq m_1\geqq 3$. This contradicts the previous result.

Noting that $\kappa(B+3K_1, W)\geqq 0$ and $B+3K_1$ is nef, we have $(B+3K_1)^2\geqq 0$ and so $(B+3K_1)^2=B^2+6B\cdot K_1+9K_1^2=3-8\delta+9K_1^2\geqq 0$. Thus $K_1^2\geqq -1/3+8\delta/9\geqq -1/3$. This implies that $K_1^2\geqq 0$. By Riemann-Roch inequality,

$$\dim|-K_1|\geqq K_1^2\geqq 0\,.$$

Therefore, $(B+3K_1, -K_1)\geqq 0$ and we have

$$(B+3K_1, B)=(B+3K_1)^2+(B+3K_1, -3K_1)\geqq 0\,.$$

But

$$(B+3K_1, B)=4\delta-3+3(1-2\delta)=-2\delta\leqq 0\,.$$

Hence, $\delta=0$. By the similar argument, $D+3K$ is nef and $|D+3K|$ is not void. Hence $(D+3K)^2\geqq 0$. $(D+3K)^2=3K\cdot(D+3K)=3+9K^2$. Again, by Riemann-Roch, $\dim|-K|\geqq K^2\geqq 0$. Then $(D+3K)^2=0$. This implies that $9K^2+3=0$; hence $K^2=-1/3$, which is absurd. This establishes $v>3$.

We shall show that $Z_v$ is nef. To show this, suppose that there exists an irreducible curve $\Gamma$ such that $Z_v\cdot\Gamma<0$. Then $D\cdot\Gamma<v/(v-2)=1+2/(v-2)\leqq 2$. Hence $D\cdot\Gamma=0$ or 1. Since $g=0$, the curve $C$ must be singular and hence,

$$1-m_1\geqq D\cdot\Gamma+m_1K\cdot\Gamma=(D+m_1K)\cdot\Gamma\geqq(C+m_1K_0)\cdot\Gamma\geqq 0\,.$$

Thus $m_1\leqq 1$, which contradicts the fact that $C$ is singular. Therefore, we have the former part of the following result.

PROPOSITION 3. *Suppose that* $(\Sigma_b, C)$ *is* ♯*-minimal.*

1. $\kappa[C]\geqq 0$.
2. *If* $g(D)>0$, *then* $D+K$ *in nef.*
3. *If* $g(D)=0$ *and* $\kappa[D]\geqq 0$, *then* $D+v/(v-2)\cdot K$ *is nef, where* $v=-D^2$.
   (a) $v\geqq 4$ *and* $P_2[D]\geqq 1$.
   (b) *If* $v=4$, *then* $\kappa[D]=0$ *or* 1.
   (c) *If* $v\geqq 5$, *then* $\kappa[D]=2$, $(D+v/(v-2)\cdot K)^2>0$ *and* $P_2[D]\geqq 2$.

Proof of the part 3). Assuming $g(D)=0$, by the Riemann-Roch formula,

$$\begin{aligned}P_2[D]&=\dim|2D+2K|+1=\dim|D+2K|+1\\&\geqq(D+2K)\cdot(D+K)/2+1=(D+K)\cdot K\,.\end{aligned}$$

From $(D+v/(v-2)\cdot K)^2\geqq 0$, it follows that

$$(D+v/(v-2)\cdot K)^2=(D+vK/(v-2))\cdot K\cdot v/(v-2)\geqq 0$$

and thus

$$K^2\geqq-\frac{v-2}{v}D\cdot K\,,$$

$$(D+K)\cdot K\geqq\left(1-\frac{v-2}{v}\right)D\cdot K=2-\frac{4}{v}\geqq 1\,.$$

If $v=4$, then $D^2=-4$, $D\cdot K=2$, $(D+2K)^2\geqq 0$ and hence $(D+2K)^2=4+4K^2\geqq 0$.

We shall verify $(D+2K)^2=0$ by deriving a contradiction under the hypothesis $(D+2K)^2>0$. Since $(D+2K)^2=4+4K^2>0$, we have $K^2\geqq 0$. By $\dim|-K|\geqq K^2\geqq 0$, we have $(D+2K)\cdot(-K)\geqq 0$. This implies $(D+2K)\cdot K=0$. Thus $(D+2K)^2=0$.

The intersection numbers are computed as follows:

$$(D+K)\cdot(D+m_1K)=(C+K_0)\cdot(C+m_1K_0)+\sum_{i=1}^{r}(m_1-m_i)(m_i-1)\,,$$

$$(C+K_0)\cdot(C+m_1K_0)=(e-(b+2)m_1)(\sigma-2)+(\sigma-2m_1)(e-\sigma b+b-2)\,.$$

Hence,

$$(C+K_0)\cdot(C+m_1K_0)\geqq 0$$

and so

$$(D+K)\cdot(D+m_1K)\geqq 0\,.$$

When the equality in the above holds, 1) $\sigma=2$ and $C$ is non-singular or

2) If $b\geqq 2$, then $m_1=m_2=\cdots=m_r$, $\sigma=2m_1$, and $e=4m_1$, $b=2$; or if $b=1$, then $\sigma=2m_1$, $e=3m_1$; or if $b=0$, then $e=\sigma=2m_1$.

In general, if a #-minimal pair $(\Sigma_b, C)$ satisfies $\sigma>2m_1$, then it is said to be *strongly #-minimal or ##-minimal*, in short.

Furthermore, if a #-minimal pair $(\Sigma_b, C)$ is birationally equivalent to a pair $(S, D)$, it is said to be a *#-minimal model* of $(S, D)$.

PROPOSITION 4. *If $(\Sigma_b, C)$ is ##-minimal, then*

$$(D+K)\cdot(D+m_1K)\geqq qm_1-2\,.$$

*Here $q=4$ where $b\neq 1$, and $q=3$ when $b=1$.*

PROOF. If $b\geqq 2$, then $e\geqq\sigma b\geqq(2m_1+1)b$ and so

$$(\sigma-2)(e-(b+2)m_1)\geqq(2m_1-1)((b-2)m_1+b)\geqq 2(2m_1-1)\,.$$

If $b=1$, then $e\geqq\sigma+m_1\geqq 3m_1+1$, and hence

$$(e-3m_1)(\sigma-2)+(\sigma-2m_1)(e-\sigma-1)\geqq 2m_1-1+m_1-1=3m_1-2\,.$$

If $b=0$, then

$$(e-2m_1)(\sigma-2)+(\sigma-2m_1)(e-2)\geqq 2(2m_1-1)\,.$$

By applying the adjunction formula, we have $D\cdot K=2g-2-D^2$, where $g$ denotes the genus of $D$. From this we obtain the following result.

COROLLARY. *If $(\Sigma_b, C)$ is ##-minimal, then*

$$D^2\leqq 2(g-1)+2g/m_1-q+K^2\,.$$

*In addition, if $K^2\leqq 3$, then*

$$D^2\leqq 2(g-1)+2g/m_1\,.$$

PROPOSITION 5. *If $C$ is $\sharp$-minimal, then*

$$D^2 \leqq 2(1+1/m_1)(g-1)+K^2 .$$

*If $K^2 \leqq -1$ and $m_1 \geqq 2$ then*

$$D^2 \leqq 3g-4 .$$

PROOF. This follows from $(D+K)\cdot(D+m_1K) \geqq 0$.

EXAMPLE. *Consider affine plane curves $C_0$ defined by $x=f(t)$, $y=g(t)$ where $f(t)=t^n+a_1t^{n-1}+\cdots+a_n$ and $g(t)=t^m+b_1t^{m-1}+\cdots+b_m$ are general polynomials such that $n>m$.* Letting $C$ denote the closure of $C_0$ in $\mathbf{P}^2$, the pair $(\mathbf{P}^2, C)$ has a non-singular model $(S, D)$. By constructing $\sharp$-minimal models, one can compute the Kodaira dimension and obtain the following result ([7]).

1. *If $n<6$ or $m<4$, then $\kappa[D]=-\infty$.*
2. *If $(n, m)=(6, 5)$ or $(6, 4)$ or $(7, 4)$ or $(8, 4)$, then $\kappa[D]=0$.*
3. *Otherwise, $\kappa[D]=2$.*

## 5. Proof of Theorem 2.

We shall enumerate all the relatively minimal pairs $(S, D)$ that satisfy the inequality $D^2 \geqq 4(g-1)$. We start with studying the case of non-singular plane curves.

If $S=\mathbf{P}^2$ and $D$ is a non-singular curve of degree $d$, then $D^2-4(g-1)=d(6-d)$. Assuming $\kappa[D]=2$ and $D^2 \geqq 4(g-1)$, we have the following three cases:

If $d=4$, then $g=3$ and $D^2=4g+4=16$.

If $d=5$, then $g=6$ and $D^2=4g+1=25$.

If $d=6$, then $g=10$ and $D^2=4g-4=36$.

If $d \geqq 7$, then $g \geqq 15$ and $D^2<4g-10$.

Except for the above cases, it suffices to study pairs under the assumption that these have $\sharp$-minimal pairs by Proposition 3.

Let $(\Sigma_b, C)$ be a $\sharp$-minimal model of $(S, D)$. Denoting the virtual genus of $C$ by $\pi$, and the number of double points by $\delta$, we have

$$\pi=(\sigma-1)(e-1)-\sigma(\sigma-1)b/2=g+\delta+\sum_{m_i>2} m_i(m_i-1)/2 ,$$

$$C^2=2\sigma e-\sigma^2 b ,$$

$$D^2=C^2-\sum_{m_i>2} m_i^2-4\delta=4g+C^2-4\pi+\sum_{m_i>2} m_i(m_i-2) .$$

Hence,

$$D^2-4g+4=C^2-4\pi+4+\sum_{m_i>2} m_i(m_i-2) .$$

Here $3 \leqq m_i \leqq \sigma/2$ and we let $t$ denote the number of $i$ such that $m_i>2$. Then

$$\sum_{m_i>2} m_i(m_i-2) \leqq t \cdot \sigma(\sigma-4)/4 = t/4 \cdot \sigma^2 - t\sigma .$$

Letting $V$ be $D^2-4g+4$, we have $V \leqq T+t/4 \cdot \sigma^2 - t\sigma$, where $T=C^2-4\pi+4$. Further, letting $Z$ be $T+t/4 \cdot \sigma^2 - t\sigma$, we have $V \leqq Z$.

We shall study the case in which $Z \geqq 0$. First, we consider the case when $\sigma=2$ or 3. Then from hypothesis on #-minimality, it follows that $m_1=1$ and so $C$ is non-singular.

If $\sigma=2$, then $D+K \sim (e-b-2)F_u$; hence $(D+K)^2=0$. In the case where $e-b-2=0$, we have the following two cases:

1) $b=2$, $e=4$ and 2) $b=0$, $e=2$.

Curves of type [2∗4, 2; 1] are birationally equivalent to a curve of type [2∗2; 1] as imbedded curves. We have $D^2=8$ and $g=1$. Moreover, any curve of type [2∗2; 1] is birationally equivalent to a plane curve of type [3; 1] as imbedded curves. Thus $P_m[D]=1$ for any $m \geqq 1$. Except for this case, we have $e-b-2>0$ and so

$$P_m[D] = \dim |(m(e-b-2)p)| + 1 = m(e-b-2)+1$$

where $p$ is a point on the base curve $\mathbf{P}^1$ and $\kappa[D]=1$. The curves have type [2∗$e$, $b$; 1]. By performing an elementary transformation of type I, a curve of type [2∗$e$, $b$; 1] is birationally equivalent to a curve of type [2∗($e-1$), $b-1$; 1] as imbedded curves. Thus these curves are birationally equivalent to curves of type [2∗($e-b+1$), 1; 1] as imbedded curves which are birationally equivalent to plane curves of type [$e-b+1$; $e-b-1$] as imbedded curves, where $e-b \geqq 3$. In this case, $g=e-b-1 \geqq 2$, $D^2=4e-4b$. Hence, $D^2=4g+4$ and $P_m[D]=m(g-1)+1$. In particular, $P_2[D]=2g-1 \geqq 3$.

In the case where $\sigma=3$, we have $D=C$ and so

$$g=\pi=2e-3b-2,\ D^2=C^2=6e-9b=3g+6 .$$

Furthermore, $(D+K)^2=2e-3b-4=g-2$, and

1) If $b \geqq 2$, then $e=3b+u$ where $u \geqq 0$, and $g=3b+2u-2 \geqq 3b-2 \geqq 4$.

2) If $b=1$, then $e=4+u$ where $u>0$ by #-minimality of $(\Sigma_1, C)$. Hence, $g=2u+3 \geqq 5$.

3) If $b=0$, then $e=3+u$, where $u \geqq 0$ and so $g=2u+4 \geqq 4$.

From this we obtain the former part of the next result.

PROPOSITION 6. *If $\sigma=3$, then $g=2e-3b-2$, $D^2=3g+6$ and $(D+K)^2=g-2$. In this case, $g \geqq 4$ and $P_m[D]=g \cdot m(m+1)/2+1-m^2$. In particular, $P_2[D]=3g-3$.*

By applying the following lemma, $P_m[D]$ will be computed.

LEMMA 1. *If $A$ is a connected curve on a rational surface $S$ with a canonical divisor $K$, then*

$$H^1(S, \mathcal{O}(A+K))=0 .$$

Here, $\mathcal{O}(D)$ denotes the sheaf associated with a divisor $D$.

PROOF. From the exact sequence

$$H^0(\mathcal{O}_S) \to H^0(\mathcal{O}_A) \to H^1(\mathcal{O}(-A)) \to H^1(\mathcal{O}_S)=0\,,$$

we have $H^1(\mathcal{O}(-A))=0$, since $H^0(\mathcal{O}_S) \to H^0(\mathcal{O}_A)$ is isomorphic. By Serre duality, the result follows.

LEMMA 2. *On a surface $S=\Sigma_b$, a divisor $L \sim \alpha\Delta_\infty + \beta F_u$ has vanishing cohomology $H^1(S, \mathcal{O}(L))=0$, if $\alpha>0$ and $\beta - b\alpha \geqq 0$. Furthermore, if $\alpha>0$ and $\beta-b\alpha>0$ then $L$ is very ample. In the case where $b=2$ or $b=1$, if $\alpha>0$ and $\beta-b\alpha=0$, then the complete linear system $|L|$ has no base points and the rational map defined by $L$ is an imbedding into a singular quadric $Q$ or $\mathbf{P}^2$ according to $b=2$ or $b=1$.*

PROOF. Let $L_0=L-K$, which is linearly equivalent to

$$\alpha\Delta_1+(\beta-b\alpha)F_u+\Delta_\infty+\Delta_1+F_u+F_u\,.$$

Here $\Delta_1$ is a section linearly equivalent to $\Delta_\infty+bF_u$. Hence $L_0$ is linearly equivalent to a connected curve. Hence by Lemma 1 we have

$$H^1(S, \mathcal{O}(L))=H^1(S, \mathcal{O}(L_0+K))=0\,.$$

By the Riemann-Roch formula,

$$\dim|L|=-b\alpha(\alpha+1)/2+\alpha+\beta+\alpha\beta\,.$$

In particular, $\dim|\Delta_1+mF_u|=b+2m+1$. It is easy to check that $|\Delta_1|$ and $|F_u|$ have no base points and that $\Delta_1+F_u$ is very ample. Since the divisor $\alpha\Delta_\infty+\beta F_u$ is linearly equivalent to $\Delta_1+F_u+(\alpha-1)\Delta_1+(\beta-b\alpha-1)F_u$, it is very ample if $\alpha>0$ and $\beta-b\alpha>0$. If $b=2$ and $\beta-2\alpha=0$, then $L\sim\alpha\Delta_1$ and the rational map defined by $L$ *is an imbedding into a singular quadric $Q$.*

Now we proceed with the study of curves $D$ on $S$ with $D^2 \geqq 4(g-1)$, where $(S, D)$ is derived from $(\Sigma_b, C)$ by minimal resolution of singularities of $C$ as imbedded curves.

If $\sigma=3$, then by Proposition 6 we have $D^2=3g+6$. Thus $V=10-g$ and we have the following list of types of curves with $\sigma=3$ and $V\geqq 0$.

1. $[3*12, 4; 1] \sim [9; 6, 2^3]$ *where* $g=10$.
2. $[3*(9+u), 3; 1] \sim [7+u; 4+u, 2^2]$ *where* $g=2u+7$ *and* $u=0, 1$.
3. $[3*(6+u), 2; 1] \sim [5+u; 2+u, 2]$ *where* $g=2u+4$ *and* $u=0, 1, 2, 3$. *Here the second singular point on* $\mathbf{P}^2$ *is an infinitely near singular point.*
4. $[3*(4+u), 1; 1] \sim [4+u; 1+u]$ *where* $g=2u+3$, $u=1, 2, 3$.
5. $[3*(3+u); 1] \sim [5+u; 2+u, 2]$ *where* $g=2u+4$, $u=0, 1, 2, 3$ *and the two singular points lie on* $\mathbf{P}^2$.

Here, to simplify the notation, we use the symbol $T_0 \sim T_1$ if curves of type $T_0$ are birationally equivalent to projective plane curves of type $T_1$.

PROPOSITION 7. *Under the previous assumption, suppose that $\sigma \geqq 4$ and $4g+4>D^2\geqq 4g-4\geqq 0$.*

1. *If* $g>1$, *then* $\sigma=4$.
2. *If* $\kappa=1$, *then* $g=1$ *and* $D^2=0$. *Moreover, the type of the curve is* a) $[2m_1*4m_1, 2; ,m_1^8]$, *or* b) $[2m_1*3m_1, 1; m_1^8]$ *or* c) $[2m_1*2m_1; m_1^8]$. *All these are birationally equivalent to plane curves of type* $[3m_1; m_1^9]$ *as imbedded curves.*
3. *If* $\kappa=2$, *then* $\sigma=4$ *and* $m_1\leqq 2$.
   a) *If* $m_1=1$, *then the type of the curve is* $[4*8, 2; 1]$ *or* $[4*6, 1; 1]$ *or* $[4*4; 1]$. *Curves of type* $[4*8, 2; 1]$ *or* $[4*4; 1]$ *are birationally equivalent to plane curves of type* $[7; 3^2]$. *And curves of type* $[4*6; 1]$ *are birationally equivalent to plane curves of type* $[6; 2]$.
   b) *If* $m_1=2$ *and* $\kappa=2$, *then types of the curves are* $[4*8, 2; 2^\delta]$ *or* $[4*6, 1; 2^\delta]$ *or* $[4*4; 2^\delta]$. *These curves are birationally equivalent to curves of type* $[6; 2^{\delta+1}]$ *as imbedded curves. We have* $g=9-\delta>0$ *and* $D^2=32-4\delta$.

PROOF. Since $K^2-(D^2-4g+4)=(D+K)^2\geqq 0$, it follows that $D^2-4g+4\leqq K^2$. Letting $V=D^2-4g+4$, we assume $V\geqq 0$. Then $0\leqq K^2$.

We consider the following cases, separately.

Case A) $b\geqq 2$. In this case, we have $e=b\sigma+u$ when $u\geqq 0$. Hence,

$$Z=(t/4-b)\sigma^2+(4+2b-t)\sigma+u(4-2\sigma)$$

and the last term of the right hand side is non-positive. Thus letting $Z_1=(t/4-b)\sigma^2+(4+2b-t)\sigma$, suppose that $q_0=4b-t>0$. As a function of $\sigma$, $Z_1=Z_1(\sigma)$ attains the maximal value at $\sigma=2+4(2-b)/q_0\leqq 2<4$. By $Z_1(4)=8(2-b)\leqq 0$, we have $V\leqq Z_1\leqq 0$. If the equality $V=0$ holds, then $u=0$, $b=2$ and $\sigma=4$. Thus the type is $[4*8, 2; 2^\delta]$ and $g=9-\delta$, $D^2=32-4\delta$.

Note that if $\kappa[D]=2$, then $0\leqq\delta\leqq 7$. If $\delta>0$, then curves of type $[4*8, 2; 2^\delta]$ are birationally equivalent to curves of plane type $[6; 2^{\delta+1}]$ as imbedded curves. But curves of type $[4*8, 2; 1]$ has the plane type $[7; 3^2]$.

Now suppose that $q_0=4b-t=0$. Then $Z_1=(4-2b)\sigma\leqq 0$. Hence, assuming that $V=0$, we have $b=2$ and $\sigma=2m_i$ for all $i$ such that $m_i>2$. From $4b-t=8-t=0$, it follows that

$$g=(2m_1-1)(4m_1-1)-2m_1(2m_1-1)-8m_1(m_1-1)/2=1$$

and

$$D^2=2\cdot 2m_1\cdot 4m_1-4m_1^2\cdot 2-8m_1^2=0\,.$$

Thus the type is $[2m_1*4m_1, 2; m_1^8]$. Since $C\cdot\Delta_\infty=e-b\cdot\sigma=0$, it follows that the singular point $p_1$ does not lie on $\Delta_\infty$. Hence, performing elementary transformation of type I with center $p_1$, the curve of type $[2m_1*4m_1, 2; m_1^8]$ is birationally equivalent to a curve of type $[2m_1*3m_1, 1; m_1^8]$.

Finally we consider the case when $q_0=4b-t<0$. Then $t>4b\geqq 8=4\cdot 2$. This contradicts $0\leqq K^2$, since $K^2=8-t-\delta$.

Case B) $b=1$. Then writing $e$ as $\sigma+m_1+u$, we get $u\geqq 0$ by #-minimality. Thus

$$Z=-\sigma^2+(6-2m_1)\sigma+4m_1+u(4-2\sigma)+t/4\cdot\sigma^2-t\sigma\,.$$

Hence, letting

$$Z_1=(t/4-1)\sigma^2+(6-2m_1-t)\sigma+4m_1\,,$$

we have $Z=Z_1+u(4-2\sigma)$. We consider $Z_1$ as a function of $\sigma$, which is indicated by $Z_1(\sigma)$. Suppose $0<t<4$. Then the maximal value is attained at $\sigma_0=2(t+2m_1-6)/(t-4)=2(-t-2(m_1-3))/(4-t)<0$. Since $\sigma\geqq 2m_1>0>\sigma_0$, we have

$$0\leqq Z_1(\sigma)\leqq Z_1(2m_1)=(t-8)m_1(m_1-2)<0\,.$$

Suppose $t=4$. Then

$$0\leqq Z_1(\sigma)=(2-2m_1)\sigma+4m_1\leqq 4(1-m_1)m_1+4m_1=4m_1(2-m_1)<0\,.$$

Suppose $t\geqq 5$. When $C$ is ##-minimal, i.e. $\sigma>2m_1$, noting that $K^2=8-t-\delta\leqq 3$, by Corollary to Proposition 4, we have

$$D^2\leqq 2(g-1)+2g/m_1\,.$$

By hypothesis, $0\leqq 4(g-1)\leqq D^2$. From these, it follows that $g\leqq m_1/(m_1-1)$. Since $m_1\geqq 3$, one has $g=1$ and hence $D^2\leqq 2/m_1$. This implies that $D^2=0$; hence $D\cdot K=0$. Again by Corollary to Proposition 4,

$$0=D^2\leqq 2(g-1)+2g/m_1-q+K^2=2/m_1-q+K^2\,.$$

Thus, $q-K^2\leqq 2/m_1<1$. Hence, $3\leqq q\leqq K^2$ and so $|-K|\neq\varnothing$ by Reimann-Roch. Hence, $-K^2=(D+K)\cdot -K\geqq 0$. This contradicts $3\leqq K^2$.

Assume that $\sigma=2m_1$. If $t<8$, then

$$Z_1(2m_1)=(t-8)m_1^2+2(8-t)m_1=(t-8)m_1(m_1-2)<0\,.$$

Hence, $V\leqq Z_1(2m_1)<0$. Supposing $t=8$, we get $Z_1(2m_1)=0$. Hence $V\leqq Z_1(2m_1)=0$. If $V=0$ then $\sigma=2m_i$ for all $i$ and so by the same reasoning as before, we have $e=3m_1$, $t=8$, $g=(2m_1-1)^2-8m_1(m_1-1)/2=1$ and $D^2=4\cdot 3m_1-4m_1^2-8m_1^2=0$. The type is $[2m_1*3m_1,\, 1;\, m_1^8]$. These have plane type $[3m_1;\, m_1^9]$.

If $t=0$, then $m_1\leqq 2$, $V=T=Z$ and

$$T=T(\sigma)=-\sigma^2+(6-2m_1)\sigma+4m_1+u(4-2\sigma)\leqq T(4)=4(2-m_1)-4u\,.$$

Therefore, if $m_1=2$, then $T\leqq 0$. Thus, $V=0$ implies that $\sigma=4$, $m_1=2$, $e=3m_1=6$. If $m_1=1$, then $C$ is itself non-singular and $T=8-(\sigma-2)^2+u(4-2\sigma)$. Thus $V\geqq 0$ implies $\sigma=4$ and $e=5+u$. Hence $V=Z=T=4-4u\leqq 0$, since $u>0$. Therefore, we have the curve of type $[4*6,\, 1;\, 1]$.

Case C) $b=0$. In this case,

$$T=4(e+\sigma)-2e\sigma=8-4(e-2)(\sigma-2)\leqq 16-4e\,.$$

Letting $e=\sigma+u$, we have

$$V\leqq Z=(t-8)/4\cdot\sigma^2+(8-t)\sigma+(4-2\sigma)u\,.$$

If $t=8$, then $Z=(4-2\sigma)u\leqq -4u$. Hence, $V\leqq 0$ and the equality implies that the type of the curve is $[2m_1*2m_1;m_1^8]$. If $0<t<8$, then $V\leqq Z_1(4)<0$ where $Z_1(\sigma)=(t-8)/4\cdot\sigma^2+(8-t)\sigma$. If $t=0$, then $V=T\leqq 16-4e$. Hence, $V\leqq 0$ and the equality implies that the type of the curve is $[4*4;\,2^\delta]$. If $\delta>0$, then the curve of type $[4*4;\,2^\delta]$ is birationally equivalent to a curve of type $[6;\,2^{\delta+1}]$ as imbedded curves.

If $m_1=1$, then letting $e=\sigma+u$, we have

$$T=T(\sigma)=2\sigma(4-\sigma)+2(2-\sigma)u\,.$$

If $u\geqq 0$, $\sigma\geqq 4$ and $T\geqq 0$, then $u=0$, $\sigma=4$, $T=V=0$ and thus the type of the curve is $[4*4;\,1]$, which is birationally equivalent to a curve of type $[7;\,3^2]$ as imbedded curves.

Theorem 2 is derived from the results obtained above.

Given a plane curve $C$ of type $[3m_1;\,m_1^9]$, we shall compute plurigenera of pairs defined by $C$. By a finite succession of blowing ups we resolve the singularities of the imbedded curve $C$. We have a non-singular pair $(S,D)$ and a birational morphism $\mu:\,S\to\mathbf{P}^2$. Then letting $L$ and $K$ be a line on $\mathbf{P}^2$ and a canonical divisor on $S$, respectively, we have

$$K\sim -3L+\mathscr{E}\,,$$

$$D\sim 3m_1L-m_1\mathscr{E}$$

where $\mathscr{E}$ is an effective divisor obtained as the total inverse image of singular points of $C$. Hence, $D\sim -m_1K$ and $D+K\sim -(m_1-1)K$.

Since $K^2=9-9=0$, by Riemann-Roch, we have an effective divisor $J\in|-K|$. Note that $D\sim m_1J$ and $D+K\sim(m_1-1)J$. Supposing that $m_1>1$, we shall show that $\kappa(S,D+K)=\kappa[D]=1$. Actually, since $D+K\sim-(m_1-1)K\sim(m_1-1)J$, it follows that $\kappa(S,D+K)=\kappa(S,D)=\kappa(S,J)$. Noting that $D\sim m_1J$ and $D$ is irreducible, we infer that $\kappa(S,D)>0$. Moreover, since $D^2=0$ and $D$ is an irreducible curve, we see $\kappa(S,D)<2$; thus $\kappa(S,D)=1$ is established. Therefore, the linear system $m(D+K)$ for some sufficiently large $m$ defines an elliptic fibering $f:\,S\to P$, whose general fiber is denoted by $A_u=f^{-1}(u)$. Then

$$(D+K)\cdot A_u=-(m_1-1)K\cdot A_u=0\,.$$

This implies $D\cdot A_u=K\cdot A_u=0$. Hence $D$ is also a fiber of the elliptic fibering. By the canonical bundle formula of elliptic surfaces by Kodaira [7], we have

$$K\sim A_u+(\nu_1-1)A_u/\nu_1=-A_u/\nu_1\,,$$

where $\nu_1$ is the multiplicity of a multiple fiber of the elliptic fibering of $S$. Since $D$ is an elliptic curve and a fiber, we have $D\sim A_u$ or $\nu_1D\sim A_u$. It is easy to derive a contradiction from the hypothesis that $\nu_1D\sim A_u$. Thus we have $D\sim A_u$ and $\nu_1=m_1$. For any integer

$j>0$, we have

$$P_j[D]=\dim|j(m_1-1)A_u/m_1|+1=[j(m_1-1)/m_1]+1 .$$

Hence if $m_1\geqq 2$ and $j\geqq 2$, then $P_j[D]\geqq 2$. In particular, if $P_2[D]=1$, then $m_1=1$.

PROPOSITION 8. *If $g(C)\geqq 1$ and $\kappa[C]\leqq 1$, then $C$ is birationally equivalent to a plane curve of type $[3m_1; m_1^9]$ as imbedded curves. Moreover, if $m_1\geqq 2$, then $P_2[C]\geqq 2$.*

## 6. Proof of Theorem 3.

If $(S, D)$ is a non-singular pair of a plane curve of degree $d>3$ and the projective plane, then $D^2-3g=(d-4)(5-d)/2+7$. In the other cases, we have a #-minimal model $(\Sigma_b, C)$ of $(S, D)$. If $\sigma=3$ then $m_1=1$ and $D^2=3g+6$. Hence, we suppose $\sigma\geqq 4$. We shall make use of the following fact due to R. Hartshorne [3].

LEMMA 3. *Let $H=2(\pi-1)\sigma-(\sigma-2)C^2$ and $R=2(\pi-1)e-(e-3)C^2$. Then $H/\sigma=2e-(2+b)\sigma$ and moreover, if $b\neq 1$, then $H\geqq 0$. If $b=1$, then $H/\sigma=2e-3\sigma$ and $R=-(e-\sigma)\cdot(2e-3\sigma)=-(e-\sigma)H/\sigma$.*

PROOF. Just by computation.

From this, under the assumption of #-minimality, we have

$$2(g-1)\sigma-(\sigma-2)D^2=H-\sum_{i=1}^{r} m_i(2m_i-\sigma)\geqq H .$$

If $b=1$, then $e\geqq\sigma+m_1\geqq 3m_i$ and hence

$$2(g-1)e-(e-3)D^2=R-\sum_{i=1}^{r} m_i(3m_i-e)\geqq R=-(e-\sigma)H/\sigma .$$

First we study the case when $H\geqq 0$. Then $2(g-1)\sigma-(\sigma-2)D^2\geqq 0$ and so

$$D^2\leqq 2(g-1)\sigma/(\sigma-2)=(g-1)(2+4/(\sigma-2)) .$$

If $\sigma\geqq 6$, then $D^2\leqq 3g-3$.

In the case where $\sigma=5$, we have $D^2-3g=C^2-3\pi-r=12+5b-2e-r$. We shall estimate $D^2-3g$ by examining the following cases.

In the subcase when $b\geqq 2$, we have $e\geqq\sigma\cdot b=5b\geqq 10$ and hence, $12+5b-2e-r=12-e+5b-e-r\leqq 12-e-r\leqq 2-r$. Thus $D^2-3g\leqq 2$.

In the subcase when $b=0$, it follows that $e\geqq\sigma=5$ and therefore

$$D^2-3g=12-2e-r\leqq 2-r .$$

In the subcase when $b=1$, from the fact $0\leqq H/\sigma=2e-3\sigma=2e-15$, we have $e\geqq 8$, and so $D^2-3g=17-2e-r\leqq -1-r$. Therefore, the hypothesis $D^2-3g\geqq 4$ implies $\sigma=4$. In this case, we have

$$D^2-3g=C^2-3\pi-r=9+2b-e-r\,.$$

In the subcase when $b\geqq 2$, it follows that $9+2b-e-r\leqq 9-2b-r\leqq 5$. Hence, $D^2-3g=5$ if and only if the type of the curve $C$ is $[4*8, 2; 1]$. The plane type of the curve is $[7; 3, 3]$. $D^2-3g=4$ if and only if the type of the curve $C$ is either $[4*9, 2; 1]$ or $[4*8, 2; 2]$. Corresponding to these, plane types of the curves are $[8; 4, 3]$ or $[6; 2, 2]$.

In the subcase when $b=0$, we have $e\geqq\sigma=4$ and therefore

$$D^2-3g=9-e-r\leqq 5-r\,.$$

Hence, $D^2-3g=5$ if and only if the type of the curve $C$ is $[4*4; 1]$. The plane type of the curve is $[7; 3, 3]$. $D^2-3g=4$ if and only if the type of the curve $C$ is either $[4*5; 1]$ or $[4*4; 2]$. Plane types of the curves are $[8; 4, 3]$ or $[6; 2, 2]$.

In the subcase when $b=1$, by $0\leqq H/\sigma=2e-3\sigma=2e-12$, we have $e\geqq 6$ and so $D^2-3g=11-e-r\leqq 5-r$. Hence, $D^2-3g=5$ if and only if the type of the curve $C$ is $[4*6, 1; 1]$. The plane type of the curve is $[6; 2]$. $D^2-3g=4$ if and only if the type of the curve $C$ is either $[4*7, 1; 1]$ or $[4*6, 1; 2]$. Plane types of the curves are $[7; 3]$ or $[6; 2, 2]$.

Second we consider the case when $H<0$. Then $b=1$ and $R=-(e-\sigma)H/\sigma>0$. Moreover, if $e\geqq 9$ then

$$D^2\leqq 2e/(e-3)\cdot(g-1)\leqq 3(g-1)\,.$$

Hence, we assume $e\leqq 8$. Letting $m_0=e-\sigma$, we have $2m_0<e-m_0$, since $2e-3\sigma<0$. Thus $m_0<e/3\leqq 8/3$, and so $m_0=2$ by #-minimality. Therefore, $e=7$ or 8 and then

$$D^2=e^2-4(1+r)\,,$$

$$g=(e-1)(e-2)/2-(1+r)\,.$$

Hence,

$$D^2-3g=(e-5)(4-e)/2+6-r\leqq 3\,.$$

Thus we readily obtain the result stated in Theorem 3.

## 7. Cases in which $g(D)=0$ or 1.

We consider the case in which $g(D)=0$ or 1. In case $g(D)=1$, the values of $D^2$ are 9, 8, 0, and certain types of negative integers.

PROPOSITION 9. *Assuming $g(D)=1$ and $\kappa[D]=2$, we have $D^2\leqq -2$.*

PROOF. By hypothesis, $(D+K)^2\geqq 1$ and $D^2+D\cdot K=2g(D)-2=0$. Hence,

$$-D\cdot K=-(D+K)\cdot K+K^2=-(D+K)^2+K^2\leqq -1+K^2\,.$$

We shall show that $K^2\leqq -1$. Indeed, if $K^2\geqq 0$, then by the Riemann-Roch formula,

$$\dim|-K|\geqq K^2\geqq 0\,.$$

Hence, $(D+K)\cdot -K\geqq 0$, since $D+K$ is nef. Thus $(D+K)\cdot K\leqq 0$. However,

$$1\leqq (D+K)^2=D^2+D\cdot K+D\cdot K+K^2=D\cdot K+K^2\leqq 0\,,$$

that is a contradiction. Therefore, $D^2=-D\cdot K\leqq K^2-1\leqq -2$. q.e.d.

EXAMPLE. A curve $D$ of type $[8*8; 4^7, 3^2]$ has the following invariants:

$$g(D)=49-7\cdot 6-2\cdot 3=1 \quad\text{and}\quad D^2=2\cdot 64-7\cdot 16-2\cdot 9=-2\,.$$

But the author does not know examples of the curve of the above type.

When $g(D)=0$, it seems interesting to study values of $D^2$.

If $\kappa[D]=0$ or 1, then $D^2=-4$.

a) Curves of the type $[12*12; 6^7, 5, 4]$ or $[10*11; 5^9]$ have the following invariants: $g(D)=0$ and $D^2=-5$.

b) Curves of the type $[8*8; 4^7, 3^2, 2]$ or $[16*16; 8^6, 7^2, 6]$ (found by Matsuda [8]) or $[6*7; 3^{10}]$ or $[20*20; 10^7, 9, 5]$ have the following invariants: $g(D)=0$ and $D^2=-6$.

In the case where $g(D)=2$, we have $D^2\leqq 4$, provided that $\kappa[D]=2$. If $D$ is obtained from a curve of type $[6; 2^8]$, then $g(D)=2$ and $D^2=4$.

In the case where $g(D)=0$ and $\kappa[D]=0$ or 1, plurigenera of $D$ are computed as follows. Repeating the similar argument to the proof of Proposition 7, one can show that $D$ is birationally equivalent to a plane curve $C$ of type $[3m_1; m_1^9, 2]$ (see Iitaka [5]). Thus one has a surface $S_0$ and a birational morphism $\varphi: S_0\to \mathbf{P}^2$ which is obtained from resolving the first nine singular points of the curve $C$. The proper inverse image of $C$ by $\varphi$ is a rational curve with one double point, denoted by $D_0$ and thus we have the relation:

$$D_0\sim -m_1K_0\,.$$

Suppose that $m_1\geqq 3$. Then $\kappa[D]=1$ and thus $S_0$ is an elliptic rational surface. $D_0$ is an irreducible and singular fiber. We write $D_0=\varphi^*(p)$ for some point $p$ on the base curve. Blowing up at the double point of $D_0$ we have a non-singular surface $S$ and a birational morphism $\mu: S\to S_0$. Since $(D+K)\cdot D=-2$ and $D^2=-4$, we see that $j_1D$ is a fixed component of the complete linear system $|j(D+K)|$, where $j_1$ is the round up of $j/2$, i.e., $j_1=-[-j/2]$.

If $j=2$, then $j_1=1$ and thus

$$D+2K\sim D_0+2K_0\sim(1-2/m_1)\varphi^*(p)\,.$$

Hence, $P_2[D]=1$ for any $m_1\geqq 2$.

If $j=3$, then $j_1=2$ and thus

$$D+3K\sim D_0+3K_0+E\sim(1-3/m_1)\varphi^*(p)+E\,,$$

where $E=\mu^{-1}(p)$. Hence $P_3[D]=1$ for any $m_1\geqq 3$.

If $j=4$, then $j_1=2$ and thus

$$2(D+2K)\sim 2(D_0+2K_0)\sim(2-4/m_1)\varphi^*(p)\,.$$

Hence $P_4[D]=2$ for any $m_1\geqq 4$. In addition, $P_4[D]=1$, if $m_1=2$ or 3. Moreover, $P_5[D]=2$ for any $m_1\geqq 5$. $P_6[D]=3$ for any $m_1\geqq 6$ and $P_6[D]=2$ for $5\geqq m_1\geqq 3$. The results for $P_6$ are derived from the formula

$$3(D+2K)\sim(3-6/m_1)\varphi^*(p)\,.$$

Therefore, we obtain the following result.

PROPOSITION 10. *If $g(D)=0$ and $P_2[D]=1$, then D is birationally equivalent to a plane curve of type $[3m_1; m_1^9, 2]$ as imbedded curves. Furthermore, if $m_1\geqq 3$, then $P_3[D]=1$. In general,*

$$P_{2i}[D]=[i-2i/m_1]+1\,,$$

$$P_{2i+1}[D]=[i-(2i+1)/m_1]+1\,.$$

As a corollary, we have the following characterization of curves of type $[6; 2^{10}]$.

*Plane curves C are birationally equivalent to curves of type $[6; 2^{10}]$ as imbedded curves if and only if $P_1[C]=0$ and $P_6[C]=1$.*

## 8. Proper birational geometry.

We shall study non-singular pairs $(S, D)$ with $\kappa[D]=-\infty$.

If $S=\mathbf{P}^2$ then $D$ is a line or conic. If $S$ is a $\mathbf{P}^1-$ bundle over $\mathbf{P}^1$, then $D$ turns out to be a fiber or a section. If $S$ is not relatively minimal, there exists a birational morphism $\mu: S\rightarrow\sum_b$ for some $b\geqq 0$. If the image of $D$ is a curve $C$, by applying the argument in the previous section, we conclude that the pair $(S, D)$ is birationally equivalent to ($\mathbf{P}^2$, line). In the remaining case, there exists a birational morphism $\mu: S\rightarrow S_0$, $S_0$ being a non-singular surface, such that the image $\mu(D)=D_0$ is an exceptional curve. The following is one of the basic results in proper birational geometry. Note that *proper birational equivalence* means that there exists a composition of proper birational morphisms and inverse of proper birational morphisms.

PROPOSITION 11. *Let p be a point on a non-singular rational surface. Then $S-\{p\}$ is proper birationally equivalent to $\mathbf{P}^2-\{point\}$.*

PROOF. In the case where $S\neq\mathbf{P}^2$, we assume that there exist no exceptional curves on $S$ which do not pass through $p$. We take $S$ such that the Picard number of $S$ is minimal among $S$ which satisfy the condition of the proposition. There exists a surjective morphism $\rho: S\rightarrow B=\mathbf{P}^1$ with a general fiber $F_u=\rho^{-1}(u)$ isomorphic to $\mathbf{P}^1$. We have a fiber $F_a$ which passes through the point $p$. Fibers other than $F_a$ is irreducible, since reducible fibers contain exceptional curves. The projection $\rho: S\rightarrow B$ has sections $\Delta$

which do not pass through $p$. We take such a section $\Delta$. If $\Delta^2 \neq -1$ then after performing elementary transformations of type I at some point which is not mapped to the point $a \in B$, we can assume that $\Delta^2 = -1$. Note that by these transformations, the Picard number of $S$ is invariant. So contracting $\Delta$ into a non-singular point $p_0$, we obtain a surface of which Picard number is smaller than that of $S$. This contradicts the hypothesis that the Picard number of $S$ is minimal. q.e.d.

By combining this with Propositions 2 and 3, we obtain the following result which was first stated by Coolidge [1].

PROPOSITION 12. *If* $P_2[D]=0$, *then* $(S, D)$ *is birationally equivalent to* $(\mathbf{P}^2$, *line).*

Proof. If $P_2[D]=0$, then by Corollary to Proposition 3, $(S, D)$ is birationally equivalent to $(S_0, E)$, $E$ being an exceptional curve on $S_0$. Hence, we assume $(S, D)=(S_0, E)$. Then by Proposition 11, $S-D$ is *proper birationally equivalent to* $\mathbf{P}^2-\{point\}$. This implies that $(S, D)$ is birationally equivalent to $(S_1, D_1)$ where $S_1-D_1=\mathbf{P}^2-\{point\}$. Hence $S_1=\Sigma_1$ and $D_1=\Delta_\infty$. It is easy to see that $(\Sigma_1, \Delta_\infty)$ is birationally equivalent to $(\mathbf{P}^2$, line). Thus we have the result.

The next result is an analog of characterizations of abelian surfaces or $K3$ surfaces by means of plurigenera.

PROPOSITION 13. *If* $g(D)=P_2[D]=1$, *then* $(S, D)$ *is birationally equivalent to a pair of plane type* [3; 1].

PROOF. Let $g$ denote $g(D)$. By Proposition 3 the hypothesis $P_2[D]=1$ implies that $\kappa[D]=0$ or 1. If $\kappa[D]=1$, then the plurigenera formula asserts that

$$P_j[D]=[j-j/m_1]+1 ,$$

where $j$ is an integer $>1$. Thus $P_2[D]=1$ implies $m_1=1$.

Propositions 10 and 13 complete the proof of Theorem 1 in the section 1.

REMARK. In p.398 of [1], Coolidge states the following result.

THEOREM 4 (Coolidge [1]). *The necessary and sufficient condition that it be possible to transform a rational curve into a straight line by means of a factorable transformation is that the conditions for special adjoints of every index should be incomplete.*

This follows immediately from Proposition 3. Moreover, Coolidge [1] gave another criterion.

THEOREM 12 (Coolidge [1]). *The necessary and sufficient condition that it be possible to change an elliptic curve into a cubic is that it should lack all special adjoints of index greater than* 1.

Note that $|jK+D|$ is said to be a special adjoint of index $j$. In order to derive Theorem 12 of Coolidge from Proposition 12, first we note that if $g(D)=1$ and $P_j[D]=1$

then $|jK+D|=\varnothing$ for $j>1$. Actually, if $|jK+D|$ contains an effective divisor $F$, then letting $\Gamma=|K+D|$, we have $(j-1)D+F\sim j\Gamma$. Since $P_j[D]=1$, then $j\Gamma=(j-1)D+F$. Hence, $j\Gamma$ has $D$ as one of irreducible components. This implies that $\Gamma$ contains $D$; thus $\Gamma-D\geqq 0$. But, $\Gamma-D\in|K|$, which is void.

Furthermore, $|2K+D|=\varnothing$ implies that $P_2[D]=1$ provided that $g(D)=1$. To verify the latter claim, take $\Gamma_0$ from $|K+D|$. Suppose that $P_2[D]>1$. Then we have $X\in|2\Gamma_0|$ which has common points with $D$. Since $X\cdot D=0$, it follows that $D$ is a component of $X$. Hence $|2K+D|\neq\varnothing$.

## 9. $\sigma$-minimality.

Let $C$ be a curve on $\Sigma_b$ of type $[\sigma*e, b; m_1, m_2, \cdots, m_r]$. Suppose that $(\Sigma_b, C)$ is #-minimal. By resolving the singularities of $C$ as an imbedded curve, we have a proper birational morphism $\mu: S\to\Sigma_b$ and a non-singular curve $D$ on $S$ which is the proper inverse image of $C$ by $\mu$. Here $(S, D)$ is relatively minimal. By $f_j$ we denote the rational map associated with the linear system $|D+jK|$ provided that $|D+jK|\neq\varnothing$ for some $j>0$. Further, let $\varphi_j$ denote the rational map associated with the linear system $|C+jK_0|$ on $\Sigma_b$. If $j\geqq m_1$, then the image of $f_j$ coincides with that of $\varphi_j$, by the next formula:

$$D+jK\sim C+jK_0+\sum_{i=1}^{r}(j-m_i)E_i$$

and

$$|D+jK|=|C+jK_0|+\sum_{i=1}^{r}(j-m_i)E_i$$

Suppose that $(\Sigma_b, C)$ is ##-minimal. Then we have $\dim f_{m_1}(S)=2$, since $\dim\varphi_{m_1}(\Sigma_b)=2$. Define $j(D)$ to be $\max\{j|\dim f_j(S)=2\}$. Then $j(D)<\sigma/2$.

Note that $j(D)$ is a birational invariant of $D$ in the sense of birational geometry of plane curves. The complete linear system $|D+j(D)K|$ is also birationally invariant.

PROPOSITION 14. *If $(\Sigma_b, C)$ is ##-minimal, then the image $W$ of $S$ by $f_j$, where $j=j(D)$, is described as follows.*

1. *$b\neq 1$. Then $W$ becomes $\Sigma_b$ except for the case where $b=2$ and $e=2\sigma$. In the exceptional case, the image $W$ is the singular quadric, which is denoted by $Q$.*
2. *$b=1$. Let $m_0$ denote $e-\sigma$. If $j<m_0$, then $W$ coincides with $\Sigma_1$. Otherwise, the image $W$ turns out to be $\mathbf{P}^2$.*

PROOF. In the case 1), defining $\alpha$ and $\beta$ by $C+jK_0\sim\alpha\Delta_\infty+\beta F_u$, we have $\alpha=\sigma-2j$ and $\beta=e-j(b+2)$. Since $W$ is a surface, both $\alpha$ and $\beta$ are positive. From Lemma 2, $\alpha\Delta_\infty+\beta F_u$ is very ample if $\beta-b\alpha>0$ and $\alpha>0$. Here, $\beta-b\alpha=e-j(b+2)-b(\sigma-2j)=e-b\sigma+jb-2j$. When $b\geqq 2$, the last term $\geqq j(b-2)\geqq 0$. Thus if $\beta-b\alpha=0$ then $b=2$ and $e=2\sigma$. When $b=0$, we get $e\geqq\sigma>2j$ and thus $W=\Sigma_0$.

In the case 2), we have

$$C+jK_0\sim(\sigma-2j)\Delta_\infty+(e-3j)F_u\,.$$

Contracting $\Delta_\infty$ into a non-singular point $p_0$ of $\mathbf{P}^2$, we have a line $L$ on $\mathbf{P}^2$, which is linearly equivalent to $\Delta_\infty+F_u$. Denoting $e-\sigma$ by $m_0$, we have

$$C+jK_0\sim(e-2j-m_0)L+(m_0-j)F_u\,.$$

By hypothesis, $\sigma-2j>0$ and $e-3j>0$ and hence, $e-m_0=\sigma>2j\geqq 2m_1$. Thus if $\beta-\alpha=m_0-j>0$ then $C+jK_0$ is very ample and $W$ turns out to be $\Sigma_1$. If $j\geqq m_0$ then let $v=j-m_0\geqq 0$ and hence,

$$(e-2j-m_0)L+(m_0-j)F_u\sim(e-3v-3m_0)L+v\Delta_\infty\,.$$

Since $e-3v-3m_0=e-3j>0$, it follows that the image $W$ becomes $\mathbf{P}^2$. q.e.d.

Letting $(S, D)$ be a non-singular model of $(\Sigma_b, C)$ where $\kappa[D]=2$, we introduce the following birational invariant:

$$j_+(D)=\sup\left\{\frac{q}{p}\;\middle|\; q\geqq p>0,\ \kappa(qK+pD, S)=2\right\}.$$

**Proposition** 15. 1. *If $S=\mathbf{P}^2$, then $j_+(D)=e/3$, where $D$ has type $[e; 1]$.*

2. *If $(S, D)$ is obtained from a ♯-minimal pair $(\Sigma_b, C)$, then*
   (a) $j_+(D)=\sigma/2$, *if* $b\neq 1$.
   (b) $j_+(D)=\min\{\sigma/2, e/3\}$, *if* $b=1$.

Before giving a proof, we introduce the notion of *$j_+$-model* of $(S, D)$ for a pair $(S, D)$ with $\kappa[D]=2$ as follows: Choose $q\geqq p>0$ such that 1) $j_+(D)>q/p$, 2) $q$ and $p$ are sufficiently large and 3) $q/p$ is sufficiently near to $j_+(D)$. By $\varphi_{q,p}$ we denote the rational map associated to $|qK+pD|$. The image $W=\varphi_{q,p}(S)$ or the pair $(\varphi_{q,p}[D], W)$ is called the $j_+$-model of $(S, D)$.

**Proposition** 16. *Suppose that $(S, D)$ satisfying that $\kappa[D]=2$ is obtained from a ♯-minimal $(\Sigma_b, C)$.*

*If it is ♯♯-minimal, then the $j_+$-model $W$ is described as follows:*

1. *If $b>2$ or $b=0$, then $W=\Sigma_b$.*
2. *If $b=2$ and $e>2\sigma$, then $W=\Sigma_2$.*
3. *If $b=2$ and $e=2\sigma$, then $W=Q$, which is a quadric cone. A minimal non-singular model of $Q$ is $\Sigma_2$.*
4. *If $b=1$ and $\sigma/2\leqq e/3$, then $W=\Sigma_1$.*
5. *If $b=1$ and $\sigma/2>e/3$, then $W=\mathbf{P}^2$.*

**Proof.** By

$$qK+pD\sim(p\sigma-2q)\Delta_\infty+(pe-q(b+2))F_u+\sum_{i=1}^{r}(q-pm_i)E_i\,,$$

if $p\sigma-2q>0$, $pe-q(b+2)>0$ and $q-pm_i\geqq 0$, then we have $\kappa(qK+pD, S)=2$.

Suppose that $p\sigma-2q>0$ and $(\Sigma_b, C)$ is ##-minimal. If $b\geqq 2$, then

$$pe-q(b+2)\geqq p\sigma b-q(b+2)>2qb-q(b+2)=q(b-2)\geqq 0\,.$$

If $b=1$, then $pe-q(b+2)=pe-3q$. We consider in the following cases, separately.

case i) $\sigma/2>e/3$. If $q$, $p$ satisfy $e/3>q/p>m_1$, then it follows that $pe-3q>0$, $p\sigma-2q>0$ and $q\geqq pm_i$. Hence $j_+(D)=e/3$.

case ii) $\sigma/2\leqq e/3$. If $\sigma/2>q/p>m_1$, then $pe-3q>0$. Hence, $j_+(D)=\sigma/2$.

If $b=0$, then $pe-q(b+2)=pe-2q\geqq p\sigma-2q>0$; thus $j_+(D)=\sigma/2$. By an argument in the proof of Proposition 14, we obtain the result.

Now we study #-minimal models of $(S, D)$ with $\kappa[D]=2$ which are not ##-minimal; i.e., $\sigma/2=m_1$. If $b=1$, then $e\geqq\sigma+m_1=3m_1$; hence, $e/3\geqq\sigma/2$. We shall verify that $j_+(D)=\sigma/2$. To do this, we let $\{p_1,\cdots,p_s\}$ be the set of (infinitely near) points $p_i$ with $m_i=m_1$. If $p_1\notin\Delta_\infty$, then performing an elementary transformation with center $p_1$, we assume $p_1\in\Delta_\infty$ and $b>0$. If there exists a singular point with multiplicity $m_1$ which is infinitely near to the point $p_1$, we say it is $p_2$. After repeating such processes, we have a sequence of singular points $\{p_2,\cdots,p_k\}$ in which each $p_{j+1}$ is infinitely near to $p_j$ for $j=1, 2,\cdots$. If $s>k$, then we assume that $p_{k+1}$ lies on $\Sigma_b$ and $p_{k+1}\in\Delta_\infty$. Therefore, we assume that if a singular point $p$ with multiplicity $m_1$ lies on $\Sigma_b$, then it belongs to $\Delta_\infty$. The number of such points is denoted by $c$. Hence, $C\cdot\Delta_\infty=e-b\sigma\geqq cm_1$. Blowing up $\Sigma_b$ at $p_1,\cdots,p_s$, we obtain a surface $Z=S_s$ and the proper transform $C_s$ of $C$. Then letting $Y_{q,p}$ denote $qK_s+pC_s$, we have

$$Y_{q,p}\sim qK_0+pC+(q-pm_1)\mathscr{E}\,,$$

where $K_s$ denotes a canonical divisor on $Z$ and $\mathscr{E}$ stands for the sum of all $E_i$ with $m_i=m_1$. Since $\kappa(S, K+D)=2$, it follows that $\kappa(Z, K_s+C_s)=2$. Hence, letting $U_s=K_s+C_s$, we have

$$U_s\sim K_0+C+(1-m_1)\mathscr{E}\sim(\sigma-2)\Delta_\infty+(e-b-2)F_u+(2-\sigma)/2\mathscr{E}$$

and $\kappa(U_s, Z)=2$.

If $1<q/p<\sigma/2$ and $q/p$ is sufficiently near to $\sigma/2$, then we claim that $\kappa(Z, Y_{q,p})=2$ and $Y_{q,p}$ is nef. Actually, as $Q$-divisors,

$$\begin{aligned}Y_{q,p}&\sim(p\sigma-2q)\Delta_\infty+(ep-q(b+2))F_u-(p\sigma-2q)/2\mathscr{E}\\&\sim(p\sigma-2q)/(\sigma-2)U_s+(ep-q(b+2))F_u-(p\sigma-2q)(e-b-2)/(\sigma-2)F_u\\&\sim(p\sigma-2q)/(\sigma-2)U_s+(q-p)(2e-2\sigma-b\sigma)/(\sigma-2)F_u\,.\end{aligned}$$

If $b\geqq 2$, then

$$2e-2\sigma-b\sigma \geqq 2(b\sigma+cm_1)-2\sigma-b\sigma \geqq (b-2)\sigma+2cm_1 \geqq 1 .$$

If $b=1$, then $e \geqq \sigma+cm_1 \geqq (2+c)m_1 \geqq 3m_1$; hence $\sigma/2=m_1 \leqq e/3$. Thus, $2e-2\sigma-b\sigma=2e-3\sigma \geqq 0$. Hence, in both cases, $\kappa(Z, Y_{q,p})=2$.

To verify that $Y_{q,p}$ is nef, first we note that any irreducible curve $\Gamma$ satisfies $U_s \cdot \Gamma \geqq 0$, if $\Gamma \neq C_s$. Thus whenever $\pi(C_s)>0$, $Y_{q,p}$ is nef. In the case where $\pi(C_s)=0$, $C_s$ is non-singular, i.e., $C_s=D$ and so $v=-D^2 \geqq 5$ by Proposition 3 since $\kappa[D]=2$. Then

$$Y_{q,p} \cdot D=q(v-2)-pv=p(v-2)(q/p-1-2/(v-2))>0$$

since $m_1 \geqq 2$ and $q/p>1+2/3 \geqq 1+2/(v-2)$. Thus $Y_{q,p}$ is nef. Combining this with the fact that $\kappa(Z, Y_{q,p})=2$, we obtain $Y_{q,p}^2>0$.

We study configuration of irreducible curves $\Gamma$ satisfying that $Y_{q,p} \cdot \Gamma=0$. For simplicity, we write as follows:

$$Y_{q,p} \sim \xi U_s+\eta F_u$$

where $\xi=(p\sigma-2q)/(\sigma-2)$ and $\eta=(q-p)(2e-2\sigma-b\sigma)/(\sigma-2)$.

Case $\eta>0$. Any irreducible curve $\Gamma$ with $Y_{q,p} \cdot \Gamma=0$, we have

$$Y_{q,p} \cdot \Gamma=\xi U_s \cdot \Gamma+\eta F_u \cdot \Gamma .$$

If $U_s \cdot \Gamma<0$ then $\pi(C_s)=0$, $C_s=D=\Gamma$ and hence $Y_{q,p} \cdot \Gamma=Y_{q,p} \cdot D>0$, which contradicts the hypothesis. Therefore, $U_s \cdot \Gamma=0$ and $F_u \cdot \Gamma=0$. Hence, $\Gamma^2=-2$, $K_s \cdot \Gamma=0$. Such curves $\Gamma$ are components of degenerate fibers of a fiber space $\rho: S_s \to P^1$ whose general fiber is $F_u=\rho^{-1}(u)$. Since $m_1=\cdots=m_s$, it follows that irreducible components of singular fibers are proper transforms of fibers of $\rho^{-1}(\rho(p_i))$ and the proper transforms $E_j'$ of exceptional curves $E_j$. Hence, it is shown that configuration of curves $\Gamma$ corresponds to a sum of Dynkin diagrams of type $A_l$ for some $l>0$ or $D_l$ for certain $l>3$.

Case $\eta=0$. From $2e=2\sigma+b\sigma$ and $e \geqq b\sigma+m_1c=m_1(2b+c)$, we have either (1) $b=c=1$, $e=3m_1$ or (2) $b=0$, $e=\sigma$ and $c=1$ or $c=2$.

In both cases, $C \sim -m_1K$ and $U_s=K_s+C_s \sim (1-m_1)K_s$. Hence, $U_s^2=m_1^2(8-s)$. Since $\kappa[D]=2$, it follows that $U_s^2>0$; hence, $s \leqq 7$.

In this case, by computation using Maple V, we can verify that the configuration of curves $\Gamma$ satisfying $U_s \cdot \Gamma=0$ corresponds to Dynkin diagrams of type $E_l$ for $l=6$, 7, 8 or of type $D_l$ for $l=4$, 5, 6, 7, 8 or of type $A_l$ for $l=1$, 2, 3, 4, 5, 6, 7, 8 or of type $A_1+A_2$ or of type $A_1+A_1$ or of type $A_1+A_5$ or of type $A_1+A_7$ or of type $A_3+2A_1$ or of type $A_1+D_6$ or of type $A_1+E_l$ for $l=6$, 7.

Contracting these curves $\Gamma$ to rational double points, we have a (possibly singular) surface $Z_0$ and a birational morphism $\mu: Z \to Z_0$. Since $Y_{q,p}$ is nef, we have a divisor $Y_0$ such that $Y_{q,p}=\mu^*(Y_0)$, where $Y_0$ is ample by Nakai's criterion on ampleness of divisors. Consequently, $Z_0$ is the $j_+$-model of $(S, D)$.

We choose $p$, $q$ such that $\sigma/2>q/p$ and $q/p$ is sufficiently near to $\sigma/2>q/p$. Then $\kappa(S, qK+pD)=\kappa(Z, qK_s+pC_s)=2$ and $\varphi_{q,p}$ factors through the map associated with

$Y_{q,p}=qK_s+pC_s$. Hence, the minimal resolution of $j_+$-model of $(S, D)$ coincides with $Z$. Therefore, we have established that $j_+(D)=\sigma/2$ in the case when $\sigma/2=m_1$ and $\kappa[D]=2$.

Given a curve $\Gamma$ on $\Sigma_\beta$, define $\sigma(\Gamma)$ to be the mapping degree of $\Gamma$ with respect to the projection of the projective bundle of $\Sigma_\beta$. Note that if $\beta=0$ then there are two projective bundle structures and in this case define $\sigma(\Gamma)$ to be the smaller degree.

The following result asserts that some numerical minimality implies geometrical minimality.

THEOREM 4. *For any birational map $h: \Sigma_b \to \Sigma_\beta$, the proper image $\Gamma=h[C]$ satisfies the following conditions.*

1. *$\sigma(C)\leqq\sigma(\Gamma)$, if $(\Sigma_b, C)$ is #-minimal.*
2. *If $(\Sigma_b, C)$ is ##-minimal and $\sigma(C)=\sigma(\Gamma)$, then $h$ is isomorphic.*

PROOF. If $(\Sigma_\beta, \Gamma)$ is not #-minimal, performing a finite number of elementary transformations of type I, II, III, we have a birational map $\lambda: (\Sigma_\beta, \Gamma)\to(\Sigma_{\beta'}, \Gamma_1)$ such that $\sigma(\Gamma)\geqq\sigma(\Gamma_1)$ and $(\Sigma_{\beta'}, \Gamma_1)$ is #-minimal. Thus we may assume that $(\Sigma_\beta, \Gamma)$ is itself #-minimal. We shall check that $\sigma(C)=\sigma(\Gamma)$ by examining the following cases, separately.

case 1). $\kappa[C]=0$ or 1. In this case, by consulting the classification of surfaces $(S, D)$ obtained from $(\Sigma_b, C)$, we verify $\sigma(C)=\sigma(\Gamma)$.

case 2). $\kappa[C]=2$. Let $(S, D)$ be a non-singular minimal model of $(\Sigma_b, C)$. Then $(S, D)$ is also a minimal model of $(\Sigma_\beta, \Gamma)$. By considering the $j_+$-model of $(S, D)$, we see that $(\Sigma_b, C)$ is ##-minimal if and only if so is $(\Sigma_\beta, \Gamma)$. In this case, by Proposition 14, $h$ is induced from the map associated to $|j(D)K+D|$; hence $h$ is isomorphic.

COROLLARY. *If a pair $(\Sigma_b, C)$ of type $[\sigma*e, b; m_1, m_2, \cdots, m_r]$ is #-minimal, then the multiplicities $m_1, m_2, \cdots, m_r$ are birational invariants.*

A curve $C$ on $\Sigma_b$ is said to be *$\sigma$-relatively minimal* if, for any birational map $h: \Sigma_b\to\Sigma_\beta$, $\sigma(C)\leqq\sigma(h[C])$ holds. Further, a $\sigma$-relatively minimal curve $C$ is called *$\sigma$-minimal*, if the assumption $\sigma(C)=\sigma(h[C])$ implies that $h$ is isomorphic.

The result in Theorem 4 is restated as follows.

*#-minimality induces $\sigma$-relative minimality and ##-minimality implies $\sigma$-minimality.*

PROPOSITION 17. *Let $(S, D)$ be a minimal pair obtained from a #-minimal pair $(C, \Sigma_b)$ of type $[\sigma*e, b; m_1, m_2, \cdots, m_r]$. Then*

1. *$(D+jK)\cdot D\geqq 0$, for all $2\leqq j\leqq m_1$,*
2. *$D+jK$ is nef for all $2\leqq j\leqq m_r$,*
3. *$D+jK$ is not nef for all $j>m_r$.*

PROOF. case 1). If $g(D)=0$, then $D^2\leqq -4$ and so $(D+jK)\cdot D=-2j-(j-1)D^2\geqq -2j+4(j-1)=2j-4\geqq 0$ where $j\geqq 2$. Thus we suppose $g=g(D)\geqq 1$. From the equality $(D+jK)\cdot D=2(g-1)+(j-1)K\cdot D$, if $D\cdot K>0$, then it follows that $(D+jK)\cdot D>0$ for any $j\geqq 1$. Thus assuming $D\cdot K\leqq 0$, we have

$$(D+\sigma/2K)\cdot D\sim\varepsilon F_u\cdot C+\sum_{i=1}^{r}(\sigma/2-m_i)m_i\,,$$

where $\varepsilon=(e-\sigma(b+2)/2)$ and $\varepsilon\geqq 0$ if $b\neq 1$. Hence, in the case when $b\neq 1$, we have $(D+\sigma/2K)\cdot D\geqq 0$ and so $(D+jK)\cdot D\geqq(D+\sigma/2K)\cdot D\geqq 0$ for $0<j\leqq\sigma/2$. In the case where $b=1$, letting $L$ be a line on the projective plane, we have

$$D+m_1K\sim(e-3m_1)L-(m_0-m_1)E_0+\sum_{i=1}^{r}(m_1-m_i)E_i\,,$$

where $m_0=e-\sigma$ and $E_0=\Delta_\infty$. Further,

$$(D+m_1K)\cdot D\geqq(e-3m_1)e-(m_0-m_1)m_0\geqq(m_0-m_1)(e-m_0)\geqq 0\,,$$

since $e\geqq m_0+2m_1$. Thus, for any $2\leqq j\leqq m_1$, it follows that $(D+jK)\cdot D\geqq 0$.

case 2). Assume that $D+jK$ is not nef for some $2\leqq j\leqq m_r$. Let $\Gamma$ be an irreducible curve such that $(D+jK)\cdot\Gamma<0$. By 1), $\Gamma$ is different from $D$. Since $|D+jK|\neq\varnothing$ and $K\cdot\Gamma<-D\cdot\Gamma/j\leqq 0$, it follows that $\Gamma^2<0$ and $K\cdot\Gamma<0$. Hence $\Gamma$ is an exceptional curve; i.e. $\Gamma^2=\Gamma\cdot K=-1$. This implies $D\cdot\Gamma<j\leqq m_r$. Noting again $(D+jK)\cdot E_i=m_i-j\geqq m_r-j\geqq 0$, we have $\Gamma\neq E_i$. Hence,

$$(D+m_1K)\cdot\Gamma=(C+m_1K_0)\cdot\Gamma+\sum_{i=1}^{r}(m_1-m_i)E_i\cdot\Gamma\geqq(C+m_1K_0)\cdot\Gamma\geqq 0\,.$$

Thus, $D\cdot\Gamma\geqq=-m_1K\cdot\Gamma=m_1\geqq m_r$, which contradicts the inequality obtained in the above.

To show the assertion 3), we notice $(D+jK)\cdot E_r=m_r-j$, which is negative if $j>m_r$. Thus $D+jK$ is not nef. q.e.d.

REMARK. In [2], Dick introduced the following invariant $\lambda$ of the pair by defining $\lambda(S,D)=\min\{\lambda\in\mathbf{Q}|K+\lambda D \text{ is nef}\}$.

By Proposition 16, for a pair $(S,D)$ obtained from a #-minimal pair of type $[\sigma*e,b;m_1,m_2,\cdots,m_r]$, $\lambda(S,D)$ is equal to $1/m_r$. Hence, if $(D+m_rK)^2>0$, then the rational map associated to the system $|n(D+m_rK)|$ for some $n>0$, is a birational morphism. On the other hand, contracting exceptional curves $E$ such that $(D+m_rK)\cdot E=0$, successively we have a pair $(S_k, C_k)$ which may appear in the process of resolving the singularities of $(S,C)$ such that

$$m_r=\cdots=m_{k+1}<m_k\leqq m_{k-1}\leqq\cdots\leqq m_1\,.$$

Letting $K_i$ be canonical divisors on $S_i$ and $C_i$ proper inverse images of $C$, we have

$$D+m_rK\sim C+m_rK_0+\sum_{i=1}^{r}(m_r-m_i)E_i\sim C_k+m_rK_k$$

and then we consider a divisor $C_k+m_kK_k$, which is clearly nef. $m_k$ is obtained from $\lambda(S_k,C_k)=1/m_k$. After contracting exceptional curves $E$ such that $(C_k+K_k)\cdot E=0$, we have a birational morphism and the pair which is the image of the pair $(S_k, C_k)$. Continuing this process, we obtain the pair which is the #-minimal model in the sense of Dick.

## 10. Generalization of a theorem of Noether.

The notion of #-minimal models is important in the general theory of birational geometry of plane curves. However, in studying singular curves, we occasionally encounter pairs $(\Sigma_b, C)$ which are not #-minimal. In order to study such pairs, taking a non-singular model $(S, D)$ of a given pair $(\Sigma_b, C)$, we consider divisors $D+m_hK$ for $h>1$. Letting $\varepsilon_i=m_i-m_h$, we have

$$D+m_hK\sim C+m_hK_0-\sum_{i=1}^{h}\varepsilon_iE_i+\sum_{i=h+1}^{r}(m_h-m_i)E_i\,,$$

$$C+m_hK_0-\sum_{i=1}^{h}\varepsilon_iE_i\sim(\sigma-2m_h)\Delta_\infty+(e-m_h(b+2))F_u-\sum_{i=1}^{h}\varepsilon_iE_i\,.$$

There exist effective divisors $G_i$ such that $F_u\sim E_i+G_i$. Hence, letting $\varepsilon=\sum_{i=1}^{h}\varepsilon_i$, we have $\varepsilon=\sum_{i=1}^{h-1}m_i-(h-1)m_h$ and

$$D+m_hK\sim(\sigma-2m_h)\Delta_\infty+(e-m_h(b+2)-\varepsilon)F_u+\sum_{i=1}^{h}\varepsilon_iG_i+\sum_{i=h+1}^{r}(m_h-m_i)E_i\,.$$

Since $e-m_h(b+2)-\varepsilon=e-\sum_{i=1}^{h-1}m_i+(h-3-b)m_h$, the following result is obtained.

PROPOSITION 18. 1. *If* $\sigma-2m_h\geqq 0$ *and* $e-\sum_{i=1}^{h-1}m_i+(h-3-b)m_h\geqq 0$, *then* $|D+m_hK|\neq\varnothing$; *hence* $\kappa[D]\geqq 0$.

2. *If* $\sigma-2m_h>0$ *and* $e-\sum_{i=1}^{h-1}m_i+(h-3-b)m_h>0$, *then* $\kappa[D]=2$.

PROOF. The last part follows from the fact that $\kappa(\Sigma_b, \alpha\Delta_\infty+\beta F_u)=2$ if $\alpha>0$ and $\beta>0$. Applying this for $b=1$ we have the following corollary.

COROLLARY. *For plane curves* $C$ *of type* $[d; m_0, m_1, \cdots, m_r]$,

1. *If* $d\geqq m_0+2m_h$ *and* $d\geqq\sum_{i=1}^{h-1}m_i-(h-4)m_h\geqq 0$, *then* $\kappa[C]=\kappa[D]\geqq 0$.
2. *If* $d>m_0+2m_h$ *and* $d>\sum_{i=1}^{h-1}m_i-(h-4)m_h>0$, *then* $\kappa[C]=\kappa[D]=2$.

REMARK. A famous theorem of Noether asserts that if $C$ is a proper image of a general line by a birational map from $\mathbf{P}^2$ into itself, then $m_0+m_1+m_2>d$ where $[d; m_0, m_1, \cdots, m_r]$ is the type of $C$. In this case, $C$ satisfies $\kappa[C]=-\infty$. Thus from the corollary the following inequalities are derived.

If $C$ is a proper image of a line by a birational map of $\mathbf{P}^2$, then $d<m_0+2m_2$ and $(d<m_0+2m_4$ or $d<m_1+m_2+m_3)$ and $(d<m_0+2m_5$ or $d<m_1+m_2+m_3+m_4-m_5)$ and so on.

## 11. (0)-minimality.

Next we shall introduce another kind of minimality for a pair $(\Sigma_b, C)$. Let $\mu: S_1 \to \Sigma_b$ be a blowing up at $p_1$ and let $Y$ denote a divisor $C_1 + m_2 K_1$ where $K_1$ is a canonical divisor of $S_1$. Our purpose here is to study when the divisor $Y$ is ample.

Assuming $p_1 \in \Delta_\infty$, we have three curves $\Delta'_\infty$, $F'$ and $E_1$ with negative selfintersection numbers. Here as in the previous sections, we let $\Delta'_\infty$, $F'$ denote proper inverse images of $\Delta_\infty$, $F$ by $\mu$, $F$ being a fiber passing through $p_1$. Since the Picard group of $S_1$ is generated by $\Delta'_\infty$, $F'$ and $E_1$, we compute the interesection numbers of $Y$ with these curves. Thus

$$Y \cdot E_1 = m_1 - m_2 ,$$

$$Y \cdot F' = Y \cdot F - (m_1 - m_2) = \sigma - m_1 - m_2 ,$$

$$Y \cdot \Delta'_\infty = e - b\sigma + m_2 b - m_2 - m_1 .$$

We say that $(\Sigma_b, C)$ satisfies the (0)-*minimality condition* (or $(\Sigma_b, C)$ is (0)-*minimal*) if $m_1 > m_2$, $\sigma > m_1 + m_2$ and $e - b\sigma + m_2 b - m_2 - m_1 > 0$ under the assumption $p_1 \in \Delta_\infty$. If $b = 0$, then the above condition turns out to be $m_1 > m_2$, $\sigma > m_1 + m_2$ and $e - m_2 - m_1 > 0$. In this case we suppose further $e \geqq \sigma$. Similarly, if $b > 0$ and $p_1 \notin \Delta_\infty$, then we have three curves $\Delta_\infty$, $F'$ and $E_1$ which have negative self intersection numbers and

$$Y \cdot \Delta_\infty = e - b\sigma + m_2 b - 2m_2 .$$

Therefore in this case, we say that $(\Sigma_b, C)$ satisfies the (0)-*minimality condition* if $m_1 > m_2$, $\sigma > m_1 + m_2$ and $e - b\sigma + m_2 b - 2m_2 > 0$.

PROPOSITION 19. *If $(\Sigma_b, C)$ is (0)-minimal, then $C_1 + m_2 K_1$ is ample.*

PROOF. Let $Y = C_1 + m_2 K_1$ and $u(b) = e - (b+1)m_2 - m_1$. First we show that $|Y|$ is not void.

If $b > 0$, we have

$$\begin{aligned} u(b) = e - (b+1)m_2 - m_1 &\geqq b\sigma - (b-2)m_2 + 1 - (b+1)m_2 - m_1 \\ &\geqq b(m_1 + m_2 + 1) - (2b-1)m_2 + 1 - m_1 \\ &\geqq (b-1)(m_1 + 1 - m_2) \geqq 0 . \end{aligned}$$

Further,

$$u(0) = e - m_1 - m_2 \geqq \sigma - m_1 - m_2 > 0 .$$

Moreover,

$$\begin{aligned} Y &\sim (\sigma - 2m_2)\Delta_\infty + (e - (b+2)m_2)F - (m_1 - m_2)E_1 \\ &\sim (\sigma - 2m_2)\Delta_\infty + (e - (b+2)m_2)F' + u(b)E_1 . \end{aligned}$$

The last divisor is a sum of curves $\Delta'_\infty$, $F'$ and $E_1$; thus $|Y|$ is not void. Suppose that there exists an irreducible curve $\Gamma$ such that $\Gamma \cdot Y \leqq 0$. If $\Gamma \cdot Y < 0$, then $\Gamma$ is one of irreducible components of $(\sigma - 2m_2)\Delta_\infty + (e-(b+2)m_2)F' + u(b)E_1$. Hence $\Gamma$ coincides with one of divisors $F'$, $\Delta'_\infty$ and $E_1$. But the intersection number of $\Gamma$ with these curves are positive by definition of (0)-minimality. This contradicts the hypothesis. Next suppose $\Gamma \cdot Y = 0$. Then $\Gamma \cdot F = \Gamma \cdot \Delta_\infty = 0$; hence $\Gamma$ does not have common points with $F$ and $\Delta_\infty$. This implies $\Gamma = E_1$, a contradiction. Therefore, by Nakai's criterion, $Y$ is shown to be ample. q.e.d.

In this case, letting $(S, D)$ be a non-singular model of $(\Sigma_b, C)$ obtained by resolving singular points successively, we have

$$D + m_2 K \sim C_1 + m_2 K_1 + \sum_{j=2}^{r} (m_2 - m_j) E_j \, .$$

Hence for any $n > 0$, the rational map associated to $|n(D + m_2 K)|$ coincides with that associated to $|n(C_1 + m_2 K_1)|$. Thus the birational map $\varphi \in \mathrm{Bir}_C(\Sigma_b) = \{h \in \mathrm{Bir}(\Sigma_b) \mid h[C] = C\}$ induces an automorphism $\psi \in \mathrm{Aut}(S_1)$ preserving $C_1$. We shall show that $\psi$ induces an automorphism of $\Sigma_b$ if $b > 1$ or if $b = 1$ and $p_1 \in \Delta_\infty$. To show this it suffices to verify the following proposition.

**Proposition 20.** *On $S_1$ all the exceptional curves of the first kind are $F'$ and $E_1$ if $b > 1$ or if $b = 1$ and $p_1 \in \Delta_\infty$. In the case when $b = 1$ and $p_1 \notin \Delta_\infty$, $\Delta_\infty$ is also an exceptional curve on $S_1$. Furthermore, if $b = 0$, then in addition to $F'$, $E_1$, there exists an exceptional curve $\Delta'_\infty$.*

**Proof.** Let $\Gamma$ be an exceptional curve on $S_1$. Suppose that $\Gamma \neq \Delta'_\infty$, $F'$, and $E_1$. Assuming $p_1 \in \Delta_\infty$, we have

$$-K_1 \sim 2\Delta'_\infty + (b+2)F' + (b+3)E_1 \, .$$

Hence,

$$1 = \Gamma \cdot (-K_1) = 2\Gamma \cdot \Delta'_\infty + (b+2)\Gamma \cdot F' + (b+3)\Gamma \cdot E_1 \, .$$

Since the intersection numbers of $\Gamma$ with $\Delta'_\infty$, $F'$, $E_1$ are non-negative, the above equation is impossible. If $p_1 \notin \Delta_\infty$, then $b > 0$ and we have

$$-K_1 \sim 2\Delta_\infty + (b+2)F' + (b+1)E_1 \, ,$$

$$1 = \Gamma \cdot (-K_1) = 2\Gamma \cdot \Delta_\infty + (b+2)\Gamma \cdot F' + (b+1)\Gamma \cdot E_1 \, .$$

From this it follows that $b = 0$, $\Gamma \cdot \Delta_\infty = 0$, $\Gamma \cdot F' = 0$ and $\Gamma \cdot E_1 = 1$, a contradiction. Except for the case when $b = 1$ and $p_1 \notin \Delta_\infty$, any automorphism of $\Sigma'_b$ preserves $E_1$. Accordingly we obtain the following result.

**Theorem 5.** *Suppose that $b > 1$ or $b = 1$ and $p_1 \in \Delta_\infty$ or $b = 0$. If $(\Sigma_b, C)$ is (0)-minimal, then*

$$\mathrm{Bir}_C(\Sigma_b) = \mathrm{Aut}_C(\Sigma_b) .$$

COROLLARY. *Let $C$ be a plane curve of type $[d; m_0, m_1, m_2, \cdots, m_r]$. If $d > m_0 + m_1 + m_2$, then a birational map of $\mathbf{P}^2$ preserving $C$ is linear.*

PROOF OF COROLLARY. If $m_1 = m_2$, then by Theorem 4 we we have the result. Otherwise we consider the blowing up at $p_0$. Thus we have the proper image $C_0$ of the curve $C$ and $(\Sigma_1, C_0)$ satisfies the (0)-minimality condition. Then a birational map of $\mathbf{P}^2$ preserving $C$ induces an automorphism of $S_1$. On $S_1$ there exist three exceptional curves $\Delta_\infty$, $F'$, $E_1$ such that $\Delta_\infty \cdot F' = 1$, $E_1 \cdot \Delta_\infty = 0$ and $F' \cdot E_1 = 1$. Contracting $\Delta_\infty$ and $E_1$, we have $\mathbf{P}^2$. Thus we obtain the result.

## References


[1] J. L. COOLIDGE, *A Treatise on Algebraic Plane Curves*, Oxford Univ. Press (1928).

[2] D. DICK, Birational Pairs according to S. Iitaka, Math. Proc. Cambridge Philos. Soc. **102** (1987), 59–69.

[3] R. HARTSHORNE, Curves with high self-intersection on algebraic surfaces, Publ. I.H.E.S. **36** (1970), 111–126.

[4] S. IITAKA, Basic structure of algebraic varieties, *Algebraic Varieties and Analytic Varieties*, Adv. Stud. Pure Math. **1** (1983), Kinokuniya, 303–316.

[5] S. IITAKA, On irreducible plane curves, Saitama Math. J. **1** (1983), 47–63.

[6] S. IITAKA, Classification of algebraic objects, *Proc. 3rd International Colloquium on finite or infinite dimensional complex analysis* (1995), 13–23.

[7] K. KODAIRA, On compact analytic surfaces II, Ann. of Math. **77** (1963), 563–626.

[8] O. MATSUDA, On birational invariants of curves on rational surfaces, *Birational Geometry of Pairs of Curves and Surfaces* (1997), Gakushuin Univ., 1–130.

[9] M. NAGATA, On rational surfaces I., Mem. Coll. Sci. Univ. Kyoto **32** (1960), 351–370.

[10] S. SUZUKI, Birational Geometry of birational pairs, Commentari Mathematici Univ. St. Paul. **32** (1983).



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