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Formulae for Relating the Modular Invariants and Defining Equations of $X_0(40)$ and $X_0(48)$

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1. Introduction.

Let N be a positive integer and let $X_0(N)$ be the modular curve over $\mathbf{Q}$ associated to the modular group $\Gamma_0(N)$. As a defining equation of $X_0(N)$, we have the modular equation of level N, which is written in the following form:

$$F_N(j, j_N)=0\ , \quad F_N(S, T)\in \mathbf{Z}[S, T]\ ,$$

where $j=j(z)$ is the modular invariant, $j_N=j_N(z)=j(Nz)$, and z is the natural coordinate on $\mathscr{H}=\{z\in\mathbf{C}\mid Im(z)>0\}$. This equation has many useful properties, but its degree and coefficients are too large to be applied to practical calculations on $X_0(N)$. In the case of a hyperelliptic modular curve, its more manageable defining equation, which we call the normal form of the hyperelliptic curve, has been given by N. Murabayashi ([7]) and M. Shimura ([11]). In particular, for a hyperelliptic curve of the type $X_0(N)$, T. Hibino and N. Murabayashi ([4]) found a certain relation between the modular equation of level N and its normal form except for $N=40$, 48. The relation gives a formula expressing j in terms of the functions x, y on $X_0(N)$ which satisfy the normal form $y^2=f(x)$, $f(T)\in\mathbf{Q}[T]$.

In this paper, we deal with the remaining cases to complete our task. To be specific, for the defining equations $y^2=x^8+8x^6-2x^4+8x^2+1$, $y^2=x^8+14x^4+1$ for $N=40$, 48, respectively, we give formulae expressing j in terms of these x, y (Theorems 4.1, 4.2).

2. Basic idea for expressing j.

In the following, we sketch our idea ([4]) which is based on the computation of the Fourier coefficients of some modular forms (cf. [3], [5], [9], [12]). Let $\mathrm{Aut}(X_0(N))$ be the group of automorphisms of $X_0(N)$ over $\mathbf{C}$. For a positive integer $d\neq 1$ dividing

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N, let w_d be the Atkin-Lehner involution on $X_0(N)$ whereas we assume that w_1 means the identity morphism over $X_0(N)$. From now on we assume that $X_0(N)$ is hyperelliptic with genus g.

Let $S_2(\Gamma_0(N))$ be the vector space over $\mathbf{C}$ of cusp forms of weight 2 for $\Gamma_0(N)$. Let $\binom{1}{0}$ denote the point of $X_0(N)$ represented by $\sqrt{-1}\infty$. If $\binom{1}{0}$ is not a Weierstrass point, one can choose a basis $h_1, \cdots, h_g$ of $S_2(\Gamma_0(N))$ with the following Fourier expansions:

$$\begin{aligned} h_1(z) &= q^g + s_1^{(g+1)} q^{g+1} + \cdots + s_1^{(i)} q^i + \cdots, \\ h_2(z) &= q^{g-1} + s_2^{(g)} q^g + \cdots + s_2^{(i)} q^i + \cdots, \\ &\ \vdots \\ h_g(z) &= q + s_g^{(2)} q^2 + \cdots + s_g^{(i)} q^i + \cdots, \end{aligned}$$

where $q = e^{2\pi\sqrt{-1}z}$ and the coefficients $s_k^{(i)}$ are rational numbers. We put $x = h_2(z)/h_1(z) = q^{-1} + \cdots$. Then x gives a covering map of degree two from $X_0(N)$ to the projective line (cf. [11]). Now we put $y = \frac{q}{h_1(z)} \frac{dx}{dq} = -q^{-(g+1)} + \cdots$. Then x and y satisfy an equation of the type as above, which is viewed as a defining equation of $X_0(N)$. Observing the Fourier coefficients of x and y, we can recursively determine the coefficients of $f(x)$.

Denote the function field of $X_0(N)$ defined over $\mathbf{Q}$ by $\mathbf{Q}(X_0(N))$. Let w_d^* be the automorphism of $\mathbf{Q}(X_0(N))$ induced by w_d. From the action of w_d on $S_2(\Gamma_0(N))$, we explicitly describe the action of w_d^* on the generators x and y of $\mathbf{Q}(X_0(N))$. Then, in the cases $N=40$, 48, we obtain the following result:

Proposition 2.1. *A defining equation of $X_0(N)$ and the action of w_d^* on x and y are given in the table below*:

N	$f(x)$	d, $(w_d^* x, w_d^* y)$
40	$x^8+8x^6-2x^4+8x^2+1$	$5, \left(-\frac{1}{x}, -\frac{y}{x^4}\right)$; $8, \left(-\frac{x-1}{x+1}, -\frac{4y}{(x+1)^4}\right)$
48	x^8+14x^4+1	$3, \left(-\frac{1}{x}, -\frac{y}{x^4}\right)$; $16, \left(-\frac{x-1}{x+1}, -\frac{4y}{(x+1)^4}\right)$

When $X_0(N)$ is hyperelliptic with N square-free, except for $N=37$, we recall the basic idea of [4] for expressing j in terms of x, y. For a positive integer M for which w_M is a hyperelliptic involution, *i.e.* $w_M^* x = x$ and $w_M^* y = -y$, we put $j_M = w_M^* j$. Then $j + j_M$ and $(j - j_M)/y$ are w_M^*-invariant. Therefore they are rational functions of x, determined explicitly by observing the pole divisors and the values at the cusps of x, y, j, and j_M, and also by comparing the Fourier expansions. Calculation of the values of x is as follows. For any cusp P on $X_0(N)$, excluding $\left(\frac{1}{0}\right)$ and $w_M\left(\left(\frac{1}{0}\right)\right)$, let us denote by w the Atkin-Lehner involution which satisfies $P=w\left(\left(\frac{1}{0}\right)\right)$. Since the pole divisors of x are $(x)_\infty=\left(\frac{1}{0}\right)+w_M\left(\left(\frac{1}{0}\right)\right)$, the value of $x(P)$ is calculated by $x(P)=x\left(w\left(\left(\frac{1}{0}\right)\right)\right)=w^*x\left(\left(\frac{1}{0}\right)\right)$, where the function w^*x is obtained as a rational function of x through the action of the Atkin-Lehner involution on $S_2(\Gamma_0(N))$.

But this method cannot be applied to the cases $N=37$, 40 or 48, because it requires that the hyperelliptic involution should be of Atkin-Lehner type, which is not the case for these three cases.

For each level N for which $X_0(N)$ is hyperelliptic, A. Ogg produced a method to check whether its hyperelliptic involution is of Atkin-Lehner type ([8]) and proved:

LEMMA 2.1 (A. Ogg). *The hyperelliptic involutions of* $X_0(40)$, $X_0(48)$ *are defined by* $\begin{pmatrix} -10 & 1 \\ -120 & 10 \end{pmatrix}$, $\begin{pmatrix} -6 & 1 \\ -48 & 6 \end{pmatrix}$, *respectively.*

3. The cases $N=40$ and 48.

In this section, we discuss the cases $N=40$, 48. In any of these cases, $\mathrm{Aut}(X_0(N))$ is not generated by the Atkin-Lehner involutions.

For a positive divisor d of N with $1<d<N$ and for an integer i prime to N, let $\left(\frac{i}{d}\right)$ denote the point of $X_0(N)$ which is represented by $\frac{i}{d}$. Then $\left(\frac{i}{d}\right)$ is defined over $\mathbf{Q}(\zeta_n)$, where $n=\gcd(d, N/d)$ and ζ_n is a primitive n-th root of unity. Reducing i modulo n, we have $\varphi(n)$ Galois-conjugate cusps associated to d. Moreover denote by $\left(\frac{0}{1}\right)$ and $\left(\frac{1}{0}\right)$ the points of $X_0(N)$ which are represented by 0 and $\sqrt{-1}\infty$, respectively.

3.1. The case $N=40$. In case $N=40$, $\mathrm{Aut}(X_0(40))$ is generated by the Atkin-Lehner involutions w_5, w_8, and the automorphism v which is induced from the matrix $\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$ (see [1] and [6]). The hyperelliptic involution S, defined by the matrix $\begin{pmatrix} -10 & 1 \\ -120 & 10 \end{pmatrix}$, is factored into vw_8vw_{40}. In the above notation, the cusps of X_0 (40) are $\left(\frac{0}{1}\right)$, $\left(\frac{1}{2}\right)$, $\left(\frac{1}{4}\right)$, $\left(\frac{1}{8}\right)$, $\left(\frac{1}{5}\right)$, $\left(\frac{1}{10}\right)$, $\left(\frac{1}{20}\right)$ and $\left(\frac{1}{0}\right)$. It is easy to see how the generators act on the cusps. These actions are listed in the table below, *e.g.* $w_5\left(\left(\frac{0}{1}\right)\right)=\left(\frac{1}{5}\right)$:

	$\left(\frac{0}{1}\right)$	$\left(\frac{1}{2}\right)$	$\left(\frac{1}{4}\right)$	$\left(\frac{1}{8}\right)$	$\left(\frac{1}{5}\right)$	$\left(\frac{1}{10}\right)$	$\left(\frac{1}{20}\right)$	$\left(\frac{1}{0}\right)$
w_5	$\left(\frac{1}{5}\right)$	$\left(\frac{1}{10}\right)$	$\left(\frac{1}{20}\right)$	$\left(\frac{1}{0}\right)$	$\left(\frac{0}{1}\right)$	$\left(\frac{1}{2}\right)$	$\left(\frac{1}{4}\right)$	$\left(\frac{1}{8}\right)$
w_8	$\left(\frac{1}{8}\right)$	$\left(\frac{1}{4}\right)$	$\left(\frac{1}{2}\right)$	$\left(\frac{0}{1}\right)$	$\left(\frac{1}{0}\right)$	$\left(\frac{1}{20}\right)$	$\left(\frac{1}{10}\right)$	$\left(\frac{1}{5}\right)$
v	$\left(\frac{1}{2}\right)$	$\left(\frac{0}{1}\right)$	$\left(\frac{1}{4}\right)$	$\left(\frac{1}{8}\right)$	$\left(\frac{1}{10}\right)$	$\left(\frac{1}{5}\right)$	$\left(\frac{1}{20}\right)$	$\left(\frac{1}{0}\right)$

Let x and y be the functions of $X_0(40)$ defined in Proposition 2.1. It is easy to see that $v^*x=-x$ and $v^*y=y$. The pole divisors of x, y are $(x)_\infty=\left(\frac{1}{0}\right)+\left(\frac{1}{4}\right)$, $(y)_\infty=4\left\{\left(\frac{1}{0}\right)+\left(\frac{1}{4}\right)\right\}$, respectively. Thus the values of x at the cusps are determined in the same way as in the square-free case:

LEMMA 3.1.

	$\left(\frac{0}{1}\right)$	$\left(\frac{1}{2}\right)$	$\left(\frac{1}{4}\right)$	$\left(\frac{1}{8}\right)$	$\left(\frac{1}{5}\right)$	$\left(\frac{1}{10}\right)$	$\left(\frac{1}{20}\right)$	$\left(\frac{1}{0}\right)$
x	1	-1	∞	0	-1	1	0	∞

On the other hand, the pole divisors of j and S^*j are

$$(j)_\infty=40\left(\frac{0}{1}\right)+10\left(\frac{1}{2}\right)+5\left(\frac{1}{4}\right)+5\left(\frac{1}{8}\right)+8\left(\frac{1}{5}\right)+2\left(\frac{1}{10}\right)+\left(\frac{1}{20}\right)+\left(\frac{1}{0}\right),$$

$$(S^*j)_\infty=40\left(\frac{1}{10}\right)+10\left(\frac{1}{5}\right)+5\left(\frac{1}{0}\right)+5\left(\frac{1}{20}\right)+8\left(\frac{1}{2}\right)+2\left(\frac{0}{1}\right)+\left(\frac{1}{8}\right)+\left(\frac{1}{4}\right),$$

$$(j\pm S^*j)_\infty=40\left\{\left(\frac{0}{1}\right)+\left(\frac{1}{10}\right)\right\}+10\left\{\left(\frac{1}{2}\right)+\left(\frac{1}{5}\right)\right\}+5\left\{\left(\frac{1}{0}\right)+\left(\frac{1}{4}\right)\right\}+5\left\{\left(\frac{1}{20}\right)+\left(\frac{1}{8}\right)\right\}.$$

Observing the pole divisors and the values of x, y, j, and S^*j at the cusps, it is easy to see that we can take polynomials F, G over $\mathbf{Q}$ which satisfy the following:

$$j+S^*j=\frac{2F(x)}{(x-1)^{40}(x+1)^{10}x^5}, \quad \frac{j-S^*j}{y}=\frac{2G(x)}{(x-1)^{40}(x+1)^{10}x^5},$$

$$F(T)=\sum_{i=0}^{60} a_i T^i \quad , \quad G(T)=\sum_{i=0}^{56} b_i T^i ,$$

$$\deg F=60 \quad , \quad \deg G=56 .$$

Therefore,

$$j=\frac{F(x)+G(x)y}{(x-1)^{40}(x+1)^{10}x^5} .$$

Observing the action of w_5^*, we see

$$j_5=\frac{F_5(x)+G_5(x)y}{(x-1)^{10}(x+1)^{40}x^5} , \quad F_5(T)=-\sum_{i=0}^{60} a_{60-i}(-T)^i ,$$

$$G_5(T)=\sum_{i=0}^{56} b_{56-i}(-T)^i .$$

Finally, using the Fourier expansions of x, y, j and j_5, we can determine the coefficients of F and G. Note that, in determining the coefficients a_i, b_i, the use of the Fourier expansion of j_5 is more effective than the use of those of S^*j since the coefficients of F are just a rearrangement of those of F_5 up to sign in reverse order and since the same goes for the coefficients b_i of the polynomials G and G_5.

3.2. The case $N=48$. In case $N=48$, $\mathrm{Aut}(X_0(48))$ is generated by the Atkin-Lehner involutions w_3, w_{16}, and the automorphism v which is induced from the matrix $\begin{pmatrix} 1 & \frac{1}{4} \\ 0 & 1 \end{pmatrix}$ (see [1] and [6]). The hyperelliptic involution S, defined by the matrix $\begin{pmatrix} -6 & 1 \\ -48 & 6 \end{pmatrix}$, is factored into $v^2 w_{16} v^2 w_{48}$. We note that v is not an involution, but so is v^2. In the notation as above, the cusps are $\left(\frac{0}{1}\right)$, $\left(\frac{1}{2}\right)$, $\left(\frac{1}{4}\right)$, $\left(\frac{3}{4}\right)$, $\left(\frac{1}{8}\right)$, $\left(\frac{1}{16}\right)$, $\left(\frac{1}{3}\right)$, $\left(\frac{1}{6}\right)$, $\left(\frac{1}{12}\right)$, $\left(\frac{7}{12}\right)$, $\left(\frac{1}{24}\right)$ and $\left(\frac{1}{0}\right)$. Let x and y be the modular functions of $X_0(48)$ defined in Proposition 2.1. It is easy to see that $v^{2*}x=-x$ and $v^{2*}y=y$. The pole divisors of x, y are $(x)_\infty=\left(\frac{1}{0}\right)+\left(\frac{1}{8}\right)$, $(y)_\infty=4\left\{\left(\frac{1}{0}\right)+\left(\frac{1}{8}\right)\right\}$. Thus the values of x at the cusps, except for $\left(\frac{1}{4}\right)$, $\left(\frac{3}{4}\right)$, $\left(\frac{1}{12}\right)$ and $\left(\frac{7}{12}\right)$, are determined in the same way as in the square-free case:

LEMMA 3.2.

	$\left(\frac{0}{1}\right)$	$\left(\frac{1}{2}\right)$	$\left(\frac{1}{8}\right)$	$\left(\frac{1}{16}\right)$	$\left(\frac{1}{3}\right)$	$\left(\frac{1}{6}\right)$	$\left(\frac{1}{24}\right)$	$\left(\frac{1}{0}\right)$
x	1	-1	∞	0	-1	1	0	∞

It is hard to see the action of v^* on x, y. Since we cannot obtain $(vw_{48})^*x$ as a rational function in x, the value $x\left(\left(\frac{1}{4}\right)\right)$ is not determined, though $\left(\frac{1}{4}\right)=vw_{48}\left(\left(\frac{1}{0}\right)\right)$. The cases $\left(\frac{3}{4}\right)$, $\left(\frac{1}{12}\right)$ and $\left(\frac{7}{12}\right)$ are in a similar situation to this. Thus we cannot determine the values of x at these four cusps by our method. However, we can obtain a few relations among the values as follows. The values $x\left(\left(\frac{1}{4}\right)\right)$ and $x\left(\left(\frac{1}{3}\right)\right)$, $x\left(\left(\frac{1}{12}\right)\right)$ and $x\left(\left(\frac{7}{12}\right)\right)$, which are in $\mathbf{Q}(\zeta_4)$, are conjugate over $\mathbf{Q}$, respectively. Since $S\left(\left(\frac{1}{4}\right)\right)=\left(\frac{1}{12}\right)$ and $w_3\left(\left(\frac{1}{4}\right)\right)=\left(\frac{1}{12}\right)$, we see that $x\left(\left(\frac{1}{4}\right)\right)=x\left(\left(\frac{1}{12}\right)\right)=x\left(w_3\left(\left(\frac{1}{4}\right)\right)\right)=w_3^*x\left(\left(\frac{1}{4}\right)\right)=-1/x\left(\left(\frac{1}{4}\right)\right)$. Then the value $x\left(\left(\frac{1}{4}\right)\right)$ satisfies the equation $x\left(\left(\frac{1}{4}\right)\right)^2+1=0$. Moreover, since $v^2\left(\left(\frac{1}{4}\right)\right)=\left(\frac{3}{4}\right)$, we see that $x\left(\left(\frac{1}{4}\right)\right)=x\left(v^2\left(\left(\frac{3}{4}\right)\right)\right)=v^{2*}x\left(\left(\frac{3}{4}\right)\right)=-x\left(\left(\frac{3}{4}\right)\right)$, *i.e.* $x\left(\left(\frac{1}{4}\right)\right)=-x\left(\left(\frac{3}{4}\right)\right)$. Similarly we obtain $x\left(\left(\frac{1}{12}\right)\right)^2+1=0$ and $x\left(\left(\frac{1}{12}\right)\right)=-x\left(\left(\frac{7}{12}\right)\right)$. On the other hand, the pole divisors of j and $j\pm S^*j$ are

$$(j)_\infty=48\left(\frac{0}{1}\right)+12\left(\frac{1}{2}\right)+16\left(\frac{1}{3}\right)+3\left(\frac{1}{4}\right)+3\left(\frac{3}{4}\right)+4\left(\frac{1}{6}\right)$$
$$+3\left(\frac{1}{8}\right)+\left(\frac{1}{12}\right)+\left(\frac{7}{12}\right)+3\left(\frac{1}{16}\right)+\left(\frac{1}{24}\right)+\left(\frac{1}{0}\right),$$
$$(j\pm S^*j)_\infty=48\left\{\left(\frac{0}{1}\right)+\left(\frac{1}{6}\right)\right\}+16\left\{\left(\frac{1}{2}\right)+\left(\frac{1}{3}\right)\right\}+3\left\{\left(\frac{1}{4}\right)+\left(\frac{1}{12}\right)\right\}$$
$$+3\left\{\left(\frac{3}{4}\right)+\left(\frac{7}{12}\right)\right\}+3\left\{\left(\frac{1}{0}\right)+\left(\frac{1}{8}\right)\right\}+3\left\{\left(\frac{1}{16}\right)+\left(\frac{1}{24}\right)\right\}.$$

Observing the pole divisors and the values at the cusps of x, y, j and S^*j, it is easy to see that we can take polynomials F, G over $\mathbf{Q}$ which satisfy the following:

$$j+S^*j=\frac{2F(x)}{(x-1)^{48}(x+1)^{16}(x^2+1)^3x^3},$$
$$\frac{j-S^*j}{y}=\frac{2G(x)}{(x-1)^{48}(x+1)^{16}(x^2+1)^3x^3},$$
$$F(T)=\sum_{i=0}^{76}a_iT^i\ ,\ G(T)=\sum_{i=0}^{72}b_iT^i\,,$$
$$\deg F=76\ ,\ \deg G=72\,.$$

Therefore,

$$j=\frac{F(x)+G(x)y}{(x-1)^{48}(x+1)^{16}(x^2+1)^3x^3}\,.$$

Observing the action of w_3^*,

$$j_3=\frac{F_3(x)+G_3(x)y}{(x+1)^{48}(x-1)^{16}(x^2+1)^3x^3}\,,\qquad F_3(T)=-\sum_{i=0}^{76}a_{76-i}(-T)^i\,,$$

$$G_3(T)=\sum_{i=0}^{72}b_{72-i}(-T)^i\,.$$

In the same way as in §3.1, by using the Fourier expansions of x, y, j and j_3, we can determine the coefficients of F and G.

4. Relations for Fricke's cases.

Displaying our results in §3 requires so much space. Instead, we give relations between our data and Fricke's work.

4.1. The case $N=40$. We define as follows:

$$p_5(t)=\frac{(t^2+10t+5)^3}{t}\,,$$

$$p_{10}(t)=\frac{t(2t+5)^2}{t+2}\,,$$

$$p_{20}(t,s)=\frac{t^2-13-s}{4}\,.$$

LEMMA 4.1 (Fricke). *We have the following sequence of covering maps between modular curves*:

$$\begin{array}{ccccccc} X_0(20) & \longrightarrow & X_0(10) & \longrightarrow & X_0(5) & \longrightarrow & X_0(1) \\ (\tau_{20},\sigma_{20}) & \longmapsto & \tau_{10} & \longmapsto & \tau_5 & \longmapsto & j\,, \end{array}$$

where $\mathbf{Q}(X_0(1))=\mathbf{Q}(j)$, $\mathbf{Q}(X_0(5))=\mathbf{Q}(\tau_5)$, $\mathbf{Q}(X_0(10))=\mathbf{Q}(\tau_{10})$, *and* $\mathbf{Q}(X_0(20))=\mathbf{Q}(\tau_{20},\sigma_{20})$ *which* (τ_{20},σ_{20}) *satisfy the equation* ${\sigma_{20}}^2={\tau_{20}}^4-12{\tau_{20}}^3+28{\tau_{20}}^2-32\tau_{20}+16$. *Moreover, the following relations hold*:

$$j=p_5(\tau_5)\,,$$

$$\tau_5=p_{10}(\tau_{10})\,,$$

$$\tau_{10}=p_{20}(\tau_{20},\sigma_{20})\,.$$

Proposition 4.1. *Writing defining equations of $X_0(20)$, $X_0(40)$ as $\sigma^2=\tau^4-12\tau^3+28\tau^2-32\tau+16$, $y^2=x^8+8x^6-2x^4+8x^2+1$, respectively, we have a covering map φ_{40} from $X_0(40)$ to $X_0(20)$ as $\varphi_{40}(x,y)=(\tau,\sigma)$, where*

$$\tau=\frac{x^4-4x^3+10x^2-4x+1-y}{2(x-1)^2x},$$

$$\sigma=(x^8-4x^7+4x^6-20x^5+22x^4-20x^3+4x^2-4x+1 \\ -(x^2+1)(x^2-4x+1)y)/(2(x-1)^4x^2).$$

Proof. In the same way as in §3.1, observing the Fourier expansions of τ, σ, x and y, we obtain the relations. □

Theorem 4.1. *Writing a defining equation of $X_0(40)$ as $y^2=x^8+8x^6-2x^4+8x^2+1$, we have a covering map from $X_0(40)$ to $X_0(1)$ as follows:*

$$j=-64(3x^{24}+580x^{23}+3132x^{22}+3580x^{21}+30278x^{20}-36180x^{19} \\ +129100x^{18}-261740x^{17}+674765x^{16}-1008280x^{15}+1343352x^{14} \\ -1319400x^{13}+1405908x^{12}-1319400x^{11}+1343352x^{10}-1008280x^9 \\ +674765x^8-261740x^7+129100x^6-36180x^5+30278x^4+3580x^3 \\ +3132x^2+580x+3+2(x^{20}+300x^{19}+1470x^{18}+1100x^{17}+7405x^{16} \\ -15120x^{15}+38760x^{14}-46160x^{13}+82450x^{12}-103960x^{11}+133044x^{10} \\ -103960x^9+82450x^8-46160x^7+38760x^6-15120x^5+7405x^4 \\ +1100x^3+1470x^2+300x+1)y)^3/((x-1)^{40}(3x^4+2x^2+3-2y)^2(x^4+2x^3 \\ -2x^2+2x+1-y)^5(x^4-10x^3+14x^2-10x+1+y)).$$

Proof. By Lemma 4.1 and Proposition 4.1, we have the relation $j=p_5\circ p_{10}\circ p_{20}\circ\varphi_{40}(x,y)$. Eliminating y^2, we obtain the formula. □

4.2. The case N=48. We define as follows:

$$p_3(t)=\frac{27(t+1)(9t+1)^3}{t},$$

$$p_6(t)=\frac{t(2t+9)^2}{27(t+4)},$$

$$p_{12}(t)=\frac{t(t+6)}{2},$$

$$p_{24}(t,s)=\frac{t^2-11-s}{2}.$$

LEMMA 4.2 (Fricke). *We have the following sequence of covering maps between modular curves*:

$$\begin{array}{ccccccccc} X_0(24) & \longrightarrow & X_0(12) & \longrightarrow & X_0(6) & \longrightarrow & X_0(3) & \longrightarrow & X_0(1) \\ (\tau_{24}, \sigma_{24}) & \longmapsto & \tau_{12} & \longmapsto & \tau_6 & \longmapsto & \tau_3 & \longmapsto & j, \end{array}$$

where $\mathbf{Q}(X_0(1))=\mathbf{Q}(j)$, $\mathbf{Q}(X_0(3))=\mathbf{Q}(\tau_3)$, $\mathbf{Q}(X_0(6))=\mathbf{Q}(\tau_6)$, $\mathbf{Q}(X_0(12))=\mathbf{Q}(\tau_{12})$, *and* $\mathbf{Q}(X_0(24))=\mathbf{Q}(\tau_{24}, \sigma_{24})$ *which* (τ_{24}, σ_{24}) *satisfy the equation* ${\sigma_{24}}^2={\tau_{24}}^4-22{\tau_{24}}^2-48\tau_{24}-23$. *Moreover, the following relations hold*:

$$j=p_3(\tau_3)\,,$$

$$\tau_3=p_6(\tau_6)\,,$$

$$\tau_6=p_{12}(\tau_{12})\,,$$

$$\tau_{12}=p_{24}(\tau_{24}, \sigma_{24})\,.$$

PROPOSITION 4.2. *Writing defining equations of* $X_0(24)$, $X_0(48)$ *as* $\sigma^2=\tau^4-22\tau^2-48\tau-23$, $y^2=x^8+14x^4+1$, *respectively, we have a covering map* φ_{48} *from* $X_0(48)$ *to* $X_0(24)$ *as* $\varphi_{48}(x, y)=(\tau, \sigma)$, *where*

$$t=\frac{x^4-4x^3+10x^2-4x+1-y}{2(x-1)^2x},$$

$$s=(x^8-4x^7+4x^6-4x^5-10x^4-4x^3+4x^2-4x+1$$
$$-(x^2+1)(x^2-4x+1)y)/(2(x-1)^4x^2)\,.$$

PROOF. Similarly with §3.1, observing the Fourier expansions of τ, σ, x and y, we obtain the relations. □

THEOREM 4.2. *Writing a defining equation of* $X_0(48)$ *as* $y^2=x^8+14x^4+1$, *we have a covering map from* $X_0(48)$ *to* $X_0(1)$ *as follows*:

$$\begin{aligned} j= &-16(x^8+12x^7-36x^6+84x^5-58x^4+84x^3-36x^2+12x \\ &+1-2(x^4+6x^2+1)y)^3(x^{24}+348x^{23}-972x^{22}+5028x^{21} \\ &-11070x^{20}+44148x^{19}-94620x^{18}+256908x^{17}-415761x^{16} \\ &+874968x^{15}-1216152x^{14}+1964328x^{13}-1765732x^{12} \\ &+1964328x^{11}-1216152x^{10}+874968x^9-415761x^8+256908x^7 \\ &-94620x^6+44148x^5-11070x^4+5028x^3-972x^2+348x+1 \\ &-2(x^4+6x^2+1)(x^{16}+168x^{15}-456x^{14}+1272x^{13}-1124x^{12} \\ &+4392x^{11}-7800x^{10}+18744x^9-14010x^8+18744x^7-7800x^6 \\ &+4392x^5-1124x^4+1272x^3-456x^2+168x+1)y)^3/((x-1)^{48}(x^4+6x^2 \\ &+1-2y)^4(x^4-2x^3+6x^2-2x+1-y)^3(2x(x^2+1)-y)^3(2(x^4 \\ &-3x^3+6x^2-3x+1)-y)(x^4-6x^3+6x^2-6x+1+y))\,. \end{aligned}$$

PROOF. By Lemma 4.2 and Proposition 4.2, we have the relation $j=p_3\circ p_6\circ p_{12}\circ p_{24}\circ\varphi_{48}(x, y)$. Eliminating y^2, we obtain the formula. □

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