Existence of Solutions of Quasilinear Integrodifferential Equations with Nonlocal Condition

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Abstract. We prove the existence and uniqueness of mild and classical solutions of a quasilinear integrodifferential equation with nonlocal condition. The results are obtained by using C_0-semigroup and the Banach fixed point theorem.

1. Introduction.

The problem of existence of solutions of evolution equations with nonlocal conditions in Banach space has been studied first by Byszewski [8]. In that paper he has established the existence and uniqueness of mild, strong and classical solutions of the following nonlocal Cauchy problem:

$$u'(t) + Au(t) = f(t, u(t)), \quad t \in (0, a] \tag{1}$$

$$u(0) + g(t_1, t_2, \cdots, t_p, u(t_1), \cdots, u(t_p)) = u_0 \tag{2}$$

where $0 < t_1 < \cdots < t_p \le a$, $-A$ is the infinitesimal generator of a C_0-semigroup in a Banach space X, $u_0 \in X$ and $f : [0, a] \times X \to X$, $g : [0, a]^p \times X^p \to X$ are given functions. For example, $g(t_i, u(t_j))$ may be given by

$$g(t_1, \cdots, t_p, u(t_1), \cdots, u(t_p)) = \sum_{i=1}^{p} c_i u(t_i)$$

where c_i $(i = 1, \cdots, p)$ are given constants. In this case (2) contains the measurements at $t = 0, t_1 \cdots, t_p$, rather than just at $t = 0$. For clarity, let us consider another example. In the theory of diffusion and heat conduction one can encounter a mathematical model of the form ([9])

$$Lu + c(x, t)u = f(x, t) \quad x \in \Omega, \quad 0 < t < T,$$

$$u(x, t) = \phi(x, t) \quad x \in \partial\Omega, \quad 0 < t < T,$$

$$u(x, 0) + \sum_{k=1}^{N} \beta_k(x)u(x, t_k) = \psi(x) \quad x \in \Omega \quad \text{with } t_k \in (0, T] \quad (k = 1, \cdots, N)$$

where Ω is a bounded domain in R^n and L is a uniformly parabolic operator with continuous and bounded coefficients. It represents the diffusion phenomenon of a small amount of gas in a transparent tube. If there is very little gas at the initial time, the measurement $u(x, 0)$ of the amount of the gas in this instant may be less precise than the measurement $u(x, 0) + \sum_{k=1}^{N} \beta_k(x)u(x, t_k)$ of the sum of the amount of this gas. Let us have one more example for hyberbolic equations ([6]). Consider the following partial differential equation

$$u_{xt}(x, t) = F(x, t, u(x, t), u_x(x, t), u_t(x, t)), \quad (x, t) \in Q$$

$$u(x, 0) + \sum_{i=1}^{p} h_i(x, t_i)u(x, t_i) = \phi(x), \quad x \in [0, a]$$

$$u(0, t) = \psi(t), \quad t \in [0, a]$$

$$\psi(0) + \sum_{i=1}^{p} \psi(t_i) = \phi(0)$$

where $Q = [0, a] \times [0, a]$, t_i, $i = 1, \cdots, p$ are finite numbers such that $0 < t_1 < t_2 < \cdots < t_p \leq a$ and $F, \phi, \psi, h_i, i = 1, \cdots, p$ are given functions with appropriate assumptions. In the theory of elasticity the sum $u(x, 0) + \sum_{i=1}^{p} h_i(x, t_i)u(x, t_i)$ is more precise to measurement of a state of a vibrating system than the only one measurement $u(x, 0)$ of the state of the vibrating system. The sum may be interpreted as the sum of the $p + 1$ measurements of positions of a vibrating elastic string and the functions $h_i(x, t_i)$ can be interpreted as the properties of the medium in which the string vibrates. For more comments and references on nonlocal conditions see [4–9, 11].

Abstract quasilinear evolution equations have been studied by many authors [1, 12–15] and well applied to partial differential equations. Recently Bahuguna [2, 3], Oka [16] and Oka and Tanaka [17] discussed the existence of solutions of quasilinear integrodifferential equations in Banach spaces. An equation of this type occurs in a nonlinear conservation law with memory

$$u(t, x) + \Psi(u(t, x))_x = \int_0^t b(t - s)\Psi(u(t, x))_x ds + f(t, x), \quad t \in [0, a], \ x \in R \quad (3)$$

$$u(0, x) = \phi(x), \quad x \in R \quad (4)$$

It is clear that if nonlocal condition (2) is introduced to (3), then it will also have better effect than the classical condition $u(0, x) = \phi(x)$. Therefore, we would like to extend the results for (1)–(2) to a class of integrodifferential equations in Banach spaces.

The aim of this paper is to prove the existence and uniqueness of the mild and classical solutions of quasilinear integrodifferential equation with nonlocal conditions of the form:

$$u'(t) + A(t, u)u(t) = f\left(t, u(t), \int_0^t k(t, s, u(s))ds\right), \quad t \in [0, a] \quad (5)$$

$$u(0) + g(u) = u_0 \quad (6)$$

where $A(t, u)$ is the infinitesimal generator of a C_0-semigroup in a Banach space X, $u_0 \in X$, $f : I \times X \times X \to X$, $k : \Delta \times X \to X$ and $g : C(I : X) \to X$ are given functions. Here $I = [0, a]$ and $\Delta = \{(t, s) : 0 \leq s \leq t \leq a\}$. Equation (5) represents an abstract formulation of many kinds of partial integrodifferential equations of hyperbolic type. These types of equations arise in the study of nonlinear behaviour of elastic strings [10] and in the theory of viscoelasticity [16]. The results obtained in this paper are generalizations of the results given by Pazy [18], Kato [13] and Bahuguna [3].

2. Preliminaries.

Let X and Y be two Banach spaces such that Y is densely and continuously embedded in X. For any Banach space Z the norm of Z is denoted by $\| \cdot \|$ or $\| \cdot \|_Z$. The space of all bounded linear operators from X to Y is denoted by $B(X, Y)$ and $B(X, X)$ is written as $B(X)$. We recall some definitions and known facts from Pazy [18].

DEFINITION 2.1. Let S be a linear operator in X and let Y be a subspace of X. The operator $\tilde{S}$ defined by $D(\tilde{S}) = \{x \in D(S) \cap Y : Sx \in Y\}$ and $\tilde{S}x = Sx$ for $x \in D(\tilde{S})$ is called the part of S in Y.

DEFINITION 2.2. Let B be a subset of X and for every $0 \leq t \leq a$ and $b \in B$, let $A(t, b)$ be the infinitesimal generator of a C_0 semigroup $S_{t,b}(s)$, $s \geq 0$, on X. The family of operators $\{A(t, b)\}$, $(t, b) \in I \times B$, is stable if there are constants $M \geq 1$ and ω such that

$$\rho(A(t, b)) \supset (\omega, \infty) \quad \text{for } (t, b) \in I \times B,$$

$$\left\| \prod_{j=1}^{k} R(\lambda : A(t_j, b_j)) \right\| \leq M(\lambda - \omega)^{-k}$$

for $\lambda > \omega$ and every finite sequences $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq a, b_j \in B, 1 \leq j \leq k$. The stability of $\{A(t, b)\}$, $(t, b) \in I \times B$ implies (see [18]) that

$$\left\| \prod_{j=1}^{k} S_{t_j, b_j}(s_j) \right\| \leq M \exp \left\{ \omega \sum_{j=1}^{k} s_j \right\}, \quad s_j \geq 0$$

and any finite sequences $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq a, b_j \in B, 1 \leq j \leq k, k = 1, 2, \cdots$.

DEFINITION 2.3. Let $S_{t,b}(s)$, $s \geq 0$ be the C_0-semigroup generated by $A(t, b)$, $(t, b) \in I \times B$. A subspace Y of X is called $A(t, b)$-admissible if Y is invariant subspace of $S_{t,b}(s)$, and the restriction of $S_{t,b}(s)$ to Y is a C_0-semigroup in Y.

Let $B \subset X$ be a subset of X such that for every $(t, b) \in I \times B$, $A(t, b)$ is the infinitesimal generator of C_0 semigroup $S_{t,b}(s)$, $s \geq 0$ on X. We make the following assumptions:

(E$_1$)   The family $\{A(t, b)\}$, $(t, b) \in I \times B$ is stable.
(E$_2$) Y is $A(t, b)$-admissible for $(t, b) \in I \times B$ and the family $\{\tilde{A}(t, b)\}$, $(t, b) \in I \times B$ of parts $\tilde{A}(t, b)$ of $A(t, b)$ in Y, is stable in Y.

(E$_3$) For $(t, b) \in I \times B$, $D(A(t, b)) \supset Y$, $A(t, b)$ is a bounded linear operator from Y to X and $t \to A(t, b)$ is continuous in the $B(Y, X)$ norm $\| \cdot \|$ for every $b \in B$.

(E$_4$) There is a constant $L > 0$ such that

$$\|A(t, b_1) - A(t, b_2)\|_{Y \to X} \leq L \|b_1 - b_2\|_X$$

holds for every $b_1, b_2 \in B$ and $0 \leq t \leq a$.

Let $B \subset X$ and $\{A(t, b)\}$, $(t, b) \in I \times B$ be a family of operators satisfying the conditions (E$_1$)–(E$_4$). If $u \in C(I : X)$ has values in B then there is a unique evolution system $U(t, s; u)$, $0 \leq s \leq t \leq a$, in X satisfying (see Theorem 5.3.1 and Lemma 6.4.2 in [18] pp. 135, 201–202)

(i) $\|U(t, s; u)\| \leq M e^{\omega(t-s)}$ for $0 \leq s \leq t \leq a$

where M and ω are stability constants.

(ii) $\dfrac{\partial^+}{\partial t} U(t, s; u)w = A(s, u(s))U(t, s; u)w$ for $w \in Y$, $0 \leq s \leq t \leq a$.

(iii) $\dfrac{\partial}{\partial s} U(t, s; u)w = -U(t, s; u)A(s, u(s))w$ for $w \in Y$, $0 \leq s \leq t \leq a$.

(E$_5$) For every $u \in C(I : X)$ satisfying $u(t) \in B$ for $0 \leq t \leq a$, we have

$$U(t, s; u)Y \subset Y, \quad 0 \leq s \leq t \leq a$$

and $U(t, s; u)$ is strongly continuous in Y for $0 \leq s \leq t \leq a$.

(E$_6$) Y is reflexive.

(E$_7$) For every $(t, b_1, b_2) \in I \times B \times B$, $f(t, b_1, b_2) \in Y$.

For a *mild solution* of (5)–(6) we mean a function $u \in C(I : X)$ with values in B and $u_0 \in X$ satisfying the integral equation

$$u(t) = U(t, 0; u)u_0 - U(t, 0; u)g(u) + \int_0^t U(t, s; u)f\left(s, u(s), \int_0^s k(s, \tau, u(\tau))d\tau\right)ds .$$

$$(7)$$

A function $u \in C(I : X)$ such that $u(t) \in D(A(t, u(t)))$ for $t \in (0, a]$, $u \in C^1((0, a] : X)$ and satisfies (5)–(6) in X is called a *classical solution* of (5)–(6) on I. Further there exists a constant $K > 0$ such that for every $u, v \in C(I : X)$ with values in B and every $w \in Y$ we have

$$\|U(t, s; u)w - U(t, s; u)w\| \leq K\|w\|_Y \int_s^t \|u(\tau) - v(\tau)\|d\tau .$$

$$(8)$$

Further we assume that

(E$_8$) $g : C(I : B) \to X$ is Lipschitz continuous in X and bounded in Y, that is, there exist constants $G > 0$ and $G_1 > 0$ such that

$$\|g(u)\|_Y \leq G ,$$

$$\|g(u) - g(v)\|_X \leq G_1 \max_{t \in I} \|u(t) - v(t)\|_X .$$

For the conditions (E$_9$) and (E$_{10}$) let Z be taken as both X and Y.

(E9) $k : \Delta \times Z \to Z$ is continuous and there exist constants $K_1 > 0$ and $K_2 > 0$ such that

$$\|k(t, s, u_1) - k(t, s, u_2)\|_Z \le K_1 \|u_1 - u_2\|_Z ,$$

$$K_2 = \max\{\|k(t, s, 0)\|_Z : (t, s) \in \Delta\} .$$

(E10) $f : I \times Z \times Z \to Z$ is continuous and there exist constants $F_1 > 0$ and $F_2 > 0$ such that

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\|_Z \le F_1(\|u_1 - v_1\|_Z + \|u_2 - v_2\|_Z) ,$$

$$F_2 = \max_{t \in I} \|f(t, 0, 0)\|_Z .$$

Let us take

$$M_0 = \max\{\|U(t, s; u)\|_{B(Z)}, 0 \le s \le t \le a, u \in B\} .$$

(E11) $M_0(\|u_0\|_Y + G) + M_0 F_1 ar + M_0 F_2 a + M_0 F_1 K_1 a^2 r + M_0 F_1 K_2 a^2 \le r$ and
$q = Ka\|u_0\|_Y + GKa + M_0 G_1 + M_0 F_1 K_1 a^2 + M_0 F_1 a + Ka(F_1 r + F_1 K_1 ar + F_1 K_2 a + F_2) < 1.$

Next we prove the existence of local classical solutions of the quasilinear problem (5)–(6).

3. Existence Theorem.

THEOREM.   *Let $u_0 \in Y$ and let $B = \{u \in X : \|u\|_Y \le r\}, r > 0$. If the assumptions (E1)–(E11) are satisfied, then the quasilinear problem (5)–(6) has a unique classical solution $u \in C([0, a] : Y) \cap C^1((0, a] : X)$.*

PROOF.   Let S be a nonempty closed subset of $C([0, a] : X)$ defined by

$$S = \{u : u \in C([0, a] : X), \|u(t)\|_Y \le r \text{ for } 0 \le t \le a\} .$$

Consider a mapping P on S defined by

$$(Pu)(t) = U(t, 0; u)u_0 - U(t, 0; u)g(u) + \int_0^t U(t, s; u)f\left(s, u(s), \int_0^s k(s, \tau, u(\tau))d\tau\right) ds .$$

We claim that P maps S into S. For $u \in S$, we have

$\|Pu(t)\|_Y$

$$= \left\| U(t, 0; u)u_0 - U(t, 0; u)g(u) \right.$$

$$\left. + \int_0^t U(t, s; u)f\left(s, u(s), \int_0^s k(s, \tau, u(\tau))d\tau\right) ds \right\|$$

$$\le \|U(t, 0; u)u_0\| + \|U(t, 0; u)g(u)\|$$

$$+ \int_0^t \left\| U(t, s; u)f\left(s, u(s), \int_o^s k(s, \tau, u(\tau))d\tau\right) \right\| ds$$

$$\le \|U(t, 0; u)u_0\| + \|U(t, 0; u)g(u)\|$$

$$+ \int_0^t \left\| U(t, s; u)\left[f\left(s, u(s), \int_0^s k(s, \tau, u(\tau))d\tau\right) - f(s, 0, 0) + f(s, 0, 0)\right] \right\| ds$$

$$\leq M_0\|u_0\|_Y + M_0 G + M_0 F_1 ar + M_0 F_1 K_1 a^2 r + M_0 F_1 K_2 a^2 + M_0 F_2 a$$

$$\leq r .$$

Therefore P maps S into itself. Moreover, if $u, v \in S$, then

$$\|Pu(t) - Pv(t)\|$$

$$\leq \|U(t, 0; u)u_0 - U(t, 0; v)u_0\| + \|U(t, 0; u)g(u) - U(t, 0; v)g(v)\|$$

$$+ \int_0^t \left\| U(t, s; u)f\left(s, u(s), \int_0^s k(s, \tau, u(\tau))d\tau\right)\right.$$

$$\left. - U(t, s; v)f\left(s, v(s), \int_0^s k(s, \tau, v(\tau))d\tau\right)\right\| ds$$

$$\leq \|U(t, 0; u)u_0 - U(t, 0; v)u_0\| + \|U(t, 0; u)g(u) - U(t, 0; v)g(u)\|$$

$$+ \|U(t, 0; v)g(u) - U(t, 0; v)g(v)\|$$

$$+ \int_0^t \left[\left\| U(t, s; u)f\left(s, u(s), \int_0^s k(s, \tau, u(\tau))d\tau\right)\right.\right.$$

$$\left. - U(t, s; v)f\left(s, u(s), \int_0^s k(s, \tau, u(\tau))d\tau\right)\right\|$$

$$+ \left\| U(t, s; v)f\left(s, u(s), \int_0^s k(s, \tau, u(\tau))d\tau\right)\right.$$

$$\left.\left. - U(t, s; v)f\left(s, v(s), \int_0^s k(s, \tau, v(\tau))d\tau\right)\right\|\right] ds$$

$$\leq Ka\|u_0\|_Y \max_{\tau\in I} \|u(\tau) - v(\tau)\|$$

$$+ GKa \max_{\tau\in I} \|u(\tau) - v(\tau)\| + M_0 G_1 \max_{\tau\in I} \|u(\tau) - v(\tau)\|$$

$$+ Ka(F_1 r + F_1 K_1 ar + F_1 K_2 a + F_2) \max_{\tau\in I} \|u(\tau) - v(\tau)\|$$

$$+ M_0 F_1 a \left(\max_{\tau\in I} \|u(\tau) - v(\tau)\|\right) + M_0 F_1 K_1 a^2 \max_{\tau\in I} \|u(\tau) - v(\tau)\|$$

$$\leq [Ka\|u_0\|_Y + GKa + M_0 G_1 + M_0 F_1 K_1 a^2 + M_0 F_1 a$$

$$+ Ka(F_1 r + F_1 K_1 ar + F_1 K_2 a + F_2)] \max_{\tau\in I} \|u(\tau) - v(\tau)\|$$

$$= q \max_{\tau\in I} \|u(\tau) - v(\tau)\|, \quad \text{where} \quad 0 < q < 1 .$$

From this inequality it follows that for any $t \in I$

$$\|Pu(t) - Pv(t)\| \leq q \max_{\tau\in I} \|u(\tau) - v(\tau)\|$$

so that P is a contraction on S. From the contraction mapping theorem it follows that P has a unique fixed point $u \in S$ which is the mild solution of (5)–(6) on $[0, a]$. Note that $u(t)$ is in $C(I : Y)$ by (E$_6$) (see [14] Lemma 7.4). In fact, $u(t)$ is weakly continuous as a Y-valued function. This implies that $u(t)$ is separably valued in Y, hence it is strongly

measurable. Then, $\|u(t)\|_Y$ is a bounded and measurable function in t. Therefore, $u(t)$ is Bochner integrable (see e.g. [19] Chap. V §§4–5). Using relation $u(t) = Pu(t)$, we conclude that $u(t)$ is in $C(I : Y)$.

Now, we consider the evolution equation

$$v'(t) + B(t)v(t) = h(t), \quad t \in [0, a] \tag{9}$$

$$v(0) = u_0 - g(u) \tag{10}$$

where $B(t) = A(t, u(t))$ and $h(t) = f(t, u(t), \int_0^t k(t, s, u(s))ds)$, $t \in [0, a]$ and u is the unique fixed point of P in S. We note that $B(t)$ satisfies (H$_1$)–(H$_3$) of [18] (Section 5.5.3) and $h \in C(I, Y)$. Theorem 5.5.2 in [18] implies that there exists a unique function $v \in C(I, Y)$ such that $v \in C^1((0, a], X)$ satisfying (9) and (10) in X and v is given by

$$v(t) = U(t, 0; u)u_0 - U(t, 0; u)g(u) + \int_0^t U(t, s; u)f\left(s, u(s), \int_0^s k(s, \tau, u(\tau))d\tau\right)ds$$

where $U(t, s; u)$ is the evolution system generated by the family $\{A(t, u(t))\}$, $t \in I$ of the linear operators in X. The uniqueness of v implies that $v = u$ on I and hence u is a classical solution of (5)–(6) and $u \in C([0, a] : Y) \cap C^1((0, a] : X)$.

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