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Reproducing Kernels Related to the Complex Sphere

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We shall introduce four Hilbert spaces with a reproducing kernel and study relations among them. These Hilbert spaces are related to Fourier transforms of analytic functionals on the complex sphere.

Introduction.

Put $\mathbf{E} = \mathbf{R}^{n+1}$ and $\tilde{\mathbf{E}} = \mathbf{C}^{n+1}$ $(n \geq 2)$. For $z, \zeta \in \tilde{\mathbf{E}}$ we put $z \cdot \zeta = z_1\zeta_1 + z_2\zeta_2 + \cdots + z_{n+1}\zeta_{n+1}$. Let $L(z) = \sqrt{\|z\|^2 + \sqrt{\|z\|^4 - |z^2|^2}}$ be the Lie norm on $\tilde{\mathbf{E}}$, where $\|z\|^2 = z \cdot \bar{z}$ and $z^2 = z \cdot z$. We denote the closed Lie ball of radius $r > 0$ by $\tilde{B}[r] = \{z \in \tilde{\mathbf{E}}; L(z) \leq r\}$ and the complex sphere of radius $\lambda \in \mathbf{C}$ by $\tilde{\mathbf{S}}_\lambda = \{z \in \tilde{\mathbf{E}}; z^2 = \lambda^2\}$ (see Hua [6], Morimoto [8] and [12] for the Lie ball and the Lie sphere). Especially for $\lambda = 0$ sometime we call $\tilde{\mathbf{S}}_0$ the complex light cone. For $|\lambda| \leq r$ put $\tilde{\mathbf{S}}_\lambda[r] = \tilde{\mathbf{S}}_\lambda \cap \tilde{B}[r]$. If $\lambda \neq 0$, then

$$\tilde{\mathbf{S}}_\lambda[r] = \{z \in \tilde{\mathbf{E}}; z^2 = \lambda^2, \|\mathrm{Im}(z/\lambda)\| \leq (r^2 - |\lambda|^2)/(2r|\lambda|)\}.$$

If $|\lambda| < r$, then $\tilde{\mathbf{S}}_\lambda[r]$ is a complex variety of complex dimension n with boundary. If $|\lambda| = r$, then $\tilde{\mathbf{S}}_\lambda[r]$ reduces to the real sphere of complex radius λ : $\mathbf{S}_\lambda = \lambda\mathbf{S}_1$, where $\mathbf{S}_1 = \{x \in \mathbf{E}; \|x\| = 1\}$. For $|\lambda| \leq r$ we put $\tilde{\mathbf{S}}_{\lambda,r} = \partial\tilde{\mathbf{S}}_\lambda[r] = \{z \in \tilde{\mathbf{S}}_\lambda; L(z) = r\}$. If $|\lambda| < r$, then $\tilde{\mathbf{S}}_{\lambda,r}$ is a compact real analytic manifold of real dimension $2n-1$. If $|\lambda| = r$, then $\tilde{\mathbf{S}}_{\lambda,r} = \tilde{\mathbf{S}}_\lambda[r] = \mathbf{S}_\lambda$. The rotation group acts transitively on $\tilde{\mathbf{S}}_{\lambda,r}$ and there is a unique normalized invariant measure on $\tilde{\mathbf{S}}_{\lambda,r}$.

We denote by $\mathcal{O}_\Delta(\tilde{B}[r])$ the space of germs of complex harmonic functions on $\tilde{B}[r]$. In §1 we consider the sesquilinear form on $\mathcal{O}_\Delta(\tilde{B}[r])$ defined by

$$(f, g)_{\tilde{\mathbf{S}}_{\lambda,r}} = \int_{\tilde{\mathbf{S}}_{\lambda,r}} f(z)\overline{g(z)}\dot{d}z\,,$$

where $\dot{d}z$ denotes the normalized invariant measure on $\tilde{\mathbf{S}}_{\lambda,r}$. We shall show that $(f, g)_{\tilde{\mathbf{S}}_{\lambda,r}}$ is an inner product on $\mathcal{O}_\Delta(\tilde{B}[r])$. We denote by $\mathfrak{h}^2_\lambda(\tilde{B}(r))$ the completion of $\mathcal{O}_\Delta(\tilde{B}[r])$ with

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respect to the inner product $(\ ,\ )_{\tilde{\mathbf{S}}_{\lambda,r}}$. If $|\lambda| = r$, then we denote $\mathfrak{h}^2(\tilde{B}(r)) = \mathfrak{h}^2_\lambda(\tilde{B}(r))$. The Poisson kernel $K_{\lambda,r}(z, w)$ is a reproducing kernel for the Hilbert space $\mathfrak{h}^2_\lambda(\tilde{B}(r))$.

We have the following inclusion relations (Theorem 1.5):

$$\mathcal{O}_\Delta(\tilde{B}[r]) \subset \mathfrak{h}^2(\tilde{B}(r)) \subset \mathfrak{h}^2_\lambda(\tilde{B}(r)) \subset \mathfrak{h}^2_0(\tilde{B}(r)) \subset \mathcal{O}_\Delta(\tilde{B}(r)) .$$

Let $\tilde{\mathbf{S}}_\lambda(r) = \tilde{\mathbf{S}}_\lambda \cap \tilde{B}(r)$, $|\lambda| < r$, be the open truncated complex sphere. We denote by $\mathcal{O}(\tilde{\mathbf{S}}_\lambda(r))$ the space of holomorphic functions on $\tilde{\mathbf{S}}_\lambda(r)$. In §2 we introduce the Hardy space $H^2(\tilde{\mathbf{S}}_\lambda(r))$ as a subspace of $\mathcal{O}(\tilde{\mathbf{S}}_\lambda(r))$ in such a way that the restriction mapping α^0_λ : $\mathfrak{h}^2_\lambda(\tilde{B}(r)) \to H^2(\tilde{\mathbf{S}}_\lambda(r))$ is a unitary isomorphism (Theorem 2.3). The restriction $\tilde{K}_{\lambda,r}(z, w)$ of $K_{\lambda,r}(z, w)$ to $\tilde{\mathbf{S}}_\lambda(r) \times \tilde{\mathbf{S}}_\lambda(r)$ is called the Cauchy kernel for $\tilde{\mathbf{S}}_\lambda(r)$. It is a reproducing kernel for $H^2(\tilde{\mathbf{S}}_\lambda(r))$.

In §§3 and 4 we shall review some results on harmonic functionals on the Lie ball and analytic functionals on the complex sphere.

In §5 we define the conical Fourier transform of $f \in \mathfrak{h}^2_\lambda(\tilde{B}(r))$ by

$$\mathcal{F}^\Delta_{\lambda,r} f(\zeta) = \int_{\tilde{\mathbf{S}}_{\lambda,r}} \exp(z \cdot \zeta)\overline{f(z)}\dot{d}z\,, \quad \zeta \in \tilde{\mathbf{S}}_0\,,$$

and denote by $\mathcal{E}^2(\tilde{\mathbf{S}}_0; \lambda, r)$ the image of $\mathfrak{h}^2_\lambda(\tilde{B}(r))$ under $\mathcal{F}^\Delta_{\lambda,r}$. If $|\lambda| = r$, then we denote $\mathcal{E}^2(\tilde{\mathbf{S}}_0; r) = \mathcal{E}^2(\tilde{\mathbf{S}}_0; \lambda, r)$. We introduce an inner product on $\mathcal{E}^2(\tilde{\mathbf{S}}_0; \lambda, r)$ in such a way that the conical Fourier transformation $\mathcal{F}^\Delta_{\lambda,r}$ is an antilinear unitary transformation from $\mathfrak{h}^2_\lambda(\tilde{B}(r))$ onto $\mathcal{E}^2(\tilde{\mathbf{S}}_0; \lambda, r)$. We shall construct the F-Poisson kernel $E^\lambda_{r,0}(\zeta, \xi)$, which is a reproducing kernel for the Hilbert space $\mathcal{E}^2(\tilde{\mathbf{S}}_0; \lambda, r)$. We shall prove the following relation (Theorem 6.4):

$$\mathrm{Exp}(\tilde{\mathbf{S}}_0; [r]) \subset \mathcal{E}^2(\tilde{\mathbf{S}}_0; 0, r) \subset \mathcal{E}^2(\tilde{\mathbf{S}}_0; \lambda, r) \subset \mathcal{E}^2(\tilde{\mathbf{S}}_0; r) \subset \mathrm{Exp}(\tilde{\mathbf{S}}_0; (r)) .$$

In §7 we review results on spaces of eigenfunctions of the Laplacian of exponential type and define the Fourier transformation $\mathcal{F}_{\lambda,r}$ for the Hardy space $H^2(\tilde{\mathbf{S}}_\lambda(r))$. We denote the image of the Hardy space $H^2(\tilde{\mathbf{S}}_\lambda(r))$ under the Fourier transformation $\mathcal{F}_{\lambda,r}$ by $\mathcal{E}^2_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; r)$.

In the last section §8, we shall construct the F-Cauchy kernel $E^\lambda_r(\zeta, \xi)$, which is a reproducing kernel for the Hilbert space $\mathcal{E}^2_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; r)$. The F-Poisson kernel $E^\lambda_{r,0}(\zeta, \xi)$ is the restriction of $E^\lambda_r(\zeta, \xi)$ to $\tilde{\mathbf{S}}_0 \times \tilde{\mathbf{S}}_0$.

The relations among our four Hilbert spaces can be summarized as the following commutative diagram (Theorem 8.1):

$$\begin{array}{ccc} \mathfrak{h}^2_\lambda(\tilde{B}(r)) & \xrightarrow{\mathcal{F}^\Delta_{\lambda,r}} & \mathcal{E}^2(\tilde{\mathbf{S}}_0; \lambda, r) \\ \downarrow{\scriptstyle \alpha^0_\lambda} & & \uparrow{\scriptstyle \beta^\lambda_0} \\ H^2(\tilde{\mathbf{S}}_\lambda(r)) & \xrightarrow{\mathcal{F}_{\lambda,r}} & \mathcal{E}^2_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; r)\,, \end{array}$$

where α^0_λ and β^λ_0 are the restriction mappings. The Hilbert spaces $H^2(\tilde{\mathbf{S}}_\lambda(r))$ and $\mathcal{E}^2_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; r)$ are discussed in Fujita [4].

In our previous papers Morimoto-Fujita [14], [15] and [16] we considered the bilinear form

$$\langle f, g\rangle_{\tilde{\mathbf{S}}_{\lambda,r}} = \int_{\tilde{\mathbf{S}}_{\lambda,r}} f(z)g(\bar{z})\dot{d}z$$

on $\mathcal{O}(\tilde{\mathbf{S}}_\lambda[r])$, but this is well-defined only for $\lambda \in \mathbf{R}$. For a general complex parameter λ we should consider the sesquilinear form $(f, g)_{\tilde{\mathbf{S}}_{\lambda,r}}$. Accordingly, the Poisson transformation and the Cauchy transformation are redefined to be antilinear mappings in this paper.

1. Complex harmonic functions.

Let $\tilde{B}(r) = \{z \in \tilde{\mathbf{E}}; L(z) < r\}$ be the open Lie ball of radius r. We denote by $\mathcal{O}(\tilde{B}(r))$ the space of holomorphic funtions on $\tilde{B}(r)$ and by $\mathcal{O}_\Delta(\tilde{B}(r)) = \{f \in \mathcal{O}(\tilde{B}(r)); \Delta_z f = 0\}$ the space of complex harmonic functions on $\tilde{B}(r)$, where

$$\Delta_z = \frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} + \cdots + \frac{\partial^2}{\partial z_{n+1}^2}$$

is the complex Laplacian. Equipped with the topology of uniform convergence on compact sets, the spaces $\mathcal{O}(\tilde{B}(r))$ and $\mathcal{O}_\Delta(\tilde{B}(r))$ are Fréchet-Schwartz spaces (FS spaces, for short), and $\mathcal{O}_\Delta(\tilde{B}(r))$ is a closed subspace of $\mathcal{O}(\tilde{B}(r))$ (for FS spaces and DFS spaces (dual Fréchet-Schwartz spaces), see, for example, Morimoto [10]).

Denote by $\mathcal{P}^k_\Delta(\tilde{\mathbf{E}})$ the space of k-homogeneous harmonic polynomials and by $N(k, n) = \dim \mathcal{P}^k_\Delta(\tilde{\mathbf{E}})$ the dimension of $\mathcal{P}^k_\Delta(\tilde{\mathbf{E}})$. We know

$$N(k, n) = \frac{(2k + n - 1)(k + n - 2)!}{k!(n - 1)!} = O(k^{n-1}).$$

Let $P_{k,n}(t)$ be the Legendre polynomial of degree k and of dimension $n + 1$ and $\gamma_{k,n}$ the principal coefficient of $P_{k,n}(t)$:

$$\gamma_{k,n} = \frac{2^k \Gamma(k + (n + 1)/2)}{k!\Gamma((n + 1)/2)N(k, n)}.$$

In our previous papers Fujita [2], [3], [4], Fujita-Morimoto [5], Morimoto [8], [9], [11], [12], Morimoto-Fujita [13], [14], [15] and [16], we defined the extended Legendre polynomial by

$$\tilde{P}_{k,n}(z, w) = (\sqrt{z^2})^k(\sqrt{w^2})^k P_{k,n}\left(\frac{z}{\sqrt{z^2}} \cdot \frac{w}{\sqrt{w^2}}\right).$$

We have $\Delta_z \tilde{P}_{k,n}(z, w) = \Delta_w \tilde{P}_{k,n}(z, w) = 0$ and $\tilde{P}_{k,n}(z, w) = \tilde{P}_{k,n}(w, z)$. $\tilde{P}_{k,n}(z, w)$ is a k-homogeneous polynomial in z and in w. Note that

$$\tilde{P}_{k,n}(z, w) = \gamma_{k,n}(z \cdot w)^k \quad \text{for} \quad z^2 = 0 \ \text{or} \ w^2 = 0.$$

In this paper, we shall use the following notation:

$$P_{k,n}(z, w) = \tilde{P}_{k,n}(z, \bar{w}).$$

The two-variable function $P_{k,n}(z, w)$ is a reproducing kernel for the finite dimensional space $\mathcal{P}^k_\Delta(\tilde{\mathbf{E}})$. The following theorem is due to Wada [17] and [18].

THEOREM 1.1. (i) *If $f_k \in \mathcal{P}^k_\Delta(\tilde{\mathbf{E}})$, $f_l \in \mathcal{P}^l_\Delta(\tilde{\mathbf{E}})$, and $k \neq l$, then we have the orthogonality*:

$$\int_{\tilde{\mathbf{S}}_{\lambda,r}} f_k(z)\overline{f_l(z)}\dot{d}z = 0,$$

where $\dot{d}z$ denotes the normalized invariant measure on $\tilde{\mathbf{S}}_{\lambda,r}$.

(ii) *If $f_k, g_k \in \mathcal{P}^k_\Delta(\tilde{\mathbf{E}})$, then we have*

$$\int_{\tilde{\mathbf{S}}_{\lambda,r}} f_k(z)\overline{g_k(z)}\dot{d}z = L_{k,\lambda,r}\int_{\mathbf{S}_1} f_k(x)\overline{g_k(x)}\dot{d}x,$$

where

$$L_{k,\lambda,r} = \begin{cases} r^{2k}, & |\lambda| = r, \\ |\lambda|^{2k} P_{k,n}((r^2/|\lambda|^2 + |\lambda|^2/r^2)/2) & 0 < |\lambda| < r, \\ 2^{-k}\gamma_{k,n} r^{2k}, & \lambda = 0. \end{cases} \tag{1}$$

Note that $L_{k,\lambda,r}$ is continuous with respect to λ and r.

(iii) *If $f_k \in \mathcal{P}^k_\Delta(\tilde{\mathbf{E}})$, then we have the reproducing formula*:

$$\begin{aligned} f_k(z) &= \frac{N(k,n)}{L_{k,\lambda,r}}\int_{\tilde{\mathbf{S}}_{\lambda,r}} f_k(w)P_{k,n}(z,w)\dot{d}w \\ &= N(k,n)\int_{\mathbf{S}_1} f_k(x)P_{k,n}(z,x)\dot{d}x. \end{aligned}$$

COROLLARY 1.2.

$$\begin{aligned} P_{k,n}(z,w) &= \frac{N(k,n)}{L_{k,\lambda,r}}\int_{\tilde{\mathbf{S}}_{\lambda,r}} P_{k,n}(z,w')P_{k,n}(w',w)\dot{d}w' \\ &= 2^k\gamma_{k,n}N(k,n)\int_{\tilde{\mathbf{S}}_{0,1}} (z\cdot\overline{w'})^k(w'\cdot\bar{w})^k\dot{d}w'. \end{aligned}$$

Suppose $f \in \mathcal{O}_\Delta(\tilde{B}(r))$ is given. Then

$$f_k(z) = \frac{N(k,n)}{L_{k,\lambda,r'}}\int_{\tilde{\mathbf{S}}'_{\lambda,r}} f(w)P_{k,n}(z,w)\dot{d}w \tag{2}$$

belongs to $\mathcal{P}^k_\Delta(\tilde{\mathbf{E}})$ and does not depend on λ, r' with $|\lambda| \le r' < r$. We call (2) the k-homogeneous harmonic component of $f \in \mathcal{O}_\Delta(\tilde{B}(r))$. We have the following theorem (Morimoto [9], Theorem 5.2):

THEOREM 1.3. *Let $f \in \mathcal{O}_\Delta(\tilde{B}(r))$ and $f_k \in \mathcal{P}^k_\Delta(\tilde{\mathbf{E}})$ be the k-homogeneous harmonic component of f. Then*

$$f(z) = \sum_{k=0}^{\infty} f_k(z), \quad z \in \tilde{B}(r) \tag{3}$$

converges uniformly on compact sets of $\tilde{B}(r)$ and we have

$$\limsup_{k\to\infty} \sqrt[k]{\|f_k\|_{\mathbf{S}_1}} \le \frac{1}{r}, \tag{4}$$

where $\| f_k \|_{\mathbf{S}_1}$ *is the* L^2 *norm on* $\mathbf{S}_1$.

Conversely, if a sequence $\{f_k\}$ *of homogeneous harmonic polynomials* $f_k \in \mathcal{P}^k_\Delta(\tilde{\mathbf{E}})$ *satisfies* (4), *then the right-hand side of* (3) *converges uniformly on compact sets of* $\tilde{B}(r)$ *and* $f(z)$ *is a complex harmonic function on* $\tilde{B}(r)$.

Put

$$\mathcal{O}(\tilde{B}[r]) = \lim\operatorname{ind}_{r'>r}\mathcal{O}(\tilde{B}(r')) \quad \text{and} \quad \mathcal{O}_\Delta(\tilde{B}[r]) = \lim\operatorname{ind}_{r'>r}\mathcal{O}_\Delta(\tilde{B}(r'))$$

and equip them with the locally convex inductive limit topology. The spaces $\mathcal{O}(\tilde{B}[r])$ and $\mathcal{O}_\Delta(\tilde{B}[r])$ are DFS spaces, and $\mathcal{O}_\Delta(\tilde{B}[r])$ is a closed subspace of $\mathcal{O}(\tilde{B}[r])$. For $f,\ g \in \mathcal{O}_\Delta(\tilde{B}[r])$ we define

$$(g, f)_{\tilde{\mathbf{S}}_{\lambda,r}} = \int_{\tilde{\mathbf{S}}_{\lambda,r}} g(w)\overline{f(w)}\dot{d}w\,.$$

$(g, f)_{\tilde{\mathbf{S}}_{\lambda,r}}$ is a sesquilinear form on $\mathcal{O}_\Delta(\tilde{B}[r])$. Let f (resp. g) $\in \mathcal{O}_\Delta(\tilde{B}[r])$ and f_k (resp. g_k) be the k-homogeneous harmonic component of f (resp. g). Then $f(z) = \sum_{k=0}^\infty f_k(z)$ and $g(z) = \sum_{k=0}^\infty g_k(z)$ converge uniformly on $\tilde{\mathbf{S}}_{\lambda,r}$. By Theorem 1.1 we have

$$\begin{aligned}(g, f)_{\tilde{\mathbf{S}}_{\lambda,r}} &= \int_{\tilde{\mathbf{S}}_{\lambda,r}} g(z)\overline{f(z)}\dot{d}z = \int_{\tilde{\mathbf{S}}_{\lambda,r}} \sum_{k=0}^\infty g_k(z)\sum_{l=0}^\infty \overline{f_l(z)}\dot{d}z \\ &= \sum_{k=0}^\infty \int_{\tilde{\mathbf{S}}_{\lambda,r}} g_k(z)\overline{f_k(z)}\dot{d}z = \sum_{k=0}^\infty L_{k,\lambda,r}(g_k, f_k)_{\mathbf{S}_1}\,.\end{aligned}$$

This implies $(g, f)_{\tilde{\mathbf{S}}_{\lambda,r}}$ is an inner product on $\mathcal{O}_\Delta(\tilde{B}[r])$.

Let $g \in \mathcal{O}_\Delta(\tilde{B}[r])$ and $f \in \mathcal{O}_\Delta(\tilde{B}(r))$. Then there is $t > 1$ such that $f(t^{-1}z)$ and $g(tz)$ belong to $\mathcal{O}_\Delta(\tilde{B}[r])$ as functions in z. Put

$$I_t = \int_{\tilde{\mathbf{S}}_{\lambda,r}} g(tz)\overline{f(t^{-1}z)}\dot{d}z\,.$$

Then $I_t = \sum_{k=0}^\infty L_{k,\lambda,r}(g_k, f_k)_{\mathbf{S}_1}$ and it is independent of $t > 1$ sufficiently close to 1. We call I_t the symbolic integral form on $\tilde{\mathbf{S}}_{\lambda,r}$ and denote it by

$$\text{s.}\int_{\tilde{\mathbf{S}}_{\lambda,r}} g(z)\overline{f(z)}\dot{d}z\,.$$

By the definition, it is a separately continuous sesquilinear form on $\mathcal{O}_\Delta(\tilde{B}[r]) \times \mathcal{O}_\Delta(\tilde{B}(r))$.

Define the Poisson kernel $K_{\lambda,r}(z, w)$ by

$$K_{\lambda,r}(z, w) = \sum_{k=0}^\infty \frac{N(k, n)}{L_{k,\lambda,r}} P_{k,n}(z, w)\,.$$

$K_{\lambda,r}(z, w)$ is holomorphic in z and antiholomorphic in w on

$$\Omega_r = \{(z, w) \in \tilde{\mathbf{E}} \times \tilde{\mathbf{E}};\ L(z)L(w) < r^2\}$$

and satisfies $\Delta_z K_{\lambda,r}(z,w) = 0$ and $K_{\lambda,r}(z,w) = \overline{K_{\lambda,r}(w,z)} = K_{\lambda,r}(\bar{w},\bar{z})$. We have the Poisson integral representation formula for $f \in \mathcal{O}_\Delta(\tilde{B}(r))$:

$$f(z) = \text{s.}\int_{\tilde{S}_{\lambda,r}} f(w)K_{\lambda,r}(z,w)\dot{d}w\,, \quad z \in \tilde{B}(r)\,.$$

Note that $K_{\lambda,r}(z,w)$ reduces to the classical Poisson kernel when $|\lambda| = r$.

LEMMA 1.4. *If* $0 < |\lambda| < |\mu| < r$ *and* $k \in \mathbf{N}$, *then we have*

$$2^{-k}\gamma_{k,n}r^{2k} < L_{k,\lambda,r} < L_{k,\mu,r} < r^{2k}\,. \tag{5}$$

PROOF. Put $a(s) = s^k P_{k,n}((s+1/s)/2)$ for $s > 0$. Then by the Laplace representation formula

$$P_{k,n}(t) = \frac{\Gamma(n/2)}{\sqrt{\pi}\,\Gamma((n-1)/2)}\int_{-1}^{1}(t + x\sqrt{t^2-1})^k(1-x^2)^{(n-3)/2}dx \quad \text{for} \quad t > 1\,,$$

for $0 < s \leq 1$ we have

$$\begin{aligned} a(s) &= \frac{\Gamma(n/2)}{\sqrt{\pi}\,\Gamma((n-1)/2)}\int_{-1}^{1} 2^{-k}((1+s^2) + x(1-s^2))^k(1-x^2)^{(n-3)/2}dx \\ &= \frac{\Gamma(n/2)}{\sqrt{\pi}\,\Gamma((n-1)/2)2^k}\int_{-1}^{1}(s^2(1-x) + (1+x))^k(1-x^2)^{(n-3)/2}dx\,. \end{aligned}$$

Then it is clear that $a(s)$ is a monotone increasing function in s with $0 < s \leq 1$. By the definition (1), we get (5). □

Let $\mathfrak{h}^2_\lambda(\tilde{B}(r))$ be the completion of $\mathcal{O}_\Delta(\tilde{B}[r])$ with respect to the norm $\|f\|_{\tilde{S}_{\lambda,r}} = \sqrt{(f,f)_{\tilde{S}_{\lambda,r}}}$. Then $\mathfrak{h}^2_\lambda(\tilde{B}(r))$ is a Hilbert space and is isomorphic to the space

$$\mathfrak{h}^2_\lambda(\tilde{B}(r)) = \left\{\{f_k\};\, f_k \in \mathcal{P}^k_\Delta(\tilde{\mathbf{E}}),\, \sum_{k=0}^{\infty} L_{k,\lambda,r}\|f_k\|^2_{\mathbf{S}_1} < \infty\right\}. \tag{6}$$

If $|\lambda| = r$, we write $\mathfrak{h}^2(\tilde{B}(r)) = \mathfrak{h}^2_\lambda(\tilde{B}(r))$. By Lemma 1.4, for $0 < |\lambda| < |\mu| < r$, we have $\mathfrak{h}^2(\tilde{B}(r)) \subset \mathfrak{h}^2_\mu(\tilde{B}(r)) \subset \mathfrak{h}^2_\lambda(\tilde{B}(r)) \subset \mathfrak{h}^2_0(\tilde{B}(r))$. If $\{f_k\} \in \mathfrak{h}^2_0(\tilde{B}(r))$, then we have

$$\sum_{k=0}^{\infty} 2^{-k}\gamma_{k,n}r^{2k}\|f_k\|^2_{\mathbf{S}_1} < \infty\,.$$

Therefore, there exists $M \geq 0$ such that $\|f_k\|_{\mathbf{S}_1} \leq Mr^{-k}(2^{-k}\gamma_{k,n})^{-1/2}$. Because of

$$2^{-k}\gamma_{k,n} = \frac{1}{N(k,n)}\frac{\Gamma(k+(n+1)/2)}{\Gamma((n+1)/2)\Gamma(k+1)} = O(k^{-(n-1)/2})\,, \tag{7}$$

we have (4). Therefore, by Theorem 1.3, $f(z) = \sum_{k=0}^{\infty} f_k(z)$ converges and complex harmonic on $\tilde{B}(r)$. With this identification we have the following theorem:

THEOREM 1.5. (i) *The Hilbert space* $\mathfrak{h}^2_\lambda(\tilde{B}(r))$ *is the Hilbert space direct sum of the finite dimensional subspaces* $\mathcal{P}^k_\Delta(\tilde{\mathbf{E}})$:

$$\mathfrak{h}^2_\lambda(\tilde{B}(r)) = \bigoplus_{k=0}^{\infty} \mathcal{P}^k_\Delta(\tilde{\mathbf{E}}).$$

(ii) *The Poisson kernel* $K_{\lambda,r}(z,w)$ *is a reproducing kernel for the Hilbert space* $\mathfrak{h}^2_\lambda(\tilde{B}(r))$; *that is, for* $f \in \mathfrak{h}^2_\lambda(\tilde{B}(r))$ *we have*

$$f(z) = (f(w), K_{\lambda,r}(w,z))_{\tilde{\mathbf{S}}_{\lambda,r}}.$$

(iii) *If* $0 < |\lambda| < |\mu| < r$, *then we have*

$$\mathcal{O}_\Delta(\tilde{B}[r]) \subset \mathfrak{h}^2(\tilde{B}(r)) \subset \mathfrak{h}^2_\mu(\tilde{B}(r)) \subset \mathfrak{h}^2_\lambda(\tilde{B}(r)) \subset \mathfrak{h}^2_0(\tilde{B}(r)) \subset \mathcal{O}_\Delta(\tilde{B}(r)).$$

2. Holomorphic functions on $\tilde{\mathbf{S}}_\lambda$.

Put $\tilde{\mathbf{S}}_\lambda(r) = \tilde{\mathbf{S}}_\lambda \cap \tilde{B}(r)$ and $\tilde{\mathbf{S}}_\lambda[r] = \tilde{\mathbf{S}}_\lambda \cap \tilde{B}[r]$. Let us denote by $\mathcal{O}(\tilde{\mathbf{S}}_\lambda(r))$ the space of holomorphic functions on the open subset $\tilde{\mathbf{S}}_\lambda(r)$ of the complex sphere $\tilde{\mathbf{S}}_\lambda$. Equipped with the topology of uniform convergence on compact sets, $\mathcal{O}(\tilde{\mathbf{S}}_\lambda(r))$ is an FS space. Put $\mathcal{H}^k(\tilde{\mathbf{S}}_\lambda) = \{\tilde{f}_k = f_k|_{\tilde{\mathbf{S}}_\lambda};\ f_k \in \mathcal{P}^k_\Delta(\tilde{\mathbf{E}})\}$ and call it the space of k-spherical harmonics on $\tilde{\mathbf{S}}_\lambda$. Especially for $\lambda = 0$, sometimes we call $\mathcal{H}^k(\tilde{\mathbf{S}}_0)$ the space of k-conical harmonics on the complex light cone $\tilde{\mathbf{S}}_0$. Note that $\dim \mathcal{H}^k(\tilde{\mathbf{S}}_\lambda) = N(k,n)$.

Put $\mathcal{O}(\tilde{\mathbf{S}}_\lambda[r]) = \lim\mathrm{ind}_{r'>r}\mathcal{O}(\tilde{\mathbf{S}}_\lambda(r'))$ and equip it with the locally convex inductive limit topology. It is a DFS space.

In general, we shall denote functions on a subset of the complex sphere $\tilde{\mathbf{S}}_\lambda$ by $\tilde{f}$, $\tilde{g}$ etc. Let $\tilde{f} \in \mathcal{O}(\tilde{\mathbf{S}}_\lambda(r))$. Then the right-hand side of

$$f_k(z) = \frac{N(k,n)}{L_{k,\lambda,r'}} \int_{\tilde{\mathbf{S}}_{\lambda,r'}} \tilde{f}(w) P_{k,n}(z,w)\dot{d}w, \quad z \in \tilde{\mathbf{E}},$$

is independent of r' with $|\lambda| \le r' < r$ and defines $f_k \in \mathcal{P}^k_\Delta(\tilde{\mathbf{E}})$. The series

$$f(z) = \sum_{k=0}^{\infty} f_k(z), \quad z \in \tilde{B}(r)$$

converges uniformly on compact sets of $\tilde{B}(r)$ and defines $f \in \mathcal{O}_\Delta(\tilde{B}(r))$, which satisfies $f|_{\tilde{\mathbf{S}}_\lambda(r)} = \tilde{f}$. We call f the harmonic extension of $\tilde{f}$ (see Morimoto [9]). If we put $\tilde{f}_k = f_k|_{\tilde{\mathbf{S}}_\lambda(r)}$, then $\tilde{f}_k \in \mathcal{H}^k(\tilde{\mathbf{S}}_\lambda)$, which will be called the k-spherical harmonic component of $\tilde{f}$. The series

$$\tilde{f}(z) = \sum_{k=0}^{\infty} \tilde{f}_k(z), \quad z \in \tilde{\mathbf{S}}_\lambda(r) \tag{8}$$

converges uniformly on compact sets of $\tilde{\mathbf{S}}_\lambda(r)$ and we have $f|_{\tilde{\mathbf{S}}_\lambda(r)} = \tilde{f}$. We call (8) the spherical harmonic expansion of $\tilde{f}$.

For the later reference, we summarize this fact as a theorem:

THEOREM 2.1. *The restriction mappings*

$$\alpha_\lambda^0 : \mathcal{O}_\Delta(\tilde{B}(r)) \to \mathcal{O}(\tilde{\mathbf{S}}_\lambda(r)) \quad \textit{and} \quad \alpha_\lambda^0 : \mathcal{O}_\Delta(\tilde{B}[r]) \to \mathcal{O}(\tilde{\mathbf{S}}_\lambda[r])$$

are topological linear isomorphisms.

PROPOSITION 2.2. *Let* $|\lambda| < r$ *and* (8) *be the expansion of* $\tilde{f}$ *in spherical harmonics.*
(i) $\tilde{f} \in \mathcal{O}(\tilde{\mathbf{S}}_\lambda(r))$ *if and only if* $\limsup_{k\to\infty} \sqrt[k]{\|\tilde{f}_k\|_{\tilde{\mathbf{S}}_{\lambda,r}}} \le 1$.
(ii) $\tilde{f} \in \mathcal{O}(\tilde{\mathbf{S}}_\lambda[r])$ *if and only if* $\limsup_{k\to\infty} \sqrt[k]{\|\tilde{f}_k\|_{\tilde{\mathbf{S}}_{\lambda,r}}} < 1$.

For $\tilde{g}, \tilde{f} \in \mathcal{O}(\tilde{\mathbf{S}}_\lambda[r])$ put

$$(\tilde{g}, \tilde{f})_{\tilde{\mathbf{S}}_{\lambda,r}} = \int_{\tilde{\mathbf{S}}_{\lambda,r}} \tilde{g}(z)\overline{\tilde{f}(z)}\dot{d}z\,.$$

Then Theorem 1.1 implies

$$(\tilde{g}, \tilde{f})_{\tilde{\mathbf{S}}_{\lambda,r}} = \sum_{k=0}^{\infty}(\tilde{g}_k, \tilde{f}_k)_{\tilde{\mathbf{S}}_{\lambda,r}} = \sum_{k=0}^{\infty}(\tilde{g}, \tilde{f}_k)_{\tilde{\mathbf{S}}_{\lambda,r}}\,. \tag{9}$$

By Proposition 2.2, the right-hand side of (9) converges for $\tilde{g} \in \mathcal{O}(\tilde{\mathbf{S}}_\lambda[r])$ and $\tilde{f} \in \mathcal{O}(\tilde{\mathbf{S}}_\lambda(r))$. So we define the sesquilinear form $(\tilde{g}, \tilde{f})_{\tilde{\mathbf{S}}_{\lambda,r}}$ by (9) and call it the symbolic integral form. We sometimes denote it by

$$(\tilde{g}, \tilde{f})_{\tilde{\mathbf{S}}_{\lambda,r}} = \text{s.}\int_{\tilde{\mathbf{S}}_{\lambda,r}} \tilde{g}(z)\overline{\tilde{f}(z)}\dot{d}z\,.$$

Let $\tilde{K}_{\lambda,r}(z, w)$ be the restriction of the Poisson kernel $K_{\lambda,r}(z, w)$ on $\tilde{\mathbf{S}}_\lambda \times \tilde{\mathbf{S}}_\lambda$. We call it the Cauchy kernel on $\tilde{\mathbf{S}}_\lambda$. $\tilde{K}_{\lambda,r}(z, w)$ is holomorphic in z and antiholomorphic in w on

$$\Omega_r = \{(z, w) \in \tilde{\mathbf{S}}_\lambda \times \tilde{\mathbf{S}}_\lambda;\ L(z)L(w) < r^2\}$$

and satisfies $\tilde{K}_{\lambda,r}(z, w) = \overline{\tilde{K}_{\lambda,r}(w, z)}$. We have the Cauchy integral representation formula for $\tilde{f} \in \mathcal{O}(\tilde{\mathbf{S}}_\lambda(r))$:

$$\tilde{f}(z) = \text{s.}\int_{\tilde{\mathbf{S}}_{\lambda,r}} \tilde{f}(w)\tilde{K}_{\lambda,r}(z, w)\dot{d}w\,, \quad z \in \tilde{\mathbf{S}}_\lambda(r)\,. \tag{10}$$

Let $\tilde{f} \in \mathcal{O}(\tilde{\mathbf{S}}_\lambda(r))$, $|\lambda| < r$. For r' with $|\lambda| < r' < r$ we put

$$\|\tilde{f}\|_{\tilde{\mathbf{S}}_{\lambda,r'}} = \left\{\int_{\tilde{\mathbf{S}}_{\lambda,r}} |\tilde{f}(z)|^2\dot{d}z\right\}^{1/2},$$

$$\|\tilde{f}\|_{(\lambda,r)} = \sup\{\|\tilde{f}\|_{\tilde{\mathbf{S}}_{\lambda,r'}};\ |\lambda| \le r' < r\}\,.$$

The Hardy space $H^2(\tilde{\mathbf{S}}_\lambda(r))$ is defined to be the class of all $\tilde{f} \in \mathcal{O}(\tilde{\mathbf{S}}_\lambda(r))$ for which $\|\tilde{f}\|_{(\lambda,r)} < \infty$. If $\tilde{f} \in \mathcal{O}(\tilde{\mathbf{S}}_\lambda[r])$, then $\|\tilde{f}\|_{(\lambda,r)} = \|\tilde{f}\|_{\tilde{\mathbf{S}}_{\lambda,r}}$. Hence, $H^2(\tilde{\mathbf{S}}_\lambda(r))$ is the completion of

$\mathcal{O}(\tilde{\mathbf{S}}_\lambda[r])$ with respect to the norm $\|\tilde{f}\|_{\tilde{\mathbf{S}}_{\lambda,r}}$. In the sequel, we denote the norm $\|\tilde{f}\|_{(\lambda,r)}$ by $\|\tilde{f}\|_{\tilde{\mathbf{S}}_{\lambda,r}}$ even for $\tilde{f} \in H^2(\tilde{\mathbf{S}}_\lambda(r))$. For $\tilde{f}, \tilde{g} \in H^2(\tilde{\mathbf{S}}_\lambda(r))$ we can define the sesquilinear form

$$(\tilde{g}, \tilde{f})_{\tilde{\mathbf{S}}_{\lambda,r}} = \int_{\tilde{\mathbf{S}}_{\lambda,r}} \tilde{g}(z)\overline{\tilde{f}(z)}\dot{d}z\,.$$

By Theorem 1.1 we have the following theorem:

THEOREM 2.3. (i) *The Hilbert space $H^2(\tilde{\mathbf{S}}_\lambda(r))$ is the Hilbert space direct sum of the finite dimensional subspaces $\mathcal{H}^k(\tilde{\mathbf{S}}_\lambda)$:*

$$H^2(\tilde{\mathbf{S}}_\lambda(r)) = \bigoplus_{k=0}^{\infty} \mathcal{H}^k(\tilde{\mathbf{S}}_\lambda)\,.$$

For $\tilde{f} \in H^2(\tilde{\mathbf{S}}_\lambda(r))$ we define the k-spherical harmonic component $\tilde{f}_k$ of $\tilde{f}$ by

$$\tilde{f}_k(z) = \frac{N(k,n)}{L_{k,\lambda,r}} \int_{\tilde{\mathbf{S}}_{\lambda,r}} \tilde{f}(w) P_{k,n}(z,w)\dot{d}w\,, \quad z \in \tilde{\mathbf{S}}_\lambda(r)\,.$$

The orthogonal projection of $H^2(\tilde{\mathbf{S}}_\lambda(r))$ onto $\mathcal{H}^k(\tilde{\mathbf{S}}_\lambda)$ is given by $\tilde{f} \mapsto \tilde{f}_k$.

(ii) *The Cauchy kernel $\tilde{K}_{\lambda,r}(z,w)$ is a reproducing kernel for the Hilbert space $H^2(\tilde{\mathbf{S}}_\lambda(r))$; that is, for $\tilde{f} \in H^2(\tilde{\mathbf{S}}_\lambda(r))$ we have*

$$\tilde{f}(z) = (\tilde{f}(w), \tilde{K}_{\lambda,r}(w,z))_{\tilde{\mathbf{S}}_{\lambda,r}}\,, \quad z \in \tilde{\mathbf{S}}_\lambda(r)\,. \tag{11}$$

(iii) *The restriction mapping $\alpha_\lambda^0 : \mathfrak{h}_\lambda^2(\tilde{B}(r)) \to H^2(\tilde{\mathbf{S}}_\lambda(r))$ is a unitary isomorphism. The inverse mapping $\tilde{f} \mapsto f$ is given by the Poisson integral formula*

$$f(z) = (\tilde{f}(w), K_{\lambda,r}(w,z))_{\tilde{\mathbf{S}}_{\lambda,r}} = \int_{\tilde{\mathbf{S}}_{\lambda,r}} \tilde{f}(w) K_{\lambda,r}(z,w)\dot{d}w\,, \quad z \in \tilde{B}(r)\,.$$

Combining (iii) with Theorem 2.1, we have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{O}_\Delta(\tilde{B}[r]) & \hookrightarrow & \mathfrak{h}_\lambda^2(\tilde{B}(r)) & \hookrightarrow & \mathcal{O}_\Delta(\tilde{B}(r)) \\ \downarrow{\scriptstyle \alpha_\lambda^0} & & \downarrow{\scriptstyle \alpha_\lambda^0} & & \downarrow{\scriptstyle \alpha_\lambda^0} \\ \mathcal{O}(\tilde{\mathbf{S}}_\lambda[r]) & \hookrightarrow & H^2(\tilde{\mathbf{S}}_\lambda(r)) & \hookrightarrow & \mathcal{O}(\tilde{\mathbf{S}}_\lambda(r))\,. \end{array}$$

3. Harmonic functionals.

Let $\mathcal{O}'_\Delta(\tilde{B}[r])$ be the dual space of $\mathcal{O}_\Delta(\tilde{B}[r])$. An element T of $\mathcal{O}'_\Delta(\tilde{B}[r])$ is called a harmonic functional on $\tilde{B}[r]$. For $T \in \mathcal{O}'_\Delta(\tilde{B}[r])$ we put

$$T_k(z) = N(k,n)\overline{\langle T_w, P_{k,n}(w,z)\rangle_w}\,, \quad z \in \tilde{\mathbf{E}}\,,$$

where $\langle\, , \rangle_w$ denotes the canonical bilinear form with a dummy parameter w. $T_k(z)$ is a k-homogeneous harmonic polynomial in z and will be called the k-homogeneous harmonic component of T. Note that T_k is antilinear with respect to T. We know

$$\limsup_{k\to\infty} \sqrt[k]{\|T_k\|_{\tilde{\mathbf{S}}_{\lambda,r}}} \le r^2 . \tag{12}$$

For $g = \sum_{k=0}^{\infty} g_k \in \mathcal{O}_\Delta(\tilde{B}[r])$, $g_k \in \mathcal{P}^k_\Delta(\tilde{\mathbf{E}})$, we have

$$\langle T, g\rangle = \sum_{k=0}^{\infty} \langle T_w, g_k(w)\rangle_w$$

by the continuity of T. Then we have

$$\begin{aligned}\langle T_w, g_k(w)\rangle_w &= \left\langle T_w, \frac{N(k,n)}{L_{k,\lambda,r}} \int_{\tilde{\mathbf{S}}_{\lambda,r}} g_k(z) P_{k,n}(w,z)\dot{d}z \right\rangle_w \\ &= \int_{\tilde{\mathbf{S}}_{\lambda,r}} g_k(z) \frac{N(k,n)}{L_{k,\lambda,r}} \langle T_w, P_{k,n}(w,z)\rangle_w \dot{d}z \\ &= \frac{1}{L_{k,\lambda,r}} \int_{\tilde{\mathbf{S}}_{\lambda,r}} g_k(z)\overline{T_k(z)}\dot{d}z = \frac{1}{L_{k,\lambda,r}} (g_k, T_k)_{\tilde{\mathbf{S}}_{\lambda,r}} .\end{aligned}$$

Therefore, we have

$$\langle T, g\rangle = \sum_{k=0}^{\infty} \frac{1}{L_{k,\lambda,r}} (g_k, T_k)_{\tilde{\mathbf{S}}_{\lambda,r}} = \sum_{k=0}^{\infty} \frac{1}{L_{k,\lambda,r}} (g, T_k)_{\tilde{\mathbf{S}}_{\lambda,r}} .$$

Conversely, if a sequence $\{T_k\}$ of homogeneous harmonic polynomials $T_k \in \mathcal{P}^k_\Delta(\tilde{\mathbf{E}})$ satisfies (12), then

$$\langle T, g\rangle = \sum_{k=0}^{\infty} \frac{1}{L_{k,\lambda,r}} (g, T_k)_{\tilde{\mathbf{S}}_{\lambda,r}} \quad \text{for} \quad g \in \mathcal{O}_\Delta(\tilde{B}[r])$$

defines a harmonic functional T on $\tilde{B}[r]$. In the sequel, we represent $T \in \mathcal{O}'_\Delta(\tilde{B}[r])$ by the sequence of its k-homogeneous harmonic components T_k and denote $T = \{T_k\}$.

For $T \in \mathcal{O}'_\Delta(\tilde{B}[r])$ we define the Poisson transform $\mathcal{P}_{\lambda,r}T$ of T by

$$\mathcal{P}_{\lambda,r}T(w) = \overline{\langle T_z, K_{\lambda,r}(z,w)\rangle}, \quad w \in \tilde{B}(r) .$$

Then $\mathcal{P}_{\lambda,r}T \in \mathcal{O}_\Delta(\tilde{B}(r))$. The mapping $\mathcal{P}_{\lambda,r} : T \mapsto \mathcal{P}_{\lambda,r}T$ is called the Poisson transformation.

THEOREM 3.1. *Let $|\lambda| < r$. The Poisson transformation $\mathcal{P}_{\lambda,r}$ establishes the topological antilinear isomorphism*

$$\mathcal{P}_{\lambda,r} : \mathcal{O}'_\Delta(\tilde{B}[r]) \to \mathcal{O}_\Delta(\tilde{B}(r)) .$$

For $T \in \mathcal{O}'_\Delta(\tilde{B}[r])$ and $g \in \mathcal{O}_\Delta(\tilde{B}[r])$ we have

$$\langle T, g\rangle = \mathrm{s.} \int_{\tilde{\mathbf{S}}_{\lambda,r}} g(w)\overline{\mathcal{P}_{\lambda,r}T(w)}\dot{d}w . \tag{13}$$

PROOF. Let $T \in \mathcal{O}'_\Delta(\tilde{B}[r])$. For $g \in \mathcal{O}_\Delta(\tilde{B}[r])$ we can find $t > 1$ sufficiently close to 1 such that

$$\langle T, g\rangle = \langle T_z, g(z)\rangle_z = \left\langle T_z, \int_{\tilde{\mathbf{S}}_{\lambda,r}} g(tw)K_{\lambda,r}(z, t^{-1}w)\dot{d}w \right\rangle_z$$
$$= \int_{\tilde{\mathbf{S}}_{\lambda,r}} g(tw)\overline{\mathcal{P}_{\lambda,r}T(t^{-1}w)}\dot{d}w = \text{s.}\int_{\tilde{\mathbf{S}}_{\lambda,r}} g(w)\overline{\mathcal{P}_{\lambda,r}T(w)}\dot{d}w\,.$$

Thus we have (13).

Now, for $f \in \mathcal{O}_\Delta(\tilde{B}(r))$ we denote by $T_f^{\lambda,r}$ the continuous linear functional on $\mathcal{O}_\Delta(\tilde{B}[r])$ defined by

$$\langle T_f^{\lambda,r}, g\rangle = \text{s.}\int_{\tilde{\mathbf{S}}_{\lambda,r}} g(z)\overline{f(z)}\dot{d}z\,, \quad g \in \mathcal{O}_\Delta(\tilde{B}[r])\,.$$

Let $f = \sum_{k=0}^{\infty} f_k \in \mathcal{O}_\Delta(\tilde{B}(r))$, $f_k \in \mathcal{P}^k_\Delta(\tilde{\mathbf{E}})$. Then the k-homogeneous harmonic component $(T_f^{\lambda,r})_k$ of the functional $T_f^{\lambda,r}$ is $L_{k,\lambda,r}f_k(z)$:

$$(T_f^{\lambda,r})_k(z) = N(k,n)\overline{\langle (T_f^{\lambda,r})_w, P_{k,n}(w,z)\rangle_w}$$
$$= N(k,n)\overline{\text{s.}\int_{\tilde{\mathbf{S}}_{\lambda,r}} P_{k,n}(w,z)\overline{f(w)}\dot{d}w} = L_{k,\lambda,r}f_k(z)\,.$$

Conversely, for $T \in \mathcal{O}'_\Delta(\tilde{B}[r])$ we denote by T_k the k-homogeneous harmonic component of T. Then we have

$$\mathcal{P}_{\lambda,r}T(w) = \overline{\left\langle T_z, \sum_{k=0}^{\infty} \frac{N(k,n)}{L_{k,\lambda,r}} P_{k,n}(z,w) \right\rangle_z} = \sum_{k=0}^{\infty} \frac{1}{L_{k,\lambda,r}} T_k(w)\,.$$

Thus the k-homogeneous harmonic component of $\mathcal{P}_{\lambda,r}T$ is $(L_{k,\lambda,r})^{-1}T_k(w)$.

We have proved that two antilinear mapppings $T \mapsto \mathcal{P}_{\lambda,r}T$ and $f \mapsto T_f^{\lambda,r}$ are inverse to each other. □

Now put

$$(\mathfrak{h}^2_\lambda(\tilde{B}(r)))' = \{T_f^{\lambda,r} \in \mathcal{O}'_\Delta(\tilde{B}[r]);\ f \in \mathfrak{h}^2_\lambda(\tilde{B}(r))\}\,.$$

Then $(\mathfrak{h}^2_\lambda(\tilde{B}(r)))'$ is antilinearly isomorphic to $\mathfrak{h}^2_\lambda(\tilde{B}(r))$ by the Riesz theorem. By the definition (6), we can characterize the space $(\mathfrak{h}^2_\lambda(\tilde{B}(r)))'$ of harmonic functionals as follows:

$$(\mathfrak{h}^2_\lambda(\tilde{B}(r)))' = \left\{ T = \{T_k\};\ T_k \in \mathcal{P}^k_\Delta(\tilde{\mathbf{E}}),\ \sum_{k=0}^{\infty} \frac{1}{L_{k,\lambda,r}} \|T_k\|^2_{\mathbf{S}_1} < \infty \right\}.$$

If $|\lambda| = r$, then we shall write $(\mathfrak{h}^2(\tilde{B}(r)))' = (\mathfrak{h}^2_\lambda(\tilde{B}(r)))'$. Finally, from Lemma 1.4 we get the following theorem:

THEOREM 3.2. *If $0 < |\lambda| < |\mu| < r$, then we have*

$$\mathcal{O}'_\Delta(\tilde{B}(r)) \subset (\mathfrak{h}^2_0(\tilde{B}(r)))' \subset (\mathfrak{h}^2_\lambda(\tilde{B}(r)))' \tag{14}$$
$$\subset (\mathfrak{h}^2_\mu(\tilde{B}(r)))' \subset (\mathfrak{h}^2(\tilde{B}(r)))' \subset \mathcal{O}'_\Delta(\tilde{B}[r])\,.$$

4. Analytic functionals on $\tilde{\mathbf{S}}_\lambda$.

Let $\mathcal{O}'(\tilde{\mathbf{S}}_\lambda[r])$ be the dual space of $\mathcal{O}(\tilde{\mathbf{S}}_\lambda[r])$. An element $\tilde{T}$ of $\mathcal{O}'(\tilde{\mathbf{S}}_\lambda[r])$ is called an analytic functional on $\tilde{\mathbf{S}}_\lambda[r]$. For $\tilde{T} \in \mathcal{O}'(\tilde{\mathbf{S}}_\lambda[r])$ we put

$$\tilde{T}_k(z) = N(k,n)\overline{\langle \tilde{T}_w, P_{k,n}(w,z)\rangle_w}, \quad z \in \tilde{\mathbf{S}}_\lambda, \tag{15}$$

where $\langle\, ,\rangle_w$ denotes the canonical bilinear form with a dummy parameter w. We call $\tilde{T}_k \in \mathcal{H}^k(\tilde{\mathbf{S}}_\lambda)$ the k-spherical harmonic component of $\tilde{T}$. Note that $\tilde{T}_k$ is antilinear with respect to $\tilde{T}$. We know

$$\limsup_{k\to\infty} \sqrt[k]{\|\tilde{T}_k\|_{\tilde{\mathbf{S}}_{\lambda,r}}} \le r^2. \tag{16}$$

For $\tilde{g} = \sum_{k=0}^{\infty} \tilde{g}_k \in \mathcal{O}(\tilde{\mathbf{S}}_\lambda[r])$, $\tilde{g}_k \in \mathcal{H}^k(\tilde{\mathbf{S}}_\lambda)$, we have

$$\langle \tilde{T}, \tilde{g}\rangle = \sum_{k=0}^{\infty} \langle \tilde{T}_w, \tilde{g}_k(w)\rangle_w$$

by the continuity of $\tilde{T}$. Then we have

$$\begin{aligned}\langle \tilde{T}_w, \tilde{g}_k(w)\rangle_w &= \left\langle \tilde{T}_w, \frac{N(k,n)}{L_{k,\lambda,r}} \int_{\tilde{\mathbf{S}}_{\lambda,r}} \tilde{g}_k(z) P_{k,n}(w,z)\dot{d}z \right\rangle_w \\ &= \int_{\tilde{\mathbf{S}}_{\lambda,r}} \tilde{g}_k(z) \frac{N(k,n)}{L_{k,\lambda,r}} \langle \tilde{T}_w, P_{k,n}(w,z)\rangle \dot{d}z \\ &= \frac{1}{L_{k,\lambda,r}} \int_{\tilde{\mathbf{S}}_{\lambda,r}} \tilde{g}_k(z)\overline{\tilde{T}_k(z)}\dot{d}z.\end{aligned}$$

Therefore, we get

$$\langle \tilde{T}, \tilde{g}\rangle = \sum_{k=0}^{\infty} \frac{1}{L_{k,\lambda,r}} (\tilde{g}_k, \tilde{T}_k)_{\tilde{\mathbf{S}}_{\lambda,r}} = \sum_{k=0}^{\infty} \frac{1}{L_{k,\lambda,r}} (\tilde{g}, \tilde{T}_k)_{\tilde{\mathbf{S}}_{\lambda,r}}.$$

Conversely, if a sequence $\{\tilde{T}_k\}$ of spherical harmonics $\tilde{T}_k \in \mathcal{H}^k(\tilde{\mathbf{S}}_\lambda)$ satisfies (16), then

$$\langle \tilde{T}, \tilde{g}\rangle = \sum_{k=0}^{\infty} \frac{1}{L_{k,\lambda,r}} (\tilde{g}_k, \tilde{T}_k)_{\tilde{\mathbf{S}}_{\lambda,r}} \quad \text{for} \quad \tilde{g} \in \mathcal{O}(\tilde{\mathbf{S}}_\lambda[r])$$

defines an analytic functional $\tilde{T}$ on $\tilde{\mathbf{S}}_\lambda[r]$ by (5) and (7). In the sequel, we represent $\tilde{T} \in \mathcal{O}'(\tilde{\mathbf{S}}_\lambda(r))$ by the sequence of its k-spherical harmonic components $\tilde{T}_k$ and denote $\tilde{T} = \{\tilde{T}_k\}$.

For $z \in \tilde{\mathbf{S}}_\lambda(r)$ the function $w \mapsto \tilde{K}_{\lambda,r}(w,z)$ is a holomorphic function in a neighborhood of $\tilde{\mathbf{S}}_\lambda[r]$. Hence, we can define the Cauchy transform $\mathcal{C}_{\lambda,r}\tilde{T}$ of $\tilde{T} \in \mathcal{O}'(\tilde{\mathbf{S}}_\lambda[r])$ by

$$\mathcal{C}_{\lambda,r}\tilde{T}(z) = \overline{\langle \tilde{T}_w, \tilde{K}_{\lambda,r}(w,z)\rangle_w}, \quad z \in \tilde{\mathbf{S}}_\lambda(r).$$

The mapping $\tilde{T} \mapsto \mathcal{C}_{\lambda,r}\tilde{T}$ is called the Cauchy transformation.

THEOREM 4.1. *Let $|\lambda| < r$. The Cauchy transformation $\mathcal{C}_{\lambda,r}$ establishes the topological antilinear isomorphism*

$$\mathcal{C}_{\lambda,r} : \mathcal{O}'(\tilde{\mathbf{S}}_\lambda[r]) \to \mathcal{O}(\tilde{\mathbf{S}}_\lambda(r)).$$

For $\tilde{T} \in \mathcal{O}'(\tilde{\mathbf{S}}[r])$ *and* $\tilde{g} \in \mathcal{O}(\tilde{\mathbf{S}}_\lambda[r])$ *we have*

$$\langle \tilde{T}, \tilde{g} \rangle = \text{s.} \int_{\tilde{\mathbf{S}}_{\lambda,r}} \tilde{g}(w)\overline{\mathcal{C}_{\lambda,r}\tilde{T}(w)}\dot{d}w \,. \tag{17}$$

PROOF. Let $\tilde{T} \in \mathcal{O}'(\tilde{\mathbf{S}}_\lambda[r])$. We have

$$\begin{aligned}
\mathcal{C}_{\lambda,r}\tilde{T}(z) &= \overline{\left\langle \tilde{T}_w, \sum_{k=0}^{\infty} \frac{N(k,n)}{L_{k,\lambda,r}} P_{k,n}(w,z) \right\rangle_w} \\
&= \sum_{k=0}^{\infty} \frac{N(k,n)}{L_{k,\lambda,r}} \overline{\langle \tilde{T}_w, P_{k,n}(w,z) \rangle_w} = \sum_{k=0}^{\infty} \frac{1}{L_{k,\lambda,r}} \tilde{T}_k(z), \quad z \in \tilde{\mathbf{S}}_\lambda(r).
\end{aligned}$$

Therefore, the Cauchy transformation $\mathcal{C}_{\lambda,r}$ maps $\mathcal{O}'(\tilde{\mathbf{S}}_\lambda[r])$ into $\mathcal{O}(\tilde{\mathbf{S}}_\lambda(r))$ antilinearly.

Let $\tilde{g} \in \mathcal{O}(\tilde{\mathbf{S}}_\lambda[r])$. Then $\tilde{g}$ can be expanded in spherical harmonics $\tilde{g}(z) = \sum_{k=0}^{\infty} \tilde{g}_k(z)$, which converges in the topology of $\mathcal{O}(\tilde{\mathbf{S}}_\lambda[r])$. Therefore, we have

$$\begin{aligned}
\langle \tilde{T}, \tilde{g} \rangle &= \sum_{k=0}^{\infty} \langle \tilde{T}_z, \tilde{g}_k(z) \rangle_z = \sum_{k=0}^{\infty} \left\langle \tilde{T}_z, \frac{N(k,n)}{L_{k,\lambda,r}} \int_{\tilde{\mathbf{S}}_{\lambda,r}} \tilde{g}_k(w) P_{k,n}(z,w)\dot{d}w \right\rangle_z \\
&= \sum_{k=0}^{\infty} \int_{\tilde{\mathbf{S}}_{\lambda,r}} \tilde{g}_k(w) \frac{1}{L_{k,\lambda,r}} \tilde{T}_k(w)\dot{d}w = \sum_{k=0}^{\infty} \int_{\tilde{\mathbf{S}}_{\lambda,r}} \tilde{g}(w) \frac{1}{L_{k,\lambda,r}} \tilde{T}_k(w)\dot{d}w \\
&= \text{s.} \int_{\tilde{\mathbf{S}}_{\lambda,r}} \tilde{g}(w)\overline{\mathcal{C}_{\lambda,r}\tilde{T}(w)}\dot{d}w \,.
\end{aligned}$$

Thus, we have (17).

Now, for $\tilde{f} \in \mathcal{O}(\tilde{\mathbf{S}}_\lambda(r))$ we denote by $\tilde{T}_{\tilde{f}}^{\lambda,r}$ the continuous linear functional on $\mathcal{O}(\tilde{\mathbf{S}}_\lambda[r])$ defined as follows:

$$\langle \tilde{T}_{\tilde{f}}^{\lambda,r}, \tilde{g} \rangle = \sum_{k=0}^{\infty} \int_{\tilde{\mathbf{S}}_{\lambda,r}} \tilde{g}(w)\overline{\tilde{f}_k(w)}\dot{d}w, \quad \tilde{g} \in \mathcal{O}(\tilde{\mathbf{S}}_\lambda[r]).$$

Then the Cauchy transform of $\tilde{T}_{\tilde{f}}^{\lambda,r}$ can be calculated as follows:

$$\mathcal{C}_{\lambda,r}\tilde{T}_{\tilde{f}}^{\lambda,r}(z) = \sum_{k=0}^{\infty} \overline{\int_{\tilde{\mathbf{S}}_{\lambda,r}} K_{\lambda,r}(w,z)\overline{\tilde{f}_k(w)}\dot{d}w} = \sum_{k=0}^{\infty} \tilde{f}_k(z) = \tilde{f}(z).$$

For $\tilde{T} \in \mathcal{O}'(\tilde{\mathbf{S}}_\lambda(r))$ we put $\tilde{f}(z) = \mathcal{C}_{\lambda,r}\tilde{T}(z)$. Then by (17) we have $\tilde{T}_{\tilde{f}}^{\lambda,r} = \tilde{T}$.

We have proved that the two antilinear mappings $\tilde{T} \mapsto \mathcal{C}_{\lambda,r}\tilde{T}$ and $\tilde{f} \mapsto T_{\tilde{f}}^{\lambda,r}$ are inverse to each other. □

COROLLARY 4.2. *We have he following commutative diagram*:

$$\begin{array}{ccccccc}
\mathcal{O}_\Delta(\tilde{B}[r]) & \hookrightarrow & \mathfrak{h}^2_\lambda(\tilde{B}(r)) & \hookrightarrow & \mathcal{O}_\Delta(\tilde{B}(r)) & \xleftarrow{\mathcal{P}_{\lambda,r}} & \mathcal{O}'_\Delta(\tilde{B}[r]) \\
\big\downarrow{\scriptstyle \alpha_\lambda^0} & & \big\downarrow{\scriptstyle \alpha_\lambda^0} & & \big\downarrow{\scriptstyle \alpha_\lambda^0} & & \big\uparrow{\scriptstyle (\alpha_\lambda^0)^*} \\
\mathcal{O}(\tilde{\mathbf{S}}_\lambda[r]) & \hookrightarrow & H^2(\tilde{\mathbf{S}}_\lambda(r)) & \hookrightarrow & \mathcal{O}(\tilde{\mathbf{S}}_\lambda(r)) & \xleftarrow{\mathcal{C}_{\lambda,r}} & \mathcal{O}'(\tilde{\mathbf{S}}_\lambda[r]).
\end{array}$$

Especially, for $\tilde{f}(z) \in \mathcal{O}(\tilde{\mathbf{S}}_\lambda(r))$ the harmonic extension $f = (\alpha_\lambda^0)^{-1}\tilde{f} = \mathcal{P}_{\lambda,r} \circ (\alpha_\lambda^0)^* \circ \mathcal{C}_{\lambda,r}^{-1}\tilde{f}$ is given by the Poisson integral

$$f(z) = \text{s.} \int_{\tilde{\mathbf{S}}_{\lambda,r}} \tilde{f}(w) K_{\lambda,r}(z, w) \dot{d}w \,.$$

Now put $(H^2(\tilde{\mathbf{S}}_\lambda(r)))' = \{T_{\tilde{f}}^{\lambda,r} \in \mathcal{O}'(\tilde{\mathbf{S}}_\lambda[r]);\ \tilde{f} \in H^2(\tilde{\mathbf{S}}_\lambda(r))\}$. Then we have

$$\begin{array}{ccccc} \mathcal{O}'_\Delta(\tilde{B}(r)) & \hookrightarrow & (\mathfrak{h}_\lambda^2(\tilde{B}(r)))' & \hookrightarrow & \mathcal{O}'_\Delta(\tilde{B}[r]) \\ \uparrow{\scriptstyle (\alpha_\lambda^0)^*} & & \uparrow{\scriptstyle (\alpha_\lambda^0)^*} & & \uparrow{\scriptstyle (\alpha_\lambda^0)^*} \\ \mathcal{O}'(\tilde{\mathbf{S}}_\lambda(r)) & \hookrightarrow & (H^2(\tilde{\mathbf{S}}_\lambda(r)))' & \hookrightarrow & \mathcal{O}'(\tilde{\mathbf{S}}_\lambda[r]) \end{array}$$

and $(H^2(\tilde{\mathbf{S}}_\lambda(r)))'$ is antilinearly isomorphic to $H^2(\tilde{\mathbf{S}}_\lambda(r))$ by the Riesz theorem.

5. Conical Fourier-Borel transformation.

Let $T \in \mathcal{O}'_\Delta(\tilde{B}[r])$. We can define the conical Fourier-Borel transform of T by

(18) $$\mathcal{F}^\Delta T(\zeta) = \langle T_z, \exp(z \cdot \zeta)\rangle$$

for $\zeta \in \tilde{\mathbf{S}}_0$ because the function $z \mapsto \exp(z \cdot \zeta)$ is complex harmonic if $\zeta \in \tilde{\mathbf{S}}_0$. We know that the conical Fourier-Borel transformation $\mathcal{F}^\Delta : T \mapsto \mathcal{F}^\Delta T$ establishes a topological linear isomorphism from $\mathcal{O}'_\Delta(\tilde{B}[r])$ onto $\mathrm{Exp}(\tilde{\mathbf{S}}_0; (r))$, where

$$\mathrm{Exp}(\tilde{\mathbf{S}}_0; (r)) = \{F \in \mathcal{O}(\tilde{\mathbf{S}}_0); \forall r' > r, \exists C \geq 0 \text{ s.t. } |F(\zeta)| \leq C \exp(r' L^*(\zeta)),\ \zeta \in \tilde{\mathbf{S}}_0\}$$

and $L^*(\zeta) = \sup\{|z \cdot \zeta|;\ L(z) \leq 1\} = \sqrt{(\|z\|^2 + |z^2|)/2}$ is the dual Lie norm. The conical Fourier-Borel transformation $\mathcal{F}^\Delta : T \mapsto \mathcal{F}^\Delta T$ also establishes a topological linear isomorphism from $\mathcal{O}'_\Delta(\tilde{B}(r))$ onto $\mathrm{Exp}(\tilde{\mathbf{S}}_0;\ [r])$, where

$$\mathrm{Exp}(\tilde{\mathbf{S}}_0;\ [r]) = \{F \in \mathcal{O}(\tilde{\mathbf{S}}_0);\ 0 < \exists r' < r, \exists C \geq 0 \text{ s.t. } |F(\zeta)| \leq C \exp(r' L^*(\zeta)),\ \zeta \in \tilde{\mathbf{S}}_0\}\,.$$

Let $T_k \in \mathcal{P}_\Delta^k(\tilde{\mathbf{E}})$ be the k-homogeneous harmonic component of $T \in \mathcal{O}'_\Delta(\tilde{B}[r])$. Then the conical Fourier-Borel transform of T can be expanded as follows:

$$\begin{aligned} \mathcal{F}^\Delta T(\zeta) = \langle T_z, \exp(z \cdot \zeta)\rangle &= \sum_{k=0}^{\infty} \int_{\mathbf{S}_1} \frac{1}{k!}(x \cdot \zeta)^k \overline{T_k(x)} \dot{d}x \\ &= \sum_{k=0}^{\infty} \int_{\mathbf{S}_1} \frac{1}{k!\gamma_{k,n}} P_{k,n}(x, \bar{\zeta}) \overline{T_k(x)} \dot{d}x = \sum_{k=0}^{\infty} \frac{1}{k!\gamma_{k,n} N(k,n)} \bar{T}_k(\zeta) \end{aligned}$$

for $\zeta \in \tilde{\mathbf{S}}_0$, where we put $\bar{T}_k(\zeta) = \overline{T_k(\bar{\zeta})}$. Now we define the conical Fourier transformation $\mathcal{F}_{\lambda,r}^\Delta$ on $\mathfrak{h}_\lambda^2(\tilde{B}(r))$. For $f \in \mathfrak{h}_\lambda^2(\tilde{B}(r))$ we put

$$\mathcal{F}_{\lambda,r}^\Delta f(\zeta) = (\exp(z \cdot \zeta),\ f(z))_{\tilde{\mathbf{S}}_{\lambda,r}}\,, \quad \zeta \in \tilde{\mathbf{S}}_0\,.$$

Then we have $\mathcal{F}^{\Delta}_{\lambda,r} f(\zeta) = \mathcal{F}^{\Delta}(T_f^{\lambda,r})(\zeta)$ for $\zeta \in \tilde{\mathbf{S}}_0$. Therefore, we have

$$\mathcal{F}^{\Delta}_{\lambda,r} f(\zeta) = \sum_{k=0}^{\infty} \frac{L_{k,\lambda,r}}{k!\gamma_{k,n} N(k,n)} \bar{f}_k(\zeta), \quad \zeta \in \tilde{\mathbf{S}}_0, \tag{19}$$

where $\bar{f}_k \in \mathcal{P}^k_{\Delta}(\tilde{\mathbf{E}})$ is defined by $\bar{f}_k(\zeta) = \overline{f_k(\bar{\zeta})}$.

Let $F, G \in \mathcal{O}(\tilde{\mathbf{S}}_0)$. We expand them as follows:

$$F(\zeta) = \sum_{k=0}^{\infty} F_k(\zeta), \quad G(\zeta) = \sum_{k=0}^{\infty} G_k(\zeta), \quad \zeta \in \tilde{\mathbf{S}}_0,$$

where $F_k, G_k \in \mathcal{H}^k(\tilde{\mathbf{S}}_0)$. We define

$$((F, G))^{\lambda,r}_{\tilde{\mathbf{S}}_0} = \sum_{k=0}^{\infty} \frac{2^k k!^2 \gamma_{k,n} N(k,n)^2}{L_{k,\lambda,r}} (F_k, G_k)_{\mathbf{S}_{0,1}}. \tag{20}$$

Put

$$\begin{aligned} E^{\lambda}_{r,0}(\zeta, \xi) &= (\exp(z \cdot \zeta), \exp(z \cdot \xi))_{\tilde{\mathbf{S}}_{\lambda,r}} \\ &= \int_{\tilde{\mathbf{S}}_{\lambda,r}} \exp(z \cdot \zeta) \overline{\exp(z \cdot \xi)} \dot{d}z, \quad \zeta, \xi \in \tilde{\mathbf{S}}_0. \end{aligned}$$

We call $E^{\lambda}_{r,0}(\zeta, \xi)$ the F-Poisson kernel on $\tilde{\mathbf{S}}_0$. Then we have

$$\begin{aligned} E^{\lambda}_{r,0}(\zeta, \xi) &= \sum_{k=0}^{\infty} \frac{1}{(k!\gamma_{k,n})^2} \int_{\tilde{\mathbf{S}}_{\lambda,r}} P_{k,n}(z, \bar{\zeta}) P_{k,n}(\bar{\xi}, z) \dot{d}z \\ &= \sum_{k=0}^{\infty} \frac{1}{(k!\gamma_{k,n})^2} \frac{L_{k,\lambda,r}}{N(k,n)} P_{k,n}(\zeta, \bar{\xi}) \\ &= \sum_{k=0}^{\infty} \frac{L_{k,\lambda,r}}{k!^2 \gamma_{k,n} N(k,n)} (\zeta \cdot \bar{\xi})^k, \quad \zeta, \xi \in \tilde{\mathbf{S}}_0. \end{aligned} \tag{21}$$

Note that $E^{\lambda}_{r,0}(\zeta, \xi) = E^{|\lambda|}_{r,0}(\zeta, \xi)$, $E^{\lambda}_{r,0}(\zeta, \xi) = \overline{E^{\lambda}_{r,0}(\xi, \zeta)}$, and the function $\zeta \mapsto E^{\lambda}_{r,0}(\zeta, \xi)$ belongs to $\mathrm{Exp}(\tilde{\mathbf{S}}_0; [r])$, hence to $\mathcal{E}^2(\tilde{\mathbf{S}}_0; \lambda, r)$.

6. Hilbert space of entire functions on $\tilde{\mathbf{S}}_0$.

Define $\mathcal{E}^2(\tilde{\mathbf{S}}_0; \lambda, r)$ to be the class of all $F \in \mathcal{O}(\tilde{\mathbf{S}}_0)$ for which $((F, F))^{\lambda,r}_{\tilde{\mathbf{S}}_0} < \infty$.

THEOREM 6.1. (i) *The Hilbert space $\mathcal{E}^2(\tilde{\mathbf{S}}_0; \lambda, r)$ is the Hilbert space direct sum of the finite dimensional subspaces $\mathcal{H}^k(\tilde{\mathbf{S}}_0)$:*

$$\mathcal{E}^2(\tilde{\mathbf{S}}_0; \lambda, r) = \bigoplus_{k=0}^{\infty} \mathcal{H}^k(\tilde{\mathbf{S}}_0). \tag{22}$$

(ii) *The F-Poisson kernel* $E^{\lambda}_{r,0}(\zeta, \xi)$ *is a reproducing kernel for the Hilbert space* $\mathcal{E}^2(\tilde{\mathbf{S}}_0; \lambda, r)$*; that is, for* $F \in \mathcal{E}^2(\tilde{\mathbf{S}}_0; \lambda, r)$ *we have*

$$F(\zeta) = ((F(\xi), E^{\lambda}_{r,0}(\xi, \zeta)))^{\lambda,r}_{\tilde{\mathbf{S}}_0}.$$

(iii) *The conical Fourier transformation*

$$\mathcal{F}^{\Delta}_{\lambda,r} : \mathfrak{h}^2_{\lambda}(\tilde{B}(r)) \to \mathcal{E}^2(\tilde{\mathbf{S}}_0; \lambda, r) \tag{23}$$

is an antilinear unitary isomorphism.

PROOF. (i) is clear by the definition.

(ii) Let $F \in \mathcal{E}^2(\tilde{\mathbf{S}}_0; \lambda, r)$. Expand F into conical harmonics:

$$F(\zeta) = \sum_{k=0}^{\infty} F_k(\zeta).$$

Then we have

$$((F, F))^{\lambda,r}_{\tilde{\mathbf{S}}_0} = \sum_{k=0}^{\infty} \frac{2^k k!^2 \gamma_{k,n} N(k,n)^2}{L_{k,\lambda,r}} \|F_k\|^2_{\tilde{\mathbf{S}}_{0,1}}.$$

By (21) and (20) we have

$$\begin{aligned} ((F(\xi), E^{\lambda}_{r,0}(\xi, \zeta)))^{\lambda,r}_{\tilde{\mathbf{S}}_0} &= \sum_{k=0}^{\infty} \frac{2^k k!^2 \gamma_{k,n} N(k,n)^2}{L_{k,\lambda,r}} \frac{1}{(k!\gamma_{k,n})^2} \frac{L_{k,\lambda,r}}{N(k,n)} (F_k(\xi), P_{k,n}(\zeta, \xi))_{\tilde{\mathbf{S}}_{0,1}} \\ &= \sum_{k=0}^{\infty} \frac{2^k N(k,n)}{\gamma_{k,n}} (F_k(\xi), P_{k,n}(\zeta, \xi))_{\tilde{\mathbf{S}}_{0,1}} \\ &= \sum_{k=0}^{\infty} F_k(\xi) = F(\xi). \end{aligned}$$

Thus we have (ii).

(iii) Let $f = \sum_{k=0}^{\infty} f_k \in \mathfrak{h}^2_{\lambda}(\tilde{B}(r))$, $f_k \in \mathcal{P}^k_{\Delta}(\tilde{\mathbf{E}})$. Then we know that

$$(f, f)_{\tilde{\mathbf{S}}_{\lambda,r}} = \sum_{k=0}^{\infty} L_{k,\lambda,r}(f_k, f_k)_{\mathbf{S}_1}.$$

Write $\mathcal{F}^{\Delta}_{\lambda,r} f(\zeta) = F(\zeta) = \sum_{k=0}^{\infty} F_k(\zeta)$, $\zeta \in \tilde{\mathbf{S}}_0$, with $F_k \in \mathcal{P}^k_{\Delta}(\tilde{\mathbf{E}})$. Then by (19) we have

$$F_k(\zeta) = \frac{L_{k,\lambda,r}}{k!\gamma_{k,n} N(k,n)} \bar{f}_k(\zeta), \quad \zeta \in \tilde{\mathbf{E}}.$$

By Theorem 1.1 we have

$$\begin{aligned}((F,F))^{\lambda,r}_{\tilde{\mathbf{S}}_0} &= \sum_{k=0}^{\infty}((F_k,F_k))^{\lambda,r}_{\tilde{\mathbf{S}}_0} = \sum_{k=0}^{\infty}\frac{2^k k!^2\gamma_{k,n}N(k,n)^2}{L_{k,\lambda,r}}(F_k,F_k)_{\tilde{\mathbf{S}}_{0,1}} \\ &= \sum_{k=0}^{\infty}\frac{(k!\gamma_{k,n}N(k,n))^2}{L_{k,\lambda,r}}(F_k,F_k)_{\mathbf{S}_1} = \sum_{k=0}^{\infty}L_{k,\lambda,r}(f_k,f_k)_{\mathbf{S}_1} \\ &= \sum_{k=0}^{\infty}(f_k,f_k)_{\tilde{\mathbf{S}}_{\lambda,r}} = (f,f)_{\tilde{\mathbf{S}}_{\lambda,r}}\,.\end{aligned}$$

Therefore, $f \mapsto F = \mathcal{F}^{\Delta}_{\lambda,r}f$ is an isometry. Because $\mathcal{F}^{\Delta}$ is a bijection from $\mathcal{O}'_{\Delta}(\tilde{B}[r])$ onto $\mathrm{Exp}(\tilde{\mathbf{S}}_0;(r))$, the composed mapping

$$\mathcal{F}^{\Delta}_{\lambda,r}: f \mapsto T^{\lambda,r}_f \mapsto \mathcal{F}^{\Delta}(T^{\lambda,r}_f)$$

is a bijection from $\mathfrak{h}^2_\lambda(\tilde{B}(r))$ onto $\mathcal{E}^2(\tilde{\mathbf{S}}_0;\lambda,r)$. Thus we have proved that $\mathcal{F}^{\Delta}_{\lambda,r}:\mathfrak{h}^2_\lambda(\tilde{B}(r)) \to \mathcal{E}^2(\tilde{\mathbf{S}}_0;\lambda,r)$ is an antilinear unitary isomorphism. □

EXAMPLE 6.2. If $w \in \tilde{B}(r)$ is fixed, then the function $f(z) = K_{\lambda,r}(z,w)$ belongs to $\mathcal{O}_{\Delta}(\tilde{B}[r])$ and we have

$$\mathcal{F}^{\Delta}_{\lambda,r}f(\zeta) = \int_{\tilde{\mathbf{S}}_{\lambda,r}}\exp(z\cdot\zeta)K_{\lambda,r}(w,z)\dot{d}z = \exp(w\cdot\zeta)\,,\qquad \zeta\in\tilde{\mathbf{S}}_0\,. \tag{24}$$

Therefore, the exponential function $\zeta \mapsto \exp(w\cdot\zeta)$ belongs to $\mathrm{Exp}(\tilde{\mathbf{S}}_0;[r])$, hence, to $\mathcal{E}^2(\tilde{\mathbf{S}}_0;\lambda,r)$ (see also Example 8.2).

EXAMPLE 6.3. If $\xi \in \tilde{\mathbf{S}}_0$ is fixed, then the function $f(z) = \exp(z\cdot\xi)$ belongs to $\mathcal{O}_{\Delta}(\tilde{\mathbf{E}})$. The $\mathcal{F}^{\Delta}_{\lambda,r}$ image of $f(z) = \exp(z\cdot\xi)$ is the F-Poisson kernel $E^{\lambda}_{r,0}(\zeta,\xi)$ (see also Example 8.3).

If $|\lambda| = r$, then we shall put $\mathcal{E}^2(\tilde{\mathbf{S}}_0;r) = \mathcal{E}^2(\tilde{\mathbf{S}}_0;\lambda,r)$. By combining Theorem 3.2 with Theorem 6.1 we have the following theorem:

THEOREM 6.4. *If $0 < |\lambda| < |\mu| < r$, then we have*

$$\begin{aligned}\mathrm{Exp}(\tilde{\mathbf{S}}_0;[r]) &\subset \mathcal{E}^2(\tilde{\mathbf{S}}_0;0,r) \subset \mathcal{E}^2(\tilde{\mathbf{S}}_0;\lambda,r) \\ &\subset \mathcal{E}^2(\tilde{\mathbf{S}}_0;\mu,r) \subset \mathcal{E}^2(\tilde{\mathbf{S}}_0;r) \subset \mathrm{Exp}(\tilde{\mathbf{S}}_0;(r))\,.\end{aligned}$$

In fact, take the $\mathcal{F}^{\Delta}$ image of (14).

Suppose $F \in \mathcal{E}^2(\tilde{\mathbf{S}}_0;\lambda,r)$. If $z \in \tilde{B}(r)$, then the function $\zeta \mapsto \exp(z\cdot\zeta)$ belongs to $\mathcal{E}^2(\tilde{\mathbf{S}}_0;\lambda,r)$. Put

$$\bar{\mathcal{F}}_{\lambda,r}F(z) = ((\exp(z\cdot\zeta),F(\zeta)))^{\lambda,r}_{\tilde{\mathbf{S}}_0}\,.$$

Then we have

$$
(25)\quad
\begin{aligned}
\bar{\mathcal{F}}_{\lambda,r}F(z) &= \sum_{k=0}^{\infty} \frac{2^k k!^2 \gamma_{k,n} N(k,n)^2}{L_{k,\lambda,r}} \frac{1}{k!\gamma_{k,n}} (P_{k,n}(z,\bar{\zeta}), F_k(\zeta))_{\tilde{\mathbf{S}}_{0,1}} \\
&= \sum_{k=0}^{\infty} \frac{2^k k! N(k,n)^2}{L_{k,\lambda,r}} \frac{2^{-k}\gamma_{k,n}}{N(k,n)} \bar{F}_k(z) = \sum_{k=0}^{\infty} \frac{k!\gamma_{k,n} N(k,n)}{L_{k,\lambda,r}} \bar{F}_k(z)\,.
\end{aligned}
$$

Therefore, $\bar{\mathcal{F}}_{\lambda,r}F$ belongs to $\mathfrak{h}^2_{\lambda}(\tilde{B}(r))$ and will be called the Fourier transform of F.

THEOREM 6.5. *The Fourier transformation*

$$\bar{\mathcal{F}}_{\lambda,r} : \mathcal{E}^2(\tilde{\mathbf{S}}_0; \lambda, r) \to \mathfrak{h}^2_{\lambda}(\tilde{B}(r))$$

is an antilinear unitary isomorphism. $\bar{\mathcal{F}}_{\lambda,r}$ *is the inverse mapping of the conical Fourier transformation* (23).

PROOF. The theorem is a consequence of (19) and (25). □

Because of (24), $\exp(w \cdot \zeta)$ is the $\mathcal{F}^{\Delta}_{\lambda,r}$ image of $K_{\lambda,r}(z,w)$. Therefore, Theorem 6.5 implies the following corollary.

COROLLARY 6.6.

$$K_{\lambda,r}(z,w) = ((\exp(z\cdot\zeta), \exp(w\cdot\zeta)))^{\lambda,r}_{\tilde{\mathbf{S}}_0}\,.$$

REMARK 6.7. Let $\rho_{\lambda,r}(s)$ be a C^{∞} function on $[0,\infty)$ satisfying

$$(26)\quad \int_0^{\infty} s^{2k+n-1}\rho_{\lambda,r}(s)ds = \frac{2^k k!^2 \gamma_{k,n} N(k,n)^2}{L_{k,\lambda,r}}\,.$$

Such a function does exist by the following theorem of Duran [1]:

THEOREM 6.8. *For any sequence* (a_k) *of complex numbers, there exists a rapidly decreasing* C^{∞} *function* ψ *such that* $\operatorname{supp}\psi \subset [0,\infty)$ *and*

$$\int_0^{\infty} s^k \psi(s)ds = a_k\,, \quad k = 0,1,2,\cdots\,.$$

In case of $|\lambda| = r$, one such $\rho_r(s) = \rho_{r,r}(s)$ was constructed by Ii [7] and Wada [17] by means of modified Bessel functions. We call $\rho_r(s)$ the Ii-Wada function. Their function $\rho_r(s)$ is not positive valued. It is not known that there is a non-negative function $\rho_{\lambda,r}(s)$ satisfying (26).

Using the function $\rho_{\lambda,r}$, we can write the inner product $((F,G))^{\lambda,r}_{\tilde{\mathbf{S}}_0}$ as follows:

$$((F,G))^{\lambda,r}_{\tilde{\mathbf{S}}_0} = \int_0^{\infty}\left(\int_{\tilde{\mathbf{S}}_{0,1}} F(s\zeta)\overline{G(s\zeta)}d\zeta\right)\rho_{\lambda,r}(s)s^{n-1}ds\,.$$

Because $\tilde{\mathbf{S}}_0 = \bigcup\{s\tilde{\mathbf{S}}_{0,1}; s \geq 0\} = \bigcup\{\tilde{\mathbf{S}}_{0,s}; s \geq 0\}$, we can understand the inner product $((F,G))^{\lambda,r}_{\tilde{\mathbf{S}}_0}$ is defined by a kind of integral over $\tilde{\mathbf{S}}_0$.

7. Fourier-Borel transformation.

Consider the space $\mathcal{O}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}) = \{\tilde{F} \in \mathcal{O}(\tilde{\mathbf{E}}); \Delta_\zeta \tilde{F}(\zeta) = \lambda^2 \tilde{F}(\zeta)\}$ of entire eigenfunctions of the complex Laplacian. For $\tilde{F} \in \mathcal{O}_{\Delta-\lambda^2}(\tilde{\mathbf{E}})$ we put

$$\begin{aligned}\tilde{F}_k(\zeta) &= 2^k \frac{N(k,n)}{\gamma_{k,n}} \int_{\tilde{\mathbf{S}}_{0,1}} \tilde{F}(\xi) P_{k,n}(\zeta, \bar{\xi}) \dot{d}\xi \\ &= 2^k N(k,n) \int_{\tilde{\mathbf{S}}_{0,1}} \tilde{F}(\xi)(\zeta \cdot \bar{\xi})^k \dot{d}\xi\,, \quad \zeta \in \tilde{\mathbf{E}}\,.\end{aligned}$$

Then $\tilde{F}_k(\zeta) \in \mathcal{P}^k_\Delta(\tilde{\mathbf{E}})$ and is called the k-homogeneous harmonic component of $\tilde{F}$. We have the following expansion formula (see Wada-Morimoto [19]):

$$\tilde{F}(\zeta) = \sum_{k=0}^{\infty} \tilde{j}_k(i\lambda\sqrt{\zeta^2})\tilde{F}_k(\zeta)\,, \quad \zeta \in \tilde{\mathbf{E}}\,, \tag{27}$$

where the convergence is uniform on compact sets of $\tilde{\mathbf{E}}$ and

$$\tilde{j}_k(t) = \tilde{J}_{k+(n-1)/2}(t)$$

is the entire Bessel function:

$$\tilde{J}_\mu(t) = \sum_{l=0}^{\infty} \frac{(-1)^l \Gamma(\mu+1)}{\Gamma(\mu+l+1)l!}\left(\frac{t}{2}\right)^{2l} = \Gamma(\mu+1)\left(\frac{t}{2}\right)^{-\mu} J_\mu(t)\,.$$

Note that

$$\tilde{J}_\mu(0) = 1\,, \quad \tilde{J}_\mu(t) = \tilde{J}_\mu(-t)\,, \quad |\tilde{J}_\mu(t)| \le \exp(|t|)\,.$$

By (27), it is clear that the following theorem holds:

THEOREM 7.1. *The restriction mapping* $\alpha_0^\lambda : \mathcal{O}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}) \to \mathcal{O}(\tilde{\mathbf{S}}_0)$ *is a topological linear isomorphism.*

EXAMPLE 7.2. The exponential function can be expanded as follows:

$$\exp(z \cdot \zeta) = \sum_{k=0}^{\infty} \frac{1}{k!\gamma_{k,n}} \tilde{j}_k(i\sqrt{z^2}\sqrt{\zeta^2}) P_{k,n}(z, \bar{\zeta})\,. \tag{28}$$

If $z^2 = 0$ or $\zeta^2 = 0$, then (28) reduces to the Taylor expansion:

$$\exp(z \cdot \zeta) = \sum_{k=0}^{\infty} \frac{1}{k!}(z \cdot \zeta)^k$$

(see Morimoto [9]).

For an analytic functional $\tilde{T} \in \mathcal{O}'(\tilde{\mathbf{S}}_\lambda[r])$ we put

$$\mathcal{F}\tilde{T}(\zeta) = \langle \tilde{T}_z, \exp(z \cdot \zeta) \rangle_z\,, \quad \zeta \in \tilde{\mathbf{E}} \tag{29}$$

and call it the Fourier-Borel transform of $\tilde{T}$.

We know that the Fourier-Borel transformation $\mathcal{F} : \tilde{T} \mapsto \mathcal{F}\tilde{T}$ establishes a topological linear isomorphism from $\mathcal{O}'(\tilde{\mathbf{S}}_\lambda[r])$ onto $\mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; (r))$, where

$$\begin{aligned}\mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; (r)) =\{&\tilde{F} \in \mathcal{O}_{\Delta-\lambda^2}(\tilde{\mathbf{E}});\\ &\forall r' > r, \exists C \geq 0, \text{ s.t. } |\tilde{F}(\zeta)| \leq C\exp(r'L^*(\zeta)), \zeta \in \tilde{\mathbf{E}}\}.\end{aligned}$$

The Fourier-Borel transformation $\mathcal{F} : \tilde{T} \mapsto \mathcal{F}\tilde{T}$ also establishes a topological linear isomorphism from $\mathcal{O}'(\tilde{\mathbf{S}}_\lambda(r))$ onto $\mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; [r])$, where

$$\begin{aligned}\mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; [r]) =\{&\tilde{F} \in \mathcal{O}_{\Delta-\lambda^2}(\tilde{\mathbf{E}});\\ &0 < \exists r' < r, \exists C \geq 0, \text{ s.t. } |\tilde{F}(\zeta)| \leq C\exp(r'L^*(\zeta)), \zeta \in \tilde{\mathbf{E}}\}\end{aligned}$$

(see Wada [17] and Morimoto-Fujita [13]). By the definitions of the Fourier-Borel transformation (29) and the conical Fourier-Borel transformation (18), we have the following commutative diagram:

$$\begin{array}{ccc}\mathcal{O}'_\Delta(\tilde{B}[r]) & \xrightarrow{\mathcal{F}^\Delta} & \mathrm{Exp}(\tilde{\mathbf{S}}_0; (r)) \\ \uparrow{\scriptstyle (\alpha_\lambda^0)^*} & & \uparrow{\scriptstyle \beta_0^\lambda} \\ \mathcal{O}'(\tilde{\mathbf{S}}_\lambda[r]) & \xrightarrow{\mathcal{F}} & \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; (r)),\end{array}$$

where β_0^λ is the restriction mapping. Hence β_0^λ is also a topological linear isomorphism (see Morimoto-Fujita [13], [14] and [16]).

Let $\tilde{T} \in \mathcal{O}'(\tilde{\mathbf{S}}_\lambda[r])$. Then we have

$$\begin{aligned}\mathcal{F}\tilde{T}(\zeta) &= \langle \tilde{T}_z, \exp(z\cdot\zeta)\rangle_z \\ &= \left\langle \tilde{T}_z, \sum_{k=0}^{\infty} \frac{1}{k!\gamma_{k,n}} \tilde{j}_k(i\lambda\sqrt{\zeta^2}) P_{k,n}(z, \bar{\zeta}) \right\rangle_z \\ &= \sum_{k=0}^{\infty} \frac{1}{k!\gamma_{k,n} N(k,n)} \tilde{j}_k(i\lambda\sqrt{\zeta^2}) \overline{\overline{T}}_k(\zeta),\end{aligned}$$

where $\tilde{T}_k(\zeta)$ is the k-homogeneous harmonic component of $\tilde{T}$ defined by (15).

For $\tilde{f} \in H^2(\tilde{\mathbf{S}}_\lambda(r))$ we define the Fourier transform $\mathcal{F}_{\lambda,r}\tilde{f}(\zeta)$ by

$$\mathcal{F}_{\lambda,r}\tilde{f}(\zeta) = (\exp(z\cdot\zeta), \tilde{f}(z))_{\tilde{\mathbf{S}}_{\lambda,r}} = \int_{\tilde{\mathbf{S}}_{\lambda,r}} \exp(z\cdot\zeta)\overline{\tilde{f}(z)}dz.$$

Then $\mathcal{F}_{\lambda,r}\tilde{f} \in \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; (r))$ and we have

$$\mathcal{F}_{\lambda,r}\tilde{f}(\zeta) = \sum_{k=0}^{\infty} \frac{L_{k,\lambda,r}}{k!\gamma_{k,n} N(k,n)} \tilde{j}_k(i\lambda\sqrt{\zeta^2}) \bar{\tilde{f}}_k(\zeta), \tag{30}$$

where $\bar{\tilde{f}}_k$ is the k-homogeneous harmonic component of $\bar{\tilde{f}}$.

We are going to characterize the image of $H^2(\tilde{\mathbf{S}}_\lambda(r))$ under the Fourier transformation $\mathcal{F}_{\lambda,r}$. Let $\tilde{F}, \tilde{G} \in \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}})$. We define

$$((\tilde{F}, \tilde{G}))^{\lambda,r} = ((\beta_0^\lambda \tilde{F}, \beta_0^\lambda \tilde{G}))_{\tilde{\mathbf{S}}_0}^{\lambda,r},$$

where $\beta_0^\lambda : \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}) \to \mathrm{Exp}(\tilde{\mathbf{S}}_0)$ is the restriction mapping.

Put

$$E_r^\lambda(\zeta, \xi) = (\exp(z\cdot\zeta), \exp(z\cdot\xi))_{\tilde{\mathbf{S}}_{\lambda,r}} = \int_{\tilde{\mathbf{S}}_{\lambda,r}} \exp(z\cdot\zeta)\overline{\exp(z\cdot\xi)}\dot{d}z\,.$$

By the orthogonality and (28), we have

$$\begin{aligned} E_r^\lambda(\zeta,\xi) &= \sum_{k=0}^{\infty} \frac{\tilde{j}_k(i\lambda\sqrt{\zeta^2})\overline{\tilde{j}_k(i\lambda\sqrt{\xi^2})}}{(k!\gamma_{k,n})^2} \int_{\tilde{\mathbf{S}}_{\lambda,r}} P_{k,n}(z,\bar{\zeta})P_{k,n}(\bar{z},\xi)\dot{d}z \\ &= \sum_{k=0}^{\infty} \frac{\tilde{j}_k(i\lambda\sqrt{\zeta^2})\overline{\tilde{j}_k(i\lambda\sqrt{\xi^2})}}{(k!\gamma_{k,n})^2} \frac{L_{k,\lambda,r}}{N(k,n)} P_{k,n}(\zeta,\xi)\,, \quad \zeta,\xi\in\tilde{\mathbf{E}}\,. \end{aligned}$$

We call $E_r^\lambda(\zeta,\xi)$ the F-Cauchy kernel, whose restriction to $\tilde{\mathbf{S}}_0 \times \tilde{\mathbf{S}}_0$ coincides with the F-Poisson kernel $E_{r,0}^\lambda(\zeta,\xi)$ (see (21)). Note that $E_r^\lambda(\zeta,\xi) = \overline{E_r^\lambda(\xi,\zeta)}$ and the function $\zeta \mapsto E_r^\lambda(\zeta,\xi)$ belongs to $\mathcal{E}^2_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; r)$, where we define $\mathcal{E}^2_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; r)$ to be the class of all eigenfunctions $\tilde{F} \in \mathcal{O}_{\Delta-\lambda^2}(\tilde{\mathbf{E}})$ for which $((\tilde{F},\tilde{F}))^{\lambda,r} < \infty$.

8. Hilbert space of eigenfunctions.

$\mathcal{E}^2_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; r)$ is a Hilbert space. Let $\mathcal{P}^k_{\Delta-\lambda^2}(\tilde{\mathbf{E}})$ be the subspace of $\mathcal{E}^2_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; r)$ defined by

$$\mathcal{P}^k_{\Delta-\lambda^2}(\tilde{\mathbf{E}}) = \{\tilde{j}_k(i\lambda\sqrt{\zeta^2})\tilde{F}_k(\zeta);\ \tilde{F}_k \in \mathcal{P}^k_\Delta(\tilde{\mathbf{E}})\}\,;$$

that is, $\mathcal{P}^k_{\Delta-\lambda^2}(\tilde{\mathbf{E}}) = (\alpha_0^\lambda)^{-1}\mathcal{H}^k(\tilde{\mathbf{S}}_0)$. We have the following theorem which is parallel to Theorem 6.1.

THEOREM 8.1. (i) *The Hilbert space $\mathcal{E}^2_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; r)$ is the Hilbert space direct sum of the finite dimensional subspaces $\mathcal{P}^k_{\Delta-\lambda^2}(\tilde{\mathbf{E}})$:*

$$\mathcal{E}^2_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; r) = \bigoplus_{k=0}^{\infty} \mathcal{P}^k_{\Delta-\lambda^2}(\tilde{\mathbf{E}})\,. \tag{31}$$

The restriction mapping $\beta_0^\lambda : \mathcal{E}^2_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; r) \to \mathcal{E}^2(\tilde{\mathbf{S}}_0; \lambda, r)$ is a unitary isomorphism, which is compatible with the decompositions (22) *and* (31).

(ii) *The F-Cauchy kernel $E_r^\lambda(\zeta,\xi)$ is a reproducing kernel for the Hilbert space $\mathcal{E}^2_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; r)$; that is, for $\tilde{F} \in \mathcal{E}^2_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; r)$ we have*

$$\tilde{F}(\zeta) = ((\tilde{F}(\xi), E_r^\lambda(\xi,\zeta)))^{\lambda,r}\,.$$

(iii) *The Fourier transformation* $\mathcal{F}_{\lambda,r} : H^2(\tilde{\mathbf{S}}_\lambda(r)) \to \mathcal{E}^2_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; r)$ *is an antilinear unitary isomorphism. We have the following commutative diagram*:

$$\begin{array}{ccc} \mathfrak{h}^2_\lambda(\tilde{B}(r)) & \xrightarrow{\mathcal{F}^{\Delta}_{\lambda,r}} & \mathcal{E}^2(\tilde{\mathbf{S}}_0; \lambda, r) \\ \downarrow{\scriptstyle \alpha^0_\lambda} & & \uparrow{\scriptstyle \beta^\lambda_0} \\ H^2(\tilde{\mathbf{S}}_\lambda(r)) & \xrightarrow{\mathcal{F}_{\lambda,r}} & \mathcal{E}^2_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; r) . \end{array}$$

PROOF. The statements are clear by the definition. For example, to prove (iii), we have to show

$$((\mathcal{F}_{\lambda,r}\tilde{f}, \mathcal{F}_{\lambda,r}\tilde{g}))^{\lambda,r} = (\tilde{f}, \tilde{g})_{\tilde{\mathbf{S}}_{\lambda,r}}$$

for $\tilde{f}, \tilde{g} \in H^2(\tilde{\mathbf{S}}_\lambda(r))$. But this can be done by a direct calculation. □

EXAMPLE 8.2. If $w \in \tilde{B}(r)$ is fixed, then the function $\tilde{f}(z) = K_{\lambda,r}(z, w)$ is holomorphic in a neighborhood of $\tilde{\mathbf{S}}_\lambda[r]$. Then for $\zeta \in \tilde{\mathbf{E}}$ we have

$$\begin{aligned} \mathcal{F}_{\lambda,r} f(\zeta) &= \int_{\tilde{\mathbf{S}}_{\lambda,r}} \exp(z \cdot \zeta) K_{\lambda,r}(w, z) \dot{d}z \\ &= \sum_{k=0}^{\infty} \int_{\tilde{\mathbf{S}}_{\lambda,r}} \frac{1}{k!\gamma_{k,n}} \tilde{j}_k(i\lambda\sqrt{\zeta^2}) P_{k,n}(z, \bar{\zeta}) K_{\lambda,r}(w, z) \dot{d}z \\ &= \sum_{k=0}^{\infty} \frac{1}{k!\gamma_{k,n}} \tilde{j}_k(i\lambda\sqrt{\zeta^2}) P_{k,n}(w, \bar{\zeta}), \quad \zeta \in \tilde{\mathbf{E}} \end{aligned}$$

(see also Example 6.2).

EXAMPLE 8.3. Let $\xi \in \tilde{\mathbf{E}}$ be fixed. The $\mathcal{F}_{\lambda,r}$ image of the function $\tilde{f}(z) = \exp(z \cdot \xi) \in \mathcal{O}(\tilde{\mathbf{S}}_\lambda)$ is the F-Cauchy kernel $E^\lambda_r(\zeta, \xi)$ (see also Example 6.3).

Combining Theorem 6.4 with Theorem 8.1, we have the following theorem (see also Fujita [3]):

THEOREM 8.4. *Let* $|\lambda| < r$. *We have the following commutative diagram*:

$$\begin{array}{ccccc} \mathrm{Exp}(\tilde{\mathbf{S}}_0; [r]) & \hookrightarrow & \mathcal{E}^2(\tilde{\mathbf{S}}_0; \lambda, r) & \hookrightarrow & \mathrm{Exp}(\tilde{\mathbf{S}}_0; (r)) \\ \uparrow{\scriptstyle \beta^\lambda_0} & & \uparrow{\scriptstyle \beta^\lambda_0} & & \uparrow{\scriptstyle \beta^\lambda_0} \\ \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; [r]) & \hookrightarrow & \mathcal{E}^2_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; r) & \hookrightarrow & \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; (r)) . \end{array}$$

If $z \in \tilde{\mathbf{S}}_\lambda(r)$, then the function $\zeta \mapsto \exp(z \cdot \zeta)$ belongs to $\mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; [r])$ and

$$\exp(z \cdot \zeta) = \sum_{k=0}^{\infty} \frac{1}{k!\gamma_{k,n}} \tilde{j}_k(i\lambda\sqrt{\zeta^2}) P_{k,n}(z, \bar{\zeta})$$

by (28). Therefore, for $\tilde{F} \in \mathcal{E}^2_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; r)$ we form

$$\bar{\mathcal{F}}^S_{\lambda,r} \tilde{F}(z) = ((\exp(z \cdot \zeta), \tilde{F}(\zeta)))^{\lambda,r}, \quad z \in \tilde{\mathbf{S}}_\lambda(r) .$$

Then we have

(32)
$$\begin{aligned}\bar{\mathcal{F}}^S_{\lambda,r}\tilde{F}(z) &= \sum_{k=0}^{\infty}\frac{(k!\gamma_{k,n}N(k,n))^2}{L_{k,\lambda,r}}\frac{1}{k!\gamma_{k,n}}(P_{k,n}(z,\bar{\zeta}),\tilde{F}_k(\zeta))_{\mathbf{S}_1}\\ &= \sum_{k=0}^{\infty}\frac{k!\gamma_{k,n}N(k,n)}{L_{k,\lambda,r}}\overline{\tilde{F}_k}(z)\,.\end{aligned}$$

The function $\bar{\mathcal{F}}^S_{\lambda,r}\tilde{F}$ belongs to $H^2(\tilde{\mathbf{S}}_\lambda(r))$ and is called the spherical Fourier transform of $\tilde{F}$.

THEOREM 8.5. (i) *The spherical Fourier transformation*
$$\bar{\mathcal{F}}^S_{\lambda,r} : \mathcal{E}^2_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; r) \to H^2(\tilde{\mathbf{S}}_\lambda(r))$$
is an antilinear unitary isomorphism. $\bar{\mathcal{F}}^S_{\lambda,r}$ *is the inverse mapping of the Fourier transformation*:
$$\mathcal{F}_{\lambda,r} : H^2(\tilde{\mathbf{S}}_\lambda(r)) \to \mathcal{E}^2_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; r)\,.$$

(ii) *We have the following commutative diagram*:
$$\begin{array}{ccc}\mathcal{E}^2(\tilde{\mathbf{S}}_0; \lambda, r) & \xrightarrow{\bar{\mathcal{F}}_{\lambda,r}} & \mathfrak{h}^2_\lambda(\tilde{B}(r))\\ \uparrow{\scriptstyle \beta^\lambda_0} & & \downarrow{\scriptstyle \alpha^0_\lambda}\\ \mathcal{E}^2_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; r) & \xrightarrow{\bar{\mathcal{F}}^S_{\lambda,r}} & H^2(\tilde{\mathbf{S}}_\lambda(r))\,.\end{array}$$

PROOF. This theorem is a consequence of (25), (30) and (32). □

COROLLARY 8.6.

(33)
$$(\alpha^0_\lambda)^{-1} = \bar{\mathcal{F}}_{\lambda,r} \circ \beta^\lambda_0 \circ \mathcal{F}_{\lambda,r}\,.$$

Let $\tilde{f} \in H^2(\tilde{\mathbf{S}}_\lambda(r))$. Calculate $f(z) = ((\alpha^0_\lambda)^{-1}\tilde{f})(z) \in \mathfrak{h}^2_\lambda(\tilde{B}(r))$ by (33). Put $F(\xi) = \mathcal{F}_{\lambda,r}\tilde{f}(\zeta) = (\exp(w\cdot\zeta), \tilde{f}(w))_{\tilde{\mathbf{S}}_{\lambda,r}}$. Then we have
$$\begin{aligned}f(z) &= ((\bar{\mathcal{F}}_{\lambda,r} \circ \beta^\lambda_0 \circ \mathcal{F}_{\lambda,r})\tilde{f})(z)\\ &= ((\bar{\mathcal{F}}_{\lambda,r} \circ \beta^\lambda_0)F)(z)\\ &= ((\exp(z\cdot\zeta), F(\zeta)))^{\lambda,r}\\ &= ((\exp(z\cdot\zeta), (\exp(w\cdot\zeta), \tilde{f}(w))_{\tilde{\mathbf{S}}_{\lambda,r}}))^{\lambda,r}\\ &= (\tilde{f}(w), ((\exp(w\cdot\zeta), \exp(z\cdot\zeta)))^{\lambda,r})_{\tilde{\mathbf{S}}_{\lambda,r}}\\ &= (\tilde{f}(w), K_{\lambda,r}(w,z))_{\tilde{\mathbf{S}}_{\lambda,r}}, \quad z \in \tilde{B}(r)\,.\end{aligned}$$

This is the Poisson integral formula (see Theorem 2.3 (iii)).

COROLLARY 8.7.

(34)
$$(\beta^\lambda_0)^{-1} = \mathcal{F}_{\lambda,r} \circ \alpha^0_\lambda \circ \bar{\mathcal{F}}_{\lambda,r}\,.$$

Let $F \in \mathcal{E}^2(\tilde{\mathbf{S}}_0; \lambda, r)$. Calculate $\tilde{F} = (\beta_0^\lambda)^{-1} F \in \mathcal{E}^2_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; r)$ by (34). Put $f(z) = \tilde{\mathcal{F}}_{\lambda,r} F(z) = ((\exp(z \cdot \zeta), F(\zeta)))^{\lambda,r} \in \mathfrak{h}^2_\lambda(\tilde{B}(r))$. We have

$$\begin{aligned}\tilde{F}(\xi) &= ((\mathcal{F}_{\lambda,r} \circ \alpha^0_\lambda \circ \tilde{\mathcal{F}}_{\lambda,r})F)(\xi)\\ &= ((\mathcal{F}_{\lambda,r} \circ \alpha^0_\lambda) f)(\xi)\\ &= (\exp(z \cdot \xi), ((\exp(z \cdot \zeta), F(\zeta)))^{\lambda,r})_{\tilde{\mathbf{S}}_{\lambda,r}}\\ &= ((F(\zeta), (\exp(z \cdot \zeta), \exp(z \cdot \xi))_{\tilde{\mathbf{S}}_{\lambda,r}}))^{\lambda,r}\\ &= ((F(\zeta), E^\lambda_r(\zeta, \xi)))^{\lambda,r}\,.\end{aligned}$$

Thus the F-Cauchy kernel $E^\lambda_r(\zeta, \xi)$ defined the inverse mapping of β^λ_0 (see Fujita-Morimoto [5]).

REMARK 8.8. Suppose $\tilde{\rho}_{\lambda,r}(s)$ is a function on $[0, \infty)$ satisfying

$$\int_0^\infty |\tilde{j}_k(i\lambda s)|^2 s^{2k+n-1} \tilde{\rho}_{\lambda,r}(s) ds = \frac{(k!\gamma_{k,n} N(k,n))^2}{L_{k,\lambda,r}}\,. \tag{35}$$

Let $\tilde{F}, \tilde{G} \in \mathcal{O}_{\Delta-\lambda^2}(\tilde{\mathbf{E}})$. Then for $s > 0$

$$\begin{aligned}\int_{\mathbf{S}_1} \tilde{F}(sx)\overline{\tilde{G}(sx)} dx &= \int_{\mathbf{S}_1} \left(\sum_{k=0}^\infty \tilde{j}_k(i\lambda s) \tilde{F}_k(sx)\right) \left(\sum_{l=0}^\infty \overline{\tilde{j}_l(i\lambda s) \tilde{G}_l(sx)}\right) dx\\ &= \sum_{k=0}^\infty |\tilde{j}_k(i\lambda s)|^2 s^{2k} (\tilde{F}_k, \tilde{G}_k)_{\mathbf{S}_1}\,.\end{aligned}$$

Therefore, by means of the function $\tilde{\rho}_{\lambda,r}$ the inner product $((\tilde{F}, \tilde{G}))^{\lambda,r}$ can be written as follows:

$$((\tilde{F}, \tilde{G}))^{\lambda,r} = \int_0^\infty \left(\int_{\mathbf{S}_1} \tilde{F}(sx)\overline{\tilde{G}(sx)} dx\right) \tilde{\rho}_{\lambda,r}(s) s^{n-1} ds\,.$$

Because $\mathbf{E} = \bigcup\{s\mathbf{S}_1; s \geq 0\} = \bigcup\{\mathbf{S}_s; s \geq 0\}$, we can understand that the inner product $((\tilde{F}, \tilde{G}))^{\lambda,r}$ is defined by a kind of integral over $\mathbf{E}$.

If $\lambda = 0$, then the condition (35) becomes as follows:

$$\int_0^\infty s^{2k+n-1} \tilde{\rho}_{0,r}(s) ds = \frac{2^k k!^2 \gamma_{k,n} N(k,n)^2}{r^{2k}}\,.$$

Therefore, we can take $\tilde{\rho}_{0,r}(s) = \rho_{r,r}(s) = \rho_r(s)$, where $\rho_r(s)$ is the Ii-Wada function. This remark is due to Fujita [2].

References

[1] A. J. DURAN, The Stieltjes moments problem for rapidly decreasing functions, Proc. AMS **107** (1989), 731–741.

[2] K. FUJITA, Hilbert spaces related to harmonic functions, Tôhoku Math. J. **48** (1996), 146–163.

[3] K. FUJITA, On some function spaces of eigenfunctions of the Laplacian, Proc. of the Fifth International Colloquium on Complex Analysis, Beijing (1997), 61–66.

[4] K. FUJITA, Hilbert spaces of eigenfunction of the Laplacian, *Reproducing Kernels and their Application*, (eds. S. Saitoh et al.), Kluwer Academic Publishers (1999), 65–80.

[5] K. FUJITA and M. MORIMOTO, Integral representation for eigenfunction of the Laplacian, J. Math. Soc. Japan **51** (1999), 699–713.

[6] L. K. HUA, *Harmonic Analysis of Functions of Several Complex Variables in Classical Domains*, (1959) (in Russian), [Transl. Math. Monographs **6** (1963), Amer. Math. Soc.].

[7] K. II, On the Bargmann-type transform and a Hilbert space of holomorphic functions, Tôhoku Math. J. **38** (1986), 57–69.

[8] M. MORIMOTO, Analytic functionals on the Lie sphere, Tokyo J. Math. **3** (1980), 1–35.

[9] M. MORIMOTO, Analytic functionals on the sphere and their Fourier-Borel transformations, *Complex Analysis*, Banach Center Publ. **11** (1983), PWN-Polish Scientific Publishers, 223–250.

[10] M. MORIMOTO, *An Introduction to Sato's Hyperfunctions*, Translations of Mathematical Monograph, **129** (1993). A. M. S.

[11] M. MORIMOTO, Entire functions of exponential type on the complex sphere, Proc. V. A. Stekkov Inst. Math. **203** (1994), 334–364, (in Russian), [Transl. Proc. Steklov Inst. Math. **203** (1995), 281–303].

[12] M. MORIMOTO, A generalization of the Cauchy-Hua integral formula on the Lie ball, Tokyo J. Math. **22** (1999), 177–192.

[13] M. MORIMOTO and K. FUJITA, Analytic functionals and entire functionals on the complex light cone, Hiroshima Math. J. **25** (1995), 493–512.

[14] M. MORIMOTO and K. FUJITA, Analytic functionals on the complex sphere and eigenfunctions of the Laplacian on the Lie ball, *Structure of Solutions of Differential Equations*, Katata/Kyoto 1995 (eds., M. Morimoto and T. Kawai), World Scientific (1996), 287–305.

[15] M. MORIMOTO and K. FUJITA, Conical Fourier-Borel transformation for harmonic functionals on the Lie ball, *Generalizations of Complex Analysis and their Applications in Physics*, (ed. J. Ławrynowicz), Banach Center Publ. **37** (1996), 95–113.

[16] M. MORIMOTO and K. FUJITA, Eigen functions of the Laplacian of exponential type, *New Trend in Microlocal Analysis* (eds. J.-M. Bony and M. Morimoto), Springer (1997), 39–58.

[17] R. WADA, On the Fourier-Borel transformations of analytic functionals on the complex sphere, Tôhoku Math. J. **38** (1986), 417–432.

[18] R. WADA, Holomorphic functions on the complex sphere, Tokyo, J. Math. **11** (1988), 205–218.

[19] R. WADA and M. MORIMOTO, A uniqueness set for the differential operator $\Delta_z + \lambda^2$, Tokyo J. Math. **10** (1987), 93–105.

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