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Dual Class of a Subvariety

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Let M be a complex manifold of dimension n and E a holomorphic vector bundle of rank k over M. If s is a regular section of E (cf. [F] B.3), it defines an analytic subspace X of pure codimension k in M. It is "well-known" that, if M is compact, then the top Chern class $c_k(E)$ of E corresponds to the homology class $[X]$ of X under the Poincaré duality $P : H^{2k}(M; \mathbf{C}) \xrightarrow{\sim} H_{2n-2k}(M; \mathbf{C})$ (in fact this holds with $\mathbf{Z}$ coefficients). The nature of the proof of this fact depends on how one defines the class $c_k(E)$ (cf. [G] §5 for the projective non-singular case, [F] §14.1 for the general case in the algebraic category and [GH] Ch. 1, §1 for the case $k = 1$ in the complex analytic category). In this article, we take up the definition of Chern classes via the Chern-Weil theory and give a relatively elementary proof of a more precise statement in the complex analytic category. Namely, we prove the following. Let V denote the support of X, then there is a canonical localization $c_k(E, s)$, in the relative cohomology $H^{2k}(M, M \setminus V; \mathbf{C})$, of $c_k(E)$ with respect to s and, if V is compact (M may not be), the class $c_k(E, s)$ corresponds to $[X]$ under the Alexander duality

$$A : H^{2k}(M, M \setminus V; \mathbf{C}) \xrightarrow{\sim} H_{2n-2k}(V; \mathbf{C})$$

(Theorem 4.2). If M is compact, we have the commutative diagram

$$\begin{tikzcd}
H^{2k}(M, M \setminus V; \mathbf{C}) \arrow[r, "j^*"] \arrow[d, "A", "\wr"'] & H^{2k}(M; \mathbf{C}) \arrow[d, "P", "\wr"'] \\
H_{2n-2k}(V; \mathbf{C}) \arrow[r, "i_*"] & H_{2n-2k}(M; \mathbf{C})
\end{tikzcd}$$

where i and j denote the inclusions $V \hookrightarrow M$ and $(M, \emptyset) \hookrightarrow (M, M \setminus V)$, respectively. Since $j^*(c_k(E, s)) = c_k(E)$, we recover the result we first mentioned. For an application, see [S2].

As related topics, we discuss intersections of analytic subspaces. We also prove a duality theorem when V as above may not be compact, considering X as a relative cycle in M modulo $M \setminus S$ for a compact connected component S of its singular set (Theorem 6.4). This fact is effectively used in [BLSS]. The proofs of the above results are done in the framework of Čech-de Rham cohomology.

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In Section 1, we recall the Čech-de Rham cohomology and integration theory on it, describe the Poincaré and Alexander dualities and define the characteristic classes in the Čech-de Rham cohomology. In Section 2, we give a short discussion on the localization of the top Chern class of a vector bundle with respect to a section and the corresponding residue. We express the residue at an isolated zero of the section in terms of the Grothendieck residue in Section 3. This is used in Section 4 to prove the duality theorem mentioned above. In Section 5, we discuss (refined) intersections of analytic subspaces. Combined with results in the previous sections, we reprove that the global intersection number of divisors intersecting at isolated points is the sum of local intersection numbers. Finally, in Section 6 we prove the other type of duality theorem mentioned above.

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1. Preliminaries.

For the background on the Čech-de Rham cohomology, we refer to [BT]. The integration theory on the Čech-de Rham cohomology is developed in [Le1–4]. For the Chern-Weil theory of characteristic classes of vector bundles, we refer to [BB], [B], [GH] and [MS]. See also [S1] for the material in this section.

(A) Čech-de Rham cohomology. Let M be a (connected) oriented C^∞ manifold of dimension m. For an open set U in M, we denote by $A^q(U)$ the space of complex valued C^∞ q-forms on U. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ be an open covering of M and set $U_{\alpha_0\cdots\alpha_p} = U_{\alpha_0}\cap\cdots\cap U_{\alpha_p}$. We assume that I is an ordered set such that, if $U_{\alpha_0\cdots\alpha_p} \neq \emptyset$, the induced order on the subset $\{\alpha_0, \cdots, \alpha_p\}$ is total. We let $I^{(p)}$ be the set of $(p+1)$-tuples $(\alpha_0, \cdots, \alpha_p)$ with $\alpha_0 < \cdots < \alpha_p$ and denote by $C^p(\mathcal{U}, A^q)$ the direct product

$$C^p(\mathcal{U}, A^q) = \prod_{(\alpha_0,\cdots,\alpha_p)\in I^{(p)}} A^q(U_{\alpha_0\cdots\alpha_p}).$$

Thus an element σ in $C^p(\mathcal{U}, A^q)$ assigns to each $(\alpha_0, \cdots, \alpha_p)$ in $I^{(p)}$ an element $\sigma_{\alpha_0\cdots\alpha_p}$ in $A^q(U_{\alpha_0\cdots\alpha_p})$. The coboundary operator $\delta : C^p(\mathcal{U}, A^q) \to C^{p+1}(\mathcal{U}, A^q)$ is defined as in the usual Čech cohomology theory. This together with the exterior derivative d makes the collection $C^\bullet(\mathcal{U}, A^\bullet)$ a double complex. The simple complex associated to this is denoted by $(A^\bullet(\mathcal{U}), D)$ or simply by $A^\bullet(\mathcal{U})$. Thus $A^r(\mathcal{U}) = \bigoplus_{p+q=r} C^p(\mathcal{U}, A^q)$ and the differential $D : A^r(\mathcal{U}) \to A^{r+1}(\mathcal{U})$ is given by

$$(D\sigma)_{\alpha_0\cdots\alpha_p} = \sum_{\nu=0}^{p}(-1)^\nu \sigma_{\alpha_0\cdots\widehat{\alpha_\nu}\cdots\alpha_p} + (-1)^p d\sigma_{\alpha_0\cdots\alpha_p}.$$

We denote by $H^r(A^\bullet(\mathcal{U}))$ the cohomology of $(A^\bullet(\mathcal{U}), D)$ and call it the Čech-de Rham cohomology associated to the covering $\mathcal{U}$. It is known (e.g., [BT]) that the restriction map

$A^r(M) \to C^0(\mathcal{U}, A^r) \subset A^r(\mathcal{U})$ induces an isomorphism

$$H^r(M; \mathbf{C}) \xrightarrow{\sim} H^r(A^\bullet(\mathcal{U}))\,, \tag{1.1}$$

where $H^r(M; \mathbf{C})$ denotes the de Rham cohomology of M.

We define the "cup product"

$$A^r(\mathcal{U}) \times A^s(\mathcal{U}) \to A^{r+s}(\mathcal{U})$$

by assigning to (σ, τ) in $A^r(\mathcal{U}) \times A^s(\mathcal{U})$ the element $\sigma \smile \tau$ in $A^{r+s}(\mathcal{U})$ given by

$$(\sigma \smile \tau)_{\alpha_0 \cdots \alpha_p} = \sum_{\nu=0}^{p} (-1)^{(r-\nu)(p-\nu)} \sigma_{\alpha_0 \cdots \alpha_\nu} \wedge \tau_{\alpha_\nu \cdots \alpha_p}\,.$$

Then $\sigma \smile \tau$ is linear in σ and τ and we have

$$D(\sigma \smile \tau) = D\sigma \smile \tau + (-1)^r \sigma \smile D\tau\,.$$

Thus it induces the cup product

$$H^r(A^\bullet(\mathcal{U})) \times H^s(A^\bullet(\mathcal{U})) \to H^{r+s}(A^\bullet(\mathcal{U}))$$

compatible, via (1.1), with the usual cup product in the de Rham cohomology.

In what follows, we use the following convention. Let $(\alpha_0, \cdots, \alpha_p)$ be an element in I^{p+1}. If $U_{\alpha_1 \cdots \alpha_p}$ is non-empty and if the α_i's are distinct, there is a permutation ρ such that $(\alpha_{\rho(0)}, \cdots, \alpha_{\rho(p)})$ is in increasing order. Then we set $\sigma_{\alpha_0 \cdots \alpha_p} = \operatorname{sign} \rho \cdot \sigma_{\alpha_{\rho(0)} \cdots \alpha_{\rho(p)}}$. Otherwise, we set $\sigma_{\alpha_0 \cdots \alpha_p} = 0$. Note that this is consistent with the definitions of the coboundary operator and the cup product.

A system of honey-comb cells adapted to $\mathcal{U}$ ([Le1–4]) is a collection $\{R_\alpha\}_{\alpha \in I}$ of m-dimensional manifolds R_α with piecewise C^∞ boundary in M satisfying the following conditions:

(1) $R_\alpha \subset U_\alpha$ and $M = \bigcup_\alpha R_\alpha$.

(2) $\operatorname{Int} R_\alpha \cap \operatorname{Int} R_\beta = \emptyset$, if $\alpha \neq \beta$.

(3) If $U_{\alpha_0 \cdots \alpha_p} \neq \emptyset$, $R_{\alpha_0 \cdots \alpha_p} = \bigcap_{\nu=0}^{p} R_{\alpha_\nu} (= \bigcap_{\nu=0}^{p} \partial R_{\alpha_\nu})$ is an $(m-p)$-dimensional manifold with piecewise C^∞ boundary.

(4) If the set $\{\alpha_0, \cdots, \alpha_p\}$ is maximal, $R_{\alpha_0 \cdots \alpha_p}$ has no boundary.

In the above, Int R denotes the interior of a subset R in M and $\{\alpha_0, \cdots, \alpha_p\}$ being maximal means that, if $U_{\alpha, \alpha_0, \cdots, \alpha_p} \neq \emptyset$, then $\alpha \in \{\alpha_0, \cdots, \alpha_p\}$. We orient $R_{\alpha_0 \cdots \alpha_p}$ by the following rules:

(1) Each R_α has the same orientation as M and the boundary is oriented so that, if $(x_1, \cdots, x_m)$ is a positive coordinate system on an open set U in M with $R_\alpha \cap U = \{x_m \geq 0\}$, then the coordinate system $(x_1, \cdots, x_{m-1})$ on ∂R_α is positive or negative according as m is even or odd.

(2) If ρ is a permutation, $R_{\alpha_{\rho(0)} \cdots \alpha_{\rho(p)}} = \operatorname{sign} \rho \cdot R_{\alpha_0 \cdots \alpha_p}$.

(3) $\partial R_{\alpha_0 \cdots \alpha_p} = \sum_\alpha R_{\alpha_0 \cdots \alpha_p \alpha}$.

Let $\{R_\alpha\}$ be a system of honey-comb cells adapted to $\mathcal{U}$. Suppose M is compact, then each R_α is compact and we define the integration

$$\int_M : A^m(\mathcal{U}) \to \mathbf{C}$$

by the sum

$$\int_M \sigma = \sum_{p=0}^{m} \left(\sum_{(\alpha_0,\cdots,\alpha_p)\in I^{(p)}} \int_{R_{\alpha_0\cdots\alpha_p}} \sigma_{\alpha_0\cdots\alpha_p} \right)$$

for σ in $A^m(\mathcal{U})$. Then we see that it induces the integration on the cohomology

$$\int_M : H^m(A^\bullet(\mathcal{U})) \to \mathbf{C},$$

which is compatible, via (1.1), with the usual integration on the de Rham cohomology.

(B) Duality theorems. If M is a compact oriented C^∞ manifold of dimension m, the bilinear pairing

$$A^l(\mathcal{U}) \times A^{m-l}(\mathcal{U}) \to A^m(\mathcal{U}) \to \mathbf{C}$$

defined as the composition of the cup product and the integration induces the Poincaré duality

$$P_M : H^l(M; \mathbf{C}) \simeq H^l(A^\bullet(\mathcal{U})) \xrightarrow{\sim} H^{m-l}(A^\bullet(\mathcal{U}))^* \simeq H_{m-l}(M; \mathbf{C}).$$

In the above isomorphism, a class $[\sigma]$ in $H^l(A^\bullet(\mathcal{U}))$ corresponds to the class $[C]$ in $H_{m-l}(M; \mathbf{C})$ such that

$$\int_M \sigma \smile \tau = \int_C \tau$$

for all τ in $A^{m-l}(\mathcal{U})$ with $D\tau = 0$, where we choose the cycle C in its homology class so that it is transverse to each $R_{\alpha_0\cdots\alpha_p}$ and the integral in the right hand side is defined by

$$\sum_{p=0}^{m} \left(\sum_{(\alpha_0,\cdots,\alpha_p)\in I^{(p)}} \int_{R_{\alpha_0\cdots\alpha_p}\cap C} \tau_{\alpha_0\cdots\alpha_p} \right).$$

We may define, for an r-chain C transverse to each $R_{\alpha_0\cdots\alpha_p}$ and an s-cochain σ in $A^s(\mathcal{U})$, an $(r-s)$-chain $C \frown \sigma$, which assigns to an $(r-s)$-cochain τ in $A^{r-s}(\mathcal{U})$ the valeu $\int_C \sigma \smile \tau$. This induces the cap product

$$H_r(M; \mathbf{C}) \times H^s(A^\bullet(\mathcal{U})) \to H_{r-s}(M; \mathbf{C}).$$

Then we may write

$$P_M([\sigma]) = [M] \frown [\sigma],$$

where $[M]$ denotes the fundamental class of M.

Now let M be an oriented manifold of dimension m again and S a closed set in M. Let $U_0 = M \setminus S$ and U_1 a neighborhood of S in M and consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M with $0 < 1$. We denote by $A^r(\mathcal{U}, U_0)$ the kernel of the canonical projection $A^r(\mathcal{U}) \to A^r(U_0)$. It is not difficult to see that

$$H^r(A^\bullet(\mathcal{U}, U_0)) \simeq H^r(M, M \setminus S; \mathbf{C}).$$

Let $\{R_0, R_1\}$ be a system of honey-comb cells adapted to $\mathcal{U}$. Recall that, if M is compact,

$$\int_M \sigma = \int_{R_0} \sigma_0 + \int_{R_1} \sigma_1 + \int_{R_{01}} \sigma_{01}$$

for $\sigma = (\sigma_0, \sigma_1, \sigma_{01})$ in $A^m(\mathcal{U})$. Now suppose that only S is compact (M may not be). Then we may assume that R_1 is compact and we still have the integration

$$\int_M : A^m(\mathcal{U}, U_0) \to \mathbf{C}$$

given by

$$\int_M \sigma = \int_{R_1} \sigma_1 + \int_{R_{01}} \sigma_{01}$$

for $\sigma = (0, \sigma_1, \sigma_{01})$ in $A^m(\mathcal{U}, U_0)$. This again induces the integration on the cohomology

$$\int_M : H^m(A^\bullet(\mathcal{U}, U_0)) \to \mathbf{C}\,.$$

In the cup product $A^l(\mathcal{U}) \times A^{m-l}(\mathcal{U}) \to A^m(\mathcal{U})$, we have

$$(\sigma_0, \sigma_1, \sigma_{01}) \smile (\tau_0, \tau_1, \tau_{01}) = (\sigma_0 \wedge \tau_0, \sigma_1 \wedge \tau_1, (-1)^r \sigma_0 \wedge \tau_{01} + \sigma_{01} \wedge \tau_1)\,.$$

Hence, if $\sigma_0 = 0$, the right hand side depends only on σ_1, σ_{01} and τ_1. Thus we have a pairing $A^l(\mathcal{U}, U_0) \times A^{m-l}(U_1) \to A^m(\mathcal{U}, U_0)$, which, followed by the integration, gives a bilinear pairing

$$A^l(\mathcal{U}, U_0) \times A^{m-l}(U_1) \to \mathbf{C}\,.$$

If we further assume that U_1 is a regular neighborhood of S, this induces the Alexander duality

$$A_{M_S} : H^l(M, M \setminus S; \mathbf{C}) \simeq H^l(A^\bullet(\mathcal{U}, U_0)) \xrightarrow{\sim} H^{m-l}(U_1; \mathbf{C})^* \simeq H_{m-l}(S; \mathbf{C})\,. \tag{1.2}$$

Similarly we have

$$H^{m-l}(S; \mathbf{C}) \simeq H^{m-l}(U_1; \mathbf{C}) \xrightarrow{\sim} H^l(A^\bullet(\mathcal{U}, U_0))^* \simeq H_l(M, M \setminus S; \mathbf{C})\,. \tag{1.3}$$

In the isomorphism (1.2), a class $[\sigma] = [(0, \sigma_1, \sigma_{01})]$ in $H^l(A^\bullet(\mathcal{U}, U_0))$ corresponds to the class $[C]$ in $H_{m-l}(S; \mathbf{C})$ such that

$$\int_{R_1} \sigma_1 \wedge \tau_1 + \int_{R_{01}} \sigma_{01} \wedge \tau_1 = \int_C \tau_1\,, \tag{1.4}$$

for all τ_1 in $A^{m-l}(U_1)$ with $d\tau_1 = 0$. Also, in the isomorphism (1.3), a class $[\tau_1]$ in $H^{m-l}(U_1; \mathbf{C})$ corresponds to the class $[C]$ in $H_l(M, M \setminus S; \mathbf{C})$ such that

$$\int_{R_1} \sigma_1 \wedge \tau_1 + \int_{R_{01}} \sigma_{01} \wedge \tau_1 = \int_{R_1 \cap C} \sigma_1 + \int_{R_{01} \cap C} \sigma_{01}\,, \tag{1.5}$$

for all $\sigma = (0, \sigma_1, \sigma_{01})$ in $A^l(\mathcal{U}, U_0)$ with $D\sigma = 0$. If S is connected, then we have $H_m(M, M \setminus S; \mathbf{C}) \simeq H^0(S; \mathbf{C}) = \mathbf{C}$. We denote by $[M_S]$ the class in $H_m(M, M \setminus S; \mathbf{C})$ corresponding to $[1]$ in $H^0(S; \mathbf{C})$. We may also define the cap product

$$H_r(M, M \setminus S; \mathbf{C}) \times H^s(M, M \setminus S; \mathbf{C}) \to H_{r-s}(S; \mathbf{C})$$

as before. Then we may write

$$A_{M_S}([\sigma]) = [M_S] \frown [\sigma].$$

When M is compact, we have the commutative diagram

$$\begin{tikzcd} H^l(M, M \setminus S; \mathbf{C}) \arrow[r, "j^*"] \arrow[d, "A_{M_S}", "\wr"'] & H^l(M; \mathbf{C}) \arrow[d, "P_M", "\wr"'] \\ H_{m-l}(S; \mathbf{C}) \arrow[r, "i_*"] & H_{m-l}(M; \mathbf{C}), \end{tikzcd} \tag{1.6}$$

where i and j denote, respectively, the inclusions $S \hookrightarrow M$ and $(M, \emptyset) \hookrightarrow (M, M \setminus S)$.

We also describe the Alexander duality in another situation we consider later. Let M be a complex manifold of (complex) dimension n and V a compact analytic subvariety (reduced analytic subspace) in M. Let $S = \mathrm{Sing}(V)$ be the singular set of V. Also, let $U_0 = M \setminus V$, U_1 a sufficiently small tubular neighborhood of $V' = V \setminus S$ and U_2 a sufficiently small regular neighborhood of S in M. We consider the coverings $\mathcal{U} = \{U_0, U_1, U_2\}$ of M and $\mathcal{U}' = \{U_1, U_2\}$ of $U = U_1 \cup U_2$, which may be assumed to be a regular neighborhood of V. An element σ in $A^l(\mathcal{U})$ is expressed as $(\sigma_0, \sigma_1, \sigma_2, \sigma_{01}, \sigma_{02}, \sigma_{12}, \sigma_{012})$. We denote by $A^l(\mathcal{U}, U_0)$ the subspace $\{\sigma \in A^l(\mathcal{U}) \mid \sigma_0 = 0\}$ of $A^l(\mathcal{U})$. The Alexander duality

$$H^l(M, M \setminus V; \mathbf{C}) \simeq H^l(A^\bullet(\mathcal{U}, U_0)) \xrightarrow{\sim} H_{2n-l}(U; \mathbf{C}) \simeq H_{2n-l}(V; \mathbf{C}) \tag{1.7}$$

is induced from the pairing

$$B : A^l(\mathcal{U}, U_0) \times A^{2n-l}(\mathcal{U}') \to \mathbf{C}$$

given by, for $\sigma = (0, \sigma_1, \sigma_2, \sigma_{01}, \sigma_{02}, \sigma_{12}, \sigma_{012})$ in $A^l(\mathcal{U}, U_0)$ and $\tau = (\tau_1, \tau_2, \tau_{12})$ in $A^{2n-l}(\mathcal{U}')$,

$$\begin{aligned} B(\sigma, \tau) = &\int_{R_1} \sigma_1 \wedge \tau_1 + \int_{R_2} \sigma_2 \wedge \tau_2 + \int_{R_{01}} \sigma_{01} \wedge \tau_1 + \int_{R_{02}} \sigma_{02} \wedge \tau_2 \\ &+ \int_{R_{12}} (\sigma_1 \wedge \tau_{12} + \sigma_{12} \wedge \tau_2) + \int_{R_{012}} (-\sigma_{01} \wedge \tau_{12} + \sigma_{012} \wedge \tau_2), \end{aligned}$$

where $\{R_0, R_1, R_2\}$ is a system of honey-comb cells adapted to $\mathcal{U}$. Thus in the Alexander duality (1.7), the class $[\sigma]$ in $H^l(A^\bullet(\mathcal{U}, U_0))$ corresponds to the class $[C]$ in $H_{2n-l}(V; \mathbf{C})$ such that

$$B(\sigma, \tau) = \int_{R_1 \cap C} \tau_1 + \int_{R_2 \cap C} \tau_2 + \int_{R_{12} \cap C} \tau_{12}. \tag{1.8}$$

for all τ in $A^{2n-l}(\mathcal{U}')$ with $D\tau = 0$.

(C) Characteristic classes in the Čech-de Rham cohomology. Let M be a C^∞ manifold of dimension m and E a C^∞ complex vector bundle of (complex) rank r on M. For a connection ∇ for E and for $i = 1, \cdots, r$, we denote by $c_i(\nabla)$ the i-th Chern form defined by ∇. Thus it is a closed $2i$-form on M and its class $[c_i(\nabla)]$ in $H^{2i}(M; \mathbf{C})$ is the i-th Chern class $c_i(E)$ of E.

If we have $p+1$ connections $\nabla_0, \cdots, \nabla_p$ for E there is a $(2i-p)$-form $c_i(\nabla_0, \cdots, \nabla_p)$ alternating in the $p+1$ entries and satisfying

$$\sum_{\nu=0}^{p}(-1)^\nu c_i(\nabla_0, \cdots, \widehat{\nabla_\nu}, \cdots, \nabla_p) + (-1)^p dc_i(\nabla_0, \cdots, \nabla_p) = 0 \tag{1.9}$$

(cf. [B]. Here we use a different sign convention, see [S1] Ch.II, (7.10)).

Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ be an open covering of M as in (A). For each α, we choose a connection ∇_α for E on U_α, and for the collection $\nabla_* = (\nabla_\alpha)_\alpha$, we define the element $c_i(\nabla_*)$ in $A^{2i}(\mathcal{U})$ by

$$c_i(\nabla_*)_{\alpha_0\cdots\alpha_p} = c_i(\nabla_{\alpha_0}, \cdots, \nabla_{\alpha_p}).$$

Then we have $Dc_i(\nabla_*) = 0$ by (1.9). Moreover, it is shown that the class of $c_i(\nabla_*)$ in $H^{2i}(A^\bullet(\mathcal{U}))$ does not depend on the choice of the collection of connections ∇_*. Comparing with the class defined by a global connection, we see tht the class $[c_i(\nabla_*)]$ corresponds to the class $c_i(E)$ in $H^{2i}(M; \mathbf{C})$ under the isomorphism (1.1).

2. Localization of the top Chern class.

Let $\pi : E \to M$ be a C^∞ complex vector bundle of rank r over an oriented C^∞ manifold M of dimension m as in the previous section. We say that a connection ∇ for E is trivial with respect to a non-vanishing section s (simply, s-trivial), if $\nabla(s) = 0$. Note that if $\nabla_0, \cdots, \nabla_p$ are s-trivial connections, we have the vanishing (cf. [S1] Ch.II, Proposition 9.1)

$$c_r(\nabla_0, \cdots, \nabla_p) = 0. \tag{2.1}$$

Let S be a closed set in M and suppose we have a C^∞ non-vanishing section s of E on $M \setminus S$. Then, from the above fact, we will see that there is a localization $c_r(E, s)$ in $H^{2r}(M, M \setminus S; \mathbf{C})$ of the top Chern class $c_r(E)$ in $H^{2r}(M; \mathbf{C})$.

Letting $U_0 = M\setminus S$ and U_1 a neighborhood of S, we consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M. Recall the Chern class $c_r(E)$ is represented by the cocycle $c_r(\nabla_*)$ in $A^{2r}(\mathcal{U})$ given by

$$c_r(\nabla_*) = (c_r(\nabla_0), c_r(\nabla_1), c_r(\nabla_0, \nabla_1)),$$

where ∇_0 and ∇_1 denote connections for E on U_0 and U_1, respectively. If we take as ∇_0 an s-trivial connection, then $c_r(\nabla_0) = 0$ and thus the cocycle is in $A^{2r}(\mathcal{U}, U_0)$ and it defines a class in the relative cohomology $H^{2r}(M, M\setminus S; \mathbf{C})$, which we denote by $c_r(E.s)$. It is sent to the class $c_r(E)$ by the canonical homomorphism $j^* : H^{2r}(M, M \setminus S; \mathbf{C}) \to H^{2r}(M; \mathbf{C})$. It does not depend on the choice of the connection ∇_1 or on the choice of the s-trivial connection ∇_0 ([S1]). We call $c_r(E, s)$ the localization of $c_r(E)$ at S with respect to the section s.

In the above situation, suppose that S is a compact set admitting a regular neighborhood. Then we have the Alexander duality (1.2)

$$A_{M_S} : H^{2r}(M, M \setminus S; \mathbf{C}) \xrightarrow{\sim} H_{m-2r}(S; \mathbf{C}).$$

Thus the class $c_r(E, s)$ defines a class in $H_{m-2r}(S; \mathbf{C})$, which we call the residue of $c_r(E)$ at S with respect to s and denote by $\operatorname{Res}_{c_r}(s, E; S)$. This residue corresponds to what is called the "localized top Chern class" of E with respect to s in [F] §14.1.

Let R_1 be an m-dimensional manifold with C^∞ boundary in U_1 containing S in its interior and set $R_0 = M \setminus \operatorname{Int} R_1$ so that $\{R_0, R_1\}$ is a system of honey-comb cells adapted to $\mathcal{U}$. Then the residue $\operatorname{Res}_{c_r}(s, E; S)$ is represented by an $(m-2r)$-cycle C in S such that

$$\int_C \tau_1 = \int_{R_1} c_r(\nabla_1) \wedge \tau_1 + \int_{R_{01}} c_r(\nabla_0, \nabla_1) \wedge \tau_1$$

for any closed $(m-2r)$-form τ_1 on U_1. In particular, if $2r = m$, the residue is a complex number given by

$$\operatorname{Res}_{c_r}(s, E; S) = \int_{R_1} c_r(\nabla_1) + \int_{R_{01}} c_r(\nabla_0, \nabla_1). \tag{2.2}$$

If we let $(S_\lambda)_\lambda$ be the connected components of S, we have

$$H_{m-2r}(C; \mathbf{C}) = \bigoplus_\lambda H_{m-2r}(S_\lambda; \mathbf{C}).$$

Hence, for each λ, $c_r(E, s)$ defines a class in $H_{m-2r}(S_\lambda; \mathbf{C})$, which we call the residue of $c_r(E)$ at S_λ with respect to s and denote by $\operatorname{Res}_{c_r}(s, E; S_\lambda)$. From the commutativity of (1.6), we have the following "residue formula".

PROPOSITION 2.3. *In the above situation, if M is compact,*

$$\sum_\lambda (i_\lambda)_* \operatorname{Res}_{c_r}(s, E; S_\lambda) = [M] \frown c_r(E) \quad \textit{in } H_{m-2r}(M; \mathbf{C}),$$

where i_λ denotes the inclusion $S_\lambda \hookrightarrow M$.

3. Residue at an isolated zero.

Let $\pi : E \to M$ be a holomorphic vector bundle of rank n over a complex manifold M of dimension n. Suppose we have a section s with an isolated zero at p in M. In this situation, we have $\operatorname{Res}_{c_n}(s, E; p)$ in $H_0(\{p\}; \mathbf{C}) = \mathbf{C}$. In the following, we compute this residue. Let U be an open neighborhood of p where the bundle E is trivial with holomorphic frame $(s_1, \cdots, s_n)$. We write $s = \sum_{i=1}^n f_i s_i$ with f_i holomorphic functions on U. In this case, we may express the residue in terms of the Grothendieck residue symbol.

THEOREM 3.1. *In the above situation, we have*

$$\operatorname{Res}_{c_n}(s, E; p) = \operatorname{Res}_p \begin{bmatrix} df_1 \wedge \cdots \wedge df_n \\ f_1, \cdots, f_n \end{bmatrix}.$$

REMARK 3.2. The right hand side above is defined as

$$\frac{1}{(2\pi\sqrt{-1})^n} \int_\Gamma \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_n}{f_n},$$

where Γ denotes the n-cycle in U defined by

$$\Gamma = \{q \in U \mid |f_1(q)| = \cdots = |f_n(q)| = \varepsilon\}$$

for a small positive number ε. If we denote by D_i the divisor in U defined by f_i, $i = 1, \cdots, n$, then this is the (local) intersection number $(D_1 \cdots D_n)_p$ of $D_1, \cdots, D_n$ at p ([GH] Ch.5).

PROOF OF THEOREM 3.1. This is done similarly as for [S1] Ch.III, Theorem 5.5. The techniques are originally due to [Le3]. Let $U_0 = U \setminus \{p\}$ and $U_1 = U$. On U_0, we let ∇_0 be an s-trivial connection for E and, on U_1, we let ∇_1 be the connection for E trivial with respect to the frame $\mathbf{s} = (s_1, \cdots, s_n)$. We set

$$R_1 = \{q \in U \mid |f_1(q)|^2 + \cdots + |f_n(q)|^2 \leq n\varepsilon^2\}$$

for a small positive number ε. Since $c_n(\nabla_1) = 0$ and $R_{01} = -\partial R_1$, from (2.2) we have

$$\operatorname{Res}_{c_n}(s, E; p) = -\int_{\partial R_1} c_n(\nabla_0, \nabla_1). \tag{3.3}$$

Now we consider the covering $\mathcal{U} = \{U^{(1)}, \cdots, U^{(n)}\}$ of $U_{01} = U_0$ defined by

$$U^{(i)} = \{q \in U_0 \mid f_i(q) \neq 0\}$$

and work on the Čech-de Rham cohomology with respect to $\mathcal{U}$. On $U^{(i)}$, we may replace s_i in the frame $\mathbf{s}$ by s to obtain a frame $\mathbf{s}^{(i)}$ for E. We denote by $\nabla^{(i)}$ the connection for E on $U^{(i)}$ trivial with respect to the frame $\mathbf{s}^{(i)}$. Then we define an element τ in $A^{2n-2}(\mathcal{U})$ by

$$\tau_{i_0 \cdots i_k} = c_n(\nabla_0, \nabla_1, \nabla^{(i_0)}, \cdots, \nabla^{(i_k)}),$$

which is a $(2n-k-2)$-form on $U^{(i_0)} \cap \cdots \cap U^{(i_k)}$. Since ∇_0 and $\nabla^{(i)}$ are all s-trivial, we have

$$c_n(\nabla_0, \nabla^{(i_0)}, \cdots, \nabla^{(i_k)}) = 0 \tag{3.4}$$

for $k \geq 0$. Also, if $0 \leq k \leq n-2$, ∇_1 and $\nabla^{(i_0)}, \cdots, \nabla^{(i_k)}$ are all s_i-trivial for some i. Hence

$$c_n(\nabla_1, \nabla^{(i_0)}, \cdots, \nabla^{(i_k)}) = 0 \quad \text{for} \quad k = 0, \cdots, n-2. \tag{3.5}$$

Now we compute $D\tau$. First for $k = 0$, we have, using (3.4) and (3.5),

$$\begin{aligned}(D\tau)_i &= dc_n(\nabla_0, \nabla_1, \nabla^{(i)}) = -c_n(\nabla_1, \nabla^{(i)}) + c_n(\nabla_0, \nabla^{(i)}) - c_n(\nabla_0, \nabla_1) \\ &= -\, c_n(\nabla_0, \nabla_1).\end{aligned}$$

For $k = 1, \cdots, n-1$, we have, by (3.4),

$$\begin{aligned}(D\tau)_{i_0 \cdots i_k} &= \sum_{\nu=0}^{k} (-1)^\nu c_n(\nabla_0, \nabla_1, \nabla^{(i_0)}, \cdots, \widehat{\nabla^{(i_\nu)}}, \cdots, \nabla^{(i_k)}) \\ &\quad + (-1)^k dc_n(\nabla_0, \nabla_1, \nabla^{(i_0)}, \cdots, \nabla^{(i_k)}) \\ &= -\, c_n(\nabla_1, \nabla^{(i_0)}, \cdots, \nabla^{(i_k)}) + c_n(\nabla_0, \nabla^{(i_0)}, \cdots, \nabla^{(i_k)}) \\ &= -\, c_n(\nabla_1, \nabla^{(i_0)}, \cdots, \nabla^{(i_k)}).\end{aligned}$$

Thus, using (3.5), we may summarize as

$$\begin{cases} (D\tau)_i & = -c_n(\nabla_0, \nabla_1) \\ (D\tau)_{i_0\cdots i_k} & = 0, \quad \text{for } k = 1, \cdots, n-2 \\ (D\tau)_{1\cdots n} & = -c_n(\nabla_1, \nabla^{(1)}, \cdots, \nabla^{(n)}) . \end{cases}$$

Denoting by ι the inclusion map $\partial R_1 \hookrightarrow U_0$, we let $\iota^*\mathcal{U}$ be the covering of ∂R_1 by the open sets $\partial R_1 \cap U^{(i)}$. Then, as a system $\{R^{(i)}\}_{i=1}^n$ of honey-comb cells adapted to $\iota^*\mathcal{U}$, we take

$$R^{(i)} = \{q \in \partial R_1 \mid |f_i(q)| \geq |f_j(q)| \text{ for } j \neq i\}$$

and, for $(i_0 \cdots i_k)$ with $1 \leq i_0 < \cdots < i_k \leq n$, we set $R^{(i_0\cdots i_k)} = R^{(i_0)} \cap \cdots \cap R^{(i_k)}$, oriented as in Section 1 (A). Considering the integration

$$\int_{\partial R_1} : A^{2n-1}(\iota^*\mathcal{U}) \to \mathbf{C},$$

we see that

$$0 = \int_{\partial R_1} D\tau = -\sum_{i=1}^n \int_{R^{(i)}} c_n(\nabla_0, \nabla_1) - \int_{R^{(1\cdots n)}} c_n(\nabla_1, \nabla^{(1)}, \cdots, \nabla^{(n)}) .$$

Hence we get, by (3.3),

$$\operatorname{Res}_{c_n}(s, E; p) = \int_{R^{(1\cdots n)}} c_n(\nabla_1, \nabla^{(1)}, \cdots, \nabla^{(n)}) .$$

If we compute the connection matrix $\theta^{(i)}$ of $\nabla^{(i)}$ with respect to the frame s, we see that $\theta^{(i)}$ is an $n \times n$ matrix whose i-th row is given by $-\frac{1}{f_i}(df_1, \cdots, df_n)$ with all other rows equal to $(0, \cdots, 0)$. Let $\tilde{\nabla}$ denote the connection for the bundle $E \times \mathbf{R}^n$ over $\bigcap_{i=1}^n U^{(i)} \times \mathbf{R}^n$ given by $\tilde{\nabla} = (1 - \sum_{i=1}^n t_i)\nabla_1 + \sum_{i=1}^n t_i \nabla^{(i)}$. Then the connection matrix $\tilde{\theta}$ of $\tilde{\nabla}$ with respect to the frame s is given by

$$\tilde{\theta} = \left(1 - \sum_{i=1}^n t_i\right)\theta_1 + \sum_{i=1}^n t_i \theta^{(i)} ,$$

where θ_1 is the connection matrix of ∇_1 with respect to the frame s and is equal to zero. Denoting by $\tilde{\kappa}$ the curvature matrix of $\tilde{\nabla}$, we compute

$$\begin{aligned} c_n(\tilde{\kappa}) &= (-1)^n n! \left(dt_1 \wedge \frac{df_1}{f_1}\right) \wedge \cdots \wedge \left(dt_n \wedge \frac{df_n}{f_n}\right) \\ &= (-1)^{n+[n/2]} n! dt_1 \wedge \cdots \wedge dt_n \wedge \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_n}{f_n} . \end{aligned}$$

We denote by Δ^n the standard n-simplex in $\mathbf{R}^n$ and by $\pi : M \times \Delta^n \to M$ the projection. Since $\int_{\Delta^n} dt_1 \wedge \cdots \wedge dt_n = 1/n!$, we get,

$$c_n(\nabla_1, \nabla^{(1)}, \cdots, \nabla^{(n)}) = \left(\frac{\sqrt{-1}}{2\pi}\right)^n \pi_*(c_n(\tilde{\kappa})) = \frac{(-1)^{[n/2]}}{(2\pi\sqrt{-1})^n} \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_n}{f_n} ,$$

where π_* denotes the integration along the fibers of π. Taking the orientations into account, we have $\Gamma = (-1)^{[n/2]} R^{(1\cdots n)}$. Hence we have the formula. □

REMARK 3.6. In the above situation, consider the C^∞ functions $\rho_i = |f_i|^2/\|f\|^2$, $\|f\|^2 = |f_1|^2 + \cdots + |f_n|^2$, on U_0. On $U^{(i)}$, we have $\nabla^{(i)}(s_l) = 0$ for $l \neq i$ and $\nabla^{(i)}(s_i) = -1/f_i \cdot \sum_{j=1}^n df_j \otimes s_j$. Thus we obtain an operator $\rho_i \nabla^{(i)}$ on U_0 by setting

$$\rho_i \nabla^{(i)}(s_l) = \begin{cases} -\dfrac{\bar{f}_i}{\|f\|^2} \displaystyle\sum_{j=1}^n df_j \otimes s_j, & \text{for } l = i \\ 0, & \text{for } l \neq i\,. \end{cases}$$

Moreover, from $\sum_{i=1}^n \rho_i \equiv 1$, we see that $\nabla_0 = \sum_{i=1}^n \rho_i \nabla^{(i)}$ is a connection for E on U_0. Note that it is s-trivial, since each $\nabla^{(i)}$ is. If we take this connection ∇_0, as in the proof of [S1] Ch.III, Theorem 4.4, we see that

$$c_n(\nabla_0, \nabla_1) = f^*\beta_n\,,$$

where $f = (f_1, \cdots, f_n)$ and β_n denotes the Bochner-Martinelli kernel on $\mathbf{C}^n$. This reproves that the Grothendieck residue in the above theorem is equal to the mapping degree of f (cf. [GH] Ch.5, 1. Lemma).

4. The duality.

Let M be a complex manifold of complex dimension n and E a holomorphic vector bundle of rank k over M. Let s be a regular section of E. This means that, at any point p in the zero set V of s, the germs of the components of s with respect to a holomorphic frame near p form a regular sequence in the ring $\mathcal{O}_{M,p}$ of germs of holomorphic functions at p (cf. [F] B.3). Let X be the analytic subspace of M defined by (the ideal generated by the components of) s. Thus, if $V \neq \emptyset$, X is a (possibly non-reduced) local complete intersection of dimension $n-k$ whose support is V. Let V_i, $i = 1, \cdots, r$, be the irreducible components of X. Then we have $V = \bigcup_{i=1}^r V_i$, which is considered as an analytic subvariety (reduced analytic subspace) of M. If V is compact, X defines a $2(n-k)$-cycle $X = \sum_{i=1}^r m_i V_i$, hence a class $[X] = \sum_{i=1}^r m_i [V_i]$ in $H_{2n-2k}(M)$ or in $H_{2n-2k}(V)$, where m_i denotes the multiplicity of V_i in X. In this situation, we prove the following

THEOREM 4.1. *If M is compact, the class $c_k(E)$ corresponds to $[X]$ under the Poincaré duality $H^{2k}(M; \mathbf{C}) \xrightarrow{\sim} H_{2n-2k}(M; \mathbf{C})$. Thus we have*

$$[M] \frown c_k(E) = [X] \quad in \quad H_{2n-2k}(M; \mathbf{C})\,.$$

In fact, this follows from the following more "precise" theorem, where the things are localized at V and we need only the compactness of V but not of M itself (cf. (1.6) and the introduction). Recall that we have the localization $c_k(E, s)$ in $H^{2k}(M, M \setminus V; \mathbf{C})$ of $c_k(E)$ with respect to the section s, as discussed in Section 2.

THEOREM 4.2. *Let X be an analytic subspace of dimension $n-k$ in M as above. If the support V of X is compact, the class $c_k(E, s)$ corresponds to $[X]$ under the Alexander duality*

$H^{2k}(M, M \setminus V; \mathbf{C}) \xrightarrow{\sim} H_{2n-2k}(V; \mathbf{C})$. *Thus we have*

$$[M_V] \frown c_k(E, s) = [X] \quad \textit{in} \quad H_{2n-2k}(V; \mathbf{C}) .$$

PROOF. Let S denote the singular set $\mathrm{Sing}(V)$ of V. Also, as in the last paragraph of Section 1 (B), let $U_0 = M \setminus V$, U_1 a sufficiently small tubular neighborhood of $V' = V \setminus S$ and U_2 a sufficiently small regular neighborhood of S in M. We consider the coverings $\mathcal{U} = \{U_0, U_1, U_2\}$ of M and $\mathcal{U}' = \{U_1, U_2\}$ of $U = U_1 \cup U_2$. It suffices to prove that there is a representative σ, $D\sigma = 0$, of $c_k(E, s)$ in $A^{2k}(\mathcal{U}, U_0)$ such that for any τ in $A^{2n-2k}(\mathcal{U}')$ with $D\tau = 0$, we have (1.8) with $C = X$. Now let σ be an element in $A^{2k}(\mathcal{U}, U_0)$ with $D\sigma = 0$ so that we have

$$\begin{aligned} &d\sigma_1 = 0, \quad d\sigma_2 = 0, \quad d\sigma_{01} = \sigma_1, \quad d\sigma_{02} = \sigma_2, \quad d\sigma_{12} = \sigma_2 - \sigma_1, \quad \text{and} \\ &d\sigma_{012} = -\sigma_{12} + \sigma_{02} - \sigma_{01} . \end{aligned} \tag{4.3}$$

Also let τ be an element in $A^{2n-2k}(\mathcal{U}')$ with $D\tau = 0$ so that we have

$$d\tau_1 = 0, \quad d\tau_2 = 0 \quad \text{and} \quad d\tau_{12} = \tau_2 - \tau_1 . \tag{4.4}$$

Thus τ_1 is a closed $2(n-k)$-form on U_1, which is a tubular neighborhood of $V' = V \setminus S$. Denoting by π the projection $U_1 \to V'$, we have an isomorphism $\pi^* : H^{2n-2k}(V'; \mathbf{C}) \xrightarrow{\sim} H^{2n-2k}(U_1; \mathbf{C})$. Hence there is a closed $2(n-k)$-form θ on V' and a $(2n-2k-1)$-form ρ_1 on U_1 such that

$$\tau_1 = \pi^*\theta + d\rho_1 . \tag{4.5}$$

Also, τ_2 is a closed $2(n-k)$-form on U_2. Since U_2 is homotopically equivalent to S, which is less than $2(n-k)$-dimensional, we have $H^{2n-2k}(U_2; \mathbf{C}) = 0$. Hence there is a $(2n-2k-1)$-form ρ_2 on U_2 such that

$$\tau_2 = d\rho_2 . \tag{4.6}$$

Let $\{R_0, R_1, R_2\}$ be a system of honey-comb cells adapted to $\mathcal{U}$ such that ∂R_2 is transverse to V. Then, using (4.3) and the Stokes theorem and noting that $\partial R_1 = -R_{01} + R_{12}$ and $\partial R_{01} = R_{012}$, we compute

$$\begin{aligned} &\int_{R_1} \sigma_1 \wedge d\rho_1 = \int_{R_1} d(\sigma_1 \wedge \rho_1) = -\int_{R_{01}} \sigma_1 \wedge \rho_1 + \int_{R_{12}} \sigma_1 \wedge \rho_1 \quad \text{and} \\ &\int_{R_{01}} \sigma_{01} \wedge d\rho_1 = \int_{R_{01}} d\sigma_{01} \wedge \rho_1 - \int_{R_{01}} d(\sigma_{01} \wedge \rho_1) = \int_{R_{01}} \sigma_1 \wedge \rho_1 - \int_{R_{012}} \sigma_{01} \wedge \rho_1 . \end{aligned}$$

Similary we have, noting that $\partial R_2 = -R_{02} - R_{12}$, $\partial R_{02} = -R_{012}$, $\partial R_{12} = R_{012}$, and $\partial R_{012} = 0$.

$$\begin{aligned}
&\int_{R_2} \sigma_2 \wedge d\rho_2 = -\int_{R_{02}} \sigma_2 \wedge \rho_2 - \int_{R_{12}} \sigma_2 \wedge \rho_2 \,,\\
&\int_{R_{02}} \sigma_{02} \wedge d\rho_2 = \int_{R_{02}} \sigma_2 \wedge \rho_2 + \int_{R_{012}} \sigma_{02} \wedge \rho_2 \,,\\
&\int_{R_{12}} \sigma_{12} \wedge d\rho_2 = \int_{R_{12}} (\sigma_2 - \sigma_1) \wedge \rho_2 - \int_{R_{012}} \sigma_{12} \wedge \rho_2 \quad \text{and}\\
&\int_{R_{012}} \sigma_{012} \wedge d\rho_2 = \int_{R_{012}} (\sigma_{12} - \sigma_{02} + \sigma_{01}) \wedge \rho_2 \,.
\end{aligned}$$

Hence, if we denote by I_1 the left hand side of (1.8), we have

$$I_1 = \int_{R_1} \sigma_1 \wedge \pi^*\theta + \int_{R_{01}} \sigma_{01} \wedge \pi^*\theta - \int_{R_{12}} \sigma_1 \wedge \rho_{12} + \int_{R_{012}} \sigma_{01} \wedge \rho_{12} \,,$$

where $\rho_{12} = \rho_2 - \rho_1 - \tau_{12}$, which is a $(2n-2k-1)$-form on $U_{12} = U_1 \cap U_2$. Note that from (4.4), (4.5) and (4.6), we have

$$d\rho_{12} = \pi^*\theta \quad \text{on} \quad U_{12} \,.$$

The chain R_{12} is in the interior of the $(2n-1)$-dimensional manifold $U_1 \cap \partial R_2$, which may be assumed to retract to $V \cap \partial R_2 = R_{12} \cap V$ by the projection π so that we have the commutative diagram

$$\begin{tikzcd}
U_1 \cap \partial R_2 \arrow[r, "\tilde{i}"] \arrow[d, "\pi"'] & U_1 \cap U_2 \arrow[d, "\pi"'] \\
V \cap \partial R_2 \arrow[r, "i"] & V' \cap U_2 \,,
\end{tikzcd}$$

where i and $\tilde{i}$ denote the inclusions. We have $d\tilde{i}^*\rho_{12} = \tilde{i}^* d\rho_{12} = \tilde{i}^*\pi^*\theta = \pi^* i^*\theta = 0$, since $i^*\theta$ is a $2(n-k)$-form on $V \cap \partial R_2$, which is a $(2n-2k-1)$-dimensional manifold. Hence we see that there exist a $(2n-2k-1)$-form ρ on $V \cap \partial R_2$ and a $(2n-2k-2)$-form ω_{12} on $U_1 \cap \partial R_2$ such that

$$\rho_{12} = \pi^*\rho + d\omega_{12} \quad \text{on} \quad U_1 \cap \partial R_2 \,. \tag{4.7}$$

We have, as before

$$\int_{R_{12}} \sigma_1 \wedge d\omega_{12} = \int_{R_{012}} \sigma_1 \wedge \omega_{12} \quad \text{and} \quad \int_{R_{012}} \sigma_{01} \wedge d\omega_{12} = \int_{R_{012}} \sigma_1 \wedge \omega_{12} \,.$$

Hence we obtain

$$I_1 = \int_{R_1} \sigma_1 \wedge \pi^*\theta + \int_{R_{01}} \sigma_{01} \wedge \pi^*\theta - \int_{R_{12}} \sigma_1 \wedge \pi^*\rho + \int_{R_{012}} \sigma_{01} \wedge \pi^*\rho \,. \tag{4.8}$$

Next, we compute the right hand side I_2 of (1.8) (with $C = X$). From

$$\int_{R_1\cap X}\tau_1=\int_{R_1\cap X}(\pi^*\theta+d\rho_1)=\int_{R_1\cap X}\theta+\int_{R_{12}\cap X}\rho_1\quad\text{and}$$

$$\int_{R_2\cap X}\tau_2=-\int_{R_{12}\cap X}\rho_2\,,$$

and using (4.7), we have

$$I_2=\int_{R_1\cap X}\theta-\int_{R_{12}\cap X}\rho\,. \tag{4.9}$$

We denote by π_1, π_{01}, π_{12} and π_{012} the restrictions of π to R_1, R_{01}, R_{12} and R_{012}, respectively. We may assume that $\pi_1 : R_1 \to R_1\cap V$ and $\pi_{12} : R_{12}\to R_{12}\cap V$ are closed $2k$-disk bundles and that $\pi_{01} : R_{01}\to R_1\cap V$ and $\pi_{012} : R_{012}\to R_{12}\cap V$ are S^{2k-1}-bundles. Recall that the orientation of R_{01} is opposite to that of ∂R_1 and that the orientation of R_{012} is same as that of ∂R_{12}. If we apply the projection formula in (4.8), we have

$$I_1=\int_{R_1\cap V}((\pi_1)_*\sigma_1+(\pi_{01})_*\sigma_{01})\cdot\theta-\int_{R_{12}\cap V}((\pi_{12})_*\sigma_1-(\pi_{012})_*\sigma_{01})\cdot\rho\,,$$

where the subscript $*$ signifies the integration along the fibers. By [S1] Ch.II, Proposition 5.2, the function $(\pi)_*\sigma_1+(\pi_{01})_*\sigma_{01}$ is locally constant and thus constant on each connected component $R_1\cap V_i$ of $R_1\cap V$. Now we let ∇_0 be an s-trivial connection for E on U_0 and let ∇_1 and ∇_2 be arbitrary connections for E on U_1 and U_2, respectively. The class $c_k(E,s)$ is then represented by the cocycle σ with $\sigma_0=c_k(\nabla_0)=0$, $\sigma_1=c_k(\nabla_1)$ and $\sigma_{01}=c_k(\nabla_0,\nabla_1)$. In fact we have $\sigma_2=c_k(\nabla_2)$ and so forth, but as we have seen above all the terms involving ∇_2 cancel out. Then the value of the function $(\pi_1)_*\sigma_1+(\pi_{01})_*\sigma_{01}$ at a point p of $R_1\cap V_i$ is exactly the residue $\mathrm{Res}_{c_k}(s\,|_{U_p}, E\,|_{U_p}; p)$, $U_p=\pi^{-1}(p)$, and, by Theorem 3.1, it is the multiplicity of V_i in X. By a similar argument, we also see that $(\pi_{12})_*\sigma_1-(\pi_{012})_*\sigma_{01}$ is constant on $R_{12}\cap V_i$ and its value is again the multiplicity of V_i in X. Comparing with (4.9), we proved the theorem. □

REMARKS 4.10. 1. Let p be a point in V'. As in the proof of [S1] Ch.III, Theorem 4.4, it is possible to choose connections ∇_0 and ∇_1 above so that we have

$$c_k(\nabla_1)=0\quad\text{and}\quad c_k(\nabla_0,\nabla_1)=f^*\beta_k\,,$$

in a neighborhood of p, where $f=(f_1,\cdots,f_k)$ denote the components of s with respect to a suitable frame of E near p and β_k the Bochner-Martinelli kernel on $\mathbf{C}^k$ (cf. Remark 3.6).

2. Theorem 4.4 in [S1] Ch.III can be also proved as above. Namely, let $\pi : E\to M$ be a C^∞ complex vector bundle of rank r over an oriented C^∞ manifold M. We denote by s_Δ the diagonal section of the pull-back bundle π^*E over E. The zero set of s_Δ is the image of the zero section of E, which is identified with M. Thus we have the localization $c_r(\pi^*E,s_\Delta)$ in $H^{2r}(E,E\setminus M;\mathbf{C})$ of $c_r(\pi^*E)$ with respect to s_Δ. Recall that we have the Thom class Ψ_E in $H^{2r}(E,E\setminus M;\mathbf{C})$ and the Euler class $e(E)$ in $H^{2r}(M;\mathbf{C})$ of E as a real bundle (cf. [S1] Ch.II, 5). In this situation, we claim

$$c_r(\pi^*E,s_\Delta)=\Psi_E\quad\text{and}\quad c_r(E)=e(E)\,.$$

In fact, the second identity follows from the first. To show the first identity, let p be an arbitrary point of M and let $i_p : E_p \hookrightarrow E$ denote the inclusin of the fiber $E_p = \pi^{-1}(p)$. Note that the restriction $\pi^*E\,|_{E_p} \simeq \mathbf{C}^r \times \mathbf{C}^r$ admits a natural complex structure so that $s_\Delta\,|_{E_p}$ is holomorphic. Then, from Theorem 3.1, we have $i_p^*c_r(\pi^*E, s_\Delta) = c_r(\pi^*E\,|_{E_p}, s_\Delta\,|_{E_p}) = 1$, which characterizes the Thom class.

The identity shows that, for a closed set S in M and a non-vanishing section s of E on $M \setminus S$,

$$c_k(E, s) = s^*\Psi_E$$

(cf. [F] Example 19.2.6).

3. Let V be the zero set of a holomorphic section s of E generically transverse to the zero section. This means that, if $(f_1, \cdots, f_k)$ denote the components of s with respect to a holomorphic frame on an open set U in M, V is given by $f_1 = \cdots = f_k = 0$ in U and $df_1 \wedge \cdots \wedge df_k \not\equiv 0$ on $V \cap U$. In this case Sing(V) is given by $df_1 \wedge \cdots \wedge df_k = 0$ in $V \cap U$ and the restriction $E|_{V'}$ of E to the regular part $V' = V \setminus \mathrm{Sing}(V)$ coincides with the normal bundle of V' in M. In fact the above condition for s is equivalent to saying that s is a regular section and that the analytic subspace X defined by s is reduced; $X = V$ (cf. [T], [Ło, VI.1.6]). In particular, V is a local complete intersection as an analytic variety. By Theorem 4.2, the class $c_k(E, s)$ in $H^{2k}(M, M \setminus V; \mathbf{C})$ is Alexander dual to $[V]$ in $H_{2n-2k}(V; \mathbf{C})$. Thus we may call $c_k(E, s)$ the Thom class of V in M (cf. [S2], where Theorem 4.2 is applied to prove the Riemann-Roch theorem for the embedding $V \hookrightarrow M$).

5. Intersection of analytic subspaces.

Let M be a complex manifold of dimension n. Also, for each $j = 1, \cdots, q$, let E_j be a holomorphic vector bundle of rank k_j over M and s_j a regular section of E_j. We denote by X_j the analytic subspace of M defined by s_j, which is pure k_j-codimensional. Denoting by V_j the support of X_j, we have the localization $c_{k_j}(E_j, s_j)$ in $H^{2k_j}(M, M \setminus V_j; \mathbf{C})$ of $c_{k_j}(E_j)$ with respect to the section s_j as in Section 2. Setting $S = \bigcap_{j=1}^q V_j$ and $k = \sum_{j=1}^q k_j$, we have the cup product

$$H^{2k_1}(M, M \setminus V_1; \mathbf{C}) \times \cdots \times H^{2k_q}(M, M \setminus V_q; \mathbf{C}) \to H^{2k}(M, M \setminus S; \mathbf{C})\,.$$

Let E be the direct sum $E = E_1 \oplus \cdots \oplus E_q$ and s the section of E given by $s = s_1 \oplus \cdots \oplus s_q$. Then the zero set of s is S and we have the localization $c_k(E, s)$ in $H^{2k}(M, M \setminus S; \mathbf{C})$ of $c_k(E)$ with respect to s. In the above cup product, we have

$$c_{k_1}(E_1, s_1) \cdots c_{k_q}(E_q, s_q) = c_k(E, s)\,.$$

Suppose S is compact (V_j may not be). Then we have the Alexander duality

$$A_{M_s} : H^{2k}(M, M \setminus S; \mathbf{C}) \xrightarrow{\sim} H_{2n-2k}(S; \mathbf{C})\,.$$

In view of Theorem 4.2, we define the (refined) intersection product $X_1 \cdots X_q$ of the analytic subspaces $X_1, \cdots, X_q$ to be the homology class $A_{M_s}(c_k(E, s)) = \mathrm{Res}_{c_k}(s, E; S)$ in

$H_{2n-2k}(S;\mathbf{C})$ (cf. [F] §8.1). Thus, if $(S_\lambda)_\lambda$ denote the connected components of S, its λ component $(X_1\cdots X_q)_\lambda$ is given by

$$(X_1\cdots X_q)_\lambda=\operatorname{Res}_{c_k}(s,E;S_\lambda)\quad\text{in}\quad H_{2n-2k}(S_\lambda;\mathbf{C})$$

and we have

$$X_1\cdots X_q=\sum_\lambda(X_1\cdots X_q)_\lambda=\sum_\lambda\operatorname{Res}_{c_k}(s,E;S_\lambda)\quad\text{in}\quad H_{2n-2k}(S;\mathbf{C})\,. \tag{5.1}$$

In particular, if $k=n$, $H_{2n-2k}(S_\lambda;\mathbf{C})=\mathbf{C}$, hence $\operatorname{Res}_{c_n}(s,E;S_\lambda)$ is a complex number. If s is also a regular section, then $X_1\cdots X_q$ is the analytic subspace defined by s and S is its support.

Recall that every divisor D on M is defined by a regular section of a line bundle. Thus, for divisors $D_1,\cdots,D_q$, we may define the intersection product $D_1\cdots D_q$ in $H_{2n-2q}(S;\mathbf{C})$, if $S=\bigcap_{j=1}^q|D_j|$ is compact, where $|D_j|$ denotes the support of D_j. From (5.1) and Theorem 3.1, we have the following

THEOREM 5.2. *Let M be a complex manifold of dimension n and let $D_1,\cdots,D_n$ be divisors on M. If $S=\bigcap_{i=1}^n|D_i|$ consists of finite isolated points, we have*

$$D_1\cdots D_n=\sum_{p\in S}(D_1\cdots D_n)_p\,.$$

where $(D_1\cdots D_n)_p$ is the local intersection number at p (see Remark 3.2).

6. Duality for non-compact varieties.

Let M be a complex manifold of dimension n, E a holomorphic vector bundle of rank k over M and X an analytic subspace of codimension k defined by a regular section s of E, as before. Also, let V_i, $i=1,\cdots,r$, be the irreducible components of X and m_i the multiplicity of V_i in X. Denoting by V the support of X, we set $V'=V\setminus\operatorname{Sing}(V)$. In this section, we do not assume that V is compact.

First we prove the following theorem. Let R be a compact C^∞ submanifold (without boundary) of $M'=M\setminus\operatorname{Sing}(V)$ of dimension d transverse to V'. Then X defines a $(d-2k)$-cycle $R\cap X=\sum_{i=1}^r m_i(R\cap V_i)$, hence a class $[R\cap X]=\sum_{i=1}^r m_i[R\cap V_i]$ in $H_{d-2k}(R)$ or in $H_{d-2k}(R\cap V)$.

THEOREM 6.1. *The class $c_k(E\,|_R, s\,|_R)$ corresponds to $[R\cap X]$ under the Alexander duality $H^{2k}(R,R\setminus(R\cap V);\mathbf{C})\xrightarrow{\sim}H_{d-2k}(R\cap V;\mathbf{C})$. Thus the class $c_k(E\,|_R)$ corresponds to $[R\cap X]$ under the Poincaré duality $H^{2k}(R;\mathbf{C})\xrightarrow{\sim}H_{d-2k}(R;\mathbf{C})$.*

PROOF. Letting $U_0=M\setminus V$ and U_1 a sufficiently small tubular neighborhood of V' in M with projection $\pi:U_1\to V'$, we consider the covering $\mathcal{U}=\{U_0,U_1\}$ of M'. Let R_1 be a closed disk bundle in U_1 and $R_0=M'\setminus\operatorname{Int}R_1$ so that $\{R_0,R_1\}$ is a system of honey-comb cells adapted to $\mathcal{U}$. Also, letting $W_i=R\cap U_i$, $i=0,1$, we consider the covering $\mathcal{W}=\{W_0,W_1\}$ of R. If we set $T_i=R\cap R_i$, $i=0,1$, $\{T_0,T_1\}$ is a system of honey-comb cells adapted to

$\mathcal{W}$. Let ∇_0 be an s-trivial connection for E on U_0 and ∇_1 an arbitrary connection for E on U_1. Then the class $c_k(E|_R, s|_R)$ is represented by the cocycle $(0, c_k(\nabla_1), c_k(\nabla_0, \nabla_1))$ on $\mathcal{U}$, restricted to R. It suffices to prove (cf. (1.4))

$$\int_{T_1} c_k(\nabla_1) \wedge \tau_1 + \int_{T_{01}} c_k(\nabla_0, \nabla_1) \wedge \tau_1 = \int_{R\cap X} \tau_1, \tag{6.2}$$

for any closed $(d-2k)$-form τ_1 on W_1. We may assume that $W_1 = \pi^{-1}(R \cap V)$. Thus we may write $\tau_1 = \pi^*\theta + d\rho_1$ for some closed $(d-2k)$-form θ on $R \cap V$ and a $(d-2k-1)$-form ρ_1 on W_1. Then it suffices to prove

$$\int_{T_1} c_k(\nabla_1) \wedge \pi^*\theta + \int_{T_{01}} c_k(\nabla_0, \nabla_1) \wedge \pi^*\theta = \int_{R\cap X} \theta$$

for any closed $(d-2k)$-form θ on $R \cap V$. If we denote by $\pi_1 : T_1 \to R \cap V$ and $\pi_{01} : T_{01} \to R \cap V$ the restrictions of π to T_1 and T_{01}, respectively, the left hand side above is equal to

$$\int_{R\cap V} ((\pi_1)_* c_k(\nabla_1) + (\pi_{01})_* c_k(\nabla_0, \nabla_1)) \cdot \theta .$$

As in the proof of Theorem 4.2, $(\pi_1)_* c_k(\nabla_1) + (\pi_{01})_* c_k(\nabla_0, \nabla_1)$ is a function on $R \cap V$, constant on each $R \cap V_i$ with value m_i, which proves the theorem. □

REMARK 6.3. Let V be the zero set of a holomorphic section s of E generically transverse to the zero section (cf. Remark 4.10.3). Then the above theorem reproves the fact that the Euler class $e(E|_R)$ of E restricted to a submanifold R in M' as above is Poincaré dual to the submanifold $R \cap V$ of R.

Now let S be a compact connected component of $\mathrm{Sing}(V)$ and U_1 a sufficiently small regular neighborhood of S in M. We may think of $[X]$ as a class in $H_{2n-2k}(M, M \setminus S; \mathbf{C})$. We denote also by $c_k(E)$ the class $c_k(E|_{U_1})$ in $H^{2k}(U_1; \mathbf{C}) \simeq H^{2k}(S; \mathbf{C})$. Recall that we have the duality (1.3)

$$H^{2k}(S; \mathbf{C}) \xrightarrow{\sim} H_{2n-2k}(M, M \setminus S; \mathbf{C}) .$$

THEOREM 6.4. *The class $c_k(E)$ corresponds to $[X]$ under the above duality.*

PROOF. Let $c_k(E)$ also denote the Chern form with respect to some connection for E on U_1. Let $U_0 = M \setminus S$ and $\{R_0, R_1\}$ a system of honey-comb cells adapted to $\mathcal{U} = \{U_0, U_1\}$ such that $R_{01} (= -\partial R_1)$ is compact and is transverse to V. It suffices to show that

$$\int_{R_1} \sigma_1 \wedge c_k(E) + \int_{R_{01}} \sigma_{01} \wedge c_k(E) = \int_{R_1\cap X} \sigma_1 + \int_{R_{01}\cap X} \sigma_{01}$$

for any $\sigma = (0, \sigma_1, \sigma_{01})$ in $A^{2n-2k}(\mathcal{U}, U_0)$ with $D\sigma = 0$ (cf. (1.5)). We have $d\sigma_1 = 0$ and may consider the class $[\sigma_1]$ in $H^{2n-2k}(U_1; \mathbf{C}) \simeq H^{2n-2k}(S; \mathbf{C})$, which is zero, since S is less than $2(n-k)$-dimensional. Hence there is a $(2n-2k-1)$-form η_1 on U_1 such that $\sigma_1 = d\eta_1$. We compute

$$\int_{R_1} \sigma_1 \wedge c_k(E) = -\int_{R_{01}} \eta_1 \wedge c_k(E) \quad \text{and} \quad \int_{R_1\cap X} \sigma_1 = -\int_{R_{01}\cap X} \eta_1 .$$

Hence it suffices to show

$$\int_{R_{01}} (\sigma_{01} - \eta_1) \wedge c_k(E) = \int_{R_{01}\cap X} (\sigma_{01} - \eta_1)\,. \tag{6.5}$$

From $\sigma_1 - d\sigma_{01} = 0$, we have $d(\sigma_{01} - \eta_1) = 0$. Therefore, (6.5) follows from the second part of Theorem 6.1 with $R = R_{01}$. □

REMARK 6.6. Let C be a relative cycle representing a class in $H_l(M, M \setminus S; \mathbf{C})$. Suppose C is transverse to R_{01} and V. Then, by a similar argument as above, we have

$$\int_{R_1\cap C} \sigma_1 \wedge c_k(E) + \int_{R_{01}\cap C} \sigma_{01} \wedge c_k(E) = \int_{R_1\cap C\cap X} \sigma_1 + \int_{R_{01}\cap C\cap X} \sigma_{01}$$

for any $\sigma = (0, \sigma_1, \sigma_{01})$ in $A^{l-2k}(\mathcal{U}, U_0)$ with $D\sigma = 0$.

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