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Affine Locally Symmetric Structures and Finiteness Theorems for Einstein-Weyl Manifolds

Mitsuhiro ITOH*

University of Tsukuba

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Introduction.

A Weyl manifold is a smooth manifold M equipped with a torsion free affine connection D and a conformal structure $[g]$ such that D preserves $[g]$, in other words, for a representative metric g within $[g]$ there is a $\omega_g \in \Omega^1(M)$ for which $Dg = \omega_g \otimes g$.

We assume $\dim M \geq 3$.

Since the affine connection D leaves the conformal structure $[g]$ invariant, D has the holonomy group contained in $CO(n)$, the conformal orthogonal group.

For a Weyl manifold $(M, D, [g])$ the Riemannian curvature tensor R^D and the Ricci tensor Ric^D are defined in terms of D. The Ricci tensor is, in general, not necessarily symmetric.

The skew-symmetric part of Ric^D coincides, up to constant, with the closed 2-form $d\omega_g$ (Lemma 1 in §1). This 2-form is an invariant of a Weyl manifold measuring the extent to which the manifold differs from a trivial Weyl manifold. Here a Weyl manifold is called trivial if the affine connection coincides with the Levi-Civita connection of a representative metric within $[g]$.

In the present paper we are interested in Weyl manifolds carrying a particular geometric structure, especially in geometry of affine locally symmetric Weyl manifolds and of Weyl manifolds whose Ricci tensor is proportional to the conformal structure.

We consider in the first part Weyl manifolds admitting a D-parallel tensor, especially Weyl manifolds with D-parallel curvature tensor. It is shown in §1 that if the 2-form $d\omega_g$ is D-parallel, then a Weyl manifold is called *locally trivial*, that is, $d\omega_g = 0$. As a direct consequence an affine locally symmetric Weyl manifold must be locally trivial, since R^D is D-parallel (see Theorem 2 in §1).

As Proposition 2 in §1 shows, the local triviality implies that an affine locally symmetric Weyl manifold $(M, D, [g])$ turns out to be locally Riemannian, that is, M has at each point an open neighborhood on which $Dg = 0$ for a certain local metric g in $[g]$.

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We remark that as Pedersen, Poon and Swann [13] showed, the local triviality holds also for Hermitian-Weyl manifolds of dimension at least 6, i.e., Weyl manifolds $(M, D, [g])$ admitting a D-parallel almost complex structure J which is Hermitian with respect to $[g]$.

The second part of this paper is devoted to showing the finiteness theorem which concerns compact Einstein-Weyl manifolds having positive Gauduchon constant.

A Weyl manifold $(M, D, [g])$ is called *Einstein-Weyl*, if the symmetric part of Ric^D is proportional to a representative metric g in $[g]$; $\mathrm{sym}(Ric^D) = \Lambda g$ for $\Lambda \in C^\infty(M)$. An Einstein-Weyl manifold is a generalization of ordinary Einstein manifold in the sense of conformal geometry.

As shown in [6], [10], [15], every compact connected Einstein-Weyl manifold admits a constant, which we call Gauduchon constant whose definition will be given in §2. Intuitively this constant behaves like the Einstein constant for Einstein metrics.

In §2 we exhibit for Einstein-Weyl 4-manifolds the finiteness theorem which asserts that, given positive numbers B and C, the collection of compact connected Einstein-Weyl 4-manifolds $(M, D, [g])$ of Gauduchon constant c satisfying $b_2(M) \leq B$, $C \leq c$ and obeying a certain scalar curvature supnorm bound has at most a finite number of diffeomorphism types.

Finiteness theorems for Einstein-Weyl n-manifolds, $n = 3$ and $n \geq 5$ are investigated in §3.

Especially, in virtue of the finiteness theorem for Einstein-Weyl 3-manifolds, the diffeomorphism finiteness of the collection of minitwistor spaces is available for compact Einstein-Weyl 3-manifolds fulfilling certain bounds. Here, due to Hitchin ([9]), the minitwistor space is the space of all oriented D-geodesics in an Einstein-Weyl 3-manifold which carries a canonically defined integrable almost complex structure.

Remark that in [17] possible geometric structures for compact Einstein-Weyl 3-manifolds are classified into four types, three of which are well known.

In the last section we investigate geometric structure of an affine locally symmetric, Einstein-Weyl manifold (refer to Theorems 6, 7 in §4).

1. Affine locally symmetric Weyl manifolds.

A manifold M with an affine connection D (with torsion tensor T^D) is called *affine locally symmetric* if each point of M has an open neighborhood on which the geodesic symmetry is an affine transformation. The following theorem is well known.

THEOREM (Helgason [7]). *An affine manifold (M, D) is affine locally symmetric if and only if $T^D = 0$ and $DR^D = 0$.*

Refer also to [12], *Chapter* XI.

The main purpose of this section is to show that every affine locally symmetric Weyl manifold is locally trivial.

We first provide several facts which hold for general Weyl manifolds.

Let $(M, D, [g])$ be a connected Weyl n-manifold. Denote by Φ^0 its restricted holonomy group, the holonomy group with respect to closed null homotopic piecewise smooth curves at a fixed point. Then

PROPOSITION 1. *$\Phi^0 \subset CO^+(n)$, where $CO^+(n) = \{\lambda A \mid \lambda \in \mathbf{R}^+, A \in SO(n)\}$ is the conformal orthogonal group.*

PROOF. While this fact is standard in Weyl geometry, we will give a proof.

Fix $p \in M$ and let $c : [0, 1] \to M$ be a closed piecewise smooth curve in M with $c(0) = c(1) = p$.

Choose a metric g in $[g]$ such that $Dg = \omega \otimes g$. The pull-back 1-form $c^*\omega = (c^*\omega)(\frac{dc}{dt})dt$ along c can be integrated as $c^*\omega = du$ where $u(t) = \int_0^t (c^*\omega)(\frac{dc}{dt})dt$. Then a conformal change metric $\tilde{g}_t = e^{-u(t)} g_{c(t)}$ is parallel along c because

$$D_X(fg) = (Xf + \omega(X))g .$$

So, for D-parallel vector fields $\{X_i(t)\}_{i=1,\cdots,n}$ along c, orthonormal at p with respect to g, we have

$$\tilde{g}_t(X_i, X_j) = e^{-u(0)}\delta_{ij} , \quad i, j = 1, \cdots , n .$$

Then, the parallel transport $\tau_c : T_pM \to T_pM$ along c is conformal with respect to $[g]$. It follows that Φ^0 is a connected Lie subgroup of $CO^+(n)$.

PROPOSITION 2. *For a Weyl n-manifold $(M, D, [g])$ the following are equivalent each other*

(i) $\Phi^0 \subset SO(n)$,

(ii) $d\omega_g = 0$,

(iii) Ric^D *is symmetric,*

(iv) *each point of M has a neighborhood on which $Dg = 0$ for a certain metric g in $[g]$.*

PROOF. (iv) $\Rightarrow$ (i) is obvious. To see (i) $\Rightarrow$ (iv) choose an orthonormal base $\{e_i\}$ at p with respect to some representative metric $\bar{g}_p$ and transport it parallelly by D to each point in a neighborhood of p. Since $\Phi^0 \subset SO(n)$, we get a metric g, in $[g]$ and $Dg = 0$. (ii) $\Leftrightarrow$ (iv) follows from the formula $D(fg) = (df + f\omega_g) \otimes g$ and the Poincaré lemma.

To see (ii) $\Leftrightarrow$ (iii) we will provide the following

LEMMA 1.

$$d\omega_g = \frac{4}{n}\,\mathrm{skew}(Ric^D)$$

PROOF. Let X, Y be vector fields satisfying $[X, Y] = 0$ at p. Then we have for any vector fields Z, W

$$Xg(Z, W) = \omega_g(X)g(Z, W) + g(D_X Z, W) + g(Z, D_X W) ,$$

$$\begin{aligned}Y(Xg(Z,W)) =&(Y\omega_g(X))g(Z,W)+\omega_g(X)\omega_g(Y)g(Z,W)\\&+\omega_g(X)g(D_YZ,W)+\omega_g(X)g(Z,D_YW)\\&+\omega_g(Y)g(D_XZ,W)+g(D_YD_XZ,W)\\&+g(D_XZ,D_YW)+\omega_g(Y)g(Z,D_XW)\\&+g(D_YZ,D_XW)+g(Z,D_YD_XW)\end{aligned}$$

and obtain the formula for $X(Yg(Z,W))$ by exchanging X and Y in the above formula.

Subtracting $Y(X(g(Z,W)))$ from $X(Y(g(Z,W)))$ yields

$$d\omega_g(X,Y)g(Z,W)=g(R^D(X,Y)Z,W)+g(Z,R^D(X,Y)W). \tag{1}$$

So

$$-nd\omega_g(X,Y)=2\sum_i g(R^D(X,Y)E_i,E_i),$$

where $\{E_i\}$ is an orthonormal base at p with respect to g. By the aid of the first Bianchi identity (notice that the connection D is torsion free so that the first Bianchi identity holds)

$$\begin{aligned}-nd\omega_g(X,Y)&=-2\sum_i\{g(R^D(Y,E_i)X,E_i)+g(R^D(E_i,X)Y,E_i)\}\\&=-2Ric^D(X,Y)+2Ric^D(Y,X).\end{aligned}$$

From this lemma we can conclude the proposition.

REMARK. (i) The formula (1) gives us the decomposition of R^D according to the Lie algebra decomposition $\mathbf{co}(n)=\mathrm{R}\oplus\mathbf{so}(n)$ for the conformal orthogonal group $CO(n)^+$ as

$$R^D=\frac{1}{2}d\omega_g\otimes\mathrm{id}_{T_pM}+R^0 \tag{2}$$

where R^0 denotes the $\mathbf{so}(n)$-part of R^D.

THEOREM 1. *Let $(M,D,[g])$ be a Weyl manifold. If the 2-form $d\omega_g$ is parallel, i.e., $D_Xd\omega_g=0$ for any tangent vector X, then $d\omega_g=0$, namely, $(M,D,[g])$ is locally trivial.*

PROOF. Assume that $d\omega_g$ is non-zero. It suffices from this assumption to deduce a contradiciton.

Take a point $p\in M$ where $d\omega_g\neq 0$. Since $D(d\omega_g)=0$, $d\omega_g$ is holonomy invariant. Namely,

$$d\omega_g(\varphi X,\varphi Y)=d\omega_g(X,Y)$$

for $X,Y\in T_pM$ and $\varphi\in\Phi^0$.

Choose a metric g representing the conformal structure $[g]$ and fix it so that we have an endomorphism $\Omega:T_pM\to T_pM$ satisfying

$$(d\omega_g)_p(X,Y)=g_p(\Omega(X),Y),\qquad X,Y\in T_pM.$$

Since $d\omega_g$ is skew-symmetric,

$$\Omega+\Omega^*=0 \tag{3}$$

for the adjoint Ω^* of Ω.

For the inner product g_p at p each $\varphi \in \Phi^0$ has as an endomorphism of T_pM the form

$$\varphi = \lambda A, \quad \lambda \in \mathbf{R}^+, \quad A \in \mathrm{SO}(n).$$

Denote by $\varphi^* : T_pM \to T_pM$ the adjoint of φ. Then,

$$\varphi^* \circ \varphi = \lambda^2 \mathrm{id}_{T_pM},$$

i.e., $g_p(\varphi^*(\varphi(X)), Y) = \lambda^2 g_p(X, Y)$.

On the other hand, we have

$$\Omega(\varphi(X)) = \frac{1}{\lambda^2}\varphi(\Omega(X)), \quad X \in T_pM,$$

or equivalently

$$\Omega(\varphi^{-1}(X)) = \lambda^2 \varphi^{-1}(\Omega(X)). \tag{4}$$

This is from the holonomy invariance of $d\omega_g$.

From (3) eigenvalues of Ω are all pure imaginary so that we write them as

$$ia_1, ia_2, \cdots, ia_r,$$

where all a_i's are real and $a_1 < \cdots < a_r$. Since $d\omega_g \neq 0$ at p, one has either $a_r > 0$ or $a_1 < 0$.

As the case $a_1 < 0$ is similar, we can assume $a_r > 0$. So, there is a complex vector $X_r \in (T_pM)^{\mathbf{C}}$ such that $\Omega(X_r) = ia_r X_r$. Then, from (4) $\varphi^{-1}(X_r)$ is an eigenvector for Ω of eigenvalue $i\lambda^2 a_r$.

That $\lambda = 1$ for $\varphi \in \Phi^0$ can be then concluded from a_r being largest among $\{a_1, \cdots, a_r\}$. It follows that all φ have only the SO(n) factor. So from Propositon 1 $(M, D, [g])$ is locally trivial, which contradicts to $d\omega_g \neq 0$.

If R^D is parallel, then Ric^D, the trace of R^D and hence $d\omega_g = \frac{4}{n}\mathrm{skew}\ Ric^D$ is also parallel. Therefore the following is immediate from Theorem 1.

THEOREM 2. *Let $(M, D, [g])$ be a connected Weyl manifold. If it is affine symmetric (or affine locally symmetric), or more generally the Ricci tensor Ric^D is D-parallel, then it must be locally trivial.*

We conclude this section by giving two applications of Proposition 2 for Weyl manifolds, one for Weyl manifold being of constant curvature in the sense of Weyl manifold and another one for Weyl 4-manifolds whose curvature tensor R^D is self-dual.

Here we say $(M, D, [g])$ is of constant curvature if it satisfies

$$R^D(X, Y)Z = \Lambda\{g(Y, Z)X - g(X, Z)Y\}, \quad \Lambda \in C^\infty,$$

for some metric g in $[g]$.

By taking trace of this equality one gets $Ric^D(X, Y) = (n-1)\Lambda g(X, Y)$, whence Ric^D is symmetric. Thus, we get from Proposition 2

COROLLARY 1. *Let $(M, D, [g])$ be a Weyl manifold. If it is of constant curvature in the sense of Weyl manifold, then it must be locally trivial.*

Consider, furthermore, an oriented Weyl 4-manifold whose R^D satisfies

$$*R^D = R^D$$

in terms of the Hodge star operator $*$. Here we regard R^D as an End(TM)-valued 2-form.

Then from (2) the diagonal component of R^D must be a self-dual 2-form so that $d\omega_g = 0$ holds from the integral $\int_M d\omega_g \wedge *d\omega_g = 0$ over M when M is compact. The condition $*R^0 = R^0$ then implies the Ricci flatness of the affine connection D. Thus it is concluded that such a 4-manifold must be locally trivial and also Ricci-flat, provided it is compact (see [11], Lemma 1.3.2 in which this self-duality curvature condition was investigated and the same conclusion was derived).

2. Gauduchon constant and compact Einstein-Weyl 4-manifolds.

We will show the diffeomorphism finiteness theorem for compact Einstein-Weyl manifolds satisfying certain boundedness.

Weyl manifolds we consider in this section and the subsequent section are assumed to be compact, connected and oriented, unless stated otherwise.

For a Weyl manifold $(M, D, [g])$ the Einstein-Weyl equation sym $(Ric^D) = \Lambda g$ can be written in terms of the Levi-Civita connection ∇ of a representative metric g of $[g]$ as ([10])

$$Ric_g + \frac{n-2}{4}(\nabla^s\omega_g + \omega_g \otimes \omega_g) = \Lambda g$$

(where Ric_g is the Ricci tensor of g and the covariant 2-tensor $\nabla^s\omega_g$ is defined by $(\nabla^s\omega_g)(X, Y) = (\nabla_X\omega_g)(Y) + (\nabla_Y\omega_g)(X)$).

By applying the gauge fixing argument given by Gauduchon ([5]) together with Tod's observation ([16]), we can choose a special metric g in $[g]$ uniquely up to homothety in such a way that the Einstein-Weyl equation splits into the Killing field equation and the simplified Einstein-Weyl equation as

$$\text{(i)}\quad \nabla^s\omega_g = 0$$

$$\text{(ii)}\quad Ric_g + \frac{n-2}{4}\omega_g \otimes \omega_g = \tilde{\Lambda} g, \quad \tilde{\Lambda} \in C^\infty(M)$$

from which it turns out that

$$c_g = s_g - \frac{n+2}{4}|\omega_g|^2$$

is constant (s_g is the scalar curvature of g) ([6], [10]). We call it the *Gauduchon constant.* We note that c_g vol$(M, g)^{2/n}$ is an invariant of an Einstein-Weyl n-manifold.

Notice that any pair (g, ω) on an n-manifold M satisfying (i), (ii) yields an Einstein-Weyl manifold $(M, D, [g])$ by defining the torsion free affine connection

$$D = \nabla + a, \quad a_X Y = \frac{1}{2}\{g(X, Y)\omega^\sharp - \omega(X)Y - \omega(Y)X\}$$

so that $Dg = \omega \otimes g$.

We call a metric g in $[g]$ satisfying the above (i), (ii) *Gauduchon metric.*

In virtue of the Killing field equation, ω_g fulfils ([10]).

$$\nabla^\star\nabla\omega_g = Ric_g(\omega_g) = \left(\frac{c_g}{n} + \frac{4-n}{4}|\omega_g|^2\right)\omega_g\,. \tag{5}$$

Further the Ricci tensor of the Gauduchon metric satisfies ([10])

$$Ric_g = \left(\frac{1}{n}c_g + \frac{1}{2}|\omega_g|^2\right)g - \frac{n-2}{4}\omega_g\otimes\omega_g\,.$$

We restrict ourselves in this section to Einstein-Weyl 4-manifolds.

The above two formulas enjoying a keystone for analyzing Einstein-Weyl manifolds then reduce to

$$\nabla^\star\nabla\omega_g = \frac{1}{4}c_g\omega_g\,, \tag{6}$$

$$Ric_g = \left(\frac{1}{4}c_g + \frac{1}{2}|\omega_g|^2\right)g - \frac{1}{2}\omega_g\otimes\omega_g\,. \tag{7}$$

So, by integrating the first equation as

$$\int_M |\nabla\omega_g|^2 dv_g = \frac{1}{4}c_g\int_M |\omega_g|^2 dv_g\,,$$

we easily obtain the following, as stated in [10].

If c_g is negative, then the Gauduchon metric g is Einstein with $\omega_g = 0$ and D coincides with the Levi-Civita connection ∇ of g.

If $c_g = 0$, then $\nabla\omega_g = 0$ and hence either (i) $\omega_g = 0$ identically or (ii) ω_g is a nonzero ∇-parallel form so that (i) g is Ricci flat or (ii) $b_1(M) = 1$ and the universal covering of (M, g) is isometric to the Riemannian product of a round 3-sphere and a straight line.

If $c_g > 0$, then Ric_g is positive definite, more precisely

$$Ric_g \geq \frac{c_g}{4}g\,, \tag{8}$$

and then $\pi_1(M)$ is finite. Actually one has

$$Ric_g(\omega_g^\sharp, \omega_g^\sharp) = \frac{c_g}{4}|\omega_g|^2\,,$$

$$Ric_g(\omega_g^\sharp, X) = 0\,,$$

$$Ric_g(X, X) = \left(\frac{c_g}{4} + \frac{1}{2}|\omega_g|^2\right)g(X, X)$$

for any tangent vector X orthogonal to $\omega_g^\sharp$.

The diffeomorphism finiteness theorem for compact Einstein-Weyl 4-manifolds is deeply based upon the following

THEOREM (Anderson and Cheeger [2]). *For positive constants Δ, v, γ and Γ the collection of closed, connected Riemannian n-manifolds (M, g) satisfying the bounds*

$$\operatorname{diam}(M, g) \leq \Delta\,, \tag{9}$$

$$\operatorname{vol}(M, g) \geq v\,, \tag{10}$$

$$|Ric_g| \leq \gamma\,, \tag{11}$$

$$\int_M |R_g|^{n/2} dv_g \leq \Gamma\,, \tag{12}$$

where R_g is the Riemannian curvature tensor, contains at most a finite number of diffeomorphism types.

This theorem was obtained by consideration of regions of three different types, regions on which R_g concentrates in $L^{n/2}$-norm, regions on which R_g does not concentrate and neck regions between those of the previous types and then applying a slightly modified standard argument in Cheeger's finiteness theorem ([4]).

Our finiteness theorem can be stated as follows.

THEOREM 3. *For positive numbers B, C and S satisfying $C \leq S$ the collection $\mathcal{EW}^4(B, C, S)$ of Einstein-Weyl 4-manifolds of positive Gauduchon constant c_g satisfying the bounds*

$$b_2(M) \leq B\,,$$
$$c_g^2 \operatorname{vol}(M, g) \geq C^2\,,$$
$$s_g^2 \operatorname{vol}(M, g) \leq S^2\,,$$

(*where g is a Gauduchon metric in* $[g]$) *contains at most a finite number of diffeomorphism types.*

PROOF. The proof is given first by identifying $\mathcal{EW}^4(B, C, S)$ with the collection $\mathcal{SEW}(B, C, S)$ of certain quadruplets (M, g, ω, c) and then applying Theorem (Anderson and Cheeger) to $\mathcal{SEW}$. Here $\mathcal{SEW}(B, C, S)$ is the collection of quadruplets (M, g, ω, c) such that c is a positive constant, and g and ω are a unit volume metric and a one-form on a compact, connected, oriented 4-manifold M satisfying

$$\nabla^s \omega = 0\,, \tag{13}$$

$$Ric_g = \left(\frac{1}{4}c + \frac{1}{2}|\omega|^2\right) g - \frac{1}{2}\omega \otimes \omega \tag{14}$$

and obeying the bounds

$$b_2(M) \leq B\,, \tag{15}$$

$$c \geq C\,, \tag{16}$$

$$s_g \leq S\,. \tag{17}$$

Let (M, g, ω, c) be a quadruplet in $\mathcal{SEW}(B, C, S)$. Then from (8)

$$Ric_g \geq \frac{C}{4} g$$

and by Myers' theorem the diameter is uniformly bounded;

$$\operatorname{diam}(M, g) \leq \frac{\sqrt{12}}{\sqrt{C}}\pi \tag{18}$$

For the upper bound for $|Ric_g|$ we have

$$|Ric_g| \leq \frac{1}{\sqrt{3}} S. \tag{19}$$

Actually,

$$3|Ric_g|^2 = s_g^2 - \frac{1}{4}c^2 - \frac{3}{4}c|\omega|^2 \tag{20}$$

so, $3|Ric_g|^2 \leq S^2$.

The uniform boundness of the L_2-norm of the Riemannian curvature tensor R_g is derived from the Chern-Gauss-Bonnet integral formula;

$$\int_M |R_g|^2 dv_g = 32\pi\chi(M) + \int_M (4|Ric_g|^2 - s_g^2) dv_g .$$

Since

$$4|Ric_g|^2 - s_g^2 \leq \frac{1}{3}s_g^2 ,$$

the square L_2-norm of R_g is estimated by $32\pi(2+B) + \frac{1}{3}S^2$. We conclude then Theorem 3.

REMARK 1. As Anderson and Cheeger state implicitly ([2]), the finiteness theorem holds also for the collection of compact Einstein 4-manifolds having the bounds in (9), (10) and (11) of Theorem (Anderson and Cheeger) together with the Betti number bound $b_2(M) \leq B$, instead of (12).

REMARK 2. The Thorpe-Hitchin inequality holds for both compact Einstein 4-manifolds and compact Einstein-Weyl 4-manifolds (see [8], [3], [14], [10]). Moreover we have $\pi_1 < \infty$ for both such 4-manifolds of positive Ricci tensor Ric_g. So, geometric aspects of compact Einstein-Weyl 4-manifolds resemble those of Einstein 4-manifolds. Note that Pedersen, Poon and Swann constructed a nontrivial compact Einstein-Weyl 4-manifolds as bundle metrics on sphere bundles over a compact 2-manifolds ([16]). These 4-manifolds are $S^2 \times S^2$ and $\mathbf{C}P^2 \sharp \overline{\mathbf{C}P^2}$ which are also equipped with Einstein metrics, the product standard metric and the Page metric, respectively ([3]).

REMARK 3. Theorem 3 states the finiteness in terms of diffeomorphism types. However, a finiteness theorem which is more geometrical is available (refer to [2], Remark 3.2). Actually we get in Theorem 3 a finite number of ε-quasiisometry types for compact Einstein-Weyl 4-manifolds of positive Gauduchon constant.

REMARK 4. If we relax the bounds of Theorem 3 as $b_2(M) \leq B$ and $C \leq c_g \mathrm{Vol}(M, g)^{1/2}$, without any assumption on the scalar curvature, then by using the estimate for the Sobolev constant we can estimate the bounds of $\|Ric_g\|_{L_4}$, $\|R_g\|_{L_2}$ in terms of B and C. So, in view of the argument given by Anderson in [1] any sequence of Einstein-Weyl 4-manifolds (M_i, g_i) with a Gauduchon metric g_i of $\mathrm{Vol}(M_i, g_i) = 1$ has a subsequence convergent to a compact Riemannian 4-orbifold having finite number of singular points.

3. Other finiteness theorems.

For three dimensional Einstein-Weyl manifolds we have also the diffeomorphism finiteness theorem.

THEOREM 4. *For given positive numbers C and S, $C \leq S$ let $\mathcal{EW}^3(C, S)$ be the collection of Einstein-Weyl 3-manifolds $(M, D, [g])$ of Gauduchon constant c_g satisfying bounds*

$$c_g \mathrm{Vol}(M, g)^{2/3} \geq C,$$
$$s_g \mathrm{Vol}(M, g)^{2/3} \leq S$$

for a Gauduchon metric g within $[g]$. Then, $\mathcal{EW}^3(C, S)$ has at most a finite number of diffeomorphism types.

This theorem then shows us that the collection of minitwistor spaces naturally arising from compact, oriented Einstein-Weyl 3-manifolds satisfying the bounds consists of a finite number of complex surfaces, up to diffeomorphism.

PROOF. Let $(M, D, [g]) \in \mathcal{EW}^3(C, S)$. Then

$$Ric_g = \left(\frac{1}{3}c_g + \frac{1}{2}|\omega_g|^2\right) g - \frac{1}{4}\omega_g \otimes \omega_g$$

for a Gauduchon metric g. Normalizing volume of the metric g one has $C \leq c_g$ and $s_g \leq S$ so that from the above

$$\frac{1}{3}Cg \leq Ric_g \leq \frac{1}{2}Sg.$$

These, in fact, follow from

$$Ric_g(\omega^\sharp, \omega^\sharp) = \left(\frac{1}{3}c_g + \frac{1}{4}|\omega_g|^2\right) |\omega^\sharp|^2,$$
$$Ric_g(X, X) = \left(\frac{1}{3}c_g + \frac{1}{2}|\omega_g|^2\right) |X|^2$$

for X orthogonal to the dual $\omega^\sharp$ of ω_g.

So, from Myers' theorem $\mathrm{diam}(M, g)$ is uniformly bounded from above.

To get the boundedness of $|R_g|^2$ we diagonalize Ric_g.

We must consider the two possibilities.

At a point where ω_g vanishes the metric is Einstein; $R_{ij} = (1/3)c_g\delta_{ij}$, so that the metric is of constant curvature and hence $|R_g|^2$ has a bound in terms of S^2.

We consider another case, namely, the bound for $|Ric_g|$ at a point p where $\omega_g \neq 0$. In fact at p the Ricci tensor takes the form

$$R_{ij} = \gamma_i \delta_{ij}, \quad i, j = 1, 2, 3$$
$$\gamma_1 = \frac{1}{3}c_g + \frac{1}{4}|\omega|^2,$$
$$\gamma_2 = \gamma_3 = \frac{1}{3}c_g + \frac{1}{2}|\omega_g|^2.$$

Then $R_{1212} = R_{1313} = (1/2)\gamma_1$ and $R_{2323} = \gamma_2 - (1/2)\gamma_1$, and hence at p

$$|R_{ijij}| \leq \frac{1}{3}S\,, \quad 1 \leq i < j \leq 3\,,$$

and other components of R_g vanish.

Therefore, $|R_g|$ is bounded above uniformly in terms of S. So we can apply Cheeger's finiteness theorem ([4]) to obtain the theorem.

For Einstein-Weyl $n(\geq 5)$-manifolds the finiteness theorem can be stated as follows in terms of a bound on the $L^{\frac{n}{2}}$-norm of the Riemannian curvature.

THEOREM 5. *For given positive numbers C_1, Δ, Γ let $\mathcal{EW}^n(C_1, \Delta, \Gamma)$ be the collection of Einstein-Weyl $n(\geq 5)$-manifolds $(M, D, [g])$ having positive Gauduchon constant c_g* fulfilling the bounds

$$c_g \operatorname{vol}(M, g)^{\frac{2}{n}} \leq C_1$$
$$\operatorname{diam}(M, g) \operatorname{vol}(M, g)^{-\frac{1}{n}} \leq \Delta\,,$$
$$\int_M |R_g|^{\frac{n}{2}} dv_g \leq \Gamma$$

for a Gauduchon metric g in $[g]$. Then $\mathcal{EW}^n(C, \Delta, \Gamma)$ has at most a finite number of diffeomorphism types.

This is verified by choosing a Gauduchon metric g of unit volume. In fact, unlike the case of $n = 3, 4$ we have the sup-norm bound on $|\omega_g|^2$ for Einstein-Weyl $n(\geq 5)$-manifolds ([10])

$$\sup |\omega_g|^2 \leq \frac{4}{n(n-4)} c_g$$

and hence from the formula (7) in §2 the sup norm of Ric_g is bounded above as

$$|Ric_g| \leq k_n C_1\,.$$

(k_n is the universal constant depending only on n.)

So, we can apply Theorem (Anderson and Cheeger) to get the theorem.

4. Affine locally symmetric Einstein-Weyl manifolds.

We are finally in a position to give the triviality remark for compact, affine locally symmetric, Einstein-Weyl manifolds.

Let $(M, D, [g])$ be an Einstein-Weyl n-manifold with a Gauduchon metric g having Gauduchon constant c_g.

If $(M, D, [g])$ is locally trivial, then the one-form ω_g for which $Dg = \omega_g \otimes g$ is harmonic, since $d^\star \omega_g = -tr_g \nabla^s \omega_g = 0$.

THEOREM 6. *Let $(M, D, [g])$ be a compact, connected and oriented, Einstein-Weyl n-manifold with a Gauduchon metric g of Gauduchon constant c_g.*

Suppose that $(M, D, [g])$ be affine locally symmetric. Then we have

(i) $(M, D, [g])$ *is globally trivial, in other words, g is Riemannian locally symmetric and D coincides with the Levi-Civita connection of g, provided that* $c_g > 0$ *and one of the following is satisfied*;

(a) $n = 3, 4$,

(b) $n \geq 5$ *and* ω_g *satisfies* $|\omega_g|^2 \leq \{4/n(n-4)\}c_g$, *but not equal identically, and*

(ii) *the same conclusion holds as that of* (i), *provided that* $c_g \leq 0$ *and one of the following is satisfied*;

(c) $n \geq 5$,

(d) $n = 4$ *and* $c_g < 0$,

(e) $n = 3$ *and* $c_g/3 + (1/4)|\omega_g|^2 \leq 0$, *but not identically equal.*

PROOF. Since an affine locally symmetric Weyl manifolds is locally trivial, the one-form ω_g for our manifold $(M, D, [g])$ is harmonic.

For (i) it suffices to show $b_1(M) = 0$. If $n = 3, 4$, then $c_g > 0$ implies $Ric_g > 0$, so $\pi_1(M) < \infty$ and hence $b_1(M) = 0$. For the case where $n \geq 5$ and ω_g satisfies the above inequality, $Ric_g \geq 0$ and $Ric_g > 0$ at some point. So, the Bochner-Weitzenböck formula yields $b_1(M) = 0$ and then $\omega_g = 0$, from which D coincides with the Levi-Civita connection of the Gauduchon metric g.

For (ii) the integral formula $\int_M |\nabla\omega_g|^2 dv_g = \int_M \left(\frac{c_g}{n} - \frac{n-4}{4}|\omega_g|^2\right)|\omega_g|^2 dv_g$ leads us the conclusion.

The following characterizes compact Einstein-Weyl manifolds which are affine locally symmetric, but not globally trivial.

THEOREM 7. *Let* $(M, D, [g])$ *be a compact, connected and oriented Einstein-Weyl n-manifold with a Gauduchon metric g of Gauduchon constant* c_g. *If* $(M, D, [g])$ *is affine locally symmetric, but not globally trivial, then the universal covering of* (M, g) *must be a Riemannian product of the straight line* $\mathbf{R}$ *and a simply connected Riemannian symmetric Einstein space of compact type.*

PROOF. Since, from (5) in §1 the one-form ω_g satisfies

$$\nabla^\star\nabla\omega_g = Ric_g(\omega_g) = \frac{1}{n}\left(c_g - \frac{n(n-4)}{4}|\omega_g|^2\right)\omega_g,$$

one has $\nabla^\star\nabla\omega_g = 0$, because ω_g is harmonic and $\Delta\omega_g = \nabla^\star\nabla\omega_g + Ric_g(\omega_g)$. So, by integrating one gets $\nabla\omega_g = 0$ and hence $|\omega_g|$ is constant. On the other hand, since $(M, D, [g])$ is not globally trivial, ω_g is non-zero and it follows from $Ric_g(\omega_g) = 0$ that $|\omega_g|^2 = \{4/n(n-4)\}c_g > 0$ for $n \neq 4$ and $c_g = 0$ for $n = 4$. Therefore, the results of Theorem 2, (iii), (v) and of Theorem 4, (iii) in [10] apply and the universal covering of (M, g) is a Riemannian product of $\mathbf{R}$ and a positive Ricci, Einstein $(n-1)$-manifold N. One can see that N is a Riemannian symmetric space.

REMARK. The possible universal covering spaces in the above theorem are $\mathbf{R}\times S^2$ for $n = 3$, $\mathbf{R}\times S^3$ for $n = 4$, and $\mathbf{R}\times G/K$ for $n \geq 5$ where G/K is a simply connected Riemannian symmetric Einstein space of compact type. In these spaces the Gauduchon metric

g is a product metric $dt^2 + g_o$ and the one-form ω_g defining the connection D is $\omega_g = adt$ (where $a > 0$ is some constant and t is the coordinate of $\mathbf{R}$, g_o is the invariant metric on the symmetric space).

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Present Address:
INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA,
TSUKUBA-SHI, IBARAKI, 305–8571 JAPAN.