TOKYO J. MATH.
VOL. 23, NO. 2, 2000

Formal Gevrey Class of Formal Power Series Solution for Singular First Order Linear Partial Differential Operators

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(Communicated by K. Uchiyama)

1. Introduction.

In this paper, we will study the following first order partial differential equation

$$Lu(z) = \sum_{i=1}^{n} a_i(z)\partial_{z_i} u(z) = F(z, u(z)) \tag{1.1}$$

where $z = (z_1, \cdots, z_n) \in \mathbf{C}^n$ and $\partial_{z_i} = \partial/\partial z_i$ for $i = 1, \cdots, n$. We assume the following conditions through this paper. The functions $a_i(z)$ and $F(z, u)$ are holomorphic functions in a neighborhood of the origin in $\mathbf{C}^n$ and $\mathbf{C}^{n+1}$ respectively, and $a_i(z)$ satisfies $a_i(0) = 0$ for $i = 1, \cdots, n$.

There are many results for (1.1). Oshima [O] and Kaplan [K] studied the existence of holomorphic solutions under some conditions.

We treat a formal power series solution for (1.1). If the solution converges, then our result becomes that of [O] and [K]. Our purpose in this paper is to give precise estimates of (1.1) in a formal Gevrey class via an appropriate coordinates change for (1.1).

We consider three examples in case $n = 2$. We put

$$P_1 = (z_1\partial_{z_1} + 1) - z_1^2\partial_{z_2}, \tag{1.2}$$

$$P_2 = (z_1\partial_{z_1} + 1) - z_2^2\partial_{z_2}, \tag{1.3}$$

$$P_3 = (z_1\partial_{z_1} + 1) - (z_1^2 + z_2^2)\partial_{z_2}. \tag{1.4}$$

The operator P_1 satisfies the conditions of [O] and [K] and $P_1 u(z) = F(z, u)$ has a unique holomorphic solution.

Next we consider P_2 and P_3. They do not satisfy the conditions of [O] and [K], while the equation

$$P_2 u(z) = \frac{z_2}{1 - z_1} \tag{1.5}$$

Received August 17, 1999
Revised February 3, 2000

has a formal power series solution $u(z)=\sum u_{\beta_1,\beta_2}z_1^{\beta_1}z_2^{\beta_2}$ with

$$u_{\beta_1,\beta_2}=\frac{(\beta_2-1)!}{(\beta_1+1)^{\beta_2-1}}. \tag{1.6}$$

We find that the solution diverges with respect to a variable z_2, while

$$\sum u_{\beta_1,\beta_2}\frac{z_1^{\beta_1}z_2^{\beta_2}}{\beta_2!} \tag{1.7}$$

converges in a neighborhood of the origin by (1.6).

Our motivation comes from the following example. We consider

$$P_3u(z)=\frac{z_2}{1-z_1}. \tag{1.8}$$

We expect that (1.8) has a formal power series solution with similar property as in (1.5). But we obtain that (1.8) has a formal power series solution with

$$u_{\beta_1,\beta_2}\geq\frac{([\beta_1/2]+\beta_2)!([\beta_1/2]+\beta_2-1)![\beta_1/2]!}{(\beta_1+1)!\beta_2}. \tag{1.9}$$

We find that this solution diverges with respect to the both variables (z_1, z_2).

We consider the following equation

$$z_1\frac{d\phi(z_1)}{dz_1}=z_1^2+(\phi(z_1))^2. \tag{1.10}$$

This equation has a holomorphic solution $\phi(z_1)$ in a neighborhood of the origin with $\phi(z_1)\equiv O(z_1^2)$. For the solution $\phi(z_1)$, we change the coordinate

$$x=z_1 \quad\text{and}\quad t=z_2+\phi(z_1). \tag{1.11}$$

Then the solution $u(z)=v(x(z),t(z))=\sum v_{\beta_1,\beta_2}x^{\beta_1}t^{\beta_2}$ has that

$$\sum v_{\beta_1,\beta_2}\frac{x^{\beta_1}t^{\beta_2}}{\beta_2!} \tag{1.12}$$

converges in a neighborhood of the origin.

In this paper, we find a good coordinate as (1.11) and give an estimate as (1.12) for (1.1).

In Section 2, we list some notations and define a formal Gevrey class and an order δ that is important to give an estimate. Lastly we give our main result. In Section 3, we give some estimates for Gamma function that are used in Section 4, and show some properties about an order δ. In Section 4, we show that a particular equation has a formal power series solution and the solution belongs to a formal Gevrey class. In Section 5, we give an existence of a holomorphic solution for a first order nonlinear system equation as (1.10). In Section 6, we find good coordinates as (1.11) and show that our equation becomes an equation of Section 4 by using results of Section 3 and Section 5.

The author would like to express his gratitude to Professor Sunao Ōuchi and Professor Hidetoshi Tahara at Sophia University for their advice and warm encouragement.

2. Notations and main result.

The sets **R**, **C** and **N** denote the set of all real numbers, complex numbers and nonnegative integers respectively. Let $z = (z_1, z_2, \cdots, z_n) \in \mathbf{C}^n$, $x = (x_1, x_2, \cdots, x_{n_0}) \in \mathbf{C}^{n_0}$, $t = (t_1, t_2, \cdots, t_{n_1-n_0}) \in \mathbf{C}^{n_1-n_0}$, and $y = (y_1, y_2, \cdots, y_{n-n_1}) \in \mathbf{C}^{n-n_1}$. The set $\mathbf{C}\{y\}[[x, t]]$ denotes the set of all formal power series $\sum_{|k|+|l|\geq 0} u_{k,l}(y)x^k t^l$ with coefficients $\{u_{k,l}(y)\}$ holomorphic functions in a common neighborhood of the origin.

DEFINITION 2.1. Let $u(x, t, y) = \sum_{|k|+|l|\geq 0} u_{k,l}(y)x^k t^l \in \mathbf{C}\{y\}[[x, t]]$. If

$$\sum_{|k|+|l|\geq 0} u_{k,l}(y)\frac{x^k t^l}{|l|!^d} \tag{2.1}$$

is a convergent power series for $d \geq 0$, then we say that $u(x, t, y)$ belongs to a formal Gevrey space $G_t^{\{d\}}(x, t, y)$.

We say that d is a formal Gevrey index and t is Gevrey variables with respect to d.

Here we consider the following partial differential equation

$$Lu(z) := \sum_{i=1}^{n} a_i(z)\partial_{z_i} u(z) = F(z, u(z)) \tag{2.2}$$

where $a_i(z)$ and $F(z, u)$ are holomorphic functions in a neighborhood of the origin in $\mathbf{C}^n$ and $\mathbf{C}^{n+1}$ respectively, and $a_i(z)$ satisfies $a_i(0) = 0$ for $i = 1, \cdots, n$.

We give the following two notations for the operator L.

: **(1)** $S = \{z \in \mathcal{U}; a_i(z) = 0 \text{ for } i = 1, 2, \cdots, n\}$ (2.3)
where $\mathcal{U}$ is a neighborhood of the origin in $\mathbf{C}^n$.

: **(2)** The matrix $\left(\dfrac{\partial a}{\partial z}(0)\right)$ denotes the Jacobian matrix of $a := (a_1(z), \cdots, a_n(z))$ at the origin.

We assume that (2.2) satisfies the following conditions (A.1)–(A.4).

: **(A.1)** S is a complex submanifold of codimension n_1 in $\mathcal{U}$ $(1 \leq n_1 \leq n)$.

If we assume (A.1), then there exist n_1-holomorphic functions $\zeta_i = \zeta_i(z)$ with $\zeta_i(0) = 0$ $(i = 1, 2, \cdots, n_1)$ that are functional independent each other such that

$$S = \{z \in \mathcal{U}; \zeta_i(z) = 0 \text{ for } i = 1, 2, \cdots, n_1\}. \tag{2.4}$$

: **(A.2)** The function $F(z, u)$ is a holomorphic function in a neighborhood of the origin of $\mathbf{C}^n \times \mathbf{C}$ with $F(z, 0) \equiv 0$ for $z \in S$.

: **(A.3)** Jordan normal form of $\left(\dfrac{\partial a}{\partial z}(0)\right)$ is

$$J(\lambda,\mu)=\begin{pmatrix}\lambda_1 & 0 & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & 0\\ \mu_1 & \lambda_2 & 0 & \cdots & \cdots & 0 & & & \vdots\\ 0 & \mu_2 & \ddots & \ddots & & \vdots & & & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & \vdots\\ \vdots & & \ddots & \ddots & \ddots & 0 & & & 0\\ 0 & \cdots & \cdots & 0 & \mu_{n_0-1} & \lambda_{n_0} & 0 & & 0\\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & & 0\\ \vdots & & & & & & \ddots & \ddots & \vdots\\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0\end{pmatrix} \tag{2.5}$$

where λ_i is the nonzero eigenvalues for $i = 1, 2, \cdots, n_0$ and $\mu_i = 0$ or 1 for $i = 1, 2, \cdots, n_0 - 1$ with $1 \le n_0 \le n_1$.

We will define the following $\mathcal{M}$ by using $\zeta_1(z), \cdots, \zeta_{n_1}(z)$ in (2.4). Let $\mathbf{C}\{z\}$ be the ring of convergent power series at the origin in the variables $\{z\}$. Then we define

$$\mathcal{M} := \sum_{i=1}^{n_1} \mathbf{C}\{z\}\zeta_i(z). \tag{2.6}$$

Therefore by (A.1), we have

$$a_i(z) \in \mathcal{M} \quad \text{for} \quad i = 1, \cdots, n. \tag{2.7}$$

We define an ideal that is constructed by some elements of $\mathcal{M}$. Let m be any positive integer and set $\{g_1(z), \cdots, g_m(z)\} \subset \mathcal{M}$. Then we define

$$\mathcal{I}\{g_1, \cdots, g_m\} := \sum_{i=1}^{m} \mathbf{C}\{z\} g_i. \tag{2.8}$$

By (A.3), we can take n_0-functions $\{a_{i_j}\}_{j=1}^{n_0}$ that are functional independent each other. If we assume (A.1) and (A.3), then we have

$$\mathcal{M} \supset \mathcal{M}^2 \supset \mathcal{M}^3 \supset \cdots \quad \text{and} \quad \mathcal{I}\{a_{i_1}, \cdots, a_{i_{n_0}}\} \subset \mathcal{M}. \tag{2.9}$$

Hence there exists δ_i such that

$$\delta_i := \sup\{d; a_i \in \mathcal{M}^d \ mod\, \mathcal{I}\{a_{i_1}, \cdots, a_{i_{n_0}}\}\} \tag{2.10}$$

for each $i = 1, \cdots, n$. If $a_i \in \mathcal{I}\{a_{i_1}, \cdots, a_{i_{n_0}}\}$, then we define $\delta_i := \infty$. Then we can define the following multiplicity δ

$$\delta := \min\{\delta_1, \delta_2, \cdots, \delta_n\} \tag{2.11}$$

and we have

$$a_i \in \mathcal{M}^\delta \ mod\, \mathcal{I}\{a_{i_1}, \cdots, a_{i_{n_0}}\} \quad \text{for} \quad i = 1, \cdots, n. \tag{2.12}$$

We assume condition (A.4).

: (A.4) $$\delta \ge 2. \tag{2.13}$$

Our main result in this paper is the following.

THEOREM 2.2. *Assume* (A.1), (A.2), (A.3) *and* (A.4). *Further assume that there exists a positive constant* σ *such that*

$$\left|\sum_{i=1}^{n_0} \lambda_i k_i - c\right| \geq \sigma(|k|+1) \quad \textit{for} \quad \forall k = (k_1, k_2, \cdots, k_{n_0}) \in \mathbf{N}^{n_0} \tag{2.14}$$

where $|k| = k_1 + k_2 + \cdots + k_{n_0}$ *and* $c = \dfrac{\partial F}{\partial u}(0,0)$. *Then we have the following two results.*

1) *There exists a unique formal power series solution* $u(z)$ *such that* (2.2).

2) *There exist local coordinates* $(x(z), t(z), y(z)) \in \mathbf{C}^{n_0} \times \mathbf{C}^{n_1 - n_0} \times \mathbf{C}^{n-n_1}$ *in a neighborhood of the origin such that*

$$S = \{z \in \mathbf{C}^n; x(z) = 0, t(z) = 0\}, \tag{2.15}$$

$$u(z) = U(x(z), t(z), y(z)) \in G_t^{\{\frac{1}{\delta-1}\}}(x(z), t(z), y(z)). \tag{2.16}$$

If $\delta = \infty$ then we have $n_0 = n_1$. We remark that the case $\delta = \infty$ are treated in [O] and [K].

3. Properties of multiplicity and estimates of Gamma function.

In this section, we give some lemmas that are needed to prove Theorem 2.2.

3.1. **Properties of multiplicity δ.** We assume conditions (A.1) and (A.3), and under two conditions we show that multiplicity δ is invariant under a coordinate change and independent of a choice of n_0 independent functions from $\{a_1, \cdots, a_n\}$. Hence we may assume that $\{a_1, \cdots, a_{n_0}\}$ are functional independent by rewriting number. Then we put

$$\delta_i := \sup\{d; a_i \in \mathcal{M}^d \ mod\, \mathcal{I}\{a_1, \cdots, a_{n_0}\}\}, \tag{3.1}$$

$$\delta := \min\{\delta_1, \delta_2, \cdots, \delta_n\}. \tag{3.2}$$

LEMMA 3.1. *Assume* (A.1) *and* (A.3). *Then the number* δ *is independent of a choice of* n_0 *independent functions from* $\{a_1, \cdots, a_n\}$.

PROOF. We assume that $a_{i_1}, \cdots, a_{i_{n_0}}$ are also functional independents. Since $a_1, \cdots, a_{n_0}$ are functional independent, we have

$$a_i = \sum_{j=1}^{n_0} c_{i,j}(z) a_j + A_i \quad \text{with} \quad A_i \in \mathcal{M}^\delta \tag{3.3}$$

for $i = 1, \cdots, n$ by (2.12). For $(a_{i_1}, \cdots, a_{i_{n_0}})$, define a matrix

$$C(0) := (c_{i_j,k}(0))_{j,k=1,\cdots,n_0}. \tag{3.4}$$

Then we have $\det C(0) \neq 0$, since $a_{i_1}, \cdots, a_{i_{n_0}}$ are functional independent. Therefore we have

$$a_i \in \mathcal{M}^\delta \ mod\, \mathcal{I}\{a_{i_1}, \cdots, a_{i_{n_0}}\} \tag{3.5}$$

for $i = 1, \cdots, n_0$. By (3.3) and (3.5), we have

$$a_i \in \mathcal{M}^\delta \; mod\, \mathcal{I}\{a_{i_1}, \cdots, a_{i_{n_0}}\} \tag{3.6}$$

for $i = 1, \cdots, n$. Hence we have the desired result. Q.E.D.

LEMMA 3.2. *Assume* (A.1) *and* (A.3). *Then the number* δ *is invariant under the coordinate change* $(Z_1, \cdots, Z_n)$.

PROOF. Let $Z_1 = Z_1(z), \cdots, Z_n = Z_n(z)$ be any coordinate change. The operator L becomes

$$L = \sum_{i=1}^{n} (LZ_i)\partial z_i \,. \tag{3.7}$$

Since $a_1, \cdots, a_{n_0}$ are functional independent, we have

$$a_i = \sum_{j=1}^{n_0} c_{i,j}(z)a_j + A_i \quad \text{with} \quad A_i \in \mathcal{M}^\delta \tag{3.8}$$

for $i = n_0 + 1, \cdots, n$ as Lemma 3.1. Therefore we have

$$LZ_i = \sum_{j=1}^{n_0} a_j \left(\partial_{z_j} + \sum_{k=n_0+1}^{n} c_{k,j}\partial_{z_k}\right) Z_i + \sum_{j=n_0+1}^{n} A_j \partial_{z_j} Z_i \tag{3.9}$$

for $i = 1, \cdots, n$. By (A.3), we can take n_0-functions that are functional independent in $\{LZ_i\}_{i=1}^{n}$. By rewriting number, we may assume that $\{LZ_i\}_{i=1}^{n_0}$ are functional independent. Let a matrix A be defined by

$$A := \left(\left(\partial_{z_j} + \sum_{k=n_0+1}^{n} c_{k,j}\partial_{z_k}\right) Z_i \bigg|_{z=0}\right)_{i,j=1,\cdots,n_0} . \tag{3.10}$$

Then we have $\det A \neq 0$, since $\{LZ_i\}_{i=1}^{n_0}$ are functional independent. Therefore we have

$$a_i \in \mathcal{M}^\delta \; mod\, \mathcal{I}\{LZ_1, \cdots, LZ_{n_0}\} \quad \text{for} \quad i = 1, \cdots, n_0 \tag{3.11}$$

by (3.8) and (3.9). Hence we have

$$LZ_i \in \mathcal{M}^\delta mod\, \mathcal{I}\{LZ_1, \cdots, LZ_{n_0}\} \tag{3.12}$$

for $i = n_0 + 1, \cdots, n$ by (3.9) and (3.11). Hence we obtain the desired result. Q.E.D.

3.2. **Estimates of Gamma function.** Here we show some lemmas needed in Section 4 in order to estimate formal power series solutions. We prove these lemmas in Section 7.

Let p, q, r, k_i, and $l_i \in \mathbf{N}$ for $i = 1, 2, \cdots, r$, $\delta \geq 2$ and $x! := \Gamma(x+1)$ for $x \geq 0$.

LEMMA 3.3. *Let* $p + \sum_{i=1}^{r} k_i = k$, *and* $q + \sum_{i=1}^{r} l_i = l$. *Then we have*

$$\prod_{i=1}^{r} \frac{\left(k_i + \frac{1}{\delta-1}l_i\right)!}{k_i!} \leq \frac{\left(k + \frac{1}{\delta-1}l\right)!}{k!} \,. \tag{3.13}$$

LEMMA 3.4. *Let $p+k_1=k$ and $q+l_1=l$. Further if $p=0$, assume $q\geq\delta$. Then we have*

$$\frac{\left(k_1+1+\frac{1}{\delta-1}l_1\right)!}{k_1!}\leq(k+1)\frac{\left(k+\frac{1}{\delta-1}l\right)!}{k!}. \tag{3.14}$$

LEMMA 3.5. *Let $p+k_1=k$, $q+l_1=l$ and $q>0$. Further if $p=0$, assume $q\geq\delta$. Then we have*

$$(l_1+1)\frac{\left(k_1+\frac{1}{\delta-1}(l_1+1)\right)!}{k_1!}\leq(\delta-1)(k+1)\frac{\left(k+\frac{1}{\delta-1}l\right)!}{k!}. \tag{3.15}$$

LEMMA 3.6. *Let $p+k_1=k$ and $q+l_1=l$. Further if $p=0$, assume $q\geq\delta$. Then we have*

$$\frac{\left(k_1+\frac{1}{\delta-1}l_1\right)!}{k_1!}\leq(k+1)\frac{\left(k+\frac{1}{\delta-1}l-1\right)!}{k!}. \tag{3.16}$$

4. Gevrey estimates.

In this section, we will study a particular equation that satisfies the assumptions of Theorem 2.2. We show that this equation has a formal power series solution that belongs to a Gevrey class. In Section 6, we reduce (2.2) to (4.2) by coordinates change and we can prove Theorem 2.2.

Let $x=(x_1,x_2,\cdots,x_{m_0})\in\mathbf{C}^{m_0}$, $t=(t_1,t_2,\cdots,t_{m_1})\in\mathbf{C}^{m_1}$ and $y=(y_1,y_2,\cdots,y_{m_2})\in\mathbf{C}^{m_2}$, when $m_0\geq 1$. We consider the following equation

$$Lu=F(x,t,y,u(x,t,y)), \tag{4.1}$$

where

$$L=\sum_{i=1}^{m_0}\{\lambda_i x_i+\mu_{i-1}x_{i-1}+a_i(x,t,y)\}\partial_{x_i}+\sum_{i=1}^{m_1}b_i(x,t,y)\partial_{t_i}+\sum_{i=1}^{m_2}c_i(x,t,y)\partial_{y_i} \tag{4.2}$$

with

$$\begin{aligned}
&a_{i_0}(x,t,y)\equiv O((|x|+|t|+|y|)^2), \quad c_{i_2}(x,t,y)\equiv O((|x|+|t|+|y|)^2),\\
&a_{i_0}(0,t,y)\equiv b_{i_1}(0,t,y)\equiv c_{i_2}(0,t,y)\equiv O(|t|^\delta) \quad \delta\geq 2,\\
&b_{i_1}(x,0,y)\equiv 0
\end{aligned} \tag{4.3}$$

for $i_0=1,2,\cdots,m_0$, $i_1=1,2,\cdots,m_1$ and $i_2=1,2,\cdots,m_2$. It follows from (4.3) that

$$\begin{aligned}
a_{i_0}(x,t,y)&=\sum_{|p|+|q|\geq 1}a_{i_0,p,q}(y)x^pt^q,\\
b_{i_1}(x,t,y)&=\sum_{|p|+|q|\geq 2}b_{i_1,p,q}(y)x^pt^q,\\
c_{i_2}(x,t,y)&=\sum_{|p|+|q|\geq 1}c_{i_2,p,q}(y)x^pt^q
\end{aligned} \tag{4.4}$$

with

$$\begin{aligned} &a_{i_0,0,q}(y) \equiv b_{i_1,0,q}(y) \equiv c_{i_2,0,q}(y) \equiv 0 \quad \text{for} \quad |q| = 1, \cdots, \delta - 1, \\ &a_{i_0,p,0}(0) = c_{i_2,p,0}(0) = 0 \quad \text{for} \quad |p| = 1, \\ &b_{i_1,p,0}(y) \equiv 0 \quad \text{for} \quad \forall p \in \mathbf{N}^{m_0}. \end{aligned} \tag{4.5}$$

The function $F(x, t, y, u)$ is a holomorphic function in a neighborhood of the origin such that

$$F(0, 0, y, 0) \equiv 0. \tag{4.6}$$

THEOREM 4.1. *Assume that there exists a positive constant σ such that*

$$\left| \sum_{i=1}^{m_0} \lambda_i k_i - c \right| \geq \sigma(|k| + 1) \quad \textit{for} \quad \forall k = (k_1, k_2, \cdots, k_{m_0}) \in \mathbf{N}^{m_0} \tag{4.7}$$

for (4.1) *where* $|k| = k_1 + k_2 + \cdots + k_{m_0}$ *and* $c = \dfrac{\partial F}{\partial u}(0, 0, 0, 0)$. *Then equation* (4.1) *has a unique formal power series solution* $u(x, t, y)$ *which belongs to* $G_t^{\{\frac{1}{\delta-1}\}}(x, t, y)$.

PROOF. Put

$$u(x, t, y) = \sum_{|k|+|l| \geq 1} u_{k,l}(y) x^k t^l, \quad F(x, t, y, u) = \sum_{|p|+|q|+r \geq 1} F_{p,q,r}(y) x^p t^q u^r, \tag{4.8}$$

$$u_{k,l}(y) = \frac{\left(|k| + \frac{1}{\delta-1}|l|\right)!}{|k|!} v_{k.l}(y). \tag{4.9}$$

Then we consider a formal power series $v(x, t, y) = \sum_{|k|+|l| \geq 1} v_{k,l}(y) x^k t^l$. In order to prove Theorem 4.1, we will show that a formal powr series $v(x, t, y)$ exists and it converges in a neighborhood of the origin. In fact, there exist positive constants A and B such that

$$\frac{\left(|k| + \frac{|l|}{\delta-1}\right)!}{|k|!|l|!^{\frac{1}{\delta-1}}} \leq AB^{|k|+|l|} \tag{4.10}$$

by Stirling's formula. Therefore $u(x, t, y)$ belongs to $G_t^{\{\frac{1}{\delta-1}\}}(x, t, y)$ by (4.9).

We define

$$\begin{aligned} &e(n, 1) = (1, 0, \cdots, 0), \cdots, e(n, n) = (0, \cdots, 0, 1) \in \mathbf{N}^n \quad \text{for} \quad \forall n = 1, 2, \cdots, \\ &k_{(i)} = (k_1^i, k_2^i, \cdots, k_{m_0}^i) \in \mathbf{N}^{m_0}, \quad l_{(i)} = (l_1^i, l_2^i, \cdots, l_{m_1}^i) \in \mathbf{N}^{m_1}, \\ &k_{\{r\}} = \left(\sum_{i=1}^r k_1^i, \cdots, \sum_{i=1}^r k_{m_0}^i\right) \quad \text{and} \quad l_{\{r\}} = \left(\sum_{i=1}^r l_1^i, \cdots, \sum_{i=1}^r l_{m_1}^i\right). \end{aligned} \tag{4.11}$$

By substituting (4.8) and (4.9) into (4.1), we have the following recurrence relations

$$\begin{aligned} &\lambda_i v_{e(m_0,i),0}(y) + \mu_i v_{e(m_0,i+1),0}(y) + \sum_{j=1}^{m_0} a_{j,e(m_0,i),0}(y) v_{e(m_0,j),0}(y) \\ &= F_{e(m_0,i),0,0}(y) + F_{0,0,1}(y) v_{e(m_0,i),0}(y) \quad \text{for} \quad i = 1, 2, \cdots, m_0, \end{aligned} \tag{4.12}$$

$$0 = F_{0,e(m_1,i),0}(y) + F_{0,0,1}(y)(1/(\delta - 1))! v_{0,e(m_1,i)}(y) \quad \text{for} \quad i = 1, 2, \cdots, m_1 \tag{4.13}$$

and for $|k|+|l|\geq 2$

$$\begin{aligned} v_{k,l}(y) &+ \sum_{i=1}^{m_0} \frac{\mu_{i-1}(k_i+1)}{(\sum_{i=1}^{m_0}\lambda_i k_i - F_{0,0,1}(y))} v_{k+e(m_0,i)-e(m_0,i-1),l}(y) \\ &+ \sum_{i=1}^{m_0} \sum_{\substack{p+k(1)=k \\ |p|=1}} \frac{(k_i^1+1)}{(\sum_{i=1}^{m_0}\lambda_i k_i - F_{0,0,1}(y))} a_{i,p,0}(y) v_{k_{(1)}+e(m_0,i),l}(y) \\ &= I_1 - I_2 - I_3 - I_4 \end{aligned} \tag{4.14}$$

where

$$\begin{aligned} I_1 = &\frac{1}{(\sum_{i=1}^{m_0}\lambda_i k_i - F_{0,0,1}(y))} \frac{|k|!}{(|k|+\frac{1}{\delta-1}|l|)!} \\ &\times \sum_{\substack{p+k_{\{r\}}=k \\ q+l_{\{r\}}=l \\ |p|+|q|+r\geq 1 \\ (p,q,r)\neq(0,0,1)}} F_{p,q,r}(y) \prod_{i=1}^{r} \frac{(|k_i|+\frac{1}{\delta-1}|l_{(i)}|)!}{|k_{(i)}|!} v_{k_{(i)},l_{(i)}}(y), \end{aligned} \tag{4.15}$$

$$\begin{aligned} I_2 = &\frac{1}{(\sum_{i=1}^{m_0}\lambda_i k_i - F_{0,0,1}(y))} \frac{|k|!}{(|k|+\frac{1}{\delta-1}|l|)!} \\ &\times \sum_{i=1}^{m_0} \sum_{\substack{p+k_{(1)}=k \\ q+l_{(1)}=l \\ |p|+|q|\geq 2}} (k_i^1+1) a_{i,p,q}(y) \frac{(|k_{(1)}|+1+\frac{1}{\delta-1}|l_{(1)}|)!}{(|k_{(1)}|+1)!} v_{k_{(1)}+e(m_0,i),l}(y), \end{aligned} \tag{4.16}$$

$$\begin{aligned} I_3 = &\frac{1}{(\sum_{i=1}^{m_0}\lambda_1 k_i - F_{0,0,1}(y))} \frac{|k|!}{(|k|+\frac{1}{\delta-1}|l|)!} \\ &\times \sum_{i=1}^{m_1} \sum_{\substack{p+k_{(1)}=k \\ q+l_{(1)}=l \\ |p|+|q|\geq 2}} (l_i^1+1) b_{i,p,q}(y) \frac{(|k_{(1)}|+\frac{1}{\delta-1}(|l_{(1)}|+1))!}{|k_{(1)}|!} v_{k_{(1)},l_{(1)}+e(m_1,i)}(y), \end{aligned} \tag{4.17}$$

$$\begin{aligned} I_4 = &\frac{1}{(\sum_{i=1}^{m_0}\lambda_1 k_i - F_{0,0,1}(y))} \frac{|k|!}{(|k|+\frac{1}{\delta-1}|l|)!} \\ &\times \sum_{i=1}^{m_2} \sum_{\substack{p+k_{(1)}=k \\ q+l_{(1)}=l \\ |p|+|q|\geq 1}} c_{i,p,q}(y) \frac{(|k_{(1)}|+\frac{1}{\delta-1}|l_{(1)}|)!}{|k_{(1)}|!} \partial_{y_i} v_{k_{(1)},l_{(1)}}(y). \end{aligned} \tag{4.18}$$

Let us show that $\{v_{k,l}(y)\}_{|k|+|l|\geq 1}$ are inductively determined.

For $v(x,t,y)=\sum_{|k|+|l|\geq 1} v_{k,l}(y)x^k t^l$, we define

$$(v)_m = \sum_{|k|+|l|=m} v_{k,l}(y)x^k t^l \quad \text{and} \quad \|(v)_m\|_{r_0} = \sum_{|k|+|l|=m} \max_{|y|\leq r_0} |v_{k,l}(y)| \tag{4.19}$$

The system equation (4.12) and (4.13) have a holomorphic solution $\{v_{k,l}(y)\}_{|k|+|l|=1}$ for sufficiently small $|y|$ by the conditions $a_{j,e(m_0,i),0}(0)=0$ and (4.7). In a word, we have $(v)_1$.

Next we consider $(v)_m$ for $m \geq 2$. For (4.14) we define

$$(Lv)_m := (v)_m + \sum_{|k|+|l|=m} \left\{ \sum_{i=1}^{m_0} \left\{ \mu_{i-1} \frac{(k_i+1)}{\sum_{j=1}^{m_0} \lambda_j k_j - F_{0,0,1}(y)} v_{k+e(m_0,i)-e(m_0,i-1),l} \right.\right.$$
$$\left.\left. + \sum_{\substack{p+k_{(1)}=k \\ |p|=1}} \frac{(k_i^1+1)}{\sum_{j=1}^{m_0} \lambda_j k_j - F_{0,0,1}(y)} a_{i,p,0} v_{k_{(1)}+e(m_0,i),l} \right\} \right\} x^k t^l . \tag{4.20}$$

Then (4.14) becomes

$$(Lv)_m = \{(v)_{m'};\ m' < m\} . \tag{4.21}$$

For $(Lv)_m$, we have the following lemma.

LEMMA 4.2. *Assume* (4.7). *Then there exist positive constants* σ_1 *and* r_0 *such that*

$$\|(Lv)_m\|_{r_0} \geq \sigma_1 \|(v)_m\|_{r_0} . \tag{4.22}$$

PROOF. By (4.7), there exists a positive constant σ_2 such that

$$\left| \sum_{i=1}^{m_0} \lambda_i k_i - F_{0,0,1}(y) \right|^{-1} \leq \sigma_2(|k|+1)^{-1} \tag{4.23}$$

for sufficiently small $|y|$. Therefore by (4.23), we have

$$\begin{aligned} &\left| \sum_{i=1}^{m_0} \lambda_i k_i - k_{0,0,1}(y) \right|^{-1} \sum_{|k|+|l|=m} \sum_{i=1}^{m_0} \mu_{i-1}(k_i+1) |v_{k+e(m_0,i)-e(m_0,i-1),l}| \\ &\leq \sigma_2 \sum_{|k|+|l|=m} \sum_{i=1}^{m_0} \mu_{i-1} |v_{k+e(m_0,i)-e(m_0,i-1),l}| . \end{aligned} \tag{4.24}$$

For $\forall \varepsilon > 0$, we may assume

$$\sum_{|k|+|l|=m} \sum_{i=1}^{m_0} \mu_{i-1} \max_{|y|\leq r_0} |v_{k+e(m_0,i)-e(m_0,i-1),l}| < \varepsilon \|(v)_m\|_{r_0} . \tag{4.25}$$

In fact, by the change of variables

$$(x_1, \cdots, x_{m_0}) \mapsto (\chi x_1, \cdots, \chi^{m_0} x_{m_0}) , \tag{4.26}$$

the left side of (4.25) becomes

$$\chi^{-1} \sum_{|k|+|l|=m} \sum_{i=1}^{m_0} \mu_{i-1} \max_{|y|\leq r_0} |v_{k+e(m_0,i)-e(m_0,i-1),l}| . \tag{4.27}$$

Therefore (4.25) holds for sufficiently large χ.

By (4.23), we have

$$\begin{aligned}\Big|\sum_{i=1}^{m_0}\lambda_i k_i - F_{0,0,1}(y)\Big|^{-1} &\sum_{|k|+|l|=m}\sum_{i=1}^{m_0}\sum_{\substack{p+k_{(1)}=k\\|p|=1}}(k_i^1+1)|a_{i,p,0}|\,|v_{k_{(1)}+e(m_0,i),l}| \\ &\le \sigma_2\sum_{|k|+|l|=m}\sum_{i=1}^{m_0}\sum_{\substack{p+k_{(1)}=k\\|p|=1}}\max_{|y|\le r_0}(|a_{i,p,0}|\,|v_{k_{(1)}+e(m_0,i),l}|) \\ &\le \sigma_2\sum_{i=1}^{m_0}\sum_{|p|=1}\max_{|y|\le r_0}|a_{i,p,,0}|\,\|(v)_m\|_{r_0}\end{aligned} \tag{4.28}$$

for $|y|\le r_0$. By the condition $a_{i,p,0}(0)=0$ with $|p|=1$, we have

$$\sigma_2\sum_{i=1}^{m_0}\sum_{|p|=1}\max_{|y|\le r_0}|a_{i,p,0}|\,\|(v)_m\|_{r_0}\le\varepsilon\|(v)_m\|_{r_0} \tag{4.29}$$

for $\forall\varepsilon>0$. By (4.25) and (4.29), we have the desired result. Q.E.D.

For $m'<m$, we assume that $(v)_{m'}$ is determined. By (4.21) and Lemma 4.2, we have $(v)_m$. Therefore $(v)_m$ is inductively determined for all $m\ge 1$. In a word, equation (4.1) has a unique formal power series solution.

Next we show that $v(x,t,y)$ converges. So we will give an estimate of $v_{k,l}(y)$. By Lemma 3.3, 3.4, 3.5 and 3.6, we have the following lemma.

LEMMA 4.3. *Assume* (4.23) *for sufficiently small* r_0. *Then we have the following four inequalities* (4.30), (4.31), (4.32) *and* (4.33).

$$|I_1|\le\sigma_2(|k|+1)^{-1}\sum_{\substack{p+k_{\{r\}}=k\\q+l_{\{r\}}=l\\|p|+|q|+r\ge1\\(p,q,r)\neq(0,0,1)}}|F_{p,q,r}(y)|\prod_{i=1}^{r}|v_{k_{(i)},l_{(i)}}(y)|, \tag{4.30}$$

$$|I_2|\le\sigma_2\sum_{i=1}^{m_0}\sum_{\substack{p+k_{(1)}=k\\q+l_{(1)}=l\\|p|+|q|\ge2}}|a_{i,p,q}(y)|\,|v_{k_{(1)}+e(m_0,i),l}(y)|, \tag{4.31}$$

$$|I_3|\le\sigma_2(\delta-1)\sum_{i=1}^{m_1}\sum_{\substack{p+k_{(1)}=k\\q+l_{(1)}=l\\|p|+|q|\ge2}}|b_{i,p,q}(y)|\,|v_{k_{(1)},l_{(1)}+e(m_1,i)}(y)|, \tag{4.32}$$

$$|I_4|\le\sigma_2\sum_{i=1}^{m_2}\sum_{\substack{p+k_{(1)}=k\\q+l_{(1)}=l\\|p|+|q|\ge1}}\frac{1}{|k|+\frac{1}{\delta-1}|l|}|c_{i,p,q}(y)|\,|\partial_{y_i}v_{k_{(1)},l_{(1)}}(y)|. \tag{4.33}$$

PROOF. Firstly let us prove (4.30). We have

$$\prod_{i=1}^{r} \frac{(|k_{(i)}| + \frac{1}{\delta-1}|l_{(i)}|)!}{|k_{(i)}|!} \leq \frac{(|k| + \frac{1}{\delta-1}|l|)!}{|k|!} \tag{4.34}$$

by Lemma 3.3. Therefore we have

$$|I_1| \leq \frac{1}{|\sum_{i=1}^{m_0} \lambda_i k_i - F_{0,0,1}(y)|} \sum_{\substack{p+k_{\{r\}}=k \\ q+l_{\{r\}}=l \\ |p|+|q|+r\geq 1 \\ (p,q,r)\neq(0,0,1)}} |F_{p,q,r}(y)| \prod_{i=1}^{r} |v_{k_{(i)},l_{(i)}}(y)|\,. \tag{4.35}$$

Hence we have (4.30) by condition (4.23). We use Lemma 3.4, 3.5 and 3.6 for (4.31), (4.32) and (4.33) respectively as (4.30). Then we have the desired result. Q.E.D.

By Lemma 4.2 and 4.3, we obtain the following inequality from (4.14)

$$\begin{aligned}
\sigma_3 \|(v)_m\|_{r_0} \leq & \sum_{\substack{|p|+|q|+m_{\{r\}}=m \\ |p|+|q|+r\geq 1 \\ (p,q,r)\neq(0,0,1)}} \max_{|y|\leq r_0} |F_{p,q,r}(y)| \prod_{i=1}^{r} \|(v)_{m_{(i)}}\|_{r_0} \\
& + \sum_{i=1}^{m_0} \sum_{\substack{|p|+|q|+m_{(1)}=m \\ |p|+|q|\geq 2}} \max_{|y|\leq r_0} |a_{i,p,q}(y)| \|(v)_{m_{(1)}+1}\|_{r_0} \\
& + (\delta-1) \sum_{i=1}^{m_1} \sum_{\substack{|p|+|q|+m_{(1)}=m \\ |p|+|q|\geq 2}} \max_{|y|\leq r_0} |b_{i,p,q}(y)|\, \|(v)_{m_{(1)}+1}\|_{r_0} \\
& + \sum_{i=1}^{m_2} \sum_{\substack{|p|+|q|+m_{(1)}=m \\ |p|+|q|\geq 1}} \frac{1}{|k| + \frac{1}{\delta-1}|l|} \max_{|y|\leq r_0} |c_{i,p,q}(y)|\, \|(\partial_{y_i} v)_{m_{(1)}}\|_{r_0}\,,
\end{aligned} \tag{4.36}$$

where $m_{\{r\}} = \sum_{i=1}^{r} m_i$ and $\sigma_3 = \sigma_1/\sigma_2$.

We define $F_{p,q,r}(R_0)$, $a_{i,p,q}(R_0)$, $b_{i,p,q}(R_0)$ and $c_{i,p,q}(R_0)$ as follows;

$$\begin{aligned}
& F_{0,0,1}(R_0) := 0\,, \\
& F_{p,q,0}(R_0) := \frac{\sigma_3 \max_{i=1,\cdots,m_2}\{\|(v)_1\|_{R_0}, \|(\partial_{y_i} v)_1\|_{R_0}\}}{m_0+m_1} \quad \text{for } |p|+|q|=1\,, \\
& F_{p,q,r}(R_0) := \max_{|y|\leq R_0} |F_{p,q,r}(y)| \quad \text{for } |p|+|q|+r\geq 2\,, \\
& a_{i,p,q}(R_0) := \max_{|y|\leq R_0} |a_{i,p,q}(y)|\,, \quad b_{i,p,q}(R_0) := \max_{|y|\leq R_0} |b_{i,p,q}(y)|\,, \\
& c_{i,p,q}(R_0) := \max_{|y|\leq R_0} |c_{i,p,q}(y)|\,.
\end{aligned} \tag{4.37}$$

Let $0 < r_0 < R_0 < 1$. We consider the following equation

$$\begin{aligned}\sigma_3 Y =& \frac{1}{R_0 - r_0} \sum_{|p|+|q|+r\geq 1} \frac{F_{p,q,r}(R_0)}{(R_0 - r_0)^{|p|+|q|+r-1}} X^{|p|+|q|} Y^r \\ &+ \frac{1}{R_0 - r_0} \sum_{i=1}^{m_0} \sum_{|p|+|q|\geq 2} \frac{a_{i,p,q}(R_0)}{(R_0 - r_0)^{|p|+|q|-2}} X^{|p|+|q|-1} Y \\ &+ \frac{\delta - 1}{R_0 - r_0} \sum_{i=1}^{m_1} \sum_{|p|+|q|\geq 2} \frac{b_{i,p,q}(R_0)}{(R_0 - r_0)^{|p|+|q|-2}} X^{|p|+|q|-1} Y \\ &+ \frac{(\delta - 1)e}{R_0 - r_0} \sum_{i=1}^{m_2} \sum_{|p|+|q|\geq 1} \frac{c_{i,p,q}(R_0)}{(R_0 - r_0)^{|p|+|q|-1}} X^{|p|+|q|} Y .\end{aligned} \tag{4.38}$$

By the condition $F_{0,0,1} = 0$ and implicit function theorem at $Y = X = 0$, equation (4.38) admits a holomorphic solution $Y(X)$.

PROPOSITION 4.4. *We obtain that* (4.38) *has a holomorphic solution* $\sum_{m\geq 1} Y_m(r_0) X^m$ *with the estimates*

$$\begin{aligned}&\|(v)_m\|_{r_0} \leq Y_m(r_0) \\ &\|(\partial_{y_i} v)_m\|_{r_0} \leq emY_m(r_0) \quad \textit{for } i = 1, \cdots, m_2 .\end{aligned} \tag{4.39}$$

We use the following Hörmander's lemma in order to prove Proposition 4.4.

LEMMA 4.5 (Hörmander). *If* $(v)_m$ *satisfies*

$$\|(v)_m\|_{r_0} \leq \frac{C}{(R_0 - r_0)^p} \quad \textit{for } 0 < r_0 < R_0 \tag{4.40}$$

for some $p \geq 0$ *and* $C > 0$, *then we have*

$$\|(\partial_{y_i} v)_m\|_{r_0} \leq \frac{(p + 1)eC}{(R_0 - r_0)^{p+1}} \quad \textit{for } i = 1, 2, \cdots, m_2 \tag{4.41}$$

where $\|(v)_m\|_{r_0} = \sum_{|k|+|l|=m} \max_{|y|\leq r_0} |v_{k,l}(y)|$.

PROOF OF PROPOSITION 4.4. By substituting $\sum_{m\geq 1} Y_m(r_0) X^m$ into (4.38), we have the following recurrence relations

$$\sigma_3 Y_1 = \sum_{|p|+|q|=1} F_{p,q,0}(R_0) \quad \text{for } m = 1 \tag{4.42}$$

and for $m \geq 2$

$$\begin{aligned}
\sigma_3(R_0 - r_0)Y_m = & \sum_{\substack{|p|+|q|+m_{\{r\}}=m \\ |p|+|q|+r\geq 1}} \frac{F_{p,q,r}(R_0)}{(R_0 - r_0)^{|p|+|q|+r-2}} \prod_{i=1}^{r} Y_{m_{(i)}} \\
& + \sum_{i=1}^{m_0} \sum_{\substack{|p|+|q|-1+m_{(1)}=m \\ |p|+|q|\geq 2}} \frac{a_{i,p,q}(R_0)}{(R_0 - r_0)^{|p|+|q|-2}} Y_{m_{(1)}} \\
& + (\delta - 1)\sum_{i=1}^{m_1} \sum_{\substack{|p|+|q|-1+m_{(1)}=m \\ |p|+|q|\geq 2}} \frac{b_{i,p,q}(R_0)}{(R_0 - r_0)^{|p|+|q|-2}} Y_{m_{(1)}} \\
& + (\delta - 1)e\sum_{i=1}^{m_2} \sum_{\substack{|p|+|q|+m_{(1)}=m \\ |p|+|q|\geq 1}} \frac{c_{i,p,q}(R_0)}{(R_0 - r_0)^{|p|+|q|-1}} Y_{m_{(1)}} .
\end{aligned} \tag{4.43}$$

Then $Y_m(r_0)$ is inductively determined for $m \geq 1$ by (4.42) and (4.43) and in the case of $(v)_m$, and we obtain that Y_m becomes a form $C_m/(R_0 - r_0)^{m-1}$ with $C_m \geq 0$ by easy calculation. By (4.37) and (4.42), we obtain (4.39) for $m = 1$. Next we assume (4.39) for $m' < m$ $(m \geq 2)$. By (4.36) and (4.43), we obtain

$$\|(v)_m\|_{r_0} \leq (R_0 - r_0)Y_m(r_0) \leq Y_m(r_0) . \tag{4.44}$$

By $\|(v)_m\|_{r_0} \leq (R_0 - r_0)Y_m = C_m/(R_0 - r_0)^{m-2}$ and Lemma 4.5, we have

$$\|(\partial_{y_i} v)_m\|_{r_0} \leq \frac{e(m-1)C_m}{(R_0 - r_0)^{m-1}} \leq emY_m(r_0) . \tag{4.45}$$

Hence we obtain Proposition 4.4 for $m \geq 1$. Q.E.D.

By Proposition 4.4, we have that $v(x, t, y)$ converges. Hence this completes the proof of Theorem 4.1. Q.E.D.

5. Holomorphic solution of system equation.

In this section, we consider the existence of a holomorphic solution for a nonlinear first order partial differential equation. By the result, we obtain the existence of coordinates change for main theorem to be reduced to the form studied in Section 4. In fact, we prove Main theorem by using the coordinate change in the next section.

Let $w = (w_1, \cdots, w_n) = (w_1, \cdots, w_{n_0}, w_{n_0+1}, \cdots, w_n) = (w', w'') \in \mathbf{C}^n$, $p = (p_1, \cdots, p_{n_0}) = \mathbf{N}^{n_0}$ and $q = (q_1, \cdots, q_m) \in \mathbf{N}^m$, and $b_{j,l}(w, \Phi)$, $c_j(w, \Phi)$ are convergent power series in a neighborhood of the origin in $\mathbf{C}^n \times \mathbf{C}^m$ where $\Phi = (\Phi_1, \cdots, \Phi_m)$ for $j = 1, \cdots, m$ and $l = 1, \cdots, n$. We assume that $b_{j,l}(w, \Phi)$, $c_j(w, \Phi)$ have the following expansion

$$
\begin{aligned}
b_{j,l}(w,\Phi) &= \sum_{|p|+|q|\geq 1} b_{j,l,p,q}(w'')\{w'\}^p\{\Phi\}^q \\
c_j(w,\Phi) &= \sum_{|p|+|q|\geq 1} c_{j,p,q}(w'')\{w'\}^p\{\Phi\}^q
\end{aligned}
\tag{5.1}
$$

where $b_{j,l,p,q}(0) = c_{j,p,q}(0) = 0$ $(|p|+|q|=1)$.

We consider the following system equation

$$
\sum_{i=1}^{n_0}(\lambda_i w_i + \mu_{i-1}w_{i-1})\partial_{w_i}\Phi_j = \sum_{l=1}^{n} b_{j,l}(w,\Phi)\partial_{w_l}\Phi_j + c_j(w,\Phi) \tag{5.2}
$$

with $j = 1, \cdots, m$.

Then we have the following proposition.

PROPOSITION 5.1. *Assume that there exists a positive constant σ_4 such that*

$$
\left|\sum_{i=1}^{n_0}\lambda_i k_i\right| \geq \sigma_4|k| \quad \textit{for } \forall k = (k_1, k_2, \cdots, k_{n_0}) \in \mathbf{N}^{n_0}. \tag{5.3}
$$

Then we obtain that (5.2) *has a tuple of unique holomorphic solution* $(\Phi_1(w), \cdots, \Phi_m(w))$ *in a neighborhood of the origin with* $\Phi_j(0, w'') \equiv 0$ *for* $j = 1, 2, \cdots, m$.

PROOF. We put $\Phi_j(w) = \sum_{|k|\geq 1}\Phi_{j,k}(w'')w'^k$,

$$
\begin{aligned}
&k_{(0)} = (k_1^{(0)}, \cdots, k_{n_0}^{(0)}) \in \mathbf{N}^{n_0}, \quad k_{(i,j)} = (k_1^{(i,j)}, \cdots, k_{n_0}^{(i,j)}) \in \mathbf{N}^{n_0}, \\
&k_{\{q\}} = \left(\sum_{i=1}^{m}\sum_{j=1}^{q_i}k_1^{(i,j)}, \cdots, \sum_{i=1}^{m}\sum_{j=1}^{q_i}k_{n_0}^{(i,j)}\right) \in \mathbf{N}^{n_0}, \quad \Phi(q, k_{\{q\}}) = \prod_{i=1}^{m}\prod_{j=1}^{q_i}\Phi_{i,k_{(i,j)}}(w'').
\end{aligned}
\tag{5.4}
$$

By substituting $\Phi_j(w)$ into (5.2), we have the following recurrence relations

$$
\begin{aligned}
\lambda_i\Phi_{j,e(n_0,i)} + \mu_i\Phi_{j,e(n_0,i+1)} =& \sum_{l=1}^{n_0} b_{j,l,e(n_0,i),0}\Phi_{j,e(n_0,l)} \\
&+ \sum_{l=1}^{n_0}\sum_{j_1=1}^{m} b_{j,l,0,e(n_0,j_1)}\Phi_{j_1,e(n_0,i)}\Phi_{j,e(n_0,l)} \\
&+ c_{j,e(n_0,i),0} + \sum_{j_1=1}^{m} c_{j,0,e(m,j_1)}\Phi_{j_1,e(n_0,i)}
\end{aligned}
\tag{5.5}
$$

for $i = 1, \cdots, n_0$ and $j = 1, \cdots, m$, and for $|k| \geq 2$

$$\begin{aligned} L_{j,k}(\Phi) =& \sum_{l=1}^{n_0} \sum_{\substack{p+k_{\{q\}}+k_{(0)}=k \\ |p|+|q|\geq 1 \\ |k_{(0)}|,|k_{(i,j)}|<|k|}} b_{j,l,p,q}\Phi(q, k_{\{q\}})(k_l^{(0)}+1)\Phi_{j,k_{(0)}+e(n_0,l)} \\ &+ \sum_{l=n_0+1}^{n} \sum_{\substack{p+k_{\{q\}}+k_{(0)}=k \\ |p|+|q|\geq 1}} b_{j,l,p,q}\Phi(q, k_{\{q\}})\partial_{w_l}\Phi_{j,k_{(0)}} \\ &+ \sum_{\substack{p+k_{\{q\}}+k_{(0)}=k \\ |p|+|q|\geq 2}} c_{j,p,q}\Phi(q, k_{\{q\}}) . \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} L_{j,k}(\Phi) := & \sum_{i=1}^{n_0}(\lambda_i k_i)\Phi_{j,k} + \sum_{i=1}^{n_0}\mu_{i-1}(k_i+1)\Phi_{j,k+e(n_0,i)-e(n_0,i-1)} \\ & -\sum_{l=1}^{n_0} \sum_{\substack{p+k_{(0)}=k \\ |p|=1}} b_{j,l,p,0}(k_l^{(0)}+1)\Phi_{j,k_{(0)}+e(n_0,l)} - \sum_{j_1=1}^{m} c_{j,0,e(m,j_1)}\Phi_{j_1,k} \\ & -\sum_{l=1}^{n_0}\sum_{j_1=1}^{m} \sum_{\substack{k_{(j_1,1)}+k_{(0)}=k \\ |k_{(0)}|=0,|k|-1}} b_{j,l,0,e(m,j_1)}\Phi_{j_1,k_{(j_1,1)}}(k_l^{(0)}+1)\Phi_{j,k_{(0)}+e(n_0,l)} . \end{aligned} \tag{5.7}$$

We remark that (5.6) is a linear equation with respect to $\{\Phi_{j,k'};\ j = 1, \cdots, m\ |k'| = |k|\}$.

We show that (5.2) has a formal power series solution. For $\sum_{|k|\geq 1}\Phi_{j,k}(w'')w'^k$, we define

$$(\Phi_j)_K := \sum_{|k|=K}\Phi_{j,k}(w'')w'^k \quad \text{and} \quad \|(\Phi_j)_K\|_r := \sum_{|k|=K}\max_{|w''|\leq r}|\Phi_{j,k}(w'')| . \tag{5.8}$$

Let us show that $(\Phi_j)_1$ is determined for $j = 1, \cdots, m$. For (5.5), implicit function theorem at $\Phi_{j,e(n_0,i)} = w'' = 0$ with $j = 1, \cdots, m$ and $i = 1, \cdots, n_0$ leads to a unique tuple of solution

$$\{\Phi_{j,e(n_0,i)}(w'');\ j = 1, \cdots, m, i = 1, 2, \cdots, n_0\} \tag{5.9}$$

by $b_{j,l,p,q}(0) = c_{j,p,q}(0) = 0$ with $|p|+|q| = 1$. Therefore we have $(\Phi_j)_1$ for $j = 1, \cdots, m$.

Next let us show that $(\Phi_j)_K$ is inductively determined for $K \geq 2$ and $j = 1, \cdots, m$. Set $(L_{j,K}\Phi)_K := \sum_{|k|=K} L_{j,k}(\Phi)w'^k$. Then (5.6) becomes

$$(L_{j,K}\Phi)_K = \{(\Phi_i)_{K'};\ K' < K, i = 1, \cdots, m\} . \tag{5.10}$$

We have the following lemma.

LEMMA 5.2. *Assume* (5.3). *Then there exist positive constants* σ_5 *and* r *such that*

$$\left\| \sum_{j=1}^{m}(L_{j,K}\Phi)_K \right\|_r \geq \sigma_5 K \sum_{j=1}^{m}\|(\Phi_j)_K\|_r . \tag{5.11}$$

PROOF. We can prove Lemma 5.2 as in the proof of Lemma 4.2 by

$$b_{j,l,p,q}(0) = c_{j,p,q}(0) = 0 \tag{5.12}$$

with $|p| + |q| = 1$ for $j = 1, \cdots, m$ and $l = 1, \cdots, n$. Q.E.D.

For $K' < K$, we assume that $(\Phi_j)_{K'}$ is determined. By (5.10) and Lemma 5.2, $(\Phi_j)_K$ is inductively determined for all K ($K \geq 2$). In a word, (5.2) has a formal power series solution with $\Phi_j(0, w'') \equiv 0$.

Next we show that the formal power series solution $\Phi_j(w)$ converges in a neighborhood of the origin. We define

$$\begin{aligned} c_{e(n_0,i),0}(R) &:= \sigma_5 \max_{l=n_0+1,\cdots,n} \left\{ \sum_{j=1}^{m} \max_{|w''| \leq R} |\Phi_{j,e(n_0,i)}(w'')|, \sum_{j=1}^{m} \max_{|w''| \leq R} |\partial_{w_l} \Phi_{j,e(n_0,i)}(w'')| \right\}, \\ b_{l,p,q}(R) &:= \sum_{j=1}^{m} \left\{ \max_{|w''| \leq R} |b_{j,l,p,q}(w'')| \right\}, \quad c_{p,q}(R) := \sum_{j=1}^{m} \left\{ \max_{|w''| \leq R} |c_{j,p,q}(w'')| \right\} \end{aligned} \tag{5.13}$$

for $|p| + |q| \geq 1$ $((|p|, |q|) \neq (1, 0))$. We have

$$\begin{aligned} &b_{l,p,q}(R) \to 0 \quad \text{for } |p| + |q| = 1, \\ &c_{0,q}(R) \to 0 \quad \text{for } |q| = 1 \end{aligned} \tag{5.14}$$

as $R \to 0$ by $b_{j,l,p,q}(0) = c_{j,p,q}(0) = 0$ with $|p| + |q| = 1$.

We consider the following equation for $0 < r < R < 1$ to show that $\Phi_j(w)$ is a holomorphic function in a neighborhood of the origin.

$$\begin{aligned} \sigma_5 Y = &\frac{1}{R-r} \sum_{l=1}^{n_0} \sum_{|p|+|q| \geq 1} \frac{b_{l,p,q}(R)}{(R-r)^{|p|+|q|-2}} X^{|p|+|q|-1} Y^{|q|+1} \\ &+ \frac{e}{R-r} \sum_{l=n_0+1}^{n} \sum_{|p|+|q| \geq 1} \frac{b_{l,p,q}(R)}{(R-r)^{|p|+|q|-1}} X^{|p|+|q|} Y^{|q|+1} \\ &+ \frac{1}{R-r} \sum_{l=1}^{n_0} \sum_{|p|+|q| \geq 1} \frac{c_{p,q}(R)}{(R-r)^{|p|+|q|-2}} X^{|p|+|q|-1} Y^{|q|}. \end{aligned} \tag{5.15}$$

Equation (5.15) admits following proposition.

PROPOSITION 5.3. *We obtain that* (5.15) *have a unique holomorphic solution* $Y = \sum_{k \geq 1} Y_K X^{K-1}$ *in a neighborhood of the origin such that*

$$\begin{aligned} &K \sum_{j=1}^{m} \|(\Phi_j)_K\|_r \leq Y_K, \\ &\sum_{j=1}^{m} \|(\partial_{w_l} \Phi_j)_K\|_r \leq e Y_K \end{aligned} \tag{5.16}$$

with sufficiently small $r > 0$ *for* $K \geq 1$ *and* $l = n_0 + 1, n_0 + 2, \cdots, n$.

PROOF. We put

$$K_{\{q\}} := \sum_{i=1}^{|q|} K_{(i)} \in \mathbf{N} \quad \text{and} \quad Y(q, K_{\{q\}}) := \prod_{i=1}^{|q|} Y_{K_{(i)}} . \tag{5.17}$$

We show that (5.15) has a holomorphic solution. By substituting $X = 0$ into (5.15), we have

$$\begin{aligned} \sigma_5 Y =& \sum_{l=1}^{n_0} \sum_{|p|=1} b_{l,p,0}(R) Y + \sum_{l=1}^{n_0} \sum_{j_1=1}^{m} b_{l,0,e(m,j_1)}(R) Y^2 \\ &+ \sum_{|p|=1} c_{p,0}(R) + \sum_{j_1=1}^{m} c_{0,e(m,j_1)}(R) Y . \end{aligned} \tag{5.18}$$

Then (5.18) has a root $Y = Y_1 > 0$ for sufficiently small $R > 0$ by (5.14). Therefore by (5.14) and implicit function theorem at $(X, Y) = (0, Y_1)$, equation (5.15) admits a holomorphic solution Y. Further the solution $Y = \sum_{K \geq 1} Y_K X^{K-1}$ satisfies $Y_K = C_K/(R-r)^{K-1}$ with $C_K \geq 0$ for sufficiently small R by easy calculation.

Next we show that (5.16) holds for $K \geq 1$. By (5.13) and (5.18), we obtain (5.16) for $K = 1$. For $K \geq 2$ we have

$$\begin{aligned} (R-r)\Bigg\{ & \sigma_5 Y_K - \sum_{l=1}^{n_0} \sum_{|p|=1} b_{l,p,0}(R) Y_K - \sum_{l=1}^{n_0} \sum_{j_1=1}^{m} 2 b_{l,0,e(m,j_1)}(R) Y_1 Y_K \\ & - \sum_{j_1=1}^{m} c_{j,0,e(m,j_1)}(R) Y_K \Bigg\} \\ = & \sum_{l=1}^{n_0} \sum_{\substack{|p|+K_{\{q\}}+K_{(0)}=K \\ |p|+|q| \geq 1 \\ K_{(i)}, K_{(0)} < K}} \frac{b_{l,p,q}(R)}{(R-r)^{|p|+|q|-2}} Y(q, K_{\{q\}}) Y_{K_{(0)}+1} \\ & + e \sum_{l=n_0+1}^{n} \sum_{\substack{|p|+K_{\{q\}}+K_{(0)}=K \\ |p|+|q| \geq 1}} \frac{b_{l,p,q}(R)}{(R-r)^{|p|+|q|-1}} Y(q, K_{\{q\}}) Y_{k_{(0)}} \\ & + \sum_{\substack{|p|+K_{\{q\}}=K \\ |p|+|q| \geq 2}} \frac{c_{p,q}(R)}{(R-r)^{|p|+|q|-2}} Y(q, K_{\{q\}}) \end{aligned} \tag{5.19}$$

for $j = 1, 2, \cdots, m$. For $K' < K$, assume that (5.16) holds. By (5.6) and (5.19), we have

$$\begin{aligned} \left\| \sum_{j=1}^{m} (L_{j,K} \Phi)_K \right\|_r \leq & (R-r)\Bigg\{ \sigma_5 Y_K - \sum_{l=1}^{n_0} \sum_{|p|=1} b_{l,p,0}(R) Y_K \\ & - \sum_{l=1}^{n_0} \sum_{j_1=1}^{m} b_{l,0,e(m,j_1)}(R)(Y_K Y_1 + Y_1 Y_K) - \sum_{j_1=1}^{m} c_{j,0,e(m,j_1)}(R) Y_K \Bigg\} . \end{aligned} \tag{5.20}$$

By Lemma 5.2 and $Y_K \geq 0$, (5.20) becomes

$$K \sum_{j=1}^{M} \|(\Phi_j)_K\|_r \leq (R-r)Y_K \leq Y_K \tag{5.21}$$

as Proposition 4.4. Therefore we have

$$\sum_{j=1}^{m} \|(\partial_{w_i}\Phi_j)_K\|_r \leq eY_K \tag{5.22}$$

as Propositon 4.4. Hence we obtain (5.16) for $K \geq 1$. Q.E.D.

By Propositon 5.3, this completes the proof of Propositon 5.1. Q.E.D.

6. Proof of Theorem.

In this section, we transform equation (2.2) to the one studied in Section 4 (Theorem 4.1) via a coordinate change. Hence Main theorem is completely proved by Theorem 4.1.

Suppose that $\eta(z) = (\eta_1(z), \cdots, \eta_n(z))$ is a local coordinate in a neighborhood of the origin. Then by $\eta = \eta(z)$, the operator L becomes

$$L = \sum_{i=1}^{n} a'_i(\eta)\partial_{\eta_i} \tag{6.1}$$

where

$$\sum_{j=1}^{n} a_j(z)\partial_{z_j}\eta_i(z) = a'_i(\eta(z)) \tag{6.2}$$

for $i = 1, \cdots, n$. Taking $\eta_i(z) = \zeta_i(z)$ for $i = 1, \cdots, n_1$ with $\zeta_i(z)$ being defined by (2.3), we have

$$a'_i(0, \cdots, 0, \eta_{n_1+1}, \cdots, \eta_n) \equiv 0 \tag{6.3}$$

for $i = 1, \cdots, n$. Therefore we may consider the following operator L

$$L = \sum_{i=1}^{n} a_i(z)\partial_{z_i} \tag{6.4}$$

where

$$a_i(0, \cdots, 0, z_{n_1+1}, \cdots, z_n) \equiv 0 \tag{6.5}$$

for $i = 1, \cdots, n$, by (6.2).

In this section, we prove Theorem 2.2 by the following steps. Firstly we show that the coefficients $\{a_i(z)\}_{i=1}^n$ of L become

$$\begin{aligned} a_i(z) &= \lambda_i z_i + \mu_{i-1}z_{i-1} + O(|z|^2) \quad &&\text{for } i = 1, \cdots, n_0 \\ a_i(z) &= O(|z|^2) &&\text{for } i = n_0+1, \cdots, n \end{aligned} \tag{6.6}$$

by a linear transformation. Secondly we find Gevrey variables t by Propositon 5.1. Lastly we seek multiplicity δ by (2.12).

PROOF OF THEOREM 2.2. Assume (A.3). Then there exists a regular matrix A such that

$$A\frac{\partial a}{\partial z}(0)A^{-1} = J(\lambda, \mu) \tag{6.7}$$

where $J(\lambda, \mu)$ in (2.5). By the condition $a_i(0, \cdots, 0, z_{n_1+1}, \cdots, z_n) \equiv 0$, we have

$$\partial_{z_j} a_i(0) = 0 \quad \text{for } j = n_1 + 1, \cdots, n. \tag{6.8}$$

By (6.8) and the form of $J(\lambda, \mu)$, we find that a form of A admits

$$\begin{pmatrix} a_{1,1} & \cdots & a_{n_1,n_1} & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \cdots & \vdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & a_{n_1+1,n_1+1} & \cdots & a_{n_1+1,n} \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n_1} & a_{n,n_1+1} & \cdots & a_{n,n} \end{pmatrix} \tag{6.9}$$

Then we have the following lemma.

LEMMA 6.1. *Assume (A.1) and (A.3). Then $a'_i(\eta)$ in (6.2) satisfies the following conditions for the coordinate change ${}^t\eta = A^t z$, where A is given by (6.9) and ${}^t(\cdot)$ denotes a transposed matrix of $(\cdot)$.*

1. $a'_i(\eta) = \lambda_i \eta_i + \mu_{i-1}\eta_{i-1} + b_i(\eta)$ *for* $i = 1, \cdots, n_0$
   $a'_i(\eta) = b_i(\eta)$ *for* $i = n_0 + 1, \cdots, n$.
2. $b_i(\eta) = O(|\eta|^2)$ *for* $i = 1, \cdots, n$
3. $b_i(0, \cdots, 0, \eta_{n_1+1}, \eta_{n_1+2}, \cdots, \eta_n) \equiv 0$ *for* $i = 1, 2, \cdots, n$.

PROOF. By the form of A, it is obvious. Q.E.D.

By Lemma 6.1 we may assume that L is in the form

$$L = \sum_{i=1}^{n_0}(\lambda_i z_i + \mu_{i-1}z_{i-1} + b_i(z))\partial_{z_i} + \sum_{i=n_0+1}^{n} b_i(z)\partial_{z_i}, \tag{6.10}$$

where

$$b_i(0, \cdots, 0, z_{n_1+1}, \cdots, z_n) \equiv 0 \quad \text{and} \quad b_i(z) = O(|z|^2) \tag{6.11}$$

for $i = 1, 2, \cdots, n$. Put $z' = (z_1, \cdots, z_{n_0})$, $z'' = (z_{n_0+1}, \cdots, z_{n_1})$ and $z''' = (z_{n_1+1}, \cdots, z_n)$.

LEMMA 6.2. *Assume that* (6.10) *satisfies that there exists a positive constant σ such that*

$$\left|\sum_{i=1}^{n_0} \lambda_i k_i - c\right| \geq \sigma(|k| + 1) \quad \textit{for some } c \textit{ and all } k \in \mathbf{N}^{n_0}. \tag{6.12}$$

Then there exists a tuple of holomorphic function $(\Phi_1(z', z'''), \cdots, \Phi_{n_1-n_0}(z', z'''))$ *with* $\Phi_j(0, z''') \equiv 0$ *for* $j = 1, \cdots, n_1 - n_0$ *such that*

$$L(z_{n_0+j} - \Phi_j(z', z''')) = \sum_{i=1}^{n_1-n_0} E_{i,j}(z)(z_{n_0+i} - \Phi_i(z', z''')) \tag{6.13}$$

where $E_{i,j}(z)$ *is a holomorphic function in a neighborhood of the origin with* $E_{i,j}(0) = 0$ *for* $i, j = 1, \cdots, n_1 - n_0$.

PROOF. We consider the following equation in order to prove Lemma 6.2

$$\begin{aligned}\sum_{i=1}^{n_0}\{\lambda_i z_i + \mu_{i-1}z_{i-1} + b_i(z', \Phi, z''')\}\partial_{z_i}\Phi_j(z', z''') \\ + \sum_{i=n_1+1}^{n} b_i(z', \Phi, z''')\partial_{z_i}\Phi_j(z', z''') = b_{n_0+j}(z', \Phi, z''')\end{aligned} \tag{6.14}$$

for $j = 1, \cdots, n_1 - n_0$, where $\Phi = (\Phi_1, \cdots, \Phi_{n_1-n_0})$. By condition (6.12), there exists a positive constant σ_4 such that

$$\left|\sum_{i=1}^{n_0} \lambda_i k_i\right| \geq \sigma_4|k| \quad \text{for } k \in \mathbf{N}^{n_0}. \tag{6.15}$$

We have that (6.14) satisfies the assumptions of Proposition 5.1 by putting $m = n_1 - n_0$, $z' \mapsto w'$ and $z''' \mapsto w''$, where m, w' and w'' in Section 5. Therefore we obtain that (6.14) has a tuple of holomorphic solution $\{\Phi_j(z', z''')\}_{j=1}^{n_1-n_0}$ with $\Phi_j(0, z''') \equiv 0$ for $j = 1, \cdots, n_1 - n_0$.

Next put $\tau_j = z_{n_0+j} - \Phi_j(z', z''')$. Then we have

$$L\tau_j = \sum_{i=1}^{n}(b_i(z', \Phi, z''') - b_i(z))\partial_{z_i}\Phi_j(z', z''') + b_{n_0+j}(z) - b_{n_0+j}(z', \Phi, z''') . \tag{6.16}$$

Further we can put

$$b_j(z', \Phi, z''') - b_j(z) = \sum_{i=1}^{n_1-n_0} e_{i,j}(z)(z_{n_0+i} - \Phi_i) = \sum_{i=1}^{n_1-n_0} e_{i,j}(z)\tau_i \tag{6.17}$$

for holomorphic functions $e_{i,j}(z)$. Therefore we have

$$L\tau_j = \sum_{i=1}^{n_1-n_0} E_{i,j}(z)\tau_i \tag{6.18}$$

and $E_{i,j}(0) = 0$ by $b_i(z) = O(|z|^2)$. Q.E.D.

By Lemma 6.2 and the coordinate change $\tau_j = z_{n_0+j} - \Phi_j(z', z''')$ with $j = 1, \cdots, n_1 - n_0$, L becomes the following form

$$L = \sum_{i=1}^{n_0} \{\lambda_i z_i + \mu_{i-1} z_{i-1} + c_i(z', \tau, z''')\}\partial_{z_i} + \sum_{i=1}^{n_1-n_0} c_{n_0+i}(z', \tau, z''')\partial_{\tau_i} + \sum_{i=n_1+1}^{n} c_i(z', \tau, z''')\partial_{z_i}, \tag{6.19}$$

where

$$\begin{aligned} &c_i(0, 0, z''') \equiv 0 \quad \text{for } i = 1, \cdots, n_0, n_1 + 1, \cdots, n \\ &c_i(z', 0, z''') \equiv 0 \quad \text{for } i = n_0 + 1, n_0 + 2, \cdots, n_1 \end{aligned} \tag{6.20}$$

and $c_i(z', \tau, z''') \equiv O((|z'| + |\tau| + |z'''|)^2)$ for $i = 1, \cdots, n$.

In the following lemma,we seek multiplicity δ. So we refer to multiplicity. We have

$$c_i(z', \tau, z''') = \sum_{j=1}^{n_0} d_{i,j}(\lambda_j z_j + \mu_{j-1} z_{j-1} + c_j) + O(|\tau|^\delta) \tag{6.21}$$

for $i = n_0 + 1, \cdots, n$ by (2.12), Lemma 3.1 and 3.2, where $d_{i,j} = d_{i,j}(z', \tau, z''')$ is a holomorphic function. Then we obtain the following result.

LEMMA 6.3. *There exist local coordinates* $(x, t, y) \in \mathbf{C}^{n_0} \times \mathbf{C}^{n_1-n_0} \times \mathbf{C}^{n-n_1}$ *such that* (6.19) *becomes the following form*

$$L = \sum_{i=1}^{n_0} \{\lambda_i x_i + \mu_{i-1} x_{i-1} + A_i(x, t, y)\}\partial_{x_i} + \sum_{i=1}^{n_1-n_0} A_{n_0+i}(x, t, y)\partial_{t_i} + \sum_{i=1}^{n-n_1} A_{n_1+i}(x, t, y)\partial_{y_i}, \tag{6.22}$$

where

$$\begin{aligned} &A_i(0, t, y) \equiv O(|t|^\delta) \quad \textit{for } i = 1, \cdots, n, \\ &A_i(0, 0, y) \equiv 0 \quad \textit{for } i = 1, \cdots, n_0, n_1 + 1, \cdots, n \\ &A_i(x, 0, y) \equiv 0 \quad \textit{for } i = n_0 + 1, n_0 + 2, \cdots, n_1. \end{aligned} \tag{6.23}$$

PROOF. Let $x_{i_0} = \lambda_{i_0} z_{i_0} + \mu_{i_0-1} z_{i_0-1} + c_{i_0}(z', \tau, z''')$, $t_{i_1} = \tau_{i_1}$ and $y_{i_2} = z_{i_2+n_1}$ for $i_0 = 1, 2, \cdots, n_0$, $i_1 = 1, 2, \cdots, n_1 - n_0$ and $i_2 = 1, 2, \cdots, n - n_1$. Then we have

$$L = \sum_{i_0=1}^{n_0} (L x_{i_0})\partial_{x_{i_0}} + \sum_{i_1=1}^{n_1-n_0} (L t_{i_1})\partial_{t_{i_1}} + \sum_{i_2=1}^{n-n_1} (L_{y_{i_2}})\partial_{y_{i_2}}. \tag{6.24}$$

For $x_i = \lambda_i z_i + \mu_{i-1} z_{i-1} + c_i(z', t, y)$ $(i = 1, 2, \cdots, n_0)$ by implicit function theorem at $x = z' = t = 0$, we obtain n_0-holomorphic functions $z' = (z_1(x, t, y), z_2(x, t, y), \cdots, z_{n_0}(x, t, y))$ with $z_i(0, 0, y) \equiv 0$ for $i = 1, 2, \cdots, n_0$. By (6.21), we have

$$c_i(z'(0, t, y), t, y) \equiv O(|t|^\delta) \tag{6.25}$$

for $i = n_0 + 1, \cdots, n$. Put

$$A_{i_0}(x,t,y) = \left\{ \sum_{i=1}^{n_0} (\lambda_i z_i + \mu_{i-1} z_{i-1} + c_i)\partial_{z_i} + \sum_{i=1}^{n_1-n_0} c_{n_0+i}\partial_{\tau_i} + \sum_{i=n_1+1}^{n} c_i \partial_{z_i} \right\} c_{i_0} \Big|_{z'=z'(x,t,y)} \tag{6.26}$$

for $i_0 = 1, 2, \cdots, n_0$ and

$$A_{i_1}(x,t,y) = c_{i_1} \big|_{z'=z'(x,t,y)} \tag{6.27}$$

for $i_1 = n_0 + 1, n_0 + 2, \cdots, n$. Then by (6.25) we have

$$A_i(0,t,y) \equiv O(|t|^{\delta}) \tag{6.28}$$

for $i = 1, \cdots, n$. Since we have

$$\begin{aligned} Lx_{i_0} =& \lambda_{i_0}(\lambda_{i_0} z_{i_0} + \mu_{i_0-1} z_{i_0-1} + c_{i_0}) + \mu_{i_0-1}(\lambda_{i_0-1} z_{i_0-1} + \mu_{i_0-2} z_{i_0-2} + c_{i_0-1}) \\ &+ \sum_{i=1}^{n_0} (\lambda_i z_i + \mu_{i-1} z_{i-1} + c_i)\partial_{z_i} c_{i_0} + \sum_{i=1}^{n_1-n_0} c_{n_0+i}\partial_{\tau_i} c_{i_0} + \sum_{i=n_1+1}^{n} c_i \partial_{z_i} c_{i_0}, \end{aligned} \tag{6.29}$$

$$Lt_{i_1} = c_{n_0+i_1} \quad \text{and} \quad Ly_{i_2} = c_{n_1+i_2}$$

for $i_0 = 1, 2, \cdots, n_0$, $i_1 = 1, 2, \cdots, n_1 - n_0$ and $i_2 = 1, 2, \cdots, n - n_1$, we obtain the desired result. Q.E.D.

By Lemma 6.3, we find that (2.2) becomes (4.2) by putting $m_0 = n_0$, $m_1 = n_1 - n_0$ and $m_2 = n - n_1$. Hence this completes the proof of Theorem 2.2 by Theorem 4.1.

7. Appendix.

PROOF OF LEMMA 3.3 We consider

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx \tag{7.1}$$

for $\alpha \geq 1$ and $\beta \geq 1$. We know the following equation

$$B(\alpha,\beta) = \frac{(\alpha-1)!(\beta-1)!}{(\alpha+\beta-1)!}. \tag{7.2}$$

By the definition of $B(\alpha,\beta)$, we have

$$B(\alpha',\beta') \leq B(\alpha,\beta) \tag{7.3}$$

for $\alpha' \geq \alpha$ and $\beta' \geq \beta$.

Putting $\beta' = \beta$ into (7.3), we have

$$\frac{(\alpha+\beta-1)!}{(\alpha-1)!} \leq \frac{(\alpha'+\beta-1)!}{(\alpha'-1)!}. \tag{7.4}$$

We show that (3.13) holds. By (7.4), we have

$$\frac{(k_1+k_2)!}{k_1!k_2!} \leq \frac{\left(k_1+k_2+\frac{1}{\delta-1}l_1\right)!}{\left(k_1+\frac{1}{\delta-1}l_1\right)!k_2!} \leq \frac{\left(k_1+k_2+\frac{1}{\delta-1}(l_1+l_2)\right)!}{\left(k_1+\frac{1}{\delta-1}l_1\right)!\left(k_2+\frac{1}{\delta-1}l_2\right)!}. \tag{7.5}$$

Therefore we have

$$\frac{\left(k_1+\frac{1}{\delta-1}l_1\right)!}{k_1!}\frac{\left(k_2+\frac{1}{\delta-1}l_2\right)!}{k_2!} \le \frac{\left(k_1+k_2+\frac{1}{\delta-1}(l_1+l_2)\right)!}{(k_1+k_2)!}. \tag{7.6}$$

Repeating this, we have

$$\prod_{i=1}^{r}\frac{\left(k_i+\frac{1}{\delta-1}l_i\right)!}{k_i!} \le \frac{\left(\sum_{i=1}^{r}k_i+\frac{1}{\delta-1}\sum_{i=1}^{r}l_i\right)!}{(\sum_{i=1}^{r}k_i)!}. \tag{7.7}$$

By (7.4), we have

$$\frac{\left(\sum_{i=1}^{r}k_i+\frac{1}{\delta-1}\sum_{i=1}^{r}l_i\right)!}{(\sum_{i=1}^{r}k_i)!} \le \frac{\left(k+\frac{1}{\delta-1}\sum_{i=1}^{r}l_i\right)!}{k!} \le \frac{\left(k+\frac{1}{\delta-1}l\right)!}{k!}. \quad \text{Q.E.D.} \tag{7.8}$$

PROOF OF LEMMA 3.4. We have

$$I_1 = \frac{\left(k_1+1+\frac{1}{\delta-1}l_1\right)!}{k_1!}\frac{1}{k+1}\frac{k!}{\left(k+\frac{1}{\delta-1}l\right)!} \le \frac{\left(k_1+\frac{1}{\delta-1}l_1+1\right)!}{\left(k_1+\frac{1}{\delta-1}l_1+p+\frac{1}{\delta-1}q\right)!}\frac{(k_1+p-1)!}{k_1!}. \tag{7.9}$$

If $p > 0$, we have

$$I_1 \le \frac{\left(k_1+\frac{1}{\delta-1}l_1+p\right)!}{\left(k_1+\frac{1}{\delta-1}l_1+p+\frac{1}{\delta-1}q\right)!} \le 1. \tag{7.10}$$

If $p = 0$, we have

$$I_1 \le \frac{\left(k_1+\frac{1}{\delta-1}l_1+1\right)!}{\left(k_1+\frac{1}{\delta-1}l_1+\frac{1}{\delta-1}q\right)!}. \tag{7.11}$$

By $q \ge \delta$, we have $q/(\delta-1) \ge \delta/(\delta-1) > 1$. Hence we have $I_1 \le 1$. Q.E.D.

PROOF OF LEMMA 3.5. We have

$$\begin{aligned} I_2 =&(l_1+1)\frac{\left(k_1+\frac{1}{\delta-1}l_1+\frac{1}{\delta-1}\right)!}{k_1!}\frac{1}{k+1}\frac{k!}{\left(k+\frac{1}{\delta-1}l\right)!} \\ \le&(l_1+1)\frac{\left(k_1+\frac{1}{\delta-1}l_1+\frac{1}{\delta-1}\right)!}{\left(k_1+\frac{1}{\delta-1}l_1+p+\frac{1}{\delta-1}q\right)!}\frac{(k_1+p-1)!}{k_1!}. \end{aligned} \tag{7.12}$$

If $p > 0$, we have

$$I_2 \le (\delta-1)\frac{\left(k_1+\frac{1}{\delta-1}l_1+p+\frac{1}{\delta-1}\right)!}{\left(k_1+\frac{1}{\delta-1}l_1+p+\frac{1}{\delta-1}q\right)!}. \tag{7.13}$$

By $q > 0$, we have $I_2 \leq \delta - 1$. If $p = 0$, we have

$$I_2 \leq (l_1 + 1)\frac{\left(k_1 + \frac{1}{\delta-1}l_1 + \frac{1}{\delta-1}\right)!}{\left(k_1 + \frac{1}{\delta-1}l_1 + \frac{1}{\delta-1}q\right)!} \leq (\delta - 1)\frac{\left(k_1 + \frac{1}{\delta-1}l_1 + \frac{\delta}{\delta-1}\right)!}{\left(k_1 + \frac{1}{\delta-1}l_1 + \frac{1}{\delta-1}q\right)!}. \tag{7.14}$$

By $q \geq \delta$, we have $q/(\delta - 1) \geq \delta/(\delta - 1)$. Hence we have $I_2 \leq \delta - 1$. Q.E.D.

PROOF OF LEMMA 3.6. We have

$$I_3 = \frac{\left(k_1 + \frac{1}{\delta-1}l_1\right)!}{k_1!}\frac{1}{k+1}\frac{k!}{\left(k + \frac{1}{\delta-1}l\right)!} \leq \frac{\left(k_1 + \frac{1}{\delta-1}l_1\right)!}{\left(k_1 + \frac{1}{\delta-1}l_1 + p + \frac{1}{\delta-1}q\right)!}\frac{(k_1 + p - 1)!}{k_1!}. \tag{7.15}$$

If $p > 0$, we have

$$I_3 \leq \frac{\left(k_1 + \frac{1}{\delta-1}l_1 + p - 1\right)!}{\left(k_1 + \frac{1}{\delta-1}l_1 + p + \frac{1}{\delta-1}q\right)!} \leq \frac{1}{k_1 + \frac{1}{\delta-1}l_1 + p + \frac{1}{\delta-1}q}. \tag{7.16}$$

If $p = 0$, we have

$$I_3 \leq \frac{\left(k_1 + \frac{1}{\delta-1}l_1\right)!}{\left(k_1 + \frac{1}{\delta-1}l_1 + \frac{1}{\delta-1}q\right)!}. \tag{7.17}$$

By $q \geq \delta$, we have

$$I_3 \leq \frac{1}{k_1 + \frac{1}{\delta-1}l_1 + \frac{1}{\delta-1}q}. \quad \text{Q.E.D.} \tag{7.18}$$

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