TOKYO J. MATH.
VOL. 23, NO. 2, 2000

The Principle of Limiting Absorption for the Non-Selfadjoint Schrödinger Operator with Energy Dependent Potential

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(Communicated by H. Ishii)

Dedicated to Professor Kiyoshi Mochizuki on his sixtieth birthday

1. Introduction.

In this paper we shall show the principle of limiting absorption for the quadratic operator pencil

$$L(\kappa) = -\Delta - i\kappa B - \kappa^2 \tag{1.1}$$

in the N-dimensional euclidean space $\mathbf{R}^N$ with $N \neq 2$. Here Δ is the N-dimensional Laplacian, $\kappa \in \mathbf{C}$ and B denotes the multiplication operator by a real-valued function $b(x)$ on $\mathbf{R}^N$. Throughout the paper we require

(1.2) *There exist constants b_0 and $\delta \in (0, 1]$ such that $0 < b_0 < \dfrac{2-\sqrt{2}}{2}\delta$ and $|b(x)| \le b_0(1+r)^{-1-2\delta}$ for any $x \in \mathbf{R}^N$, where $r = |x|$.*

The operator $L(\kappa)$ is derived from the wave equation

$$w_{tt}(x,t) - \Delta w(x,t) + b(x)w_t(x,t) = 0 \tag{1.3}$$

where $x \in \mathbf{R}^N$, $t \ge 0$, $w_t = \partial w/\partial t$, $w_{tt} = \partial^2 w/\partial t^2$ and $b(x) \in C^1(\mathbf{R}^N)$. If $b(x) \ge 0$, $b(x)w_t$ represents a friction of viscous type. As is known in Mochizuki [14], under a more general assumption than (1.2) with $b(x) \ge 0$, the solutions of (1.3) are asymptotically equal for $t \to +\infty$ to those of free wave equations

$$w_{0tt}(x,t) - \Delta w_0(x,t) = 0\,. \tag{1.4}$$

If we consider the solutions of the form $w(x,t) = u(x)e^{-i\kappa t}$ in (1.3), then $u(x)$ satisfies

$$L(\kappa)u(x) = (-\Delta - i\kappa b(x) - \kappa^2)u(x) = 0\,. \tag{1.5}$$

The principle of limiting absorption states that there exist two Hilbert spaces

$$L^{2,(1+\delta)/2}(\mathbf{R}^N) \quad \text{and} \quad L^{2,-(1+\delta)/2}(\mathbf{R}^N)$$

Received July 14, 1999

such that

$$L^{2,(1+\delta)/2}(\mathbf{R}^N) \subset L^2(\mathbf{R}^N) \subset L^{2,-(1+\delta)/2}(\mathbf{R}^N)$$

(the definition of these spaces is given in section 3), and $R((\sigma \pm i\tau)^2) \equiv L(\sigma \pm i\tau)^{-1}$ is continuously extended to $\tau = 0$ as an operator from $L^{2,(1+\delta)/2}(\mathbf{R}^N)$ to $L^{2,-(1+\delta)/2}(\mathbf{R}^N)$. More precisely,

MAIN THEOREM. *Assume $N \neq 2$ and* (1.2). *Let*

$$K_+^+ = \{\kappa = \sigma + i\tau \in \mathbf{C} \setminus \{0\} \,|\, \sigma \in I = (a_1, a_2), \tau \in (0, 1)\}$$

with $0 < a_1 < a_2 < \infty$. For $\sigma + i\tau \in K_+^+$, and $f \in L^{2,(1+\delta)/2}(\mathbf{R}^N)$ $(0 < \delta \leq 1)$, we define

$$u(\sigma, \tau) = R((\sigma + i\tau)^2)f\,.$$

Then $u(\sigma, \tau) \in L^{2,-(1+\delta)/2}(\mathbf{R}^N)$ converges in $L^{2,-(1+\delta)/2}(\mathbf{R}^N)$ as $\tau \downarrow 0$. Moreover this convergence is uniform with respect to $\sigma \in I$. If we denote this limit by $u(\sigma)$, then $u(\sigma)$ is a solution of $L(\sigma)u(x) = f(x)$ and is uniformly continuous with respect to $\sigma \in I$ in $L^{2,-(1+\delta)/2}(\mathbf{R}^N)$.

REMARK. In this theorem, we can replace K_+^+ by K_-^+, K_+^- or K_-^-:

$$K_-^+ = \{\kappa = \sigma - i\tau \in \mathbf{C} \setminus \{0\} \,|\, \sigma \in I = (a_1, a_2), \tau \in (0, 1)\},$$
$$K_+^- = \{\kappa = \sigma + i\tau \in \mathbf{C} \setminus \{0\} \,|\, \sigma \in I = (-a_2, -a_1), \tau \in (0, 1)\},$$
$$K_-^- = \{\kappa = \sigma - i\tau \in \mathbf{C} \setminus \{0\} \,|\, \sigma \in I = (-a_2, -a_1), \tau \in (0, 1)\}.$$

The principle of limiting absorption has been studied for the Schrödinger equation

$$(-\Delta + V(x) - \kappa^2)u(x) = f(x) \quad \text{in} \quad \mathbf{R}^N\,. \tag{1.6}$$

After the pioneering work of Éĭdus [2], Agmon [1], Ikebe-Saito [4] and Mochizuki [13] extended it to a wider class of real valued potentials. The spectral theory is also developed for complex potentials by Kato [7], Mochizuki [11], [12] and Saito [18], [19], and Saito extended the principle there to complex potentials.

The methods developed in these works are not directly applied to our problem since $i\kappa b(x)$ depends on the energy κ. In order to overcome the difficulty we suppose the smallness condition (1.2). Then Mochizuki's method [14] directly applies to obtain the key a-priori estimate (Theorem 3.5 (3.18)). From this, the uniqueness of solutions for $L(\kappa)u(x) = f(x)$ $(\mathrm{Im}\,\kappa \gtreqless 0)$ which play an important role in the proof of Main Theorem follows.

As for the wave equations with dissipative term, Mizohata-Mochizuki [10] and Iwasaki [5] proved the limiting amplitude principle. Energy decay, non-decay and asymptotic behavior of solutions were shown by Mochizuki [15] and Mochizuki-Nakazawa [16], [17]. Among them, in [17] the energy decay (in an exterior domain) was shown under the dissipation effective only in exterior of the large ball in $\mathbf{R}^N$ $(N \neq 2)$. Recently, Kadowaki [6] and Matsuyama [9] also treated the dissipative wave equation. Kadowaki [6] showed the existence of scattering states in stratified media and Matsuyama [9] proved the energy non-decay and local energy decay in an exterior domain without any geometrical condition of the boundary under the dissipation effective only near the boundary.

The contents of the present paper will be outlined as follows. In section 2 we study the spectral structure of $L(\kappa)$. The key a-priori estimate is shown in section 3 (Theorem 3.5 (3.18)). As a corollary, non-existence of positive eigenvalue for $L(\kappa)$ follows (Propositon 2.6). The Main Theorem is proved in section 4. In the final section 5, we shall state the limiting absorption principle for the operator $i\begin{pmatrix} 0 & 1 \\ \Delta & -B \end{pmatrix}$.

ACKOWLEDGEMENT. The author wishes to express his sincere thanks to Professor K. Mochizuki for his helpful advices and encouragement and to Professor M. Kadowaki for his kind advices and discussions. The author would also like to thank to the referee for his careful reading and useful comments.

2. Spectral structure of $L(\kappa)$.

In this section, for the sake of simplicity, we assume the non-negativity of $b(x)$ and study the spectral structure of $L(\kappa)$. We define the linear operators H_0 by

$$\mathfrak{D}(H_0) = H^2(\mathbf{R}^N) \quad H_0 = -\Delta\,. \tag{2.1}$$

As is well known, H_0 is a selfadjoint operator and we have

$$\rho(H_0) = \mathbf{C} \setminus [0, \infty)\,, \quad \sigma(H_0) = \sigma_{ess}(H_0) = \sigma_c(H_0) = [0, \infty)\,, \quad \sigma_r(H_0) = \sigma_p(H_0) = \emptyset\,,$$

where $\rho(H_0)$, $\sigma(H_0)$, $\sigma_{ess}(H_0)$, $\sigma_c(H_0)$, $\sigma_r(H_0)$ and $\sigma_p(H_0)$ denote the resolvent set, the spectrum, the essential spectrum, the continuous spectrum, the residual spectrum and the point spectrum, respectively.

For $\kappa \in \mathbf{C} \setminus \{0\}$, we define the linear operator $L(\kappa)$ by

$$L(\kappa) = H_0 - i\kappa B - \kappa^2\,, \quad \mathfrak{D}(L(\kappa)) = H^2(\mathbf{R}^N)\,. \tag{2.2}$$

This is a non-selfadjoint but closed operator. In this section, we consider $L(\kappa)$ in $\operatorname{Im}\kappa \geq 0$.

The adjoint operator $L^*(\kappa)$ of $L(\kappa)$ is given by

$$L^*(\kappa) = H_0 + i\bar{\kappa} B - \bar{\kappa}^2\,, \quad \mathfrak{D}(L^*(\kappa)) = H^2(\mathbf{R}^N)\,.$$

The resolvent set, the spectrum, the essential spectrum, the continuous spectrum, the residual spectrum and the point spectrum of $L(\kappa)$ are defined as follows (cf. Markus [8] p. 56, Saito [18] p. 406):

$$\rho(L(\kappa)) = \{\kappa^2 \in \mathbf{C} \,|\, Ker(L(\kappa)) = \{0\}, \overline{\mathfrak{R}(L(\kappa))} = L^2(\mathbf{R}^N), (L(\kappa))^{-1} \in \mathfrak{L}(L^2(\mathbf{R}^N))\}\,,$$

$$\sigma(L(\kappa)) = \mathbf{C} \setminus \rho(L(\kappa))\,,$$

$$\sigma_{ess}(L(\kappa)) = \Big\{\kappa^2 \in \sigma(L(\kappa)) \,\big|\, {}^{\exists}\{f_n\} \subset \mathfrak{D}(L(\kappa)) \text{ s.t } \|f_n\|_{L^2} = 1,$$
$$L(\kappa) f_n \underset{s}{\to} 0\, (n \to \infty),\ f_n \underset{w}{\to} 0\, (n \to \infty)\Big\}\,,$$

(where $\underset{s}{\to}$ and $\underset{w}{\to}$ denote the strongly and weakly convergence in L^2 respectively),

$$\sigma_c(L(\kappa)) = \{\kappa^2 \in \sigma(L(\kappa)) \,|\, Ker(L(\kappa)) = \{0\},\ \overline{\mathfrak{R}(L(\kappa))} = L^2(\mathbf{R}^N),\ (L(\kappa))^{-1} \notin \mathfrak{L}(L^2(\mathbf{R}^N))\},$$

$$\sigma_r(L(\kappa)) = \{\kappa^2 \in \sigma(L(\kappa)) \,|\, Ker(L(\kappa)) = \{0\},\ \overline{\mathfrak{R}(L(\kappa))} \neq L^2(\mathbf{R}^N)\},$$

$$\sigma_p(L(\kappa)) = \{\kappa^2 \in \sigma(L(\kappa)) \,|\, Ker(L(\kappa)) \neq \{0\}\},$$

where $Ker(L(\kappa)) = \{f \in \mathfrak{D}(L(\kappa)) \,|\, L(\kappa)f = 0\}$, $\mathfrak{R}(L(\kappa))$ is the range of the operator $L(\kappa)$, $\overline{\mathfrak{R}(L(\kappa))}$ denotes the closure of $\mathfrak{R}(L(\kappa))$ in $L^2(\mathbf{R}^N)$, and $\mathfrak{L}(L^2(\mathbf{R}^N))$ is the set of the bounded linear operator on $L^2(\mathbf{R}^N)$. For $\kappa^2 \in \rho(L(\kappa))$, we put $R(\kappa^2) = (L(\kappa))^{-1}$ and call it resolvent of $L(\kappa)$. We can define the spectrum sets for $L^*(\kappa)$ similarly as above. As in the case of the usual definition, we see

$$\kappa^2 \in \sigma_r(L(\kappa)) \Leftrightarrow \bar{\kappa}^2 \in \sigma_p(L^*(\kappa)), \quad \kappa^2 \notin \sigma_p(L(\kappa)). \tag{2.3}$$

It follows from the second resolvent equation that

$$R(\kappa^2) = R_0(\kappa^2) + i\kappa R_0(\kappa^2)BR(\kappa^2) \tag{2.4}$$

for any $\kappa^2 \in \rho(H_0) \cap \rho(L(\kappa))$, where $R_0(\kappa^2) = (H_0 - \kappa^2)^{-1}$.

Let A be the operator of multiplication by $\sqrt{b(x)}$. Then since $B = A^2$, we see from (2.4)

$$AR(\kappa^2) = AR_0(\kappa^2) + i\kappa AR_0(\kappa^2)A \cdot AR(\kappa^2). \tag{2.5}$$

Now we define the operator $Q(\kappa)$ by

$$Q(\kappa) = i\kappa AR_0(\kappa^2)A. \tag{2.6}$$

Then we have from (2.5)

$$[I - Q(\kappa)]AR(\kappa^2) = AR_0(\kappa^2). \tag{2.7}$$

If the bounded inverse of $I - Q(\kappa)$ exists, then

$$AR(\kappa^2) = [I - Q(\kappa)]^{-1}AR_0(\kappa^2). \tag{2.8}$$

Therefore we obtain from (2.4) and (2.8) that

$$R(\kappa^2) = R_0(\kappa^2) + i\kappa R_0(\kappa^2)A[I - Q(\kappa)]^{-1}AR_0(\kappa^2). \tag{2.9}$$

LEMMA 2.1. *Assume that* $0 \le b(x) \le b_0(1+r)^{-\varepsilon}$ *for some positive constants* b_0 *and* ε. *Then* $Q(\kappa)$ *and* $i\kappa BR_0(\kappa^2)$ *are compact operators on* $L^2(\mathbf{R}^N)$ *for any* $\kappa \in \mathbf{C} \setminus \{0\}$ *with* $\operatorname{Im}\kappa > 0$.

For the proof of this lemma, it suffices only to check the compactness of $AR_0(\kappa^2)$. But it is easily verified since $b(x)$ satisfies the well-known Stummel condition.

Moreover, noting that $i\kappa B$ is H_0- and $(H_0 - i\kappa B)$-compact, we can follow the same line of proof as Schechter [20] Chapter 3 to show the following proposition:

PROPOSITION 2.2. $\sigma_{ess}(L(\kappa)) = \sigma_{ess}(H_0) = [0, \infty)$.

Next, we shall study the invertibility of $I - Q(\kappa)$ for $\operatorname{Im}\kappa > 0$. To do so, we give

DEFINITION 2.3. $\Sigma = \{\kappa \in \mathbf{C} \setminus \{0\}$ *with* $\operatorname{Im}\kappa > 0 \mid I - Q(\kappa)$ *is not invertible*$\}$.

Then we obtain (cf. Mochizuki [12] Theorem 2.1, Ikebe [3] Lemma 2.4)

LEMMA 2.4. *Let* $\kappa \in \mathbf{C} \setminus \{0\}$ *with* $\operatorname{Im}\kappa > 0$. *Then* κ *belongs to* Σ *if and only if* κ^2 *is an eigenvalue of* $L(\kappa)$ *of finite multiplicity.*

PROOF. ($\Rightarrow$) If $\kappa \not\equiv 0$ belongs to Σ then there exists $f(\not\equiv 0) \in L^2(\mathbf{R}^N)$ such that $Q(\kappa)f = f$, i.e., $f = i\kappa A R_0(\kappa^2)Af$. Operating A from the left, we have $Af = i\kappa A^2 R_0(\kappa^2)Af$. If we set $\varphi = Af$ then $\varphi(\not\equiv 0) \in L^2(\mathbf{R}^N)$ and $\varphi = i\kappa A^2 R_0(\kappa^2)\varphi$. Operating $R_0(\kappa^2)$ from the left, we obtain $R_0(\kappa^2)\varphi = i\kappa R_0(\kappa^2)A^2 R_0(\kappa^2)\varphi$. If we put $\psi = R_0(\kappa^2)\psi$, then $\psi(\not\equiv 0) \in H^2(\mathbf{R}^N)$ and $\psi = i\kappa R_0(\kappa^2)A^2\psi$. Operating $H_0 - \kappa^2$ from the left, we obtain $(H_0 - \kappa^2)\psi = i\kappa A^2\psi$, i.e., $L(\kappa)\psi = 0$ with $\psi(\not\equiv 0) \in H^2(\mathbf{R}^N)$. Therefore, $\kappa^2 \in \sigma_p(L(\kappa))$. ($\Leftarrow$) We can reverse the above argument. □

As for Σ, the following lemma holds (cf. Mizohata-Mochizuki [10] Lemma 3.4, Iwasaki [5] pp. 388).

LEMMA 2.5.

$$\Sigma = \emptyset . \tag{2.10}$$

PROOF. By the Fredholm alternative, it is sufficient to prove that $Q(\kappa)\varphi = \varphi$, $\varphi \in L^2(\mathbf{R}^N)$ implies $\varphi \equiv 0$. Since A is selfadjoint, we have

$$0 = ((I - Q(\kappa))\varphi, \varphi) = \|\varphi\|^2 - i\kappa(R_0(\kappa^2)A\varphi, A\varphi) \tag{2.11}$$

where $(\cdot, \cdot)$ denotes the inner-product of $L^2(\mathbf{R}^N)$. By spectral resolution, we see

$$-i\kappa(R_0(\kappa^2)A\varphi, A\varphi) = -i\kappa \int_0^\infty \frac{1}{\lambda - \kappa^2} d(E_0(\lambda)A\varphi, A\varphi),$$

where $E_0(\lambda)$ is the spectral measure for H_0. If we write $\kappa = \sigma + i\tau$ ($\sigma, \tau \in \mathbf{R}$), then

$$\frac{-i\kappa}{\lambda - \kappa^2} = \frac{\tau(\lambda + \sigma^2 + \tau^2) + i\sigma(\lambda - \sigma^2 - \tau^2)}{(\lambda - \sigma^2 + \tau^2)^2 + (2\sigma\tau)^2}.$$

Therefore taking the real part of (2.11), we have

$$0 = \|\varphi\|^2 + \int_0^\infty \frac{\tau(\lambda + \sigma^2 + \tau^2)}{(\lambda - \sigma^2 + \tau^2)^2 + (2\sigma\tau)^2} d(E_0(\lambda)A\varphi, A\varphi). \tag{2.12}$$

Since $\tau = \operatorname{Im}\kappa > 0$ by assumption, the second term of the right-hand side of (2.12) is nonnegative. Therefore, we obtain $\varphi \equiv 0$. □

From this lemma, we see that $\sigma_p(L(\kappa)) \cap (\mathbf{C} \setminus [0, \infty)) = \emptyset$. Moreover, if we assume $N \neq 2$ and

(2.13) *There exist positive constants* b_0 *and* δ *such that* $0 < b_0 < \delta \leq 1$ *and* $0 \leq b(x) \leq b_0(1 + r)^{-1-2\delta}$ *for any* $x \in \mathbf{R}^N$,

then the absence of positive eigenvalue of $L(\kappa)$ is obtained by the modifications of the proof of (3.18) of Theorem 3.5. Therefore we have

PROPOSITON 2.6. *Assume that* $N \neq 2$ *and* (2.13). *Then* $\sigma_p(L(\kappa)) = \emptyset$.

REMARK. The condition (2.13) is slightly weaker than (1.2) if $b(x) \geq 0$.

We give the proof after Theorem 3.5 in Section 3.
Next we examine the absence of residual spectrum of $L(\kappa)$.

PROPOSITION 2.7. $\sigma(L(\kappa)) = \emptyset$.

PROOF. This is easily obtaind from (2.3). □

Summarizing what has been discussed so far, one can now state the following theorem on the spectral structure of $L(\kappa)$.

THEOREM 2.8. *Assume that* $N \neq 2$ *and* (2.13). *Then the following assertions hold*:

$$\sigma(L(\kappa)) = \sigma_{ess}(L(\kappa)) = \sigma_c(L(\kappa)) = [0, \infty),$$
$$\sigma_p(L(\kappa)) = \sigma_r(L(\kappa)) = \emptyset, \quad \rho(L(\kappa)) = \mathbf{C} \setminus [0, \infty).$$

3. Some inequalities—resolvent estimate.

In this section we shall show some inequalities for L^2-solutions of the Schrödinger equation of the form

$$-\Delta u - i\kappa b(x)u - \kappa^2 u = f(x) \quad \text{in} \quad \mathbf{R}^N, \tag{3.1}$$

where $\kappa = \sigma + i\tau$ $(\sigma(\neq 0), \tau \in \mathbf{R}) \in \mathbf{C} \setminus \{0\}$ and $f(x) \in L^{2,(1+\delta)/2}(\mathbf{R}^N)$, where

$$L^{2,\alpha}(G) = \left\{ f(x) \,\middle|\, \|f\|_{\alpha,G} = \left(\int_G (1 + |x|)^{2\alpha} |f(x)|^2 dx \right)^{1/2} < \infty \right\}.$$

If $G = \mathbf{R}^N$, we shall omit the second subscript G.

We define two operators $\mathcal{D}^{\pm}$ and $\mathcal{D}_r^{\pm}$ as follows (cf. Ikebe-Saito [4], Mochizuki [13], [14], [15], Saito [18], [19]):

$$\mathcal{D}^{\pm}u = \nabla u + \frac{N-1}{2r} u \frac{x}{r} \mp i\kappa u \frac{x}{r} \quad (\operatorname{Im}\kappa \gtrless 0),$$

$$\mathcal{D}_r^{\pm}u = \frac{x}{r} \cdot \mathcal{D}^{\pm}u = u_r + \frac{N-1}{2r} u \mp i\kappa u \quad (\operatorname{Im}\kappa \gtrless 0).$$

LEMMA 3.1. *If* u *is a solution of* (3.1), *then the following identity holds*:

$$\begin{aligned}
&\frac{1}{2}\int_{S_\rho} \left(\left| u_r + \frac{N-1}{2r} u \right|^2 + |\kappa|^2 |u|^2 \right) dS \pm \int_{S_\rho} \frac{\tau(N-1)}{2r} |u|^2 dS \\
&\pm \int_{B_\rho} \{ \tau(|\nabla u|^2 + |\kappa|^2 |u|^2) + |\kappa|^2 b(x) |u|^2 \} dx \\
&= \frac{1}{2}\int_{S_\rho} |\mathcal{D}_r^{\pm} u|^2 dS \mp \operatorname{Re} \int_{B_\rho} f \overline{i\kappa u} dx \quad (\tau = \operatorname{Im}\kappa \gtrless 0),
\end{aligned} \tag{3.2}$$

where $S_\rho = \{x \in \mathbf{R}^N \mid |x| = \rho\}$, $B_\rho = \{x \in \mathbf{R}^N \mid |x| < \rho\}$ *for* $\rho > 0$ *and* Re *means the real part.*

PROOF (cf. Mochizuki [14] Lemma 2.1). Multiplying by $\overline{i\kappa u}$ on the both sides of (3.1), we have

$$-\nabla \cdot (\overline{i\kappa u}\nabla u) + \overline{i\kappa}|\nabla u|^2 - |\kappa|^2 b(x)|u|^2 + i\kappa|\kappa|^2|u|^2 = f\overline{i\kappa u}\,. \tag{3.3}$$

Integration on B_ρ of the both sides of (3.3) gives

$$-\int_{S_\rho} u_r \overline{i\kappa u} dS + \int_{B_\rho} \{\overline{i\kappa}|\nabla u|^2 - |\kappa|^2 b(x)|u|^2 + i\kappa|\kappa|^2|u|^2\} dx = \int_{B\rho} f\overline{i\kappa u} dx\,. \tag{3.4}$$

Noting that the identity

$$\mathrm{Re}(-\overline{i\kappa u}u_r) = \pm\frac{1}{2}|\mathcal{D}_r^{\pm}u|^2 \mp \frac{1}{2}\left(\left|u_r + \frac{N-1}{2r}u\right|^2 + |\kappa|^2|u|^2\right) - \frac{\tau(N-1)}{2r}|u|^2 \tag{3.5}$$

and taking the real part of the both sides of (3.4), we obtain the desired identity. □

From this lemma, we have

PROPOSITION 3.2. *If* u *is the solution of* (3.1), *then the following inequality holds*:

$$\begin{aligned} &\int_{\mathbf{R}^N} (1+r)^{-1-\delta}\left|u_r + \frac{N-1}{2r}u\right|^2 dx + |\kappa|^2 \int_{\mathbf{R}^N} \left\{(1+r)^{-1-\delta} - \frac{2|b(x)|}{\delta}\right\}|u|^2 dx \\ &\quad \leqq \int_{\mathbf{R}^N} (1+r)^{-1-\delta}|\mathcal{D}_r^{\pm}u|^2 dx + \frac{2}{\delta}\int_{\mathbf{R}^N} |f\overline{i\kappa u}| dx \quad (\mathrm{Im}\,\kappa \gtrless 0)\,. \end{aligned} \tag{3.6}$$

PROOF (cf. Mochizuki [14] Lemma 2.1). It follows from (3.2) that

$$\begin{aligned} &\int_{S_\rho} \left(\left|u_r + \frac{N-1}{2r}u\right|^2 + |\kappa|^2|u|^2\right) dS - 2|\kappa|^2 \int_{\mathbf{R}^N} |b(x)|\,|u|^2 dx \\ &\quad \leq \int_{S_\rho} |\mathcal{D}^{\pm}u|^2 dS + 2\int_{\mathbf{R}^N} |f\overline{i\kappa u}| dx\,. \end{aligned} \tag{3.7}$$

Multiplying by $(1+\rho)^{-1-\delta}$ on the both sides of (3.7) and integrating over $[0, +\infty)$, we obtain (3.6). □

LEMMA 3.3. *Let* $\varphi = \varphi(r)$ *be a real-valued* C^1*-function. If* u *is a solution of* (3.1), *then the following identity holds*:

$$\begin{aligned} &\int_{\mathbf{R}^N} \left\{\left(\pm\tau\varphi + \frac{\varphi_r}{2}\right)|\mathcal{D}^{\pm}u|^2 + \left(\frac{\varphi}{r} - \varphi_r\right)(|\mathcal{D}^{\pm}u|^2 - |\mathcal{D}_r^{\pm}u|^2)\right\} dx \\ &\quad \pm \int_{\mathbf{R}^N} \frac{b(x)\varphi}{2}\left(|\mathcal{D}_r^{\pm}u|^2 + |\kappa|^2|u|^2 - \left|u_r + \frac{N-1}{2r}u\right|^2\right) dx \\ &\quad + \int_{\mathbf{R}^N} \left\{-\frac{(N-1)(N-3)}{8}\left(\frac{\varphi}{r^2}\right)_r |u|^2 \pm \frac{\tau(N-1)(N-3)}{4r^2}\varphi|u|^2\right\} dx \\ &= \mathrm{Re}\int_{\mathbf{R}^N} f\varphi\overline{\mathcal{D}_r^{\pm}u} dx \quad (\mathrm{Im}\,\kappa \gtrless 0)\,. \end{aligned} \tag{3.8}$$

PROOF (cf. Mochizuki [13] pp. 37, [14] Lemma 2.2, and [15] pp. 43. See also Ikebe-Saito [4] Lemma 2.2.).

Put $v(x;\kappa)=e^{\rho(r;\kappa)}u(x)$ where $\rho(r;\kappa)=\mp i\kappa r+((N-1)/2)\log r$. Then v satisfies

$$-\Delta v+2\nabla\rho\cdot\nabla v+\tilde{b}(x;\kappa)v=e^{\rho}f\,, \tag{3.9}$$

where $\tilde{b}(x;\kappa)=-i\kappa b(x)+(N-1)(N-3)/4r^2$. Multiplying by $\overline{v_r}$ on the both sides of (3.9) and using the relation $\nabla v\cdot\nabla(\overline{v_r})=\nabla\cdot((|\nabla v|^2/2)(x/r))-(N-3)/2r|\nabla v|^2-|v_r|^2/r$, we see

$$\begin{aligned}&-\nabla\cdot\left(\nabla v\overline{v_r}-\frac{|\nabla v|^2}{2}\frac{x}{r}\right)-\frac{(N-3)}{2r}|\nabla v|^2-\frac{|v_r|^2}{r}\\&+2\nabla\rho\cdot\nabla v\overline{v_r}+\tilde{b}(x;\kappa)v\overline{v_r}=e^{\rho}f\overline{v_r}\,.\end{aligned} \tag{3.10}$$

Noting that $\nabla v=e^{\rho}\mathcal{D}^{\pm}u$, $v_r=e^{\rho}\mathcal{D}_r^{\pm}u$ and dividing the both sides of (3.10) by $e^{\mp 2\tau r}r^{N-1}$, we obtain

$$\begin{aligned}&-\nabla\cdot\left(\mathcal{D}^{\pm}u\overline{\mathcal{D}_r^{\pm}u}-\frac{|\mathcal{D}^{\pm}u|^2}{2}\frac{x}{r}\right)+\frac{|\mathcal{D}^{\pm}u|^2-|\mathcal{D}_r^{\pm}u|^2}{r}\\&\pm\tau|\mathcal{D}^{\pm}u|^2+2i\sigma|\mathcal{D}_r^{\pm}u|^2+\tilde{b}(x;\kappa)u\overline{\mathcal{D}_r^{\pm}u}=f\overline{\mathcal{D}_r^{\pm}u}\,.\end{aligned} \tag{3.11}$$

Multiplying $\varphi=\varphi(r)$ on the both sides of (3.11), integrating over $\mathbf{R}^N$ and taking the real part, we have

$$\begin{aligned}&\int_{\mathbf{R}^N}\left\{\left(\pm\tau\varphi+\frac{\varphi_r}{2}\right)|\mathcal{D}^{\pm}u|^2+\left(\frac{\varphi}{r}-\varphi_r\right)(|\mathcal{D}^{\pm}u|^2-|\mathcal{D}_r^{\pm}u|^2)\right\}dx\\&+\mathrm{Re}\int_{\mathbf{R}^N}\varphi\tilde{b}(x;\kappa)u\overline{\mathcal{D}_r^{\pm}u}dx=\mathrm{Re}\int_{\mathbf{R}^N}\varphi f\overline{\mathcal{D}_r^{\pm}u}dx\,.\end{aligned} \tag{3.12}$$

We write $u_1=\mathrm{Re}\,u$, $u_2=\mathrm{Im}\,u$ (i.e., $u=u_1+iu_2$). Then

(3.13)

$$\begin{aligned}&\mathrm{Re}\int_{\mathbf{R}^N}\varphi\tilde{b}(x;\kappa)u\overline{\mathcal{D}_r^{\pm}u}dx\\&=\int_{\mathbf{R}^N}b(x)\varphi\left\{\frac{\tau(N-1)}{2r}|u|^2+|\kappa|^2|u|^2+\sigma(u_2u_{1r}-u_1u_{2r})+\tau(u_1u_{1r}+u_2u_{2r})\right\}dx\\&+\int_{\mathbf{R}^N}\frac{(N-1)(N-3)}{4r^2}\varphi\left\{(u_1u_{1r}+u_2u_{2r})+\frac{N-1}{2r}|u|^2\pm\tau|u|^2\right\}dx\end{aligned}$$

where $\kappa=\sigma+i\tau$. Note that

$$\frac{\varphi}{r^2}(u_1u_{1r}+u_2u_{2r})=\nabla\cdot\left(\frac{\varphi|u|^2}{2r^2}\frac{x}{r}\right)-\frac{N-1}{r}\frac{\varphi|u|^2}{2r^2}-\left(\frac{\varphi}{r^2}\right)_r\frac{|u|^2}{2}\,.$$

And we recall (3.5) to observe

$$\begin{aligned}&\sigma(u_2u_{1r}-u_1u_{2r})+\tau(u_1u_{1r}+u_2u_{2r})=\mathrm{Re}(-i\overline{\kappa u}u_r)\\&=\pm\frac{1}{2}|\mathcal{D}^{\pm}u|^2\mp\frac{1}{2}\left(\left|u_r+\frac{N-1}{2r}u\right|^2+|\kappa|^2|u|^2\right)-\frac{\tau(N-1)}{2r}|u|^2\,.\end{aligned}$$

Substituting these relations to (3.13), we obtain

$$\begin{aligned} &\mathrm{Re}\int_{\mathbf{R}^N} \varphi\tilde{b}(x;\kappa)u\overline{\mathcal{D}_r^{\pm}u}dx \\ &= \pm\int_{\mathbf{R}^N} \frac{b(x)\varphi}{2}\left(|\mathcal{D}_r^{\pm}u|^2 + |\kappa|^2|u|^2 - \left|u_r + \frac{N-1}{2r}u\right|^2\right)dx \\ &+\int_{\mathbf{R}^N}\left\{-\frac{(N-1)(N-3)}{8}\left(\frac{\varphi}{r^2}\right)_r|u|^2 \pm \frac{\tau(N-1)(N-3)}{4r^2}\varphi|u|^2\right\}dx\,. \end{aligned} \tag{3.14}$$

(3.12) and (3.14) give the desired identity. □

From this lemma, we obtain the next inequality.

PROPOSITON 3.4. *Assume that* $N \neq 2$. *If* u *is a solution of* (3.1), *then the following inequality holds*:

$$\begin{aligned} &\int_{\mathbf{R}^N}(1+r)^{-1+\delta}|\mathcal{D}^{\pm}u|^2dx \\ &-\frac{1}{\delta}\int_{\mathbf{R}^N}|b(x)|(1+r)^{\delta}\left(|\mathcal{D}^{\pm}u|^2 + |\kappa|^2|u|^2 + \left|u_r + \frac{N-1}{2r}u\right|^2\right)dx \\ &\leq \frac{2}{\delta}\int_{\mathbf{R}^N}(1+r)^{\delta}|f\overline{\mathcal{D}_r^{\pm}u}|dx \quad (\mathrm{Im}\,\kappa \gtreqless 0)\,. \end{aligned} \tag{3.15}$$

PROOF. Put $\varphi = \varphi(r) = (1+r)^{\delta}$ in Lemma 3.3. Then since $\delta \leq 1$, we have

$$\frac{\varphi}{r} - \varphi_r \geq 0\,, \tag{3.16}$$

$$\left(\frac{\varphi}{r^2}\right)_r \leq 0\,. \tag{3.17}$$

(3.16), (3.17), $N \neq 2$, and $|\mathcal{D}_r^{\pm}u| \leq |\mathcal{D}^{\pm}u|$ give the desired inequality. □

THEOREM 3.5. *Assume that* $N \neq 2$ *and* (1.2). *If* u *is a solution of* (3.1) *then there exist positive constants* C_1 *and* C_2 *independent of* κ *such that the following inequalities hold*:

$$|\kappa|\|u\|_{-(1+\delta)/2} \leq C_1\|f\|_{(1+\delta)/2} \quad (\mathrm{Im}\,\kappa \gtreqless 0)\,, \tag{3.18}$$

$$\|\mathcal{D}^{\pm}u\|_{(-1+\delta)/2} \leq C_2\|f\|_{(1+\delta)/2} \quad (\mathrm{Im}\,\kappa \gtreqless 0)\,. \tag{3.19}$$

PROOF. Since $b_0 < ((2-\sqrt{2})/2)\delta$, there exists a constant ε such that

$$0 < \varepsilon < \frac{2b_0^2 - 4\delta b_0 + \delta^2}{\delta^2} (< 1)\,. \tag{3.20}$$

Then

$$1 - \frac{b_0}{\delta} > 1 - \varepsilon - \frac{2b_0}{\delta} > \frac{2b_0(\delta - b_0)}{\delta^2} > 0\,. \tag{3.21}$$

By (3.20) and (3.21), there exists a constant φ_0 (> 0) such that

$$\frac{1+\varepsilon}{1 - b_0/\delta} < \varphi_0 < \frac{\delta}{b_0}\left(1 - \varepsilon - \frac{2b_0}{\delta}\right)\,. \tag{3.22}$$

Adding (3.6) to φ_0-times (3.15), we obtain

$$
\begin{aligned}
&|\kappa|^2 \int_{\mathbf{R}^N} \left\{ (1+r)^{-1+\delta} - \frac{2|b(x)|}{\delta} - \frac{\varphi_0 |b(x)|}{\delta}(1+r)^{\delta} \right\} |u|^2 dx \\
&+ \int_{\mathbf{R}^N} \left\{ \varphi_0 (1+r)^{-1+\delta} - \frac{\varphi_0 |b(x)|}{\delta}(1+r)^{\delta} - (1+r)^{-1-\delta} \right\} |\mathcal{D}^{\pm} u|^2 dx \\
&+ \int_{\mathbf{R}^N} \left\{ (1+r)^{-1-\delta} - \frac{\varphi_0 |b(x)| (1+r)^{\delta}}{\delta} \right\} \left| u_r + \frac{N-1}{2r} u \right|^2 dx \\
&\le \frac{2}{\delta} \int_{\mathbf{R}^N} |f \overline{i\kappa u}| dx + \frac{2\varphi_0}{\delta} \int_{\mathbf{R}^N} (1+r)^{\delta} |f \overline{\mathcal{D}_r^{\pm} u}| dx .
\end{aligned} \tag{3.23}
$$

Note that by the Schwarz inequality,

$$
\frac{2}{\delta} \int_{\mathbf{R}^N} |f \overline{i\kappa u}| dx \le \frac{1}{\varepsilon \delta^2} \int_{\mathbf{R}^N} (1+r)^{1+\delta} |f|^2 dx + \varepsilon |\kappa|^2 \int_{\mathbf{R}^N} (1+r)^{-1-\delta} |u|^2 dx , \tag{3.24}
$$

$$
\begin{aligned}
&\frac{2\varphi_0}{\delta} \int_{\mathbf{R}^N} (1+r)^{\delta} |f \overline{\mathcal{D}_r^{\pm} u}| dx \\
&\le \frac{\varphi_0^2}{\varepsilon \delta^2} \int_{\mathbf{R}^N} (1+r)^{1+\delta} |f|^2 dx + \varepsilon \int_{\mathbf{R}^N} (1+r)^{-1+\delta} |\mathcal{D}^{\pm} u|^2 dx .
\end{aligned} \tag{3.25}
$$

Then (3.23), (3.24) and (3.25) give

(3.26)

$$
\begin{aligned}
&|\kappa|^2 \int_{\mathbf{R}^N} \left\{ (1+r)^{-1-\delta} - \frac{2|b(x)|}{\delta} - \frac{\varphi_0 |b(x)|}{\delta}(1+r)^{\delta} - \varepsilon (1+r)^{-1-\delta} \right\} |u|^2 dx \\
&+ \int_{\mathbf{R}^N} \left\{ \varphi_0 (1+r)^{-1+\delta} - \frac{\varphi_0 |b(x)|}{\delta}(1+r)^{\delta} - (1+r)^{-1-\delta} - \varepsilon (1+r)^{-1+\delta} \right\} |\mathcal{D}^{\pm} u|^2 dx \\
&+ \int_{\mathbf{R}^N} \left\{ (1+r)^{-1-\delta} - \frac{\varphi_0 |b(x)| (1+r)^{\delta}}{\delta} \right\} \left| u_r + \frac{N-1}{2r} u \right|^2 dx \\
&\le \frac{1+\varphi_0^2}{\varepsilon \delta^2} \int_{\mathbf{R}^N} (1+r)^{1+\delta} |f|^2 dx .
\end{aligned}
$$

By (3.22) and (1.3), we see

$$
\begin{aligned}
&(1+r)^{-1-\delta} - \frac{2|b(x)|}{\delta} - \frac{\varphi_0 |b(x)|}{\delta}(1+r)^{\delta} - \varepsilon (1+r)^{-1-\delta} \\
&\ge \left(1 - \frac{2b_0}{\delta} - \frac{\varphi_0 b_0}{\delta} - \varepsilon \right) (1+r)^{-1-\delta} > 0 ,
\end{aligned} \tag{3.27}
$$

$$
\begin{aligned}
&\varphi_0 (1+r)^{-1+\delta} - \frac{\varphi_0 |b(x)|}{\delta}(1+r)^{\delta} - (1+r)^{-1-\delta} - \varepsilon (1+r)^{-1+\delta} \\
&\ge \left(\varphi_0 - \frac{\varphi_0 b_0}{\delta} - 1 - \varepsilon \right) (1+r)^{-1+\delta} > 0
\end{aligned} \tag{3.28}
$$

and

$$
(1+r)^{-1-\delta} - \frac{\varphi_0 |b(x)| (1+r)^{\delta}}{\delta} \ge \left(1 - \frac{\varphi_0 b_0}{\delta} \right) (1+r)^{-1-\delta} > 0 . \tag{3.29}
$$

Therefore by (3.26), (3.27), (3.28) and (3.29), we obtain (3.18) and (3.19). □

PROOF OF PROPOSITON 2.6. It suffices to show that $\sigma_p(L(\kappa)) \cap (0, \infty) = \emptyset$ with $b(x) \geq 0$. To do so, we have only to show the inequality (3.18) for $\operatorname{Im} \kappa = 0$ under the assumption in Proposition 2.6.

Similary with the proof of Propositon 3.2 and 3.4, we can show the following inequalities:

$$\begin{aligned}&\int_{\mathbf{R}^N} (1+r)^{-1-\delta} \left(\left| u_r + \frac{N-1}{2r} u \right|^2 + |\kappa|^2 |u|^2 \right) dx \\ &\leqq \int_{\mathbf{R}^N} (1+r)^{-1-\delta} |\mathcal{D}_r^+ u|^2 dx + \frac{2}{\delta} \int_{\mathbf{R}^N} |f \overline{i\kappa u}| dx ,\end{aligned} \tag{3.30}$$

$$\begin{aligned}&\int_{\mathbf{R}^N} (1+r)^{-1+\delta} |\mathcal{D}^+ u|^2 dx - \frac{1}{\delta} \int_{\mathbf{R}^N} b(x)(1+r)^\delta \left| u_r + \frac{N-1}{2r} u \right|^2 dx \\ &\leq \frac{2}{\delta} \int_{\mathbf{R}^N} (1+r)^\delta |f \overline{\mathcal{D}_r^+ u}| dx ,\end{aligned} \tag{3.31}$$

for $\operatorname{Im} \kappa = 0$. Since $b_0 < \delta$, there exists a constant φ_0 such that $1 < \varphi_0 < \delta / b_0$. Then

$$1 - \frac{b_0 \varphi_0}{\delta} > 0 . \tag{3.32}$$

Next since $\varphi_0 - 1 > 0$, there exists a constnat ε such that

$$0 < \varepsilon < \min\{\varphi_0 - 1, 1\} . \tag{3.33}$$

Adding (3.30) to φ_0-times (3.31) and using (3.24) and (3.25), we obtain

$$\begin{aligned}&(1-\varepsilon)|\kappa|^2 \int_{\mathbf{R}^N} (1+r)^{-1-\delta} |u|^2 dx + (\varphi_0 - 1 - \varepsilon) \int_{\mathbf{R}^N} (1+r)^{-1+\delta} |\mathcal{D}^+ u|^2 dx \\ &+ \int_{\mathbf{R}^N} \left\{ (1+r)^{-1-\delta} - \frac{\varphi_0 b(x)(1+r)^\delta}{\delta} \right\} \left| u_r + \frac{N-1}{2r} u \right|^2 dx \\ &\leq \frac{1+\varphi_0^2}{\varepsilon \delta^2} \int_{\mathbf{R}^N} (1+r)^{1+\delta} |f|^2 dx .\end{aligned} \tag{3.34}$$

By the condition $0 \leq b(x) \leq b_0 (1+r)^{-1-2\delta}$ and (3.32), we have

$$(1+r)^{-1-\delta} - \frac{\varphi_0 b(x)(1+r)^\delta}{\delta} \geq \left(1 - \frac{\varphi_0 b_0}{\delta} \right) (1+r)^{-1-\delta} > 0 . \tag{3.35}$$

Therefore, noting (3.34) and (3.35), we obtain (3.18) for $\operatorname{Im} \kappa = 0$. □

From Theorem 3.5, we have

COROLLARY 3.6. *Assume that $N \neq 2$ and* (1.3).

(1) *There exists a unique solution u of* (3.1) *in $L^{2,-(1+\delta)/2}(\mathbf{R}^N)$ for* $\operatorname{Im} \kappa \gtreqless 0$.

(2) *There exists a positive constant C_3 independent of κ such that the following inequality holds*:

$$|\kappa|^2 \|u\|^2_{-(1+\delta)/2, E_R} \leq C_3 (1+R)^{-\delta} \|f\|^2_{(1+\delta)/2} \quad (\operatorname{Im} \kappa \gtreqless 0) , \tag{3.36}$$

where $E_R = \{x \in \mathbf{R}^N \mid |x| > R\}$ *for* $R > 0$.

PROOF. (1) is a direct consequence of Theorem 3.5 (3.18). Therefore, we shall show the assertion (2).

It follows from (3.7) that

$$|\kappa|^2 \int_{S_\rho} |u|^2 dS - |\kappa|^2 \int_{\mathbf{R}^N} |b(x)|\,|u|^2 dx \le \int_{S_\rho} |\mathcal{D}^\pm u|^2 dS + 2\int_{\mathbf{R}^N} |f\overline{i\kappa u}|dx\,.$$

Multiplying by $(1+\rho)^{-1-\delta}$ on the both sides of this and integrating on $\rho \in [R, +\infty)$, we have

$$\begin{aligned}(3.37)\qquad |\kappa|^2 \int_{|x|>R} (1+r)^{-1-\delta}|u|^2 dx &\le \frac{(1+R)^{-\delta}}{\delta}|\kappa|^2 \int_{\mathbf{R}^N} |b(x)|\,|u|^2 dx \\ &+ \int_{|x|>R} (1+r)^{-1-\delta}|\mathcal{D}_r^\pm u|^2 dx + \frac{2}{\delta}(1+R)^{-\delta}\int_{\mathbf{R}^N} |f\overline{i\kappa u}|dx\,.\end{aligned}$$

Note that the following inequalities:

$$\begin{aligned}(3.38)\qquad \int_{|x|>R} (1+r)^{-1-\delta}|\mathcal{D}_r^\pm u|^2 dx &\le (1+R)^{-2\delta}\int_{\mathbf{R}^N} (1+r)^{-1+\delta}|\mathcal{D}^\pm u|^2 dx \\ &\le C_2^2 (1+R)^{-\delta}\int_{\mathbf{R}^N} (1+r)^{1+\delta}|f|^2 dx\,,\end{aligned}$$

$$(3.39)\qquad \int_{\mathbf{R}^N} |f\overline{i\kappa u}|dx \le C_1 \int_{\mathbf{R}^N} (1+r)^{1+\delta}|f|^2 dx\,,$$

where we have used (3.19) and (3.18) in (3.38) and (3.39), respectively. Moreover it follows from (1.3) and (3.18) that

$$(3.40)\qquad |\kappa|^2 \int_{\mathbf{R}^N} |b(x)|\,|u|^2 dx \le C_1^2 b_0 \int_{\mathbf{R}^N} (1+r)^{1+\delta}|f|^2 dx\,.$$

Using (3.38), (3.39) and (3.40) in (3.37), we obtain (3.36). □

From Theorem 3.5 and Corollary 3.6 (1), we can show the invertibility of $I - Q(\kappa)$ with $\mathrm{Im}\,\kappa \gtreqless 0$.

THEOREM 3.7. *Assume that* $N \ne 2$ *and* (2.13). *If* $\varepsilon = \mathrm{Im}\,\kappa = 0$, *we define* $Q(\sigma \pm i0) = \lim_{\varepsilon\to 0} Q(\sigma \pm i\varepsilon)$, *where* lim *means the limit in the uniform operator topology. Then for* $\mathrm{Im}\,\kappa \gtreqless 0$, $[I - Q(\kappa)]^{-1} \in \mathfrak{L}(L^2(\mathbf{R}^N))$ *holds, where in general* $\lim_{\varepsilon\to 0} Q(\sigma + i\varepsilon) \ne \lim_{\varepsilon\to 0} Q(\sigma - i\varepsilon)$.

PROOF. Firstly, we shall show that $I - Q(\kappa) \in \mathfrak{L}(L^2(\mathbf{R}^N))$. For any $f \in L^2(\mathbf{R}^N)$, we see $Af \in L^{2,(1+\delta)/2}(\mathbf{R}^N)$. Then we find from (3.18) of Theorem 3.5 that $R(\kappa^2)Af \in L^{2,-(1+\delta)/2}(\mathbf{R}^N)$ holds. Since it follows from this and the resolvent estimate (cf. Mochizuki [14] or Theorem 3.5 with $b(x) \equiv 0$) that $\|Q(\kappa)f\| \le C\|f\|$ for $\mathrm{Im}\,\kappa \gtreqless 0$ (where C is some positive constant independent of κ), we obtain the conclusion.

Next we shall show that $I - Q(\kappa)$ is injection. If $[1 - Q(\kappa)]f = 0$ for any $f \in L^2(\mathbf{R}^N)$, then we see $Af - i\kappa A^2 R_0(\kappa^2)Af = 0$ ($Af \in L^{2,(1+\delta)/2}(\mathbf{R}^N)$). Operating $R_0(\kappa^2)$ from the left, we have $g - i\kappa R_0(\kappa^2)A^2 g = 0$ ($g \equiv R_0(\kappa^2)Af \in L^{2,-(1+\delta)/2}(\mathbf{R}^N)$). Operating $H_0 - \kappa^2$ from the left, we obtain $L(\kappa)g = 0$. By Corollary 3.6 (1), $g = 0$, therefore, $f = 0$ follows.

Finally, we shall show that $I - Q(\kappa)$ is surjection. To do so, it suffices to show that $L^2(\mathbf{R}^N) \subseteq \mathfrak{R}(I - Q(\kappa))$, i.e., for any $f \in L^2(\mathbf{R}^N)$, there exists $g \in L^2(\mathbf{R}^N)$ such that $f = [I - Q(\kappa)]g$ holds. Since $f \in L^2(\mathbf{R}^N)$, we see $Af \in L^{2,(1+\delta)/2}(\mathbf{R}^N)$. Consider the equation: $L(\kappa)\varphi = Af$. Then, there exists unique $\varphi \in L^{2,-(1+\delta)/2}(\mathbf{R}^N)$ by Corollary 3.6 (1). Therefore, we find $A\varphi \in L^2(\mathbf{R}^N)$. Now we define g by $g = i\kappa A\varphi + f$. Then we have $g \in L^2(\mathbf{R}^N)$. Moreover the equation above is equivalent to $(H_0 - \kappa^2)\varphi = Ag$. Since $g \in L^2(\mathbf{R}^N)$, $Ag \in L^{2,(1+\delta)/2}(\mathbf{R}^N)$ holds. Therefore, we find $\varphi = R_0(\kappa^2)Ag$, that is, $i\kappa A\varphi = Q(\kappa)g$. From this and the definition of g, we obtain $[I - Q(\kappa)]g = f$.

Thus, the conclusion follows from the inverse mapping principle. □

Before closing this section, we derive an elliptic estimate for (3.1) required in the proof of the limiting absorption principle.

PROPOSITION 3.8. *Let $\kappa = \sigma + i\tau \in \mathbf{C} \setminus \{0\}$ with $\sigma \in I = (a_1, a_2)$ $(0 < a_1 < a_2 < \infty)$ or $I = (-a_2, -a_1)$, $|\tau| \le \tau_0 < \infty$ for some positive τ_0 and let G, G' are bounded domains such that $G \Subset G' \subset \mathbf{R}^N$ where $G \Subset G'$ means $\bar{G}$* (closure of G) $\subset G'$. *Then there exists a positive constant $C = C(I, \tau_0, G')$ such that*

$$\int_G |\nabla u|^2 dx \le C \int_{G'} (|u|^2 + |f|^2) dx\,. \tag{3.41}$$

PROOF. We may assume $0 \in G$ without loss of generality. Multyplying by $\psi \bar{u}$ on the both sides of (3.1) where $\psi = \psi(r) \ge 0$, we have

$$\begin{aligned} &\psi|\nabla u|^2 - \nabla\cdot(\psi\nabla u\bar{u}) + \nabla\cdot\left(\frac{\psi_r}{2}|u|^2\frac{x}{r}\right) - \frac{N-1}{2r}\psi_r|u|^2 \\ &-\frac{\psi_{rr}}{2}|u|^2 - i\kappa b(x)\psi|u|^2 - \kappa^2\psi|u|^2 = f\psi\bar{u}\,. \end{aligned} \tag{3.42}$$

Choose $\psi \in C_0^\infty([0,\infty))$ such that

$$\psi(r) = \begin{cases} 1 & \text{on } G \\ 0 & \text{on } \mathbf{R}^N \setminus G' \end{cases} \tag{3.43}$$

and integrate (3.42) by parts on $B(R)$, where $B(R)$ is large ball in $\mathbf{R}^N$ such that $G \subset G' \subset B(R)$ for sufficiently large R (> 0). Then

$$\begin{aligned} &\int_G |\nabla u|^2 dx - \int_{S_R} \psi u_r \bar{u}\, dS + \int_{S_R} \frac{\psi_r}{2}|u|^2 dS - \int_{G'} \frac{N-1}{2r}\psi_r|u|^2 dx \\ &- \int_{G'} \frac{\psi_{rr}}{2}|u|^2 dx - i\kappa\int_G b(x)|u|^2 dx - \kappa^2\int_G |u|^2 dx = \int_G f\bar{u}\, dx\,. \end{aligned} \tag{3.44}$$

The second and third terms of the left-hand side of (3.44) are 0 by (3.43). Then taking the real part of (3.44), we obtain

$$\int_G |\nabla u|^2 dx - \int_{G'} \frac{N-1}{2r}\psi_r |u|^2 dx - \int_{G'} \frac{\psi_{rr}}{2}|u|^2 dx + \tau \int_G b(x)|u|^2 dx - (\sigma^2 - \tau^2)\int_G |u|^2 dx = \mathrm{Re}\int_G f\bar{u}dx\,.$$

It follows from this that

(3.45)

$$\int_G |\nabla u|^2 dx \le \int_{G'} \left(\frac{N-1}{2r}|\psi_r| + \frac{|\psi_{rr}|}{2}\right)|u|^2 dx + \int_G |f\bar{u}|dx + (\sigma^2 + \tau_0 b_0)\int_G |u|^2 dx\,.$$

Noting that $\psi_r = \psi_{rr} = 0$ on G, we find

$$\begin{aligned}\int_{G'} \left(\frac{N-1}{2r}|\psi_r| + \frac{|\psi_{rr}|}{2}\right)|u|^2 dx &= \int_{G'\setminus G} \left(\frac{N-1}{2r}|\psi_r| + \frac{|\psi_{rr}|}{2}\right)|u|^2 dx \\ &\le \left(\frac{N-1}{2r_0}\max_{G'\setminus G}|\psi_r| + \frac{1}{2}\max_{G'\setminus G}|\psi_{rr}|\right)\int_{G'}|u|^2 dx\,,\end{aligned} \tag{3.46}$$

where $r_0 = \max_{x\in G}|x|$. Moreover, by the Schwarz inequality

$$\int_G |f\bar{u}|dx \le \frac{1}{2}\int_{G'}(|u|^2 + |f|^2)dx\,. \tag{3.47}$$

Then (3.45), (3.46) and (3.47) give the desired inequality. □

4. Proof of Main Theorem.

We shall abbreviate $u(\sigma,\tau) = R((\sigma + i\tau)^2)f$ by $u(\tau)$. Let $\{\tau_n\}$ be any sequence such that $\tau_n \downarrow 0$. It follows from (3.18) of Theorem 3.5 that

$$\|u(\tau_n)\|_{-(1+\delta)/2} \le \frac{C_1}{|\sigma|}\|f\|_{(1+\delta)/2}\,. \tag{4.1}$$

Firstly, we shall show that among $\{u(\tau_n)\}$ we can choose a strong convergent subsequence $\{u(\tau_{n_k})\}$ in $L^{2,-(1+\delta)/2}(\mathbf{R}^N)$.

If we note Corollary 3.6 (2), for given any $\varepsilon > 0$, we can choose $R > 0$ to satisfy

$$\|u(\tau_n)\|^2_{-(1+\delta)/2,E_R} \le \frac{\varepsilon}{4}\,.$$

In the following we fix such an R. For this R, we see

$$\|u(\tau_n) - u(\tau_m)\|_{-(1+\delta)/2} \le \|u(\tau_n) - u(\tau_m)\|_{B_R} + \frac{\varepsilon}{2}\,. \tag{4.2}$$

Since $\{u(\tau_n)\}$ is a bounded in $L^2(B_R)$ by (4.1), it follows from Propositon 3.8 and the Rellich theorem that $\{u(\tau_n)\}$ becomes compact in $L^2(B_R)$, namely there exists subsequence $\{u(\tau_{n_k})\} \subseteq \{u(\tau_n)\}$ such that $\{u(\tau_{n_k})\}$ becomes Cauchy sequence in $L^2(B_R)$. This and (4.2) give the conclusion.

Next, when $\{\tau_n\}$, $\{\tau'_n\}$ are sequences which converge to 0 and if

$$u(\tau_n) \underset{S}{\rightarrow} u_0,\ u(\tau'_n) \underset{S}{\rightarrow} u'_0, \quad \text{in } L^{2,-(1+\delta)/2}(\mathbf{R}^N) \quad \text{as } n \to \infty \tag{4.3}$$

then $u_0 = u'_0$ holds.

Let us denote $\kappa_n = \sigma + i\tau_n$ and $\kappa'_n = \sigma + i\tau'_n$, then $u(\tau_n)$ and $u(\tau'_n)$ satisfy (3.1) with $\kappa = \kappa_n$ and $\kappa = \kappa'_n$ respectively. If we put $v_n = u(\tau_n) - u(\tau'_n)$, then v_n satisfies

$$L(\kappa_n)v_n = ib(x)(\kappa_n - \kappa'_n)u(\kappa'_n) + (\kappa_n{}^2 - \kappa'^2_n)u(\kappa'_n)\,. \tag{4.4}$$

Since $\kappa_n - \kappa'_n$ and $\kappa_n{}^2 - \kappa'^2_n \to 0$ as $n \to \infty$, we find the right-hand side of (4.4) goes to 0 as $n \to \infty$. Thus we see

$$L(\sigma)(u_0 - u'_0) = 0\,.$$

Noting (3.18) of Theorem 3.5, we obtain the conclusion.

Finally, we shall show that the convergence of $u(\sigma, \tau)$ to $u(\sigma)$ in $L^{2,-(1+\delta)/2}$ is uniform with respect to $\sigma \in I$.

If it is not uniform, there exist a positive number μ, $\sigma^{(n)} \in I$ and $\{\tau_n\}$ ($\tau_n \downarrow 0$) such that

$$\|u(\sigma^{(n)}, \tau_n) - u(\sigma^{(n)})\|_{-(1+\delta)/2} \geq \mu \tag{4.5}$$

holds. Without loss of generality we may assume that $\{\sigma^{(n)}\}$ converge $\sigma \in \bar{I}$. On the other hand, since $u(\sigma^{(n)}, \tau) \underset{S}{\rightarrow} u(\sigma^{(n)})$ in $L^{2,-(1+\delta)/2}(\mathbf{R}^N)$ as $\tau \downarrow 0$, there exists $\{\tau'_n\}$ ($\tau'_n \downarrow 0$) such that

$$\|u(\sigma^{(n)}, \tau'_n) - u(\sigma^{(n)})\|_{-(1+\delta)/2} \leq \frac{\mu}{2^n} \tag{4.6}$$

for any $n \in \mathbf{N}$. Since we have $\sigma^{(n)} + i\tau_n \to \sigma$ and $\sigma^{(n)} + i\tau'_n \to \sigma$ as $n \to \infty$, both $\{u(\sigma^{(n)}, \tau_n)\}$ and $\{u(\sigma^{(n)}, \tau'_n)\}$ converge to the same limit $u(\sigma)$. Moreover we also obtain that $u(\sigma^{(n)}) \underset{S}{\rightarrow} u(\sigma)$ in $L^{2,-(1+\delta)/2}(\mathbf{R}^N)$ by (4.6). Therefore $u(\sigma^{(n)}, \tau_n) \underset{S}{\rightarrow} u(\sigma)$ in $L^{2,-(1+\delta)/2}(\mathbf{R}^N)$, and it is a contradiction to (4.5).

This completes the proof of Theorem 4.1 □

5. The principle of limiting absorption for $i \begin{pmatrix} 0 & 1 \\ \Delta & -B \end{pmatrix}$.

First of all, note that (1.3) with initial data $w(x, 0) = w_1$, $w_t(x, 0) = w_2$ is equivalent to

$$i\begin{pmatrix} w \\ w_t \end{pmatrix}_t - i\begin{pmatrix} 0 & 1 \\ \Delta & -B \end{pmatrix}\begin{pmatrix} w \\ w_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} w(0) \\ w_t(0) \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \tag{5.1}$$

We define the Hilbert space $\mathcal{H} \equiv \dot{H}^1(\mathbf{R}^N) \times L^2(\mathbf{R}^N)$ ($\dot{H}^1(\mathbf{R}^N)$ being the Beppo-Levi space) with inner product

$$(\tilde{f}, \tilde{g})_{\mathcal{H}} = \int_{\mathbf{R}^N} (\nabla f_1 \cdot \overline{\nabla g_1} + f_2 \cdot \overline{g_2})dx$$

for $\tilde{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, $\tilde{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \mathcal{H}$.

If $b(x) \geq 0$ then the solution of (5.1) is given by

$$\begin{pmatrix} w \\ w_t \end{pmatrix} = e^{-iWt} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

where $W = i\begin{pmatrix} 0 & 1 \\ \Delta & -B \end{pmatrix}$ is the generator of the contraction semi-group $\{e^{-iWt}\}_{t\geq 0}$ on $\mathcal{H}$ (cf. Mochizuki [14] p. 384, or Kadowaki [6]).

The limiting absorption principle for W can be stated as follows:

THEOREM 5.1. *Assume that $N \neq 2$, (1.2) and $\tilde{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{H}_1 \equiv \dot{H}^1(\mathbf{R}^N) \cap L^{2,(1+\delta)/2}(\mathbf{R}^N) \times L^{2,(1+\delta)/2}(\mathbf{R}^N)$ $(\subset \mathcal{H})$. If $\tilde{u} = \tilde{u}_{\sigma\pm i\tau} = \begin{pmatrix} u_{1,\sigma\pm i\tau} \\ u_{2,\sigma\pm i\tau} \end{pmatrix}$ is a solution of the equation*:

$$W\tilde{u} - (\sigma \pm i\tau)\tilde{u} = \tilde{f} \tag{5.2}$$

$(\sigma \in \mathbf{R} \setminus \{0\}, \tau > 0)$, *then there exists the limit*

$$\lim_{\tau\downarrow 0} \tilde{u}_{\sigma\pm i\tau} = \widetilde{u_\pm}$$

in $\mathcal{H}_2 \equiv H^{1,-(1+\delta)/2}(\mathbf{R}^N) \times L^{2,-(1+\delta)/2}(\mathbf{R}^N)$ $(\supset \mathcal{H})$ and $\widetilde{u_\pm}$ solves the equation

$$W\widetilde{u_\pm} - \sigma\widetilde{u_\pm} = \tilde{f},$$

where

$$H^{1,-(1+\delta)/2}(\mathbf{R}^N) = \left\{ f(x) \,\middle|\, \int_{\mathbf{R}^N} (1+r)^{-1-\delta}(|\nabla f(x)|^2 + |f(x)|^2)dx < \infty \right\}.$$

PROOF. Put $\kappa = \sigma \pm i\tau$. From (5.2), we obtain

$$L(\kappa)u_{1,\sigma\pm i\tau} = F(\kappa), \tag{5.3}$$

$$u_{2,\sigma\pm i\tau} = -i\kappa u_{1,\sigma\pm i\tau} - if_1, \tag{5.4}$$

where $F(\kappa) = (ib(x) + \kappa)f_1 - f_2$. Note that $F(\kappa) \in L^{2,(1+\delta)/2}$. Then by the Main Theorem, $u_{1,\sigma\pm i\tau}$ converge $u_{1,\pm}$ in $L^{2,-(1+\delta)/2}$ as $\tau \downarrow 0$ and $u_{1,\pm}$ is the solution of the equation:

$$L(\sigma)u_{1,\pm} = F(\sigma). \tag{5.5}$$

Thus we find that $v \equiv u_{1,\sigma\pm i\tau} - u_{1,\pm}$ is the solution of the equation:

$$L(\kappa)v = G(x;\tau), \tag{5.6}$$

where

$$G(x;\tau) = \mp\tau b(x)u_{1,\pm} + (\pm 2\sigma\tau i - \tau^2)u_{1,\pm} \mp i\tau f_1. \tag{5.7}$$

Multiplying by $\phi\bar{v}$ on the both sides of (5.6) $(\phi = \phi(r) = (1+r)^{-1-\delta})$, we see

$$-\nabla\cdot(\phi\nabla v\bar{v}) + \phi_r v_r \bar{v} + \phi|\nabla v|^2 - i\kappa b(x)\phi|v|^2 - \kappa^2\phi|v|^2 = G(x;\tau)\phi\bar{v}.$$

Integration on $\mathbf{R}^N$ by parts gives

$$\int_{\mathbf{R}^N} \phi|\nabla v|^2 dx \le \int_{\mathbf{R}^N} (|\phi_r v_r \bar{v}| + |\kappa|\,|b(x)|\phi|v|^2 + |\kappa|^2\phi|v|^2 + \phi|G(x;\tau)\bar{v}|)dx\,. \tag{5.8}$$

We choose ε as $0 < \varepsilon < 1$. Noting $|\phi_r| \le (1+\delta)(1+r)^{-2-\delta}$ and (1.2), we obtain

$$\int_{\mathbf{R}^N} |\phi_r v_r \bar{v}|dx \le \varepsilon \int_{\mathbf{R}^N} (1+r)^{-1-\delta}|\nabla v|^2 dx + \frac{(1+\delta)^2}{4\varepsilon}\int_{\mathbf{R}^N}(1+r)^{-1-\delta}|v|^2 dx\,, \tag{5.9}$$

$$|\kappa|\int_{\mathbf{R}^N} |b(x)|\phi|v|^2 dx \le b_0|\kappa|\int_{\mathbf{R}^N}(1+r)^{-1-\delta}|v|^2 dx\,, \tag{5.10}$$

(5.11)

$$\int_{\mathbf{R}^N} \phi|G(x;\tau)\bar{v}|dx \le \left(\int_{\mathbf{R}^N}(1+r)^{-1-\delta}|G(x;\tau)|^2 dx\right)^{1/2}\left(\int_{\mathbf{R}^N}(1+r)^{-1-\delta}|v|^2 dx\right)^{1/2}.$$

Using (5.9), (5.10) and (5.11) in (5.8), we find

$$\begin{aligned}(1-\varepsilon)&\int_{\mathbf{R}^N}(1+r)^{-1-\delta}|\nabla v|^2 dx\\ &\le \left(\frac{(1+\delta)^2}{4\varepsilon} + b_0|\kappa| + |\kappa|^2\right)\int_{\mathbf{R}^N}(1+r)^{-1-\delta}|v|^2 dx\\ &\quad + \left(\int_{\mathbf{R}^N}(1+r)^{-1-\delta}|G(x;\tau)|^2 dx\right)^{1/2}\left(\int_{\mathbf{R}^N}(1+r)^{-1-\delta}|v|^2 dx\right)^{1/2}.\end{aligned} \tag{5.12}$$

By the Main Theorem, we see

$$\int_{\mathbf{R}^N}(1+r)^{-1-\delta}|v|^2 dx \to 0 \quad (\tau \downarrow 0)\,.$$

Thus it follows from (5.12) that

$$\|\nabla u_{1,\sigma\pm i\tau} - \nabla u_{1,\pm}\|_{-(1+\delta)/2} \to 0 \quad (\tau \downarrow 0)\,.$$

On the other hand, as for $u_{2,\sigma\pm i\tau}$,

$$\lim_{\tau\downarrow 0} u_{2,\sigma\pm i\tau} = -i\sigma u_{1,\pm} - if_1 \equiv u_{2,\pm} \quad \text{in } L^{2,-(1+\delta)/2}\,. \quad \Box$$

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