TOKYO J. MATH.
VOL. 23, NO. 2, 2000

Homotopically Energy-Minimizing Harmonic Maps of Tori into $\mathbf{RP}^3$

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(Communicated by M. Oka)

Abstract. We determine homotopically energy-minimizing harmonic maps of tori into the 3-dimensional real projective space $\mathbf{RP}^3$ of constant sectional curvature 1.

Introduction.

Let N be a compact Riemannian manifold with $\pi_2(N) = 0$ and ϕ a continuous map of a Riemann surface M into N. Then Sacks and Uhlenbeck [S-U] proved that there exists a homotopically energy-minimizing harmonic map in the homotopy class of ϕ. The energy and the number of the energy-minimizing harmonic maps are not however explicit.

In this paper, in the case $M = T^2$ (a flat torus), $N = \mathbf{RP}^3$, we determine the energy and the number of the energy-minimizing harmonic maps $T^2 \to \mathbf{RP}^3$.

A flat torus is represented by $\mathbf{R}^2/[1, z]$, where $1, z$ are lattice vectors such that $\mathrm{Im}\, z > 0$, that is, $z \in H$ (the upper half plane). Let $\langle 1\rangle$, $\langle z\rangle$ denote the generator of $\pi_1(\mathbf{R}^2/[1, z])$ represented by $1, z$. Since $\pi_1(\mathbf{RP}^3)$ is $\mathbf{Z}_2(= \{0, 1\})$, there exist k, l such that

$$\phi(\langle 1\rangle) = k, \quad \phi(\langle z\rangle) = l,$$

where $k, l = 0$ or 1. So the homotopy set of maps of the torus into $\mathbf{RP}^3$ are classified according to

$$(k, l) = (0, 0), \quad (1, 0), \quad (0, 1), \quad (1, 1).$$

If $(k, l) = (0, 0)$, then ϕ is null-homotopic and hence the harmonic maps corresponding to ϕ are constant maps. If $(k, l) = (1, 0)$, then $\mathbf{R}^2/[1, -1/z]$ is homothetic $\mathbf{R}^2/[1, z]$ and the map $\tilde{\phi}$ of $\mathbf{R}^2/[1, -1/z]$ into $\mathbf{RP}^3$ corresponding to ϕ satisfies

$$\tilde{\phi}(\langle 1\rangle) = 0, \quad \tilde{\phi}\left(\left\langle -\frac{1}{z}\right\rangle\right) = 1.$$

Received July 12, 1999

If $(k, l) = (1, 1)$, then we have the homothety of $\mathbf{R}^2/[1, z]$ onto $\mathbf{R}^2/[1, z/(z+1)]$ and corresponding to $\tilde{\phi}$ again satisfies

$$\tilde{\phi}(\langle 1\rangle) = 0, \quad \tilde{\phi}\left(\left\langle \frac{z}{z+1}\right\rangle\right) = 1 .$$

Thus it is enough to consider the case where $(k, l) = (0, 1)$ in the homotopy set and hence determine homotopically energy-minimizing harmonic maps φ of $\mathbf{R}^2/[1, z]$ into $\mathbf{RP}^3$ such that $\varphi(\langle 1\rangle)$ is null-homotopic and $\varphi(\langle z\rangle)$ is not null-homotopic in $\mathbf{RP}^3$.

So, in this paper, we assume that homotopically enegy-minimizing harmonic maps are in the homotopy class corresponding to $(k, l) = (0, 1)$.

Let $SL(2, \mathbf{Z})$ be the modular group acting on H and Γ' the subgroup defined by

$$\begin{pmatrix} l & k \\ n & m \end{pmatrix}$$

with l odd and n even (so m is odd). Then Ω defined by

$$\left\{z \in H : \left|z - \frac{1}{2}\right| \geq \frac{1}{2}, \quad 0 \leq \operatorname{Re} z \leq 1\right\}$$

is a fundamental domain of Γ'. Furthermore we denote by Υ

$$\left\{z \in H : \left|z - \frac{1}{2}\right| = \frac{1}{2}\right\} .$$

We obtain the following on the number of homotopically energy-minimizing harmonic maps of a flat torus into $\mathbf{RP}^3$:

THEOREM A. (i) *The number of homotopically energy-minimizing harmonic maps* φ *of* $\mathbf{R}^2/[1, z]$ *for* $z \in H$ *and* $z \notin \Gamma'\Upsilon$ *such that* $\varphi(\langle 1\rangle)$ *is null-homotopic and* $\varphi(\langle z\rangle)$ *is not null-homotopic in* $\mathbf{RP}^3$ *is one up to isometries of* $\mathbf{RP}^3$ *and the image is a geodesic.* (ii) *The number of homotopically energy-minimizing harmonic maps* φ *of* $\mathbf{R}^2/[1, z]$ *for* $z \in \Gamma'\Upsilon$ *such that* $\varphi(\langle 1\rangle)$ *is null-homotopic and* $\varphi(\langle z\rangle)$ *is not mull-homotopic in* $\mathbf{RP}^3$ *is infinity up to isometries of* $\mathbf{RP}^3$*. More precisely, two of these have geodesics as their images and the others are a one parameter family of homotopically energy-minimizing harmonic maps with all Clifford tori (in* $\mathbf{RP}^3$*) as images. Furthermore, the limits of the one parameter family are the above two maps (whose images are geodesics).*

Note that the space of homotopically energy-minimizing harmonic maps of a flat torus into $\mathbf{RP}^3$ is path-connected. Mukai [M] has studied a one parameter family of harmonic maps of the square torus into $S^3(1)$ whose images are Clifford tori in $S^3(1)$ and has determined the Jacobi fields and their integrability and hence a connected component containing the above harmonic maps in the moduli of harmonic maps of the square torus into $S^3(1)$.

Let $E(z)$ denote the energy $E(\varphi)$ of a homotopically energy-minimizing harmonic map φ of $\mathbf{R}^2/[1, z]$ into $\mathbf{RP}^3$. Then $E(z)$ is a function on H and has the following property:

THEOREM B. (i) $E(z) = \pi^2/(2 \operatorname{Im} z)$ *for* $z \in \Omega$. (ii) $E(z)$ *is invariant by* Γ' *and is not smooth on* $\Gamma'\Upsilon$.

1. The Clifford minimal surface.

Let $\mathbf{R}^4$ be the 4-dimensional Euclidean space and (X, Y, Z, W) a canonical coordinate system of $\mathbf{R}^4$. Let $S^3(1)$ be the 3-dimensional unit sphere with center at the origin in $\mathbf{R}^4$ and P the stereographic projection of $S^3(1)\backslash(0,0,0,1)$ onto the (X, Y, Z)-plane. We denote by (x, y, z) the image of $(X, Y, Z, W) \in S^3(1)\backslash(0,0,0,1)$, so that

$$x = \frac{X}{1-W}, \quad y = \frac{Y}{1-W}, \quad z = \frac{Z}{1-W}.$$

Let ϕ be the Clifford minimal embedding of the torus $S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2})$ into $S^3(1)$ given by

$$\phi(s,t) = \left(\frac{1}{\sqrt{2}}\cos\sqrt{2}s, \frac{1}{\sqrt{2}}\sin\sqrt{2}s, \frac{1}{\sqrt{2}}\cos\sqrt{2}t, \frac{1}{\sqrt{2}}\sin\sqrt{2}t\right).$$

Then we get an embedding

$$P\phi(s,t) = \left(\frac{\cos\sqrt{2}s}{\sqrt{2}-\sin\sqrt{2}t}, \frac{\sin\sqrt{2}s}{\sqrt{2}-\sin\sqrt{2}t}, \frac{\cos\sqrt{2}t}{\sqrt{2}-\sin\sqrt{2}t}\right),$$

for which the following is well known [S-T]:

LEMMA 1. *$P\phi$ is an embedding of $S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2})$ into the (x, y, z)-plane and the image is a surface of revolution about the z-axis of a circle of center $(\sqrt{2}, 0)$ and radius 1 in the (x, z)-plane.*

Since the Clifford minimal torus $S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2})$ is invariant under the antipodal map of $S^3(1)$, it admits and isometry. Using lattice vectos $(\sqrt{2}\pi, 0)$, $(0, \sqrt{2}\pi)$, we can identify $\mathbf{R}^2/[(\sqrt{2}\pi, 0), (0, \sqrt{2}\pi)]$ with $S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2})$, and the above isometry is given by

$$[s,t] \mapsto \left[s + \frac{1}{\sqrt{2}}\pi, t + \frac{1}{\sqrt{2}}\pi\right].$$

We now have a map ψ of a torus $\mathbf{R}^2/[e, f]$ with the lattice generated by

$$e = (\sqrt{2}\pi, 0), \quad f = \left(\frac{1}{\sqrt{2}}\pi, \frac{1}{\sqrt{2}}\pi\right)$$

into $\mathbf{RP}^3$. We shall also call this the Clifford minimal surface.

Now we can study the homotopy class and the energy of ψ as follows:

LEMMA 2. *The curve $P\phi(s, 0)$ is a circle of center $(0, 0)$ and radius $\pi/\sqrt{2}$ in the plane $z = 1/\sqrt{2}$, and so $\psi(\langle e\rangle)$ is null-homotopic in $\mathbf{RP}^3$. The curve $P\phi(t,t)$ is a circle of center $(0, 1)$ and radius $\sqrt{2}$ in the $(\tilde{x}, \tilde{y})$-plane defined by an orthonormal basis $\{1/\sqrt{2}(1,0,1), (0,1,0)\}$, and so $\psi(\langle f\rangle)$ is the generator of $\pi_1(\mathbf{RP}^3)$. Furthermore $\psi(\langle f\rangle)$ is a geodesic in $\mathbf{RP}^3$, and the energy of ψ equals π^2.*

Since ψ is an isometric minimal embedding, ψ is a harmonic map. Similarly, using the Clifford embedding of $S^1(1/r_1) \times S^1(1/r_2)$ into $S^3(1)$, that is,

$$(s,t) \mapsto \left(\frac{1}{r_1}\cos r_1 s, \frac{1}{r_1}\sin r_1 s, \frac{1}{r_2}\cos r_2 t, \frac{1}{r_2}\sin r_2 t\right),$$

where $(1/r_1)^2 + (1/r_2)^2 = 1$, we also obtain an isometric embedding $\widetilde{\psi_{r_1}}$ of $\mathbf{R}^2/[(2\pi/r_1, 0), (\pi/r_1, \pi/r_2)]$ into $\mathbf{RP}^3$. Note that $\widetilde{\psi_{r_1}}$ is not a harmonic map except when $r_1 = \sqrt{2}$, because it is not minimal except when $r_1 = \sqrt{2}$ (the Clifford minimal surface). Changing the flat metric of $\mathbf{R}^2/[(2\pi/r_1, 0), (\pi/r_1, \pi/r_2)]$, we shall find the flat metric such that $\widetilde{\psi_{r_1}}$ is harmonic.

We consider flat metrics on $\mathbf{R}^2/[(2\pi/r_1, 0), (\pi/r_1, \pi/r_2)]$ defined by

$$\alpha ds^2 + 2\beta ds dt + \gamma dt^2,$$

where $\alpha > 0$ and $\alpha\gamma - \beta^2 > 0$. Since the harmonicity is conformally invariant, we may assume $\alpha = 1$. Our problem is as follows:

PROBLEM. *When is $\widetilde{\psi_{r_1}}$ harmonic with respect to the above flat metric given by β and γ?*

We define a diffeomorphism $T_{a,b}$ of the torus $\mathbf{R}^2/[(2\pi/r_1, 0), (a, b)]$ $(b > 0)$ onto $\mathbf{R}^2/[(2\pi/r_1, 0), (\pi/r_1, \pi/r_2)]$ by

$$T_{a,b}(\tilde{s}, \tilde{t}) = \left(\tilde{s} + \frac{1}{b}\left(\frac{\pi}{r_1} - a\right)\tilde{t}, \frac{\pi}{br_2}\tilde{t}\right).$$

Then the flat metric on $\mathbf{R}^2/[(2\pi/r_1, 0), (a, b)]$ $(b > 0)$ induces the flat metric on $\mathbf{R}^2/[(2\pi/r_1, 0), (\pi/r_1, \pi/r_2)]$ by $T_{a,b}$, which is given by

$$ds^2 + 2\left(-\left(\frac{\pi}{r_1} - a\right)\frac{r_2}{\pi}\right)ds dt + \left(\left(\left(\frac{\pi}{r_1} - a\right)\frac{r_2}{\pi}\right)^2 + \left(\frac{br_2}{\pi}\right)^2\right)dt^2.$$

When $a = \pi/r_1 + (\pi/r_2)\beta$ and $b = (\pi/r_2)\sqrt{\gamma - \beta^2}$, the induced metric is $ds^2 + 2\beta ds dt + \gamma dt^2$. Thus the problem is reduced to studying whether $\psi_{r_1,a,b} = \psi_{r_1} T_{a,b}$ for a and $b > 0$ is harmonic.

$\psi_{r_1,a,b}$ is given by

$$\psi_{r_1,a,b}(\tilde{s}, \tilde{t}) = \left[\frac{1}{r_1}\cos r_1\left(\tilde{s} + \frac{1}{b}\left(\frac{\pi}{r_1} - a\right)\tilde{t}\right), \frac{1}{r_1}\sin r_1\left(\tilde{s} + \frac{1}{b}\left(\frac{\pi}{r_1} - a\right)\tilde{t}\right),\right.$$
$$\left.\frac{1}{r_2}\cos r_2\frac{\pi}{br_2}\tilde{t}, \frac{1}{r_2}\sin r_2\frac{\pi}{br_2}\tilde{t}\right].$$

By a simple calculation, we obtain the following:

PROPOSITION 3. *$\psi_{r_1,a,b}$ is a harmonic map if and only if*

$$\left(a - \frac{\pi}{r_1}\right)^2 + b^2 = \left(\frac{\pi}{r_1}\right)^2.$$

Then $E(\psi_{r_1,a,b})$ is given by $(1/2)(\pi^2/b)(2\pi/r_1)$.

We set $z = r_1 a/2\pi + i r_1 b/2\pi$, then $z \in \Upsilon$ and $\mathbf{R}^2/[1, z]$ is homotheitc to $\mathbf{R}^2/[(2\pi/r_1, 0), (a, b)]$ $(b > 0)$ and hence we can define a one parameter family of hamonic maps ψ_{z,r_1} of $\mathbf{R}^2/[1, z]$ with $E(\psi_{z,r_1}) = \pi^2/(2\operatorname{Im} z)$ into $\mathbf{RP}^3$ by $\psi_{r_1,a,b}$ as follows:

$$\psi_{z,r_1}(s,t) = \left[\frac{1}{r_1}\cos 2\pi\left(s + \frac{1-2\operatorname{Re} z}{2\operatorname{Im} z}t\right), \frac{1}{r_1}\sin 2\pi\left(s + \frac{1-2\operatorname{Re} z}{2\operatorname{Im} z}t\right),\right.$$
$$\left.\frac{1}{r_2}\cos 2\pi\left(\frac{1}{2\operatorname{Im} z}t\right), \frac{1}{r_2}\sin 2\pi\left(\frac{1}{2\operatorname{Im} z}t\right)\right],$$

where $1 < r_1$. Note that $\psi_{z,r_1}(\langle 1\rangle)$ is null-homotopic and $\psi_{z,r_1}(\langle z\rangle)$ is not null-homotopic.

Now we can answer our problem.

COROLLARY 4. *The* ψ_{z,r_1} $(r_1 > 1)$ *are precisely the harmonic maps which we seek. The conformal structures are given by* $1, z$ $(z \in \Upsilon)$, *and* $E(\psi_{z,r_1}) = \pi^2/(2\operatorname{Im} z)$.

We may consider that $\langle z\rangle, \langle z\rangle - \langle 1\rangle$ also express closed geodesics for the homology cycles. Since $\psi_{z,r_1}(\langle z\rangle)$ and $\psi_{z,r_1}(\langle z\rangle - \langle 1\rangle)$ are geodesics in $\mathbf{RP}^3$, we note that only $\langle z\rangle$ and $\langle z\rangle - \langle 1\rangle$ are asymptotic curves on a Clifford surface. This fact is used in Section 2.

REMARK. We refer to [D] on the terminology (asymptotic curve, second fundamental form, etc.) of the geometry of submanifolds.

REMARK. ψ_{z,r_1} induces a harmonic map of $\mathbf{R}^2/[1, 2z]$ into $S^3(1)$, which has a constant energy density. Harmonic maps with constant energy density into spheres were studied by Tóth [T].

2. An energy estimate.

We shall obtain an energy inequality.

We consider lattice vectors 1 and z, where $0 \le \operatorname{Re} z \le 1$ and define two diffeomorphisms F and $\tilde{F}$ of a torus $\mathbf{R}^2/[(1,0), (0, \operatorname{Im} z)]$ onto $\mathbf{R}^2/[1, z]$ by

$$F(u,v) = \left(u + \frac{\operatorname{Re} z}{\operatorname{Im} z}v, v\right), \quad \tilde{F}(u,v) = \left(u - \frac{1-\operatorname{Re} z}{\operatorname{Im} z}v, v\right).$$

Then

$$F_*\frac{\partial}{\partial u} = \frac{\partial}{\partial s}, \quad F_*\frac{\partial}{\partial v} = \frac{\operatorname{Re} z}{\operatorname{Im} z}\frac{\partial}{\partial s} + \frac{\partial}{\partial t}$$

and the Riemannian metric g_{ij} induced by F is as follows:

$$g_{11} = 1, \quad g_{12} = \frac{\operatorname{Re} z}{\operatorname{Im} z}, \quad g_{22} = 1 + \left(\frac{\operatorname{Re} z}{\operatorname{Im} z}\right)^2$$

and F is hence an area element preserving map. Similarly so is $\tilde{F}$.

Let φ be a C^1-map of $\mathbf{R}^2/[1, z]$ into $\mathbf{RP}^3$ such that $\varphi(\langle 1\rangle)$ is null-homotopic and $\varphi(\langle z\rangle)$ is not null-homotopic. Then, since the curve $\varphi F(u, v)$, whre u is fixed, is not null-homotopic in $\mathbf{RP}^3$, the length is greater than or equal to π. Namely,

$$\pi \leq \int_0^{\operatorname{Im} z} \left|\frac{\partial \varphi F}{\partial v}\right| dv$$

holds. So, Schwarz's inequality yields

$$\pi^2 \leq (\operatorname{Im} z) \int_0^{\operatorname{Im} z} \left|\frac{\partial \varphi F}{\partial v}\right|^2 dv\,,$$

which implies

$$\int_0^1 \frac{\pi^2}{\operatorname{Im} z} du \leq \int_0^1 \int_0^{\operatorname{Im} z} \left|\frac{\partial \varphi F}{\partial v}\right|^2 dudv\,.$$

Since F is an area element preserving map,

$$\frac{\pi^2}{\operatorname{Im} z} \leq \iint_{\mathbf{R}^2/[1,z]} \left| \left(\frac{\operatorname{Re} z}{\operatorname{Im} z}\right) \frac{\partial \varphi}{\partial s} + \frac{\partial \varphi}{\partial t}\right|^2 dsdt\,.$$

Namely,

$$\frac{\pi^2}{\operatorname{Im} z} \leq \iint_{\mathbf{R}^2/[1,z]} \left(\left(\frac{\operatorname{Re} z}{\operatorname{Im} z}\right)^2 \left|\frac{\partial \varphi}{\partial s}\right|^2 + \frac{2\operatorname{Re} z}{\operatorname{Im} z} \left\langle \frac{\partial \varphi}{\partial s}, \frac{\partial \varphi}{\partial t}\right\rangle + \left|\frac{\partial \varphi}{\partial t}\right|^2 \right) dsdt\,. \tag{2.1}$$

The equality holds if and only if

$$\left|\frac{\operatorname{Re} z}{\operatorname{Im} z}\frac{\partial \varphi}{\partial s} + \frac{\partial \varphi}{\partial t}\right| = \frac{\pi}{\operatorname{Im} z}\,. \tag{2.2}$$

Using $\tilde{F}$, we obtain the following similar to (2.1):

(2.3)

$$\frac{\pi^2}{\operatorname{Im} z} \leq \iint_{\mathbf{R}^2/[1,z]} \left(\left(\frac{(1-\operatorname{Re} z)}{\operatorname{Im} z}\right)^2 \left|\frac{\partial \varphi}{\partial s}\right|^2 - \frac{2(1-\operatorname{Re} z)}{\operatorname{Im} z} \left\langle \frac{\partial \varphi}{\partial s}, \frac{\partial \varphi}{\partial t}\right\rangle + \left|\frac{\partial \varphi}{\partial t}\right|^2 \right) dsdt\,.$$

The equality holds if and only if

$$\left| -\frac{1-\operatorname{Re} z}{\operatorname{Im} z}\frac{\partial \varphi}{\partial s} + \frac{\partial \varphi}{\partial t}\right| = \frac{\pi}{\operatorname{Im} z}\,. \tag{2.4}$$

Summing up (2.1), (2.3), we obtain an inequality on $E(\varphi)$:

$$\frac{1}{2}\left(\frac{1}{\operatorname{Re} z} + \frac{1}{(1-\operatorname{Re} z)}\right)\pi^2 \leq \max\left\{\frac{1}{\operatorname{Im} z}, \frac{\operatorname{Im} z}{\operatorname{Re} z} + \frac{\operatorname{Im} z}{(1-\operatorname{Re} z)}\right\} \times E(\varphi) \quad (\operatorname{Re} z \neq 0, 1)\,,$$

$$\frac{\pi^2}{2\operatorname{Im} z} \leq E(\varphi) \quad (\operatorname{Re} z = 0, 1)\,.$$

Consequently, we obtain the following energy estimate:

PROPOSITION 5. $\displaystyle \frac{1}{2}\frac{\pi^2}{\max\left(\frac{\operatorname{Re} z(1-\operatorname{Re} z)}{\operatorname{Im} z}, \operatorname{Im} z\right)} \leq E(\varphi)\,.$

In particular, if $|z - 1/2| \geq 1/2$ *and* $0 \leq \operatorname{Re} z \leq 1$, *then* $E(\varphi)$ *is greater than or equal to* $\pi^2/(2\operatorname{Im} z)$.

Proposition 5, together with Corollary 4, implies

COROLLARY 6. *The* ψ_{z,r_1} $(r_1 > 1)$ *are homotopically energy-minimizing harmonic maps.*

Let S^1 be a geodesic with length π of $\mathbf{RP}^3$, which is a one dimensional torus $\mathbf{R}/[\pi]$. We define a map of $\mathbf{R}^2/[1, z]$ into a geodesic $\mathbf{R}/[\pi] \subset \mathbf{RP}^3$ by

$$(s, t) \mapsto \left[\frac{\pi}{\operatorname{Im} z} t\right].$$

Then the energy is equal to $\pi^2/(2 \operatorname{Im} z)$. It follows from Proposition 5 that this map is a homotopically energy-minimizing harmonic map if $|z - 1/2| \geq 1/2$ and $0 \leq \operatorname{Re} z \leq 1$. We shall investigate the stability of a harmonic map of a torus into a geodesic in $\mathbf{RP}^3$ in Section 3.

We shall determine homotopically energy-minimizing harmonic maps of $\mathbf{R}^2/[1, z]$ $(z \in \Omega)$ into $\mathbf{RP}^3$ whose image is not a geodesic in $\mathbf{RP}^3$.

If φ satisfies the equality in Proposition 5 for

$$\left|z - \frac{1}{2}\right| > \frac{1}{2},$$

then the differentiation in the direction of s vanishes, that is, φ is a harmonic map into a geodesic in $\mathbf{RP}^3$. Using the classification (Lemma 9 in Section 3) of harmonic maps on $\mathbf{R}^2/[1, z]$ into $\mathbf{R}/[\pi]$, we find that φ is

$$(s, t) \mapsto \left[\pm\frac{\pi}{\operatorname{Im} z} t\right].$$

Note that $(s, t) \mapsto [-(\pi/\operatorname{Im} z)t]$ is congruent to $(s, t) \mapsto [(\pi/\operatorname{Im} z)t]$. Thus we obtain the following:

COROLLARY 7. *The only homotopically energy-minimizing harmonic maps of* $\mathbf{R}^2/[1, z]$ *with* $z \in \Omega$ *and* $z \notin \Upsilon$ *into* $\mathbf{RP}^3$ *is*

$$(s, t) \mapsto \left[\frac{\pi}{\operatorname{Im} z} t\right].$$

Next we consider the case where $z \in \Upsilon$, that is, $|z - 1/2| = 1/2$. (2.2) and (2.4) imply that

$$\left\langle\frac{\partial\varphi}{\partial s}, \frac{\partial\varphi}{\partial s}\right\rangle + \left\langle\frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial t}\right\rangle = \frac{\pi^2}{\operatorname{Re} z(1 - \operatorname{Re} z)} \quad (\operatorname{Re} z \neq 0, 1). \tag{2.5}$$

$$\left\langle\frac{\partial\varphi}{\partial s}, \frac{\partial\varphi}{\partial s}\right\rangle = 0, \quad \left\langle\frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial t}\right\rangle = \frac{\pi^2}{(\operatorname{Im} z)^2} \quad (\operatorname{Re} z = 0, 1). \tag{2.6}$$

On the other hand, since ψ is a harmonic map, the quadratic differential

$$\left\langle\frac{\partial\varphi}{\partial z}, \frac{\partial\varphi}{\partial z}\right\rangle dz^2$$

is holomorphic and hence is of the form ηdz^2, where $z = s + it$ and η is a constant. This implies that

$$\left\langle \frac{\partial\varphi}{\partial s}, \frac{\partial\varphi}{\partial s} \right\rangle - \left\langle \frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial t} \right\rangle = 4\,\mathrm{Re}\,\eta\,, \quad \left\langle \frac{\partial\varphi}{\partial s}, \frac{\partial\varphi}{\partial t} \right\rangle = -2\,\mathrm{Im}\,\eta\,. \tag{2.7}$$

(2.5) and (2.7) state that

$$\left\langle \frac{\partial\varphi}{\partial s}, \frac{\partial\varphi}{\partial s} \right\rangle, \quad \left\langle \frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial t} \right\rangle$$

are also constants and so is the rank of φ.

We have two possibilities, according to whether the rank of φ is one or two.

If the rank of φ is one, then φ is again a map into a geodesic. We shall determine energy-minimizing harmonic maps of $\mathbf{R}^2/[1, z]$ ($z \in \Upsilon$) into a geodesic $\mathbf{R}/[\pi] \subset \mathbf{RP}^3$ (Corollary 10 in Section 3).

Assume tht the rank of φ is two. Then φ is a flat immersion of $\mathbf{R}^2/[1, z]$ into $\mathbf{RP}^3$ and hence φ defines a surface in $\mathbf{RP}^3$. We denote by σ the second fundamental form of the surface. Using the harmonicity of φ, we obtain

$$\sigma\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) + \sigma\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = 0\,. \tag{2.8}$$

Let e_1, e_2 be an orthonormal parallel fields with respect to the metric induced by φ. Then there exist constants a, b, c, d such that

$$\frac{\partial}{\partial s} = ae_1 + be_2\,, \quad \frac{\partial}{\partial t} = ce_1 + de_2\,,$$

which, together with (2.8), imply

$$(a^2 + c^2)\sigma_{11} + 2(ab + cd)\sigma_{12} + (b^2 + d^2)\sigma_{22} = 0\,, \tag{2.9}$$

where $\sigma_{11} = \sigma(e_1, e_1)$, etc.

$$\sigma_{11}\sigma_{22} - (\sigma_{12})^2 = -1 \tag{2.10}$$

is the Gauss equation of the flat immersion of φ. The Codazzi equation of φ is given by

$$\sigma_{12,1} - \sigma_{11,2} = \sigma_{21,2} - \sigma_{22,1} = 0\,, \tag{2.11}$$

where $\sigma_{11,2}$ means $(\nabla_{e_1}\sigma)(e_1, e_2)$. Note that $(\nabla_X\sigma)(Y, Z)$ is defined by

$$(\nabla_X\sigma)(Y, Z) = X\sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)\,.$$

Covariantly differentiating (2.9), (2.10) by e_1, e_2, and using (2.11), we obtain the following homogeneous linear equations in $\sigma_{11,1}, \sigma_{11,2}, \sigma_{22,1}, \sigma_{22,2}$:

$$\sigma_{22}\sigma_{11,1} - 2\sigma_{12}\sigma_{11,2} + \sigma_{11}\sigma_{22,1} = 0\,,$$

$$\sigma_{22}\sigma_{11,2} - 2\sigma_{12}\sigma_{22,1} + \sigma_{11}\sigma_{22,2} = 0\,,$$

$$(a^2 + c^2)\sigma_{11,1} + 2(ab + cd)\sigma_{11,2} + (b^2 + d^2)\sigma_{22,1} = 0\,,$$

$$(a^2 + c^2)\sigma_{11,2} + 2(ab + cd)\sigma_{22,1} + (b^2 + d^2)\sigma_{22,2} = 0\,.$$

It follows from (2.9), (2.10) that the determinant of the coefficient matrix of the equations is

$$4((a^2+c^2)(b^2+d^2)-(ab+cd)^2)$$

and hence is positive by the assumption of the rank of φ. Thus φ has a parallel second fundamental form, that is, $\nabla\sigma = 0$.

In the geometry of submanifolds, submanifolds with parallel second fundamental form in space forms have been classified (see, for example, [F]). In particular, Lawson [L] proved that compact surfaces with parallel second fundamental forms in $S^3(1)$ are a totally geodesic surface, Clifford tori and their covering spaces up to isometries of $S^3(1)$. Since φ has a parallel second fundamental form, the image of φ must be a totally geodesic surface $\mathbf{RP}^2$ or a Clifford surface $\mathbf{R}^2/[(2\pi/r_1, 0), (\pi/r_1, \pi/r_2)]$ in $\mathbf{RP}^3$. Since φ is a flat immersion, φ induces a covering map of $\mathbf{R}^2/[1, z]$ onto $\mathbf{RP}^2$ or $\mathbf{R}^2/[(2\pi/r_1, 0), (\pi/r_1, \pi/r_2)]$. We obtin only a covering map θ of $\mathbf{R}^2/[1, z]$ onto $\mathbf{R}^2/[(2\pi/r_1, 0), (\pi/r_1, \pi/r_2)]$ such that $\varphi = \widetilde{\psi_{r_1}}\theta$. Indeed, there does not exist a covering map of a torus onto $\mathbf{RP}^2$. Moreover, θ is a flat immersion. Changing the flat metric on $\mathbf{R}^2/[(2\pi/r_1, 0), (\pi/r_1, \pi/r_2)]$ as in Section 1, we may consider that θ is homothetic. So $\widetilde{\psi_{r_1}}$ is harmonic with respect to the new flat metric and hence, there exists $z' \in \Upsilon$ such that $\varphi = \psi_{z', r_1}\theta$.

We shall prove the injectivity of θ as follows:
There exist integers a, b, c, d such that

$$\theta(\langle 1\rangle) = a\langle 1\rangle + b\langle z'\rangle\,, \quad \theta(\langle z\rangle) = c\langle 1\rangle + d\langle z'\rangle\,.$$

Since $\varphi(\langle z\rangle)$ and $\varphi(\langle z\rangle - \langle 1\rangle)$ are geodesics with length π in $\mathbf{RP}^3$,

$$\theta(\langle z\rangle) = c\langle 1\rangle + d\langle z'\rangle\,, \quad \theta(\langle z\rangle - \langle 1\rangle) = (c-a)\langle 1\rangle + (d-b)\langle z'\rangle$$

are bijectively mapped geodesics with length π in $\mathbf{RP}^3$ by ψ_{z', r_1} and hence are asymptotic curves. As only $\langle z'\rangle$, $\langle z'\rangle - \langle 1\rangle$ express two asymptotic curves through each point (see Section 1), there exist four possibilities:

$$c\langle 1\rangle + d\langle z'\rangle = \pm\langle z'\rangle\,, \quad (c-a)\langle 1\rangle + (d-b)\langle z'\rangle = \pm(\langle z'\rangle - \langle 1\rangle)\,; \tag{2.12}$$

$$c\langle 1\rangle + d\langle z'\rangle = \pm(\langle z'\rangle - \langle 1\rangle)\,, \quad (c-a)\langle 1\rangle + (d-b)\langle z'\rangle = \mp\langle z'\rangle\,; \tag{2.13}$$

$$c\langle 1\rangle + d\langle z'\rangle = \pm\langle z'\rangle\,, \quad (c-a)\langle 1\rangle + (d-b)\langle z'\rangle = \mp(\langle z'\rangle - \langle 1\rangle)\,; \tag{2.14}$$

$$c\langle 1\rangle + d\langle z'\rangle = \pm(\langle z'\rangle - \langle 1\rangle)\,, \quad (c-a)\langle 1\rangle + (d-b)\langle z'\rangle = \pm\langle z'\rangle\,; \tag{2.15}$$

which yield

$$(a, b, c, d) = (\pm 1, 0, 0, \pm 1)\,, \quad (\mp 1, \pm 2, \mp 1, \pm 1)\,, \quad (\mp 1, \pm 2, 0, \pm 1)\,, \quad (\mp 1, 0, \mp 1, \pm 1)\,.$$

Thus θ is injective and orientation-preserving for (2.12) and (2.13), orientation-reversing for (2.14) and (2.15).

We shall determine $\psi_{z', r_1}\theta$ for the above cases.

For (2.12), we may consider that θ is the identity.

For (2.13), we may consider $z = (z' - 1)/(2z' - 1)$ and hence

$$\theta(s, t) = ((2\operatorname{Re} z' - 1)s - (2\operatorname{Im} z')t, (2\operatorname{Im} z')s + (2\operatorname{Re} z' - 1)t).$$

Then $\psi_{z',r_1}\theta$ is given by

$$\left[\frac{1}{r_1}\cos 2\pi\left(-\frac{1}{2\operatorname{Im} z}t\right), \frac{1}{r_1}\sin 2\pi\left(-\frac{1}{2\operatorname{Im} z}t\right),\right.$$
$$\left.\frac{1}{r_2}\cos 2\pi\left(s + \frac{1-2\operatorname{Re} z}{2\operatorname{Im} z}t\right), \frac{1}{r_2}\sin 2\pi\left(s + \frac{1-2\operatorname{Re} z}{2\operatorname{Im} z}t\right)\right],$$

which is congruent to ψ_{z,r_2}.

For (2.14), we may consider that $z = z'$ and

$$\theta(s, t) = ((2\operatorname{Re} z' - 1)s + (2\operatorname{Im} z')t, (2\operatorname{Im} z')s - (2\operatorname{Re} z' - 1)t).$$

Then $\psi_{z',r_1}\theta$ is congruent to ψ_{z,r_2}.

For (2.15), we may consider that $z = 1 - \overline{z'}$ and

$$\theta(s, t) = (s, -t).$$

Then $\psi_{z',r_1}\theta$ is congruent to ψ_{z,r_1}.

By Corollary 6, we obtain the following:

PROPOSITION 8. *The ψ_{z,r_1} $(r_1 > 1)$ are the only homotopically energy-minimizing harmonic maps of $\mathbf{R}^2/[1, z]$ $(z \in \Upsilon)$ into $\mathbf{RP}^3$ whose images are not geodesics in $\mathbf{RP}^3$.*

3. The stability of harmonic maps of tori into a geodesic in $\mathbf{RP}^3$.

First of all, we shall determine harmonic maps f of $\mathbf{R}^2/[1, z]$ into $\mathbf{R}/[\pi]$.

Since df is a harmonic 1-form,there exist constants α, β such that

$$df = \alpha ds + \beta dt.$$

As df should define a map of $\mathbf{R}^2/[1, z]$ onto $\mathbf{R}/[\pi]$, the periods of df satisfy

$$\alpha, \quad \alpha\operatorname{Re} z + \beta\operatorname{Im} z = 0 \pmod{\pi},$$

that is, $(1/\pi)(\alpha, \beta)$ is a dual lattice vector. Note that the space L^* of dual lattice vectors for $1, z$ is generated by

$$\hat{e} = \left(1, -\frac{\operatorname{Re} z}{\operatorname{Im} z}\right), \quad \hat{f} = \left(0, \frac{1}{\operatorname{Im} z}\right).$$

We obtain the following classification of harmonic maps of $\mathbf{R}^2/[1, z]$ into $\mathbf{R}/[\pi]$:

LEMMA 9. *A harmonic map χ_μ of $\mathbf{R}^2/[1, z]$ into $\mathbf{R}/[\pi]$ is given by*

$$\pi\langle\mu, (s, t)\rangle \pmod{\pi},$$

where $\mu \in L^*$. In particular, $\chi_\mu(\langle 1\rangle)$ is null-homotopic, $\chi_\mu(\langle z\rangle)$ is not null-homotopic in $\mathbf{RP}^3$ if and only if

$$\mu = 2n\hat{e} + (2m + 1)\hat{f}, \tag{3.1}$$

where m and n are integers. The map is given by

$$\pi\left(2ns+\left(\frac{(2m+1)-2n\operatorname{Re}z}{\operatorname{Im}z}\right)t\right) \tag{3.2}$$

with the energy

$$\frac{1}{2}\pi^2\left((2n)^2+\left(\frac{(2m+1)-2n\operatorname{Re}z}{\operatorname{Im}z}\right)^2\right)\times\operatorname{Im}z. \tag{3.3}$$

Now we can determine harmonic maps of $\mathbf{R}^2/[1,z]$ into a geodesic $\mathbf{R}/[\pi]$ with length π in $\mathbf{RP}^3$ with the energy $\pi^2/(2\operatorname{Im}z)$ for $z\in\Upsilon$.

Since m, n such that

$$\frac{1}{2}\pi^2\left((2n)^2+\left(\frac{(2m+1)-2n\operatorname{Re}z}{\operatorname{Im}z}\right)^2\right)\times\operatorname{Im}z=\frac{\pi^2}{2\operatorname{Im}z}$$

are (0,0), $(-1,0)$, (0,1), $(-1,-1)$, we obtain the following:

COROLLARY 10. *For* $z\in\Upsilon$,

$$(s,t)\mapsto\left[\frac{\pi}{\operatorname{Im}z}t\right], \tag{3.4}$$

$$(s,t)\mapsto\left[\pi\left(2s+\frac{1-2\operatorname{Re}z}{\operatorname{Im}z}t\right)\right] \tag{3.5}$$

are the only homotopically energy-minimizing harmonic maps of $\mathbf{R}^2/[1,z]$ *into a geodesic in* $\mathbf{RP}^3$ *up to isometries of* $\mathbf{RP}^3$.

Thus we can determine homotopically energy-minimizing harmonic maps of $\mathbf{R}^2/[1,z]$ into $\mathbf{RP}^3$ for $z\in\Omega$ by Corollaries 7, 10 and Proposition 8. In particular, $\psi_{z,r_1}\mapsto$ (3.4), (3.5) if $r_1\mapsto\infty$, 1, respectively.

In the remaining part of this section, we shall investigate the stability of χ_μ where μ satisfies (3.1), as a harmonic map into $\mathbf{RP}^3$. It is not necessary to prove Theorems A and B, but we shall find that χ_μ is energy-minimizing if it is stable.

Let $\tilde{\Delta}$ be the Laplacian of $\chi_\mu^*T\mathbf{RP}^3$. Then, since $\mathbf{RP}^3$ has constant sectional curvatures 1, the Jacobi operator J of χ_μ is given by

$$\begin{aligned}Ju=&-\tilde{\Delta}u-\left\langle\chi_{\mu*}\frac{\partial}{\partial s},\chi_{\mu*}\frac{\partial}{\partial s}\right\rangle u+\left\langle\chi_{\mu*}\frac{\partial}{\partial s},u\right\rangle\chi_{\mu*}\frac{\partial}{\partial s}\\&-\left\langle\chi_{\mu*}\frac{\partial}{\partial t},\chi_{\mu*}\frac{\partial}{\partial t}\right\rangle u+\left\langle\chi_{\mu*}\frac{\partial}{\partial t},u\right\rangle\chi_{\mu*}\frac{\partial}{\partial t},\end{aligned} \tag{3.6}$$

where u is a section of $\chi_\mu^*T\mathbf{RP}^3$ (see, for example, [E-L]).

Over a geodesic S^1 of $\mathbf{RP}^3$, $T\mathbf{RP}^3$ decomposes as the sum of the tangent bundle TS^1 and the normal bundle NS^1, and NS^1 has the decomposition $N_1S^1+N_2S^1$ by parallel transport. Moreover, TS^1 has a flat connection with trivial holonomy and N_1S^1 and N_2S^1 have flat connections with $\mathbf{Z}_2$ holonomy. Thus $\chi_\mu^*T\mathbf{RP}^3$ decomposes into $\chi_\mu^*TS^1$ and $\chi_\mu^*N_1S^1+\chi_\mu^*N_2S^1$, where $\chi_\mu^* TS^1$ has a trivial holonomy and $\chi_\mu^* N_1 S^1$ and $\chi_\mu^* N_2 S^1$ has a non-trivial holonomy whose representation ρ is given by

$$\rho(\langle 1\rangle) = I\,, \quad \rho(\langle z\rangle) = -I\,.$$

So $\chi_\mu^* N_1 S^1$ and $\chi_\mu^* N_2 S^1$ are the flat bundle E_ρ on $\mathbf{R}^2/[1, z]$ for ρ (see [S1). Let Δ_ρ be the Laplacian of E_ρ and Δ_0 the Laplacian of $\chi_\mu^* TS^1$. Then, using (3.6), we obtain the following:

PROPOSITION 11. *It follows from the decomposition $u = u_0 + u_1 + u_2$, where u_0 is a section of $\chi_\mu^* TS^1$, u_1 is a section of $\chi_\mu^* N_1 S^1$ and u_2 is a section of $\chi_\mu^* N_2 S^1$ that*

$$\begin{aligned} Ju = &- \Delta_0 u_0 - \Delta_\rho u_1 - \pi^2\left((2n)^2 + \left(\frac{(2m+1) - 2n\operatorname{Re} z}{\operatorname{Im} z}\right)^2\right) u_1 \\ &- \Delta_\rho u_2 - \pi^2\left((2n)^2 + \left(\frac{(2m+1) - 2n\operatorname{Re} z}{\operatorname{Im} z}\right)^2\right) u_2\,. \end{aligned}$$

So we should know the eigenvalues of $-\Delta_\rho$ to determine of the stability of χ_μ. Following [S2], we must calculate a dual lattice vector α such that

$$1 = \exp 2\pi i \langle (1, 0), \alpha\rangle\,, \quad -1 = \exp 2\pi i \langle (\operatorname{Re} z, \operatorname{Im} z), \alpha\rangle\,,$$

and hence α is given by

$$\alpha = \left(N, \frac{1 + 2M - 2N\operatorname{Re} z}{2\operatorname{Im} z}\right)\,,$$

where N, M are integers. For example, we can set $N = M = 0$ and hence $\alpha = (0, 1/(2\operatorname{Im} z))$, which implies the following:

PROPOSITION 12 ([S2]). *The eigenvalues of $-\Delta_\rho$ are given by*

$$\left\{4\pi^2\left|\mu + \left(0, \frac{1}{2\operatorname{Im} z}\right)\right|^2 : \mu \in L^*\right\}\,,$$

that is,

$$\pi^2\left((2q)^2 + \left(\frac{(2p+1) - 2q\operatorname{Re} z}{\operatorname{Im} z}\right)^2\right)\,, \tag{3.7}$$

where p and q are integers.

PROOF. Sunada's proof is as follows:

$$\varphi_\mu(s, t) = A^{-1/2}\exp 2\pi i \langle (s, t), \mu\rangle$$

for $\mu \in L^*$, where A is the area of $\mathbf{R}^2/[1, z]$, is an orthonormal basis of $L^2(\mathbf{R}^2/[1, z])$ which satisfies $-\Delta_0\varphi_\mu = 4\pi^2|\mu|^2\varphi_\mu^2$. Hence $\{4\pi^2|\mu|^2 : \mu \in L^*\}$ are eigenvalues of $-\Delta_0$. For $f \in L^2(\mathbf{R}^2/[1, z])$,

$$u(s, t) = \exp 2\pi i \langle (s, t), \alpha\rangle f(s, t)$$

is an element of $L^2(E_\rho)$, because $u((s,t)+\sigma) = \rho(\sigma)u(s,t)$ holds. This correspondence $f \mapsto u$ is isometric and hence

$$s_\mu(s,t) = A^{-1/2}\exp 2\pi i\langle (s,t), \mu+\alpha\rangle$$

for $\mu \in L^*$ is an orthonormal basis on $L^2(E_\rho)$. Since

$$-\Delta_\rho s_\mu = 4\pi^2|\mu+\alpha|^2 s_\mu\,,$$

$\{4\pi^2|\mu+\alpha|^2 : \mu \in L^*\}$ are eigenvalues. Q.E.D.

Thus by Propositions 11, 12, we can determine the satability of χ_μ.

COROLLARY 13. *χ_μ, where $\mu = 2n\hat{e} + (2m+1)\hat{f}$, is stable if and only if m and n satisfy*

$$(2n)^2 + \left(\frac{(2m+1)-2n\operatorname{Re} z}{\operatorname{Im} z}\right)^2 \le (2q)^2 + \left(\frac{(2p+1)-2q\operatorname{Re} z}{\operatorname{Im} z}\right)^2$$

for all integers p and q.

Let m and n be integers which minimize

$$(2n)^2 + \left(\frac{(2m+1)-2n\operatorname{Re} z}{\operatorname{Im} z}\right)^2$$

for a fixed $z \in H$. Then the map χ_μ for m and n is a stable harmonic map and other maps are unstable by Corollary 13. In particular, since

$$2^2 + \left(\frac{1-2\operatorname{Re} z}{\operatorname{Im} z}\right)^2 < \left(\frac{1}{\operatorname{Im} z}\right)^2,$$

we know the following:

COROLLARY 14. $(s,t) \mapsto \left[\dfrac{\pi}{\operatorname{Im} z}t\right]$
is unstable for z such that $|z-1/2| < 1/2$.

Let $\chi_{z,m,n}$ denote the stable map with above m and n for z. Then we obtain the following:

THEOREM 15. *$\chi_{z,m,n}$ is a homotopically energy-minimizing harmonic map.*

PROOF. As m and n are integers which minimize

$$(2n)^2 + \left(\frac{(2m+1)-2n\operatorname{Re} z}{\operatorname{Im} z}\right)^2,$$

$2n$ and $2m+1$ are coprime, and there exist integers p and q such that

$$p(2n) + q(2m+1) = 1\,.$$

Since

$$\begin{pmatrix} q & p \\ -2n & 2m+1 \end{pmatrix} \in SL(2,\mathbf{Z})\,,$$

$$e' = (2m+1)(1,0) - 2n(\operatorname{Re} z, \operatorname{Im} z), \qquad f' = p(1,0) + q(\operatorname{Re} z, \operatorname{Im} z)$$

is a generator of the lattice vectors.

Considering e' and f' as complex numbers, we denote by z' the complex number f'/e'. Then $\chi_{z,m,n}$ may be a stable map of $\mathbf{R}^2/[1, z']$. Since the curve $\chi_{z,m,n}(e')$ is a constant and a curve $\chi_{z,m,n}(f')$ is a geodesic with length π, the map is $\chi_{z',m',n'}$, where $m' = 0$ or -1 and $n' = 0$. If $0 \leq \operatorname{Re} z \leq 1$, then $\chi_{z',m',n'}$ is homotopically energy-minimizing by Corollaries 7, 14 and so is $\chi_{z,m,n}$. If $\operatorname{Re} z < 0$ or $\operatorname{Re} z > 1$, then we obtain $\hat{z}$ such that $0 \leq \operatorname{Re}\hat{z} \leq 1$ and $z' = \hat{z} + l$, where l is an integer, then $\chi_{\hat{z},m',n'} = \chi_{z',m',n'}$. Since $\chi_{\hat{z},m',n'}$ is stable and hence homotopically energy-minimizing as the above, $\chi_{z,m,n}$ is again homotopically energy-minimizing. Q.E.D.

4. An energy function and the proofs of Theorems A, B.

Let $SL(2, \mathbf{Z})$ be the modular group acting on H. Let $\Gamma(2)$ be the principal congruence subgroup of level 2 of $SL(2, \mathbf{Z})$, that is, the set of

$$\begin{pmatrix} l & k \\ n & m \end{pmatrix} \in SL(2, \mathbf{Z})$$

which satisfies

$$\begin{pmatrix} l & k \\ n & m \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\operatorname{mod} 2)\,.$$

Then $\Gamma(2)$ is generated by

$$z \mapsto z + 2, \quad z \mapsto \frac{z}{2z+1}$$

and a fundamental domain is given by

$$\left\{ z \in H : 0 \leq \operatorname{Re} z \leq 2, \quad \left| z - \frac{1}{2} \right| \geq \frac{1}{2}, \quad \left| z - \frac{3}{2} \right| \geq \frac{1}{2} \right\}.$$

Next we define a subgroup Γ' as the set of

$$\begin{pmatrix} l & k \\ n & m \end{pmatrix}$$

of $SL(2, \mathbf{Z})$ with l odd and n even (so m is odd). Then Γ' is a subgroup of $SL(2, \mathbf{Z})$ which contains $\Gamma(2)$ and is generated by

$$z \mapsto z + 1, \quad z \mapsto \frac{z}{2z+1}\,.$$

Moreover Ω is a fundamental domain of Γ'.

We consider the energy $E(\varphi)$ of a homotopically energy-minimizing harmonic map φ of $\mathbf{R}^2/[1, z]$ into $\mathbf{RP}^3$ such that $\varphi(\langle 1 \rangle)$ is null-homotopic and $\varphi(\langle z \rangle)$ is not null-homotopic in $\mathbf{RP}^3$. This gives a function $E(z)$ on H which we call the energy function.

We shall investigate $E(z)$.

For $z \in \Omega$, we have determined homotopically energy-minimizing harmonic maps of $\mathbf{R}^2/[1, z]$ of $\mathbf{RP}^3$ in Section 3. For z in other fundamental domains of Γ', we can determine homotopically energy-minimizing harmonic maps as follows:

For

$$\omega = \begin{pmatrix} l & k \\ n & m \end{pmatrix} \in SL(2, \mathbf{R}),$$

the lattice vectors $nz + m$ and $lz + k$ form 1 and z. Note that $\varphi(\langle nz + m \rangle)$ is null-homotopic and $\varphi(\langle lz + k \rangle)$ is not null-homotopic if and only if $\omega \in \Gamma'$. Then φ is considered as a harmonic map of the parallelogram spanned by $nz + m$ and $lz + k$, which define an energy-minimizing harmonic map φ' of $\mathbf{R}^2/[1, z']$, where $z' = (lz + k)/(nz + m)$. Moreover, $\varphi'(\langle 1 \rangle)$ is null homotopic and $\varphi'(\langle z' \rangle)$ is not null-homotopic. Since there exists ω such that $z' \in \Omega$, we can use our classification of energy-minimizing harmonic maps of $\mathbf{R}^2/[1, z']$ into $\mathbf{RP}^3$. Namely, homotopically energy-minimizing harmonic maps φ of $\mathbf{R}^2/[1, z]$ such that $\varphi(\langle 1 \rangle) = 0$ and $\varphi(\langle z \rangle) \neq 0$ are made from homotopically energy-minimizing harmonic maps φ' of $\mathbf{R}^2/[1, z']$ such that $\varphi'(\langle 1 \rangle) = 0$ and $\varphi'(\langle z' \rangle) \neq 0$ by using ω^{-1}. In particular, if $z' \notin \Upsilon$, then the number of homotopically energy-minimizing harmonic maps of $\mathbf{R}^2/[1, z]$ is one up to isometries of $\mathbf{RP}^3$ and the image is a geodesic, if $z' \in \Upsilon$, then we obtain a one parameter family of homotopically energy-minimizing harmonic maps of $\mathbf{R}^2/[1, z]$ with all Clifford tori as images, whose limits are harmonic maps in geodesics in $\mathbf{RP}^3$.

Thus we obtain Theorems A and B except for the assertion that E is not smooth on $\Gamma\Upsilon$.

Note that, for an interior point z of other fundamental domain Ω' of Γ'

$$\Omega' = \left\{ z \in H : \left| z - \frac{1}{2} \right| \leq \frac{1}{2}, \quad \left| z - \frac{1}{4} \right| \geq \frac{1}{4}, \quad \left| z - \frac{3}{4} \right| \geq \frac{1}{4} \right\},$$

the energy-minimizing harmonic map φ of $\mathbf{R}^2/[1, z]$ into $\mathbf{RP}^3$ such that $\varphi(\langle 1 \rangle)$ is null-homotopic and $\varphi(\langle z \rangle)$ is not null-homotopic is given by (3.5), that is,

$$(s, t) \mapsto \left[\left(\pi(2s + \frac{1 - 2\operatorname{Re} z}{\operatorname{Im} z} t \right) \right]$$

with the energy

$$\frac{\pi^2}{2} \left(4 + \left(\frac{1 - 2\operatorname{Re} z}{\operatorname{Im} z} \right)^2 \right) \operatorname{Im} z .$$

This is suggested by Corollaries 7, 10, 14 and Theorem A. We directly state the reason as follows:

Ω' is the image of Ω by $(z-1)/(2z-1)$ and hence there exists an interior point z' of Ω such that

$$z = \frac{z' - 1}{2z' - 1} .$$

Thus, using the homotopically energy-minimizing harmonic map of $\mathbf{R}^2/[1, z']$ into $\mathbf{RP}^3$:

$$(\tilde{s}, \tilde{t}) \mapsto \left[\frac{\pi}{\operatorname{Im} z'} \tilde{t} \right],$$

$$\tilde{s} = (2\operatorname{Re} z' - 1)s - (2\operatorname{Im} z')t, \quad \tilde{t} = (2\operatorname{Im} z')s + (2\operatorname{Re} z' - 1)t,$$

we obtain the homotopically enegy-minimizing harmonic map of $\mathbf{R}^2/[1, z]$ into $\mathbf{RP}^3$:

$$(s, t) \mapsto \left[\frac{\pi}{\operatorname{Im} z'} ((2\operatorname{Im} z')s + (2\operatorname{Re} z' - 1)t) \right],$$

which is equal to

$$(s, t) \mapsto \left[\pi\left(2s + \frac{1 - 2\operatorname{Re} z}{\operatorname{Im} z} t\right)\right]$$

by

$$\frac{2\operatorname{Re} z' - 1}{\operatorname{Im} z'} = -\frac{2\operatorname{Re} z - 1}{\operatorname{Im} z}.$$

Finally we give the proof of the last part of (ii) in Theorem B: Since $E(z) = \pi^2/(2\operatorname{Im} z)$ for $z \in \Omega$ and

$$E(z) = \frac{\pi^2}{2}\left(4 + \left(\frac{1 - 2\operatorname{Re} z}{\operatorname{Im} z}\right)^2\right)\operatorname{Im} z$$

for $z \in \Omega'$, E is not smooth at each point of Υ.

ACKNOWLEDGEMENT. During the author's stay at Warwick University (1988–1989), Professors J. Eells and M. Micallef asked him whether the Clifford minimal surface in $\mathbf{RP}^3$ is a homotopically energy-minimizing harmonic map. He expresses gratitude to Professors J. Eells and M. Micallef.

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