TOKYO J. MATH.
VOL. 23, NO. 2, 2000

Deformation of Current Lie Algebra with Type Euclidean Group

Faouzi AMMAR

Universite de Sfax

(Communicated by Y. Maeda)

Introduction.

Let E be a vector space of dimension n, we note $\langle\, ,\rangle$ the non degenerate bilinear, symmetric form and the Lie algebra of orthogonal group

$$\theta(E) = \{M \in M_n(K)/\langle MX, Y\rangle + \langle X, MY\rangle = 0; \forall X, Y \in E\}.$$

There is an isomorphism between $\theta(E)$ and $\wedge^2 E$ (see section 2).

We consider $\mathfrak{g} = \theta(E)| \times E$ the semi direct product of $\theta(E)$ by E with Lie group G and $\mathfrak{g}_F$ the current Lie algebra over a smooth manifold M with type G.

We note $H_{\mathrm{loc}}(\mathfrak{g}_F, \mathfrak{g}_F)$ the cohomology of the Chevalley local cochains of the adjoint representation of $\mathfrak{g}_F$.

Our main purpose in this paper is to compute the second group of local cohomology of $\mathfrak{g}_F$ and determine the local deformations associated to it.

1. Some notation and definitions.

Let $m = \dim M$, we denote by $\mathfrak{g}_\infty(m)$ the Lie algebra of formal power series in m indeterminates $t^1, \cdots, t^m$ with coefficients in $\mathfrak{g}$. We denote by $\sum_\alpha t^\alpha x_\alpha$ a generic element of $\mathfrak{g}_\infty(m)$; here $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbf{N}^m$ is a multi-index, t^α is the monomial $(t^1)^{\alpha_1} \cdots (t^m)^{\alpha_m}$, we denote by $|\alpha|$ the weight of α, let $|\alpha| = \sum_{i=1}^m \alpha_i$.

The adjoint representation of $\mathfrak{g}$ induces a representation $(\sum_\alpha t^\alpha x_\alpha) \to ad_{x_o}$.

For multi-indices $\alpha_1, \cdots, \alpha_p$, we define the $(\alpha_1, \cdots, \alpha_p)$ component of a $\mathfrak{g}$-valued p-cochain C on $\mathfrak{g}_\infty(m)$ to be the p-linear map from $\mathfrak{g}^p$ into $\mathfrak{g}$:

$$C_{\alpha_1\cdots\alpha_p}(x_1, \cdots, x_p) = C(t^{\alpha_1}x_1, \cdots, t^{\alpha_p}x_p).$$

The cochain C is said to be homogeneous of weight k if $\sum_{i=1}^p |\alpha_i| \neq k \Rightarrow C_{\alpha_1\cdots\alpha_p} = 0$.

Denote by $\bar{\mathfrak{g}}_\infty(m)$ the ideal of formal power series in m indeterminates, without constant term and with coefficient in $\mathfrak{g}$. The space of homogeneous $\mathfrak{g}$-valued cochains of order k on

Received June 1, 1999
Revised September 23, 1999

$\bar{\mathfrak{g}}_\infty(m)$ is denoted by $\wedge_k(\mathfrak{g}_\infty(m), \mathfrak{g})$. It is a subcomplex of the Chevalley of the representation of $\mathfrak{g}_\infty(m)$ on $\mathfrak{g}$.

Observe that the space of continuous $\mathfrak{g}$-valued cochains on $\mathfrak{g}_\infty(m)$ (with respect to the m-adic topology of $\mathfrak{g}_\infty(m)$) is $\wedge_c(\mathfrak{g}_\infty(m), \mathfrak{g}) = \bigoplus_{k\geq 0} \wedge_k(\mathfrak{g}_\infty(m), \mathfrak{g})$ and that its cohomology simply reads $H_c(\mathfrak{g}_\infty(m), \mathfrak{g}) = \bigoplus_{k\geq 0} H(\mathfrak{g}_\infty(m), \mathfrak{g})_k$.

Recall that a $\mathfrak{g}_F$-valued p-cochain c on $\mathfrak{g}_F$ is local, if

$$\operatorname{supp}[c(s_1, \cdots, s_p)] \subset \bigcap_{i=1}^{i=p} \operatorname{supp} s_i \quad \text{for all } s_1, \cdots, s_p \in \mathfrak{g}_F ,$$

where supp denotes the support.

As easily seen, the space $\wedge_{\text{loc}}(\mathfrak{g}_F, \mathfrak{g}_F)$ of local $\mathfrak{g}_F$-valued cochains on $\mathfrak{g}_F$ is a subcomplex of the Chevalley complex of the adjoint representation of $\mathfrak{g}_F$. We write $H_{\text{loc}}(\mathfrak{g}_F, \mathfrak{g}_F)$ for its cohomology.

P. Lecomte and C. Roger showed that $H_{\text{loc}}(\mathfrak{g}_F, \mathfrak{g}_F)$ may be computed in terms of the cohomology of the continuous cochains of $\mathfrak{g}_\infty(m)$ on $\mathfrak{g}$ [4].

PROPOSITION 1.1 (P. Lecomte and C. Roger [4]). *If the manifold M is parallelizable, then* $H_{\text{loc}}(\mathfrak{g}_F, \mathfrak{g}_F) = C_\infty(M, \mathbf{R}) \otimes H_c(\mathfrak{g}_\infty(m), \mathfrak{g})$. *We consider in this paper the vector space E of dimension* $n > 2$, *the case* $n = 2$ *will be studied* [2].

2. Computing the space $H^*(\mathfrak{g}_F, \mathfrak{g}_F)$.

The isomorphism between $\theta(E)$ and $\wedge^2 E$ is given by

$$\wedge^2 E \to \theta(E)$$

$v \wedge w \to \phi_{v\wedge w}$ for v and w in E, where $\phi_{v\wedge w}$ is defined by

$$\phi_{v\wedge w}(a) = (\langle w, a\rangle v - \langle v, a\rangle w) .$$

This map induces an isomorphism between the Lie algebra $(\theta(E), [\,,\,])$ and $(\wedge^2 E, b)$ where $[\,,\,]$ is the usual braket of $\theta(E)$ and b is defined by

$$b(v \wedge w, \alpha \wedge \beta) = \langle w, \alpha\rangle v \wedge \beta + \langle v, \beta\rangle w \wedge \alpha - \langle v, \alpha\rangle w \wedge \beta - \langle w, \beta\rangle v \wedge \alpha .$$

Let us take some preliminary results.

PROPOSITION 2.1. *Let* $s : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *be a bilinear* $\mathfrak{g}$*-invariant mapping.*

i) *If s is symmetric then* $s = 0$.

ii) *Assume that s is skew-symmetric then*

$$s(e_i, e_j) = a e_i \wedge e_j \quad i, j = 1, \cdots, n;\ a \in \mathbf{R} \text{ for all } e_i, e_j \in E$$

$$s(e_i \wedge e_j, e_k) = [e_i \wedge e_j, e_k] \quad i, j, k = 1, \cdots, n \text{ for all } e_i, e_j, e_k \in E .$$

PROOF. See [4] for the case $\mathfrak{g} = \theta(E)$ for the other case see [6].

PROPOSITION 2.2. $H^1(\mathfrak{g}, Hom(\mathfrak{g}, \mathfrak{g})) = 0$.

PROOF. From the theorem of Hochshild and Serre [3] we have $H^1(\mathfrak{g}, Hom(\mathfrak{g}, \mathfrak{g})) = Inv_{\mathfrak{g}} H^1(E, Hom(\mathfrak{g}, \mathfrak{g})) = H^1(Inv_{\theta(E)} C^1(E, Hom(\mathfrak{g}, \mathfrak{g})))$ since $\theta(E)$ is semi simple Lie algebra, we identify $C^1(E, Hom(\mathfrak{g}, \mathfrak{g}))$ to the space of bilinear maps from $E \times \mathfrak{g}$ to $\mathfrak{g}$. Let $C \in Inv_{\theta(E)} C^1(E, Hom(\mathfrak{g}, \mathfrak{g}))$, from the vanishing of the coboundary of C, one easily sees that $H^1(\mathfrak{g}, Hom(\mathfrak{g}, \mathfrak{g}))$ is identically zero. Hence the result follows.

We study the cohomology $H^*(\mathfrak{g}_\infty(m), \mathfrak{g})$ and we consider the filtration of $\wedge(\mathfrak{g}_\infty(m), \mathfrak{g})$ deduced from the Hochshild-Serre filtration induced on $\wedge(\mathfrak{g}_\infty(m), \mathfrak{g})_k$ by the extension:

$$0 \to \bar{\mathfrak{g}}_\infty(m) \to \mathfrak{g}_\infty(m) \to \mathfrak{g} \to 0\,.$$

The term of the corresponding spectral sequence reads [3] $E_2 = H(\mathfrak{g}, H(\bar{\mathfrak{g}}_\infty(m), \mathfrak{g}))$ and one has following.

PROPOSITION 2.3. $H^2(\mathfrak{g}_\infty(m), \mathfrak{g}) = H^2(\mathfrak{g}, \mathfrak{g}) \oplus S^2\mathbf{R}^m$.

PROOF. The E_2 terms of the spectral sequence can be identified as

$$E_2^{o,2} = H^2(\mathfrak{g}, H_c^o(\bar{\mathfrak{g}}_\infty(m), \mathfrak{g})) = H^2(\mathfrak{g}, \mathfrak{g})$$

and $E_2^{1,1} = H^1(\mathfrak{g}, H_c^1(\bar{\mathfrak{g}}_\infty(m), \mathfrak{g})) = H^1(\mathfrak{g}, Hom(\mathfrak{g}, \mathfrak{g}) \otimes \mathbf{R}^m)$ this space is identically zero, (see Proposition 2.2). $E_2^{o,2} = H^o(\mathfrak{g}, H^2(\bar{\mathfrak{g}}_\infty(m), \mathfrak{g}))_k = 0$ if $k > 2$ see [1].

If $k = 2$ we have $Inv_{\mathfrak{g}} H^2(\bar{\mathfrak{g}}_\infty(m), \mathfrak{g})_2 = H^2(Inv_{\theta(E)} C^2(\bar{\mathfrak{g}}_\infty(m), \mathfrak{g}))$ by semi simplicity again. By Proposition 2.1 the components of a $\mathfrak{g}$-invariant 2-cocycle C on $\bar{\mathfrak{g}}_\infty(m)$ valued in $\mathfrak{g}$ have the form: $C_{\alpha\beta}(e_i, e_j) = a^{\alpha\beta} e_i \wedge e_j$, $1 \le i < j \le n$.

While $a^{\alpha\beta}$ are coefficients of a 2-tensor $(a^{\alpha\beta}) \in \wedge^2\mathbf{R}^m$ one easily shows that the spectral sequence degenerates. Hence the result follows.

Let observe the local cohomology of current Lie algebra. The correspondence between $H_{\text{loc}}(\mathfrak{g}_F, \mathfrak{g}_F)$ and $H_c(\mathfrak{g}_\infty(m), \mathfrak{g})$ can be viewed on the space of cochains as

$$I : C^p(\mathfrak{g}_\infty(m), \mathfrak{g}) \otimes C_\infty(M, \mathbf{R}) \to C^p(\mathfrak{g}_F, \mathfrak{g}_F)\,.$$

$I(s \otimes a)(f_1, \cdots, f_p)(x) = a(x) s(j_\infty f_1(x), \cdots, j_\infty f_p(x))$, $i = 1, \cdots, p$, where $j_\infty f_i(x)$ is the jets at x of f_i.

From the relation between $H_{\text{loc}}(\mathfrak{g}_F, \mathfrak{g}_F)$ and $H_c(\mathfrak{g}_\infty(m), \mathfrak{g})$ we have

THEOREM 2.4. $H^2_{\text{loc}}(\mathfrak{g}_F, \mathfrak{g}_F) = H^2(\mathfrak{g}, \mathfrak{g}) \otimes C^\infty(M) \oplus S^2(M)$.

Let $S^2(M)$ the space of 2-contravariant-symmetric tensor field on M. Let $s \in S^2(M)$. As is easily verified, there is a unique bilinear local skew symmetric map $C_s : \mathfrak{g}_F \times \mathfrak{g}_F \to \mathfrak{g}_F$ such that on each chart $(U, (x^1, \cdots, x^m))$ of M, one has

$$C_s(f, g)(x) = \sum_{\alpha,\beta} s^{\alpha\beta} C(\partial_\alpha f(x), \partial_\beta g(x))\,.$$

3. Construction of deformations.

a) Since $H^2(\mathfrak{g}, \mathfrak{g})$ is included in $H^2_{\text{loc}}(\mathfrak{g}_F, \mathfrak{g}_F)$ the deformations of $\mathfrak{g}$ can be understood as deformation of order 0 of $\mathfrak{g}_F$.

b) Let studied the deformation corresponding to the tensors space S^2M.

Let C_s the cocycle determined by an arbitrary 2-tensor of S^2M. By the special nature of the invariant C (see the proof of the Proposition 2.3), we show that the bracket of Richardson-Nijenhuis [5] of C_s by it self is zero $[|C_s, C_s|] = 0$.

Every 2-cocycle C_s is indeed a formal deformation of order 1 (called true deformation) of the current Lie algebras.

$$[f, g]_t(x) = [f, g](x) + tC_s(f(x), g(x)).$$

ACKNOWLEDGMENTS. I thank Professor Claude Roger for several illuminating discussions.

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Present Address:
DEPARTMENT DE MATHÉMATIQUES, FACULTÉ DES SCIENCES, UNIVERSITE DE SFAX,
3038 SFAX, TUNISIE.