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A Representation of $Spin(4)$ on the Eigenspinors of the Dirac Operator on S^3

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Abstract. We construct the eigenspinors of the Dirac Operator D_3 on S^3 from a representation theoretical point of view and give a representation of *Spin*(4) on them explicitly. These eigenspinors are extended to zero mode spinors of the Dirac operator $D_4^{\pm}$ on upper or lower hemisphere of S^4.

0. Introduction.

In this paper we construct the eigenspinors of the Dirac operator D_3 on the 3-dimensional sphere S^3 from a representation theoretical point of view and show that each eigenspace gives a highest weight representation of *Spin*(4). Moreover, because we represent the eigenspinors by using matrix components of irreducible representations of $SU(2)$, we can calculate the actions of *Spin*(4) and D_3 on the eigenspinors explicitly. When we think of S^3 as the equator of the 4-dimensional sphere S^4, D_3 is the tangential part of the Dirac operator $D_4^{\pm}$ on S^4, where the (total) Dirac operator D_4 splits as $D_4 = \begin{pmatrix} 0 & D_4^- \\ D_4^+ & 0 \end{pmatrix}$. It follows that we extend our eigenspinors of D_3 to zero mode spinors of $D_4^{\pm}$ on upper or lower hemisphere of S^4. This extension is important in (1 + 3)-dimensional quantum field theory (see [11]) and the Dirac boundary value problem (see [4]).

In the case of the eigenvalue problem on S^1, the Dirac operator is $-i(d/d\theta)$ on $L^2(S^1, \mathbf{C})$ and the eigenspinors are $\{e^{im\theta}\}_{m\in\mathbf{Z}}$. Here, each eigenspace gives an irreducible representation of $U(1)$. Furthermore, $L^2(S^1, \mathbf{C})$ splits as the direct sum of the spaces of the positive and the negative spinors, that is, $L^2(S^1, \mathbf{C}) = H_+ \oplus H_-$, where $H_+ = \bigoplus_{m\geq 0} \mathbf{C}(e^{im\theta})$ and $H_- = \bigoplus_{m>0} \mathbf{C}(e^{-im\theta})$. The positive eigenspinors $\{e^{im\theta}\}_{m\geq 0}$ are extended to holomorphic functions (zero mode spinors) $\{z^m\}_{m\geq 0}$ on $\mathbf{C}^1 \subset P^1(\mathbf{C})$. The negative spinors are extended to $\{w^m\}_{m>0}$, where $w = 1/z$. Our results give analogues of these facts for the higher dimensional case.

Section 1 and 2 are preliminary. In section 1 we explain the spin bundle and the spinor bundle over S^n as homogeneous bundles. To decompose the space of spinors as a representation space of $Spin(n + 1)$, we employ Frobenius reciprocity. In particular, we construct the spinor bundle on S^3 and give its trivializations. By Frobenius reciprocity, we obtain an

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irreducible decomposition of the space of spinors on S^3 with respect to *Spin*(4). In section 2, we give a formula of the Dirac operator D_n on S^n as a homogeneous differential operator. Under the trivializations given in section 1, we obtain an explicit formula of D_3 on S^3, which is represented by using right invariant vector fields on $SU(2) = S^3$. In section 3 we solve the eigenvalue problem of the Dirac operator D_3 and show that each eigenspace gives an irreducible representation of *Spin*(4). Moreover, we construct the eigenspinors explicitly and calculate the action of *Spin*(4) on them. As a result we obtain the main theorem 3.5. Some results in section 1–3 are known ([3], [7] and [9]). But we describe them from a new point of view based on representation theory or the theory of homogeneous vector bundles. In the last section we extend our eigenspinors to zero mode spinors of $D_4^{\pm}$ on $\mathbf{R}^4 \subset S^4$. We consider the following embedding of S^3 into S^4 as the equator:

$$S^3 = SU(2) \times SU(2)/\mathrm{diag}\, SU(2) \ni [(p, q)] \mapsto [(p, q)] \in P^1(\mathbf{H}) = S^4 , \tag{0.1}$$

where we use $SU(2) = Sp(1) \subset \mathbf{H}$. This embedding is related to the trivializations of the spin bundle on S^3 given in section 1. We have a local trivialization of the spinor bundle on S^4 and a local formula of the Dirac operator D_4. In this situation we give a polar decomposition of $D_4^{\pm}$ such that its tangential part is D_3. It follows that the positive (resp. negative) spinors of D_3 can be extended to zero mode spinors of D_4^{+} on upper (resp. lower) hemisphere of S^4.

1. Spinor bundles over S^n as homogeneous vector bundles.

Let G be the spin group $Spin(n+1)$. The action of G on $\mathbf{R}^{n+1}$ is defined by $Ad(g)x = g \cdot x \cdot g^{-1}$ for x in $\mathbf{R}^{n+1}$ and g in G, where the multiplications are the Clifford multiplication. Then the orbit of base point $e = (0, \cdots, 0, 1)$ is the n-dimensional sphere S^n in $\mathbf{R}^{n+1}$ and its isotropy subgroup is $Spin(n)$. We denote this subgroup $Spin(n)$ by H. Thus we have the Riemannian symmetric space $S^n = G/H$. We also have the principal H-bundle G on S^n. This principal bundle gives a spin structure for S^n as follows: by the linear isotropy representation $Ad : H \to SO(T_e(S^n))$, we obtain the orthonormal frame bundle of S^n, $G \times_{Ad} SO(n)$. Since the representation Ad gives a double covering of $SO(n)$, the following bundle double covering gives a spin structure for S^n:

$$G \ni g \mapsto [g, id] \in G \times_{Ad} SO(n) . \tag{1.1}$$

Since S^n admits only one spin structure for $n \geq 2$, we denote this principal spin bundle G by $\mathbf{Spin}(S^n)$.

Now, we consider the complex (unitary) spinor representation,

$$\Delta_n : Spin(n) \to U(W_n) . \tag{1.2}$$

The representation space W_n is $2^{[n/2]}$-dimensional complex vector space with an Hermitian metric and a Clifford module structure. The spinor representation Δ_n induces the spinor bundle on S^n as a homogeneous vector bundle, that is,

$$\mathbf{S}(S^n) := G \times_{\Delta_n} W_n . \tag{1.3}$$

For $n = 2m$, the representation (Δ_n, W_n) decomposes as the direct sum of two inequivalent irreducible representations, (Δ_n^+, W_n^+) and (Δ_n^-, W_n^-), with each dimension $2^{[n/2]-1}$. Hence the spinor bundle $\mathbf{S}(S^n)$ also splits as the direct sum of $\mathbf{S}^+(S^n)$ and $\mathbf{S}^-(S^n)$, where $\mathbf{S}^\pm(S^n)$ is $G \times_{\Delta_n^\pm} W_n^\pm$.

The spinor sections are given by the H-equivariant functions from G to W_n,

$$C^\infty(G; \Delta_n) := \{\Psi : G \to W_n \mid \Psi(gh) = \Delta_n(h^{-1})\Psi(g), \text{ for any } h \in H\}. \tag{1.4}$$

If we define an action of G on these spinor sections by $(\Theta(g_0)\Psi)(g) = \Psi(g_0^{-1}g)$ for g_0 in G, then the representation Θ becomes a unitary representation of G on $L^2(G; \Delta_n)$, where $L^2(G; \Delta_n)$ is the L^2-completion of $C^\infty(G; \Delta_n)$. We shall use Frobenius reciprocity to obtain an irreducible decomposition of $(\Theta, L^2(G; \Delta_n))$. So we prepare some objects of representation theory. First, let $\hat{G}$ be the set of all equivalent classes of irreducible unitary representations of G. For γ in $\hat{G}$, (π_γ, V_γ) denotes a representative of γ. Secondly, let $\mathrm{Hom}_H(V_\gamma, W_n)$ be the vector space of all H-module homomorphisms from V_γ to W_n. If A is in $\mathrm{Hom}_H(V_\gamma, W_n)$, then the following diagram is commutative for any h in H:

$$\begin{array}{ccc} V_\gamma & \xrightarrow{A} & W_n \\ {\scriptstyle \pi_\gamma|_H(h)}\downarrow & & \downarrow{\scriptstyle \Delta_n(h)} \\ V_\gamma & \xrightarrow{A} & W_n \end{array} \tag{1.5}$$

where $\pi_\gamma|_H$ is the restriction of π_γ to H. We put $E_\gamma := V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, W_n)$ and define an action Π of G on E_γ by $\Pi(g_0)(v \otimes A) := \pi_\gamma(g_0)v \otimes A$ for $v \otimes A$ in E_γ and g_0 in G. Then the direct sum of E_γ for every γ in $\hat{G}$ gives an irreducible decomposition of $(\Theta, L^2(G; \Delta_n))$.

THEOREM 1.1 (Frobenius reciprocity). *Let $(\Theta, L^2(G; \Delta_n))$ and $(\Pi, \overline{\bigoplus_{\gamma\in\hat{G}} E_\gamma})$ be representations of G as above. Then $(\Theta, L^2(G; \Delta_n))$ is unitarily equivalent to $(\Pi, \overline{\bigoplus_{\gamma\in\hat{G}} E_\gamma})$ by the mapping χ*

$$\chi : \overline{\bigoplus_{\gamma\in\hat{G}} E_\gamma} \ni v \otimes A \mapsto A\pi_\gamma(g^{-1})v \in L^2(G; \Delta_n), \tag{1.6}$$

where $v \otimes A$ is in E_γ. In particular, $\chi(E_\gamma)$ consists of smooth sections and the multiplicity of γ in $L^2(G; \Delta_n)$ coincides with the dimension of $\mathrm{Hom}_H(V_\gamma, W_n)$.

It is easy to see that $A\pi_\gamma(g^{-1})v$ is a H-equivariant function from G to W_n and that $\Theta(g_0)\chi = \chi\Pi(g_0)$. For a proof of this theorem see [13].

Now, we shall investigate the case $n = 3$, where $G = Spin(4)$ and $H = Spin(3)$. It is well-known that *Spin*(4) and *Spin*(3) are isomorphic to $SU(2) \times SU(2)$ and $SU(2)$ respectively. To obtain the action of *Spin*(4) on S^3 explicitly, we realize S^3 as $SU(2)$:

$$S^3 \ni x = (x_1, x_2, x_3, x_4) \mapsto h = \begin{pmatrix} x_4 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_4 - ix_1 \end{pmatrix} \in SU(2), \tag{1.7}$$

where we remark that the base point $e = (0, 0, 0, 1)$ corresponds to the identity matrix I. Then we know that the action of *Spin*(4) can be written by the following matrix multiplication: for

(p, q) in $SU(2) \times SU(2) = Spin(4)$,

$$Spin(4) \times S^3 \ni ((p, q), h) \mapsto phq^{-1} \in S^3 . \tag{1.8}$$

Here, it is clear that the isotropy subgroup $Spin(3)$ is $\operatorname{diag} SU(2)$ in $SU(2) \times SU(2)$, where the map 'diag' is

$$\operatorname{diag} : SU(2) \ni h \mapsto (h, h) \in SU(2) \times SU(2) . \tag{1.9}$$

Thus S^3 is the homogeneous space $(SU(2) \times SU(2))/\operatorname{diag} SU(2)$ and the bundle projection of $\mathbf{Spin}(S^3)$ is given by

$$\mathbf{Spin}(S^3) = Spin(4) \ni (p, q) \mapsto pq^{-1} \in S^3 . \tag{1.10}$$

Since the spinor representation (Δ_3, W_3) is the natural representation of $SU(2)$ denoted by (ρ_1, V_1), the spinor bundle $\mathbf{S}(S^3)$ is $Spin(4) \times_{\rho_1} V_1$.

We can trivialize the bundle $\mathbf{Spin}(S^3)$ and $\mathbf{S}(S^3)$. In fact, we have two natural trivializations,

$$\Phi_+ : \mathbf{Spin}(S^3) = Spin(4) \ni (p, q) \mapsto (pq^{-1}, p) \in S^3 \times Spin(3) , \tag{1.11}$$

$$\Phi_- : \mathbf{Spin}(S^3) = Spin(4) \ni (p, q) \mapsto (pq^{-1}, q) \in S^3 \times Spin(3) . \tag{1.12}$$

The maps Φ_+ and Φ_- induce trivializations of the associated bundles, that is, the Clifford bundle, the spinor bundle, and so on. Then we trivialize $\mathbf{S}(S^3)$ as follows:

$$\phi_+ : \mathbf{S}(S^3) = Spin(4) \times_{\rho_1} V_1 \ni [(p, q), v] \mapsto (pq^{-1}, pv) \in S^3 \times \mathbf{C}^2 , \tag{1.13}$$

$$\phi_- : \mathbf{S}(S^3) = Spin(4) \times_{\rho_1} V_1 \ni [(p, q), v] \mapsto (pq^{-1}, qv) \in S^3 \times \mathbf{C}^2 . \tag{1.14}$$

We call the first trivialization (1.13) 'Δ^+-trivialization' and the second one (1.14) 'Δ^--trivialization'. We shall explain why we call them so, for this fact is related to the extension problem in section 4. We consider the 4-dimensional sphere S^4 including S^3 as the equator. Although $\mathbf{Spin}(S^4)$ is not a trivial bundle, if we restrict $\mathbf{Spin}(S^4)$ to $\mathbf{R}^4 = S^4 \setminus \{\text{north pole}\}$, then we can trivialize it, that is, $\mathbf{Spin}(S^4)|_{\mathbf{R}^4} = \mathbf{R}^4 \times Spin(4)$ and have the following inclusion from $\mathbf{Spin}(S^3)$ into $\mathbf{R}^4 \times Spin(4)$ as a spin bundle:

$$\mathbf{Spin}(S^3) = Spin(4) \ni (p, q) \mapsto (pq^{-1}, (p, q)) \in \mathbf{R}^4 \times Spin(4) . \tag{1.15}$$

Here we realize $\mathbf{R}^4$ by

$$x = \begin{pmatrix} x_4 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_4 - ix_1 \end{pmatrix} \in \mathbf{R}^4 . \tag{1.16}$$

The spinor representation Δ_4 is given by $\Delta_4(p, q) = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$ on $\mathbf{C}^4$ and decomposes as the direct sum of $\Delta_4^+(p, q) = p$ on $\mathbf{C}^2$ and $\Delta_4^-(p, q) = q$ on $\mathbf{C}^2$. Then we get the spinor bundles $\mathbf{R}^4 \times (Spin(4) \times_{\Delta_4^\pm} \mathbf{C}^4)$ and their trivializations:

$$\mathbf{R}^4 \times (Spin(4) \times_{\Delta_4^+} \mathbf{C}^2) \ni (x, [(p, q), v]) \mapsto (x, pv) \in \mathbf{R}^4 \times \mathbf{C}^2 , \tag{1.17}$$

$$\mathbf{R}^4 \times (Spin(4) \times_{\Delta_4^-} \mathbf{C}^2) \ni (x, [(p, q), v]) \mapsto (x, qv) \in \mathbf{R}^4 \times \mathbf{C}^2 . \tag{1.18}$$

These trivializations correspond to (1.13) and (1.14) respectively. So we call the trivialization (1.13) Δ^+-trivialization and (1.14) Δ^--trivialization.

We shall see how spinor sections are expressed under $\Delta^{\pm}$-trivialization. If Ψ is in $C^{\infty}(Spin(4); \Delta_4)$, that is, Ψ is a $Spin(3)$-equivariant function from $Spin(4)$ to V_1, then we get a $\mathbf{C}^2$-valued (V_1-valued) function on S^3 for Δ^+-trivialization;

$$\begin{aligned}\phi_+([(p,q),\Psi(p,q)]) &= (pq^{-1}, p\Psi(p,q)) \\ &= (pq^{-1}, \Psi((p,q)p^{-1})) \\ &= (pq^{-1}, \Psi(I, qp^{-1}))\,.\end{aligned}$$

So we define a $\mathbf{C}^2$-valued function ψ^+ associated to Ψ by

$$\psi^+ : S^3 \ni h \mapsto \Psi(I, h^{-1}) \in \mathbf{C}^2\,. \tag{1.19}$$

We also have $\psi^-(h) := \Psi(h, I)$ for Δ^--trivialization and obtain a relation between ψ^+ and ψ^-,

$$\psi^+(h) = \Psi(I, h^{-1}) = h\Psi(h, I) = h\psi^-(h)\,. \tag{1.20}$$

In other words, the multiplication h in (1.20) gives a bundle automorphism for $\mathbf{S}(S^3)$.

Now, to describe Theorem 1.1 for $G = Spin(4)$ and $H = Spin(3)$, we need all the equivalent classes of irreducible unitary representations of $Spin(4)$. As $Spin(4)$ is a product Lie group $SU(2) \times SU(2)$, we need $\widehat{SU(2)}$ first. It is well-known that $\widehat{SU(2)}$ are given by $\{(\rho_m, V_m)\}_{m\geq 0}$, where V_m is the $(m+1)$-dimensional vector space of all complex polynomials of degree $\leq m$ in z and the action ρ_m is defined by $\rho_m(h)z^k = (bz+d)^{m-k}(az+c)^k$ for $h = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$ in $SU(2)$. Furthermore, if we introduce an inner product on V_m by

$$\left(\frac{z^k}{\sqrt{k!(m-k)!}}, \frac{z^l}{\sqrt{l!(m-l)!}}\right) = \delta_{kl}\,, \tag{1.21}$$

then (ρ_m, V_m) induces a unitary representation. All the irreducible unitary representations of $SU(2) \times SU(2)$ are given by $(\rho_m \hat{\otimes} \rho_n, V_m \hat{\otimes} V_n)$ for $m, n \geq 0$, that is,

$$\widehat{Spin(4)} = \{(\rho_m \hat{\otimes} \rho_n, V_m \hat{\otimes} V_n) \mid m, n \geq 0\}\,. \tag{1.22}$$

Here $V_m \hat{\otimes} V_n$ is the $(m+1)(n+1)$-dimensional vector space whose basis is $\{z_1^k z_2^l \mid 0 \leq k \leq m, 0 \leq l \leq n\}$ and the inner product is

$$\left(\frac{z_1^k z_2^l}{\sqrt{k!(m-k)!l!(n-l)!}}, \frac{z_1^{k'} z_2^{l'}}{\sqrt{k'!(m-k')!l'!(n-l')!}}\right) = \delta_{kk'}\delta_{ll'}\,. \tag{1.23}$$

We shall investigate $\mathrm{Hom}_{SU(2)}(V_\gamma, W_3)$ for any γ in $\widehat{Spin(4)}$. We have known that the spinor representation (Δ_3, W_3) is the natural representation (ρ_1, V_1) of $SU(2)$ and (π_γ, V_γ) is given by $(\rho_m \hat{\otimes} \rho_n, V_m \hat{\otimes} V_n)$. So we shall consider $\mathrm{Hom}_{SU(2)}(V_m \hat{\otimes} V_n, V_1)$ for any $m, n \geq 0$. If A is in $\mathrm{Hom}_{SU(2)}(V_m \hat{\otimes} V_n, V_1)$, then the following diagram is commutative for any h in $SU(2)$:

$$\begin{array}{ccc} V_m \hat{\otimes} V_n & \xrightarrow{\ A\ } & V_1 \\ {\scriptstyle \rho_m \hat{\otimes} \rho_n|_{SU(2)}(h)}\downarrow & & \downarrow{\scriptstyle \rho_1(h)} \\ V_m \hat{\otimes} V_n & \xrightarrow{\ A\ } & V_1 \end{array} \tag{1.24}$$

Because $\rho_m \hat{\otimes} \rho_n|_{SU(2)}$ is the usual tensor representation $\rho_m \otimes \rho_n$, we use Clebsch-Gordan formula, $\rho_m \otimes \rho_n \simeq \rho_{m+n} \oplus \cdots \oplus \rho_{|m-n|}$. For $|m-n| = 1$, that is, for $n = m \pm 1$, only the last part $\rho_{|m-n|}$ is (ρ_1, V_1). By Schur's lemma, A is a scalar on $V_{|m-n|} = V_1$ and zero on other parts. It follows that $\dim \mathrm{Hom}_{SU(2)}(V_m \hat{\otimes} V_{m\pm 1}, V_1) = 1$. On the other hand, for $|m-n| \neq 1$, ρ_1 does not appear in $\rho_{m+n} \oplus \cdots \oplus \rho_{|m-n|}$ and A is zero on $V_m \hat{\otimes} V_n$. Hence $\dim \mathrm{Hom}_{SU(2)}(V_m \hat{\otimes} V_n, V_1) = 0$.

Thus we have obtained the following proposition.

PROPOSITION 1.2. *Put*

$$E_{-m} := (V_m \hat{\otimes} V_{m+1}) \otimes \mathrm{Hom}_{SU(2)}(V_m \hat{\otimes} V_{m+1}, V_1), \tag{1.25}$$

$$E_m := (V_{m+1} \hat{\otimes} V_m) \otimes \mathrm{Hom}_{SU(2)}(V_{m+1} \hat{\otimes} V_m, V_1). \tag{1.26}$$

Then $L^2(Spin(4); \Delta_3)$ has the following decomposition as a representation space of $Spin(4)$:

$$L^2(Spin(4); \Delta_3) = \overline{\bigoplus_{m \geq 0} E_{-m} \oplus E_m}, \tag{1.27}$$

where each dimension of E_{-m} and E_m is $(m+1)(m+2)$.

In section 3 we will show that the irreducible component $E_{\pm m}$ is an eigenspace of the Dirac operator. So the above decomposition is nothing but the eigenspace decomposition of the Dirac operator.

2. A formula for the Dirac operator.

The Dirac operator D_n on S^n is a first order differential operator acting on $\mathbf{S}(S^n)$. We represent D_n as $\sum e_i \cdot \nabla_{e_i}$ locally, where ∇ is the spin connection induced by the Levi-Civita connection and $\{e_i\}$ is a local orthonormal frame of S^n. When we think of $\mathbf{S}(S^n)$ as a homogeneous bundle, D_n is a homogeneous differential operator, that is, D_n commutes with the action of G.

We decompose the Lie algebra $\mathfrak{g}$ of G into the Lie algebra $\mathfrak{h}$ of H and its orthonormal complement $\mathfrak{p}$; $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. Since $\mathfrak{p}$ is isomorphic to $T_e(S^n)$, we can find a basis $\{X_i\}_{1 \leq i \leq n}$ of $\mathfrak{p}$ corresponding to an orthonormal basis $\{e_i\}_{1 \leq i \leq n}$ of $T_e(S^n)$. Then D_n is realized as a differential operator acting on $C^\infty(G; \Delta_n)$ and hence $L^2(G; \Delta_n)$ as follows:

THEOREM 2.1. *For Ψ in $C^\infty(G; \Delta_n)$,*

$$D_n[g, \Psi(g)]|_{g=g_0} = \left[g_0, \sum_{i=1}^{n} e_i \cdot X_i \big|_{g=g_0} \Psi \right], \tag{2.1}$$

where $X_i|_{g=g_0}\Psi = \frac{d}{dt}(g_0 \exp tX_i)|_{t=0}$ and $e_i \cdot$ is the Clifford multiplication by e_i.

It is easy to show that D_n is a homogeneous differential operator. Therefore, the vector space $E_\gamma = V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, W_n)$ for any γ in $\hat{G}$ is invariant subspace with respect to D_n. In facts, we see the action of D_n on E_γ.

COROLLARY 2.2. *In the decomposition* $L^2(G; \Delta_n) = \overline{\bigoplus_{\gamma \in \hat{G}} E_\gamma}$, *the restriction of the Dirac operator* D_n *to* $E_\gamma = V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, W_n)$ *is given by* $id \otimes D_n^\gamma$, *where*

$$D_n^\gamma(A) := -\sum_{i=1}^{n} e_i \cdot A\pi_{\gamma*}(X_i) \quad \textit{for } A \in \mathrm{Hom}_H(V_\gamma, W_n)\,, \tag{2.2}$$

and $\pi_{\gamma*}$ *is the infinitesimal representation of* π_γ.

This corollary is important to calculate the eigenvalues of D_n. The above theorem and corollary follows from a general result provided in [3].

Now, we shall investigate the case $n = 3$ and give an explicit formula of D_3 on S^3. Since $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\mathfrak{h} = \mathrm{diag}\,\mathfrak{su}(2)$, if we fix an orthonormal basis of $\mathfrak{su}(2)$ as

$$\frac{\sigma_1}{2} := \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \frac{\sigma_2}{2} := \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \frac{\sigma_3}{2} := \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{2.3}$$

then we have orthonormal bases of $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{p}$ as follows:

$$\text{basis of } \mathfrak{g} := \left\{ \left(\frac{\sigma_i}{2}, 0\right), \left(0, \frac{\sigma_j}{2}\right) \right\}_{1 \le i,j \le 3}, \tag{2.4}$$

$$\text{basis of } \mathfrak{h} := \left\{ \left(\frac{\sigma_i}{2}, \frac{\sigma_i}{2}\right) \right\}_{1 \le i \le 3}, \tag{2.5}$$

$$\text{basis of } \mathfrak{p} := \{X_i\}_i = \left\{ \left(\frac{\sigma_i}{2}, -\frac{\sigma_i}{2}\right) \right\}_{1 \le i \le 3}. \tag{2.6}$$

Since the standard basis $\{e_i\}$ of $T_e(S^3)$ is $\{\sigma_i\}$ under the realization (1.7) of S^3, we see that the above $\{X_i\}$ corresponds to $\{e_i\}$ under $\mathfrak{p} \simeq T_e(S^3)$:

$$\begin{aligned} \frac{d}{dt}(\exp tX_i)e\Big|_{t=0} &= \frac{d}{dt}\left(\exp t\frac{\sigma_i}{2}\right) I \left(\exp\left(-t\frac{\sigma_i}{2}\right)\right)^{-1}\Big|_{t=0} \\ &= \frac{\sigma_i}{2} + \frac{\sigma_i}{2} \\ &= \sigma_i = e_i \in T_e(S^3)\,. \end{aligned}$$

We shall show that the Clifford multiplication $e_i \cdot$ is given by σ_i as a matrix multiplication. The (complex) spinor representation is obtained by restricting a complex irreducible Clifford representation to the spin group. We realize the complex Clifford algebra $Cl_3 \otimes \mathbf{C}$ as $\mathbf{C}(2) \oplus \mathbf{C}(2)$ by setting $e_i = (\sigma_i, -\sigma_i)$ for $1 \le i \le 3$, where $\mathbf{C}(k)$ is the set of $k \times k$ complex matrices. Then we have two inequivalent irreducible Clifford representations,

$$\mathbf{C}(2) \oplus \mathbf{C}(2) \ni (\alpha, \beta) \mapsto \alpha \in \mathbf{C}(2)\,, \tag{2.7}$$

$$\mathbf{C}(2) \oplus \mathbf{C}(2) \ni (\alpha, \beta) \mapsto \beta \in \mathbf{C}(2)\,. \tag{2.8}$$

If we restrict them to *Spin*(3), then we know that these representations are equivalent to each other because *Spin*(3) is diag $SU(2)$ in $\mathbf{C}(2) \oplus \mathbf{C}(2)$. Thus we have the spinor representation (Δ_3, W_3) where the Clifford multiplication $e_i \cdot$ is given by σ_i or $-\sigma_i$. So we choose σ_i as the Clifford multiplication $e_i \cdot$ for $1 \le i \le 3$.

We are now in a position to obtain a formula of D_3. We use Δ^+-trivialization to represent D_3 explicitly, where D_3 is a differential operator acting on $\mathbf{C}^2$-valued functions.

PROPOSITION 2.3. *For Δ^+-trivialization* (1.13), *the Dirac operator D_3 on $\mathbf{S}(S^3)$ is given by*

$$D_3 = \frac{3}{2}I + \sigma_1 Z_1 + \sigma_2 Z_2 + \sigma_3 Z_3 . \tag{2.9}$$

Here Z_1, Z_2, and Z_3 are differential operators on S^3 defined by

$$\begin{aligned} Z_1 &= -x_1\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_1} - x_3\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_3}, \\ Z_2 &= -x_2\frac{\partial}{\partial x_4} + x_3\frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial x_3}, \\ Z_3 &= -x_3\frac{\partial}{\partial x_4} - x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3}. \end{aligned} \tag{2.10}$$

PROOF. Let $\psi^+(h) = \Psi(I, h^{-1})$ be a spinor section of $\mathbf{S}(S^3)$ for Δ^+-trivialization. We calculate $(\sigma_i X_i \Psi)(I, h^{-1})$ for $1 \le i \le 3$.

$$\begin{aligned} (X_i\Psi)(I, h^{-1}) &= \frac{d}{dt}\Psi((I, h^{-1})\exp tX_i)\Big|_{t=0} \\ &= \frac{d}{dt}\Psi\left(\exp\frac{\sigma_i}{2}t, h^{-1}\exp\left(-\frac{\sigma_i}{2}t\right)\right)\Big|_{t=0} \\ &= \frac{d}{dt}\Psi\left((I, h^{-1}\exp(-\sigma_i t))\exp\frac{\sigma_i}{2}t\right)\Big|_{t=0} \\ &= \frac{d}{dt}\left(\exp\left(-\frac{\sigma_i}{2}t\right)\right)\Psi(I, h^{-1}\exp(-\sigma_i t))\Big|_{t=0} \\ &= -\frac{\sigma_i}{2}\Psi(I, h^{-1}) + \frac{d}{dt}\Psi((I, h^{-1}\exp(-\sigma_i t))\Big|_{t=0} \\ &= -\frac{\sigma_i}{2}\Psi(I, h^{-1}) + (Z_i\Psi)(I, h^{-1}) \\ &= -\frac{\sigma_i}{2}\psi^+ + Z_i\psi^+ , \end{aligned}$$

where Z_i is a right invariant vector field on $S^3 = SU(2)$ corresponding to σ_i, that is,

$$(Z_i f)(h) = \frac{d}{dt}f((\exp t\sigma_i)h)\Big|_{t=0} \quad \text{for } f \in C^\infty(SU(2)).$$

Then we have $\sigma_i X_i = (1/2)I + \sigma_i Z_i$ and complete the proof.

REMARK 2.1. For Δ^--trivialization, D_3 is represented by using the left invariant vector fields. Our formula of the Dirac operator coincides with the one discussed in [7] and [9].

3. The eigenspinors on S^3.

In this section we shall solve the eigenvalue problem of the Dirac operator D_3 and construct the eigenspinors. A way of solving the eigenvalue problem is to use Corollary 2.2.

So we shall investigate $\mathrm{Hom}_{SU(2)}(V_m \hat{\otimes} V_{m+1}, V_1)$ and $\mathrm{Hom}_{SU(2)}(V_{m+1} \hat{\otimes} V_m, V_1)$ more precisely. As a first step we need the following lemma.

LEMMA 3.1. *Let $\{z_1^k\}_{0 \le k \le m}$ be a basis of V_m and $\{z_2^k\}_{0 \le k \le m+n}$ a basis of V_{m+n}.*

1. *In the irreducible decomposition of $V_m \otimes V_{m+n}$, the basis of the irreducible component V_n is given by*

$$\{z_2^{n+1-i}(z_2 - z_1)^m\}_{1 \le i \le n+1} . \tag{3.1}$$

In particular, for $n = 1$, we have the following unitary basis of V_1 in $V_m \otimes V_{m+1}$

$$\omega_1 = \frac{z_2(z_2 - z_1)^m}{m!\sqrt{(m+1)(m+2)/2}} \quad and \quad \omega_2 = \frac{(z_2 - z_1)^m}{m!\sqrt{(m+1)(m+2)/2}}, \tag{3.2}$$

where the inner product is given by (1.23).

2. *The same result holds for $V_{m+n} \otimes V_m$ if we exchange z_1 and z_2.*

PROOF. We prove only the case $n = 1$. Let h be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SU(2)$. Then

$$\begin{aligned}((\rho_m \otimes \rho_{m+1})(h))(z_2 - z_1)^m &= (bz_1 + d)^m (bz_2 + d)^{m+1} \left(\frac{az_2 + c}{bz_2 + d} - \frac{az_1 + c}{bz_1 + d} \right)^m \\ &= (bz_2 + d)\{(az_2 + c)(bz_1 + d) - (az_1 + c)(bz_2 + d)\}^m \\ &= (bz_2 + d)(z_2 - z_1)^m .\end{aligned}$$

Similarly we have $((\rho_m \otimes \rho_{m+1})(h))z_2(z_2 - z_1)^m = (az_2 + c)(z_2 - z_1)^m$. Thus we get the natural representation (ρ_1, V_1) in $V_m \otimes V_{m+1}$:

$$((\rho_m \otimes \rho_{m+1})(h))(\omega_1, \omega_2) = (\omega_1, \omega_2) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \rho_1(h)(\omega_1, \omega_2) .$$

The unitarity is clear and we have proved the lemma.

This lemma implies that there is a $SU(2)$-module homomorphism A from $V_m \hat{\otimes} V_{m+1}$ to V_1 such that, under the decomposition $V_m \otimes V_{m+1} \simeq V_{2m+1} \oplus \cdots \oplus V_1$,

$$A|_{V_{2m+1} \oplus \cdots \oplus V_3} = 0 \quad \text{and} \quad A|_{V_1}(\omega_i) = v_i \quad (i = 1, 2) , \tag{3.3}$$

where $v_1 = (1, 0)^t$ and $v_2 = (0, 1)^t$ are in $\mathbf{C}^2 \simeq V_1$.

If we put $n = 3$ and $(\pi_\gamma, V_\gamma) = (\rho_m \hat{\otimes} \rho_{m+1}, V_m \hat{\otimes} V_{m+1})$ in Corollary 2.2, then we find a constant c_γ such that $D_3^\gamma(A) = c_\gamma A$ because of $\dim \mathrm{Hom}_{SU(2)}(V_\gamma, V_1) = 1$. We carry out a similar discussion for $\pi_\gamma = \rho_{m+1} \hat{\otimes} \rho_m$ and have the following lemma.

LEMMA 3.2. *We consider the case $n = 3$ in Corollary 2.2. Then, for $\pi_\gamma = \rho_m \hat{\otimes} \rho_{m+1}$,*

$$D_3^\gamma = -\left(\frac{3}{2} + m\right) id \quad on \quad \mathrm{Hom}_{SU(2)}(V_m \hat{\otimes} V_{m+1}, V_1) , \tag{3.4}$$

and, for $\pi_\gamma = \rho_{m+1} \hat{\otimes} \rho_m$,

$$D_3^\gamma = \left(\frac{3}{2} + m\right) id \quad on \quad \mathrm{Hom}_{SU(2)}(V_{m+1} \hat{\otimes} V_m, V_1) . \tag{3.5}$$

By virtue of this lemma we see how the Dirac operator D_3 acts on the irreducible component $E_{\pm m}$.

PROPOSITION 3.3. *In the irreducible decomposition* (1.27) *of* $L^2(Spin(4), \Delta_3)$, *the irreducible component* E_m (*resp.* E_{-m}) *is the eigenspace with the eigenvalue* $m + 3/2$ (*resp.* $-(m + 3/2)$) *of* D_3. *In particular, the multiplicity of the eigenvalue* $\pm(m + 3/2)$ *is* $(m + 1)(m + 2)$.

We call E_m (*resp.* E_{-m}) *positive* (*resp. negative*) *eigenspace of spinors for any* $m \geq 0$.

PROOF OF LEMMA 3.2. The infinitesimal representation (ρ_{m*}, V_m) of (ρ_m, V_m) is given as follows:

$$\begin{aligned} \rho_{m*}\left(\frac{\sigma_1}{2}\right) z^k &= i\left(k - \frac{m}{2}\right) z^k, \\ \rho_{m*}\left(\frac{\sigma_2}{2}\right) z^k &= \frac{m-k}{2} z^{k+1} - \frac{k}{2} z^{k-1}, \\ \rho_{m*}\left(\frac{\sigma_3}{2}\right) z^k &= i\frac{m-k}{2} z^{k+1} + i\frac{k}{2} z^{k-1}. \end{aligned} \tag{3.6}$$

Hence we have the following formula for $((\rho_m \hat{\otimes} \rho_{m+1})_*, V_m \hat{\otimes} V_{m+1})$:

$$\begin{aligned} (\rho_m \hat{\otimes} \rho_{m+1})_*(X_1) z_1^k z_2^l &= i\left(k - l + \frac{1}{2}\right) z_1^k z_2^l, \\ (\rho_m \hat{\otimes} \rho_{m+1})_*(X_2 + iX_3) z_1^k z_2^l &= -k z_1^{k-1} z_2^l + l z_1^k z_2^{l-1}, \\ (\rho_m \hat{\otimes} \rho_{m+1})_*(X_2 - iX_3) z_1^k z_2^l &= (m-k) z_1^{k-1} z_2^l - (m+1-l) z_1^k z_2^{l-1}, \end{aligned} \tag{3.7}$$

where $\{X_i\}_{1 \leq i \leq 3}$ is the basis of $\mathfrak{p}$ given by (2.6). In Corollary 2.2, let π_γ be $\rho_m \hat{\otimes} \rho_{m+1}$ and A be the $SU(2)$-module homomorphism from $V_m \hat{\otimes} V_{m+1}$ to V_1 satisfying (3.3). We shall look for the constant c_γ satisfying $D_3^\gamma(A) = -\sum_{i=1}^3 e_i \cdot A\pi_{\gamma*}(X_i) = c_\gamma A$. If we substitute ω_2 to this equation, then we obtain $D_3^\gamma(A)(\omega_2) = c_\gamma A(\omega_2) = c_\gamma v_2$. We calculate $D_3^\gamma(A)(\omega_2)$.

$$\begin{aligned} D_3^\gamma(A)(\omega_2) &= -\sum_{i=1}^3 e_i \cdot A\pi_{\gamma*}(X_i)(\omega_2) \\ &= -\sum_{i=1}^3 \sigma_i\{(\pi_{\gamma*}(X_i)(\omega_2), \omega_1)v_1 + (\pi_{\gamma*}(X_i)(\omega_2), \omega_2)v_2\} \\ &= -(\pi_{\gamma*}(X_1)(\omega_2), \omega_1)iv_1 + (\pi_{\gamma*}(X_1)(\omega_2), \omega_2)iv_2 - (\pi_{\gamma*}(X_2)(\omega_2), \omega_2)v_1 \\ &\quad + (\pi_{\gamma*}(X_2)(\omega_2), \omega_1)v_2 - (\pi_{\gamma*}(X_3)(\omega_2), \omega_2)iv_1 - (\pi_{\gamma*}(X_3)(\omega_2), \omega_1)iv_2 \\ &= \{i(\pi_{\gamma*}(X_1)(\omega_2), \omega_2) + (\pi_{\gamma*}(X_2 - iX_3)(\omega_2), \omega_1)\}v_2, \end{aligned}$$

where (,) denotes the inner product on $V_m \hat{\otimes} V_{m+1}$. We have

$$\begin{aligned} i(\pi_{\gamma*}(X_1)(\omega_2), \omega_2) &= i\alpha_m^2(\pi_{\gamma*}(X_1)(z_2 - z_1)^m, (z_2 - z_1)^m) \\ &= i\alpha_m^2\left(\pi_{\gamma*}(X_1)\sum_{k=0}^m \binom{m}{k}(-1)^k z_1^k z_2^{m-k}, \sum_{l=0}^m \binom{m}{l}(-1)^l z_1^l z_2^{m-l}\right) \end{aligned}$$

$$= i\alpha_m^2 \sum_{k,l} \binom{m}{k}\binom{m}{l}(-1)^{k+l}\left(2k - m + \frac{1}{2}\right) i(z_1^k z_2^{m-k}, z_1^l z_2^{m-l})$$
$$= -\alpha_m^2 (m!)^2 \sum_k \left(2k - m + \frac{1}{2}\right)(k+1)$$
$$= -\frac{2m+3}{6},$$

where $\alpha_m = \left(m!\sqrt{(m+1)(m+2)/2}\right)^{-1}$. In the same way we have

$$(\pi_{\gamma *}(X_2 - iX_3)(\omega_2), \omega_1) = -2\frac{2m+3}{6}.$$

Thus we conclude that $D_3^\gamma(A)(\omega_2) = -(m+3/2)v_2$ and hence $c_\gamma = -(m+3/2)$. We carry out a similar calculation for $\pi_\gamma = \rho_{m+1}\hat{\otimes}\rho_m$ and obtain $c_\gamma = m + 3/2$. We have thus proved the lemma.

Now, we shall construct the eigenspinors. We know that the negative eigenspace E_{-m} has the following basis:

$$\{v_{kl} \otimes A\}_{kl} \in E_{-m} = (V_m \hat{\otimes} V_{m+1}) \otimes \mathrm{Hom}_{SU(2)}(V_m \hat{\otimes} V_{m+1}, V_1), \tag{3.8}$$

where $0 \le k \le m, 0 \le l \le m+1$ and

$$v_{kl} = \frac{z_1^k z_2^l}{\sqrt{k!(m-k)!l!(m+1-l)!}}. \tag{3.9}$$

So we get the eigenspinors with the eigenvalue $-(m+3/2)$ as $SU(2)$-equivariant V_1-valued functions on *Spin*(4):

$$\Psi_{-m(k,l)}(p,q) := A\pi_\gamma(p^{-1}, q^{-1})v_{kl}. \tag{3.10}$$

We use Δ^+-trivialization to represent the eigenspinors as $\mathbf{C}^2$-valued (V_1-valued) functions on S^3.

$$\begin{aligned}\psi^+_{-m(k,l)}(h) &:= \Psi_{-m(k,l)}(I, h^{-1}) \\ &= A\pi_\gamma(I,h)v_{kl} \\ &= (\pi_\gamma(I,h)v_{kl}, \omega_1)v_1 + (\pi_\gamma(I,h)v_{kl}, \omega_2)v_2 \\ &= \begin{pmatrix}(\pi_\gamma(I,h)v_{kl}, \omega_1) \\ (\pi_\gamma(I,h)v_{kl}, \omega_2)\end{pmatrix} \in \mathbf{C}^2.\end{aligned} \tag{3.11}$$

We calculate each entry $(\pi_\gamma(I,h)v_{kl}, \omega_1)$ and $(\pi_\gamma(I,h)v_{kl}, \omega_2)$:

$$\begin{aligned}(\pi_\gamma(I,h)v_{kl}, \omega_1) &= \alpha_m\left(\frac{z_1^k \rho_{m+1}(h) z_2^l}{\sqrt{k!(m-k)!l!(m+1-l)!}}, z_2(z_2 - z_1)^m\right) \\ &= \frac{(-1)^k\sqrt{m+1-k}}{\sqrt{(m+1)(m+2)/2}}\left(\frac{\rho_{m+1}(h)z_2^l}{\sqrt{l!(m+1-l)!}}, \frac{z_2^{m+1-k}}{\sqrt{(m+1-k)!k!}}\right) \\ &= \frac{(-1)^k\sqrt{m+1-k}}{\sqrt{(m+1)(m+2)/2}} v^{m+1}_{l,m+1-k}(h),\end{aligned} \tag{3.12}$$

$$(\pi_\gamma(I,h)v_{kl}, \omega_2) = \frac{(-1)^k\sqrt{k+1}}{\sqrt{(m+1)(m+2)/2}} v^{m+1}_{l,m-k}(h), \tag{3.13}$$

where $v^m_{i,j}$ is a matrix component of the representation (ρ_m, V_m) of $SU(2)$, that is,

$$v^m_{i,j}(h) := \left(\frac{\rho_m(h)z^i}{\sqrt{i!(m-i)!}}, \frac{z^j}{\sqrt{j!(m-j)!}}\right). \tag{3.14}$$

We carry out a similar calculation for the positive eigenspace E_m. Then we get the following eigenspinors for Δ^+-trivialization:

$$\psi^+_{-m(k,l)}(h) = \begin{pmatrix} \sqrt{m+1-k}\,v^{m+1}_{l,m+1-k}(h) \\ \sqrt{k+1}\,v^{m+1}_{l,m-k}(h) \end{pmatrix} \quad (0 \le k \le m,\ 0 \le l \le m+1). \tag{3.15}$$

$$\psi^+_{m(k,l)}(h) = \begin{pmatrix} -\sqrt{k}\,v^m_{l,m+1-k}(h) \\ \sqrt{m+1-k}\,v^m_{l,m-k}(h) \end{pmatrix} \quad (0 \le k \le m+1,\ 0 \le l \le m). \tag{3.16}$$

In the same way we obtain the eigenspinors for Δ^--trivialization:

$$\psi^-_{-m(k,l)}(h) = \begin{pmatrix} -\sqrt{l}\,v^m_{k,m-l+1}(h^{-1}) \\ \sqrt{m+1-l}\,v^m_{k,m-l}(h^{-1}) \end{pmatrix} \quad (0 \le k \le m,\ 0 \le l \le m+1), \tag{3.17}$$

$$\psi^-_{m(k,l)}(h) = \begin{pmatrix} \sqrt{m+1-l}\,v^{m+1}_{k,m+1-l}(h^{-1}) \\ \sqrt{l+1}\,v^{m+1}_{k,m-l}(h^{-1}) \end{pmatrix} \quad (0 \le k \le m+1,\ 0 \le l \le m). \tag{3.18}$$

REMARK 3.1. From the above formula, we have

$$\psi^-_{-m(k,l)}(h) = \psi^+_{m(l,k)}(h^{-1}) \quad \text{and} \quad \psi^-_{m(k,l)}(h) = \psi^+_{-m(l,k)}(h^{-1}). \tag{3.19}$$

Moreover, we have known the relation (1.20) between Δ^+- and Δ^--trivialization. As a result we have

$$\psi^+_{-m(l,k)}(h) = h\psi^+_{m(k,l)}(h^{-1}) \quad \text{and} \quad \psi^+_{m(l,k)}(h) = h\psi^+_{-m(k,l)}(h^{-1}). \tag{3.20}$$

These relations are interesting because they are analogous to the following relations for the S^1 case. If we denote the eigenspinor on S^1 with the eigenvalue m by $\psi_m(\theta) = e^{im\theta}$, then we get relations among the eigenspinors, that is, $\psi_{-m}(\theta) = \psi_m(-\theta)$. Here $-\theta$ is the inverse element of θ in $S^1 = U(1)$. In our situation an extra term 'h' appears in (3.20), because the spinor bundles $\mathbf{S}^+(S^4)$ and $\mathbf{S}^-(S^4)$ are not trivial bundles and 'h' is used as a transition function for them.

We have the following proposition.

PROPOSITION 3.4. *We have the following orthonormal basis of* $L^2(S^3, \mathbf{S}(S^3)) = L^2(Spin(4); \Delta_3)$ *for* Δ^+*-trivialization.*

1. *The positive eigenspinors with the eigenvalue* $m + 3/2$ *for* $m \ge 0$,

$$\psi^+_{m(k,l)}(h) \in E_m \quad \textit{for } 0 \le k \le m+1,\ 0 \le l \le m. \tag{3.21}$$

2. *The negative eigenspinors with the eigenvalue* $-(m+3/2)$ *for* $m \ge 0$,

$$\psi^+_{-m(l,k)}(h) = h\psi^+_{m(k,l)}(h^{-1}) \in E_{-m} \quad \textit{for } 0 \le k \le m+1,\ 0 \le l \le m. \tag{3.22}$$

Let us check the orthonormality of our eigenspinors. Of course, the orthonormality is clear by Frobenius reciprocity, but we can verify it by Peter-Weyl theorem for $SU(2)$. From

Peter-Weyl theorem we have the relations among $\{v^m_{i,j}\}$,

$$\int_{SU(2)} v^m_{i,j}(h)\overline{v^n_{k,l}(h)}dh = \frac{1}{m+1}\delta_{mn}\delta_{ik}\delta_{jl}\,. \tag{3.23}$$

Then we can easily show that our eigenspinors compose an orthonormal basis of $L^2(S^3, \mathbf{S}(S^3))$. For example,

$$\begin{aligned}(\psi^+_{m(k,l)}, \psi^+_{m(k,l)})_{L^2} &= \int_{S^3}(\psi^+_{m(k,l)}(h), \psi^+_{m(k,l)}(h))dh \\ &= \int_{SU(2)}\{k v^m_{l,m+1-k}(h)\overline{v^m_{l,m+1-k}(h)} \\ &\quad +(m+1-k)v^m_{l,m-k}(h)\overline{v^m_{l,m-k}(h)}\}dh \\ &= \frac{k}{m+1} + \frac{m+1-k}{m+1} \\ &= 1\,.\end{aligned} \tag{3.24}$$

Now, we investigate the action of *Spin*(4) on our eigenspinors. Let (p_0, q_0) be in *Spin*(4) and $\psi^+(h)$ be a section of $\mathbf{S}(S^3)$ for Δ^+-trivialization. Then the action of *Spin*(4) is given by

$$\begin{aligned}(\Theta(p_0, q_0)\psi^+)(h) &= (\Theta(p_0, q_0)\Psi)(I, h^{-1}) \\ &= \Psi(p_0^{-1}, q_0^{-1}h^{-1}) \\ &= \Psi((I, q_0^{-1}h^{-1}p_0)p_0^{-1}) \\ &= p_0\Psi(I, (p_0^{-1}hq_0)^{-1}) \\ &= p_0\psi^+(p_0^{-1}hq_0)\,.\end{aligned} \tag{3.25}$$

It follows that the basis (2.3) of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ corresponds to the set of differential operators on $C^\infty(S^3, \mathbf{S}(S^3))$:

$$\left(\frac{\sigma_i}{2}, 0\right) \mapsto \frac{\sigma_i}{2} - \frac{Z_i}{2} \quad 1 \le i \le 3\,, \tag{3.26}$$

$$\left(0, \frac{\sigma_i}{2}\right) \mapsto \frac{\tilde{Z}_i}{2} \quad 1 \le i \le 3\,, \tag{3.27}$$

where Z_i is the right invariant vector field on S^3 as (2.10) and $\tilde{Z}_i$ is the left invariant vector field corresponding to σ_i;

$$\begin{aligned}\tilde{Z}_1 &= -x_1\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3}\,, \\ \tilde{Z}_2 &= -x_2\frac{\partial}{\partial x_4} - x_3\frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3}\,, \\ \tilde{Z}_3 &= -x_3\frac{\partial}{\partial x_4} + x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3}\,,\end{aligned} \tag{3.28}$$

Then we can calculate the action of *Spin*(4) on the eigenspinors explicitly. Now we can state our main theorem:

THEOREM 3.5. 1. *The space of spinors on S^3 has an irreducible decomposition as a representation space of Spin(4):*

$$L^2(S^3, \mathbf{S}(S^3)) = \overline{\bigoplus_{m\geq 0} E_{-m} \oplus E_m}\,. \tag{3.29}$$

Here, the irreducible component $E_{\pm m}$ is the eigenspace of the Dirac operator with eigenvalue $\pm(m+3/2)$ and $\dim E_{\pm m} = (m+1)(m+2)$.

2. (a) *The positive eigenspace E_m gives the highest weight representation with highest weight vector $\psi^+_{m(m+1,m)}$, where the orthonormal basis of E_m is $\{\psi^+_{m(k,l)}(h) \mid 0 \leq k \leq m+1,\ 0 \leq l \leq m\}$ given in* (3.16). *The action of* $\mathfrak{spin}(4)$ *on E_m is given by*

$$\begin{aligned}
\left\{\frac{\sigma_1}{2} - \frac{Z_1}{2}\right\}\psi^+_{m(k,l)} &= i\left(k - \frac{m+1}{2}\right)\psi^+_{m(k,l)}\,,\\
\left\{\frac{\sigma_2 + i\sigma_3}{2} - \frac{Z_2 + iZ_3}{2}\right\}\psi^+_{m(k,l)} &= \sqrt{k}\sqrt{m+1-(k-1)}\,\psi^+_{m(k-1,l)}\,,\\
\left\{\frac{\sigma_2 - i\sigma_3}{2} - \frac{Z_2 - iZ_3}{2}\right\}\psi^+_{m(k,l)} &= -\sqrt{k+1}\sqrt{m+1-k}\,\psi^+_{m(k+1,l)}\,,\\
\frac{\tilde{Z}_1}{2}\psi^+_{m(k,l)} &= i\left(l - \frac{m}{2}\right)\psi^+_{m(k,l)}\,,\\
\frac{\tilde{Z}_2 + i\tilde{Z}_3}{2}\psi^+_{m(k,l)} &= -\sqrt{l}\sqrt{m-l+1}\,\psi^+_{m(k,l-1)}\,,\\
\frac{\tilde{Z}_2 - i\tilde{Z}_3}{2}\psi^+_{m(k,l)} &= \sqrt{l+1}\sqrt{m-l}\,\psi^+_{m(k,l+1)}\,,
\end{aligned} \tag{3.30}$$

where the basis of $\mathfrak{spin}(4)$ *corresponds to differential operators on S^3 given in* (3.26) *and* (3.27).

(b) *The negative eigenspace E_{-m} gives the highest weight representation with highest weight vector $\psi^+_{-m(m,m+1)}$, where the orthonormal basis of E_{-m} is $\{\psi^+_{-m(k,l)}(h) \mid 0 \leq k \leq m,\ 0 \leq l \leq m+1\}$ given in* (3.15) *and the action of* $\mathfrak{spin}(4)$ *on E_{-m} is given by*

$$\begin{aligned}
\left\{\frac{\sigma_1}{2} - \frac{Z_1}{2}\right\}\psi^+_{-m(k,l)} &= i\left(k - \frac{m}{2}\right)\psi^+_{-m(k,l)}\,,\\
\left\{\frac{\sigma_2 + i\sigma_3}{2} - \frac{Z_2 + iZ_3}{2}\right\}\psi^+_{-m(k,l)} &= \sqrt{k}\sqrt{m-(k-1)}\,\psi^+_{-m(k-1,l)}\,,\\
\left\{\frac{\sigma_2 - i\sigma_3}{2} - \frac{Z_2 - iZ_3}{2}\right\}\psi^+_{-m(k,l)} &= -\sqrt{k+1}\sqrt{m-k}\,\psi^+_{-m(k+1,l)}\,,\\
\frac{\tilde{Z}_1}{2}\psi^+_{-m(k,l)} &= i\left(l - \frac{m+1}{2}\right)\psi^+_{-m(k,l)}\,,\\
\frac{\tilde{Z}_2 + i\tilde{Z}_3}{2}\psi^+_{-m(k,l)} &= -\sqrt{l}\sqrt{(m+1)-l+1}\,\psi^+_{-m(k,l-1)}\,,\\
\frac{\tilde{Z}_2 - i\tilde{Z}_3}{2}\psi^+_{-m(k,l)} &= \sqrt{l+1}\sqrt{(m+1)-l}\,\psi^+_{-m(k,l+1)}\,.
\end{aligned} \tag{3.31}$$

PROOF. It remains to calculate the action of $\mathfrak{spin}(4)$ on the eigenspinors. We have the following formulas from Peter-Weyl theorem for $SU(2)$:

$$\begin{aligned}(Z_1 v^m_{k,l})(h) &= i(2l-m)v^m_{k,l}(h)\,,\\ ((Z_2+iZ_3)v^m_{k,l})(h) &= -2\sqrt{m-1}\sqrt{l+1}v^m_{k,l+1}(h)\,,\\ ((Z_2-iZ_3)v^m_{k,l})(h) &= 2\sqrt{m-l+1}\sqrt{l}v^m_{k,l-1}(h)\,,\\ (\tilde{Z}_1 v^m_{k,l})(h) &= i(2k-m)v^m_{k,l}(h)\,,\\ ((\tilde{Z}_2+i\tilde{Z}_3)v^m_{k,l})(h) &= -2\sqrt{m-k+1}\sqrt{k}v^m_{k-1,l}(h)\,,\\ ((\tilde{Z}_2-i\tilde{Z}_3)v^m_{k,l})(h) &= 2\sqrt{m-k}\sqrt{k+1}v^m_{k+1,l}(h)\,.\end{aligned} \tag{3.32}$$

By these formulas we prove the equation (3.30) and (3.31).

REMARK 3.2. We can calculate the action of D_3 on the eigenspinors by the above formula and show that $\psi_{\pm m(k,l)}$ is the eigenspinor with the eigenvalue $\pm(m+3/2)$.

4. The extension problem.

In this section we solve 'the extension problem', that is, the problem of extending a given spinor ψ on S^3 to a zero mode spinor of $D_4^{\pm}$ on upper (or lower) hemisphere of S^4, where S^3 is the boundary of the upper hemisphere. A spinor ψ on upper hemisphere is said to be zero mode spinor of D_4^+ if ψ satisfies that $D_4^+\psi = 0$. Our method using a polar decomposition of D_4^+ follows Kori's paper [9]. We have constructed the spin bundle and the spinor bundles over S^4 in section 1, that is, $\mathbf{Spin}(S^4) = Spin(5) \to Spin(5)/Spin(4)$ and $\mathbf{S}^{\pm}(S^4) = Spin(5) \times_{\Delta_4^{\pm}} W_4^{\pm}$. The local trivializations of these bundles are given in the below. We realize S^4 by the patching of $\mathbf{R}^4$ and $\widehat{\mathbf{R}^4}$, where $\mathbf{R}^4$ is $S^4\setminus\{\text{north pole}\}$ and $\widehat{\mathbf{R}^4}$ is $S^4\setminus\{\text{south pole}\}$. Let x be a coordinate of $\mathbf{R}^4$ represented by

$$x = \begin{pmatrix} x_4+ix_1 & x_2+ix_3 \\ -x_2+ix_3 & x_4-ix_1 \end{pmatrix} \in \mathbf{R}^4\,, \tag{4.1}$$

and y a coordinate of $\widehat{\mathbf{R}^4}$ represented by

$$y = \begin{pmatrix} y_4+iy_1 & y_2+iy_3 \\ -y_2+iy_3 & y_4-iy_1 \end{pmatrix} \in \widehat{\mathbf{R}^4}\,. \tag{4.2}$$

On this local coordinate system, the coordinate transformation is given by

$$y = \frac{x^*}{|x|^2} \quad \text{for } x \in \mathbf{R}^4_0\,, \tag{4.3}$$

where x^* denotes the transposed conjugate of x and $\mathbf{R}^4_0$ denotes $\mathbf{R}^4 \setminus \{x=0\}$. We use the Clifford algebra Cl_4 to get a local trivialization of $\mathbf{Spin}(S^4)$ (see [1]). We realize the Clifford algebra $\mathbf{C}l_4 = Cl_4 \otimes \mathbf{C}$ as $\mathbf{C}(4)$ by setting

$$\begin{aligned} e_1 &= \begin{pmatrix} 0 & -\sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -\sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \\ e_3 &= \begin{pmatrix} 0 & -\sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & -id \\ id & 0 \end{pmatrix}. \end{aligned} \tag{4.4}$$

Then we can decompose $\mathbf{C}l_4$ to the direct sum of the even part $\mathbf{C}l_4^0$ and the odd part $\mathbf{C}l_4^1$:

$$\mathbf{C}l_4^0 = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in \mathbf{C}l_4 \,\middle|\, \alpha, \beta \in \mathbf{C}(2) \right\}, \tag{4.5}$$

$$\mathbf{C}l_4^1 = \left\{ \begin{pmatrix} 0 & \delta \\ \gamma & 0 \end{pmatrix} \in \mathbf{C}l_4 \,\middle|\, \delta, \gamma \in \mathbf{C}(2) \right\}, \tag{4.6}$$

$$\mathbf{C}l_4 = \mathbf{C}l_4^0 \oplus \mathbf{C}l_4^1 = \left\{ \begin{pmatrix} \alpha & \delta \\ \gamma & \beta \end{pmatrix} \,\middle|\, \alpha, \beta, \delta, \gamma \in \mathbf{C}(2) \right\}. \tag{4.7}$$

Let $Pin(4)$ be the pin group, that is, the double covering group of $O(4)$. $Pin(4)$ has two connected components, $Pin^0(4)$ and $Pin^1(4)$. We realize these groups in $\mathbf{C}l_4$ as follows:

$$Pin^0(4) = Spin(4) = \left\{ \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \in \mathbf{C}l_4^0 \,\middle|\, p, q \in SU(2) \right\} \simeq SU(2) \times SU(2), \tag{4.8}$$

$$Pin^1(4) = \left\{ \begin{pmatrix} 0 & r \\ s & 0 \end{pmatrix} \in \mathbf{C}l_4^1 \,\middle|\, r, s \in SU(2) \right\}. \tag{4.9}$$

In this situation we prove that the spin bundle $\mathbf{Spin}(S^4)$ is isomorphic to the bundle given by the identification

$$\begin{aligned} \mathbf{R}_0^4 \times Pin^0(4) \ni (x, g) &\mapsto \left(\frac{x^*}{|x|^2}, \frac{1}{|x|} \sum x_i e_i \cdot g \right) \\ &= \left(\frac{x^*}{|x|^2}, \begin{pmatrix} 0 & -\frac{x}{|x|} \\ \frac{x^*}{|x|} & 0 \end{pmatrix} g \right) \in \widehat{\mathbf{R}_0^4} \times Pin^1(4). \end{aligned} \tag{4.10}$$

Now, we consider a Z_2 graded $\mathbf{C}l_4$-module (ρ, M), that is, a $\mathbf{C}l_4$-module $M = M^0 \oplus M^1$ such that $\rho(\mathbf{C}l_4^i) M^j = M^{i+j \pmod 2}$. If we have such a module, then we get a spinor bundle $\mathbf{Spin}(S^4) \times_\rho M^0$ whose bundle patching is given by

$$\mathbf{R}_0^4 \times M^0 \ni (x, v) \mapsto \left(\frac{x^*}{|x|^2}, \rho\left(\frac{1}{|x|} \sum x_i e_i \right) v \right) \in \widehat{\mathbf{R}_0^4} \times M^1. \tag{4.11}$$

To get the spinor bundle $\mathbf{S}^\pm(S^4)$, we choose the following Z_2 graded $\mathbf{C}l_4$-modules $(\rho_4^\pm, M)$: we define the action of $\mathbf{C}l_4^0$ on $M^0 := \mathbf{C}^2$ by

$$\rho_4^+ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \alpha \quad \text{or} \quad \rho_4^- \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \beta. \tag{4.12}$$

Then we have the Z_2 graded module $M := \mathbf{C}l_4 \otimes_{\mathbf{C}l_4^0} M^0$ with the action ρ_4^+ or ρ_4^-. Since the restriction of $\rho_4^\pm$ to $Spin(4)$ is $\Delta_4^\pm$, we see that the spinor bundles $\mathbf{S}^+(S^4)$ and $\mathbf{S}^-(S^4)$ are isomorphic to the bundles given by the identifications

$$\mathbf{R}_0^4 \times \mathbf{C}^2 \ni (x, v) \mapsto \left(\frac{x^*}{|x|^2}, \frac{x^*}{|x|} v \right) \in \widehat{\mathbf{R}_0^4} \times \mathbf{C}^2 , \tag{4.13}$$

$$\mathbf{R}_0^4 \times \mathbf{C}^2 \ni (x, v) \mapsto \left(\frac{x^*}{|x|^2}, -\frac{x}{|x|} v \right) \in \widehat{\mathbf{R}_0^4} \times \mathbf{C}^2 , \tag{4.14}$$

respectively. Thus we have local trivializations of the spin bundle and the spinor bundles over S^4.

REMARK 4.1. If we restrict $\mathbf{S}^{\pm}(S^4)$ to $S^3 \subset \mathbf{R}^4$, then we get $\Delta^{\pm}$-trivialization of $\mathbf{S}(S^3)$ discussed in section 1.

Now, from [5], we have a local formula of the Dirac operator D_4,

$$\begin{aligned} D_4 &= \begin{pmatrix} 0 & D_4^- \\ D_4^+ & 0 \end{pmatrix} \\ &= \sum_{i=1}^{4} \left\{ (1 + |x|^2) e_i \frac{\partial}{\partial x_i} - 3 x_i e_i \right\} \quad \text{on } \mathbf{R}^4 , \end{aligned} \tag{4.15}$$

where $D_4^{\pm}$ is a differential operator from $C^\infty(S^4, \mathbf{S}^{\pm}(S^4))$ to $C^\infty(S^4, \mathbf{S}^{\mp}(S^4))$. We rewrite the above formula by matrices σ_i instead of e_i:

$$D_4^+ = (1 + |x|^2) \left(\frac{\partial}{\partial x_4} - \sigma_1 \frac{\partial}{\partial x_1} - \sigma_2 \frac{\partial}{\partial x_2} - \sigma_3 \frac{\partial}{\partial x_3} \right) - 3x^* , \tag{4.16}$$

$$D_4^- = -(1 + |x|^2) \left(\frac{\partial}{\partial x_4} + \sigma_1 \frac{\partial}{\partial x_1} + \sigma_2 \frac{\partial}{\partial x_2} + \sigma_3 \frac{\partial}{\partial x_3} \right) + 3x . \tag{4.17}$$

The same formula holds on $\widehat{\mathbf{R}^4}$ if we replace x by y. From now on, we consider only D_4^+. We shall find a polar decomposition of D_4^+, that is, the decomposition of D_4^+ to a sum of normal derivative and tangential derivative. We can easily show that

$$-\frac{x}{r} D_4^+ = (1 + r^2) \frac{\partial}{\partial r} - \frac{1 + r^2}{r} (\sigma_1 Z_1 + \sigma_2 Z_2 + \sigma_3 Z_3) - 3r , \tag{4.18}$$

where $r = |x|$ is a coordinate of the normal direction. Then we have a polar decomposition of D_4^+,

$$D_4^+ = \frac{x^*}{r} \left\{ (1 + r^2) \frac{\partial}{\partial r} - \frac{1 + r^2}{r} (\sigma_1 Z_1 + \sigma_2 Z_2 + \sigma_3 Z_3) - 3r \right\} . \tag{4.19}$$

This polar decomposition is important because the extension problem is reduced to finding a scalar function $\phi(r)$ such that $D_4^+(\phi(r)\psi^+(h)) = 0$ for a given spinor $\psi^+(h)$ on S^3. First, we shall extend the positive spinor $\psi^+_{m(k,l)}(h)$ with eigenvalue $m + 3/2$. By using (2.9) we have

$$\sum \sigma_i Z_i \psi^+_{m(k,l)}(h) = m \psi^+_{m(k,l)}(h) . \tag{4.20}$$

It follows that the equation $D_4^+(\phi(r)\psi^+_{m(k,l)}(h)) = 0$ reduces to the ordinary differential equation,

$$(1 + r^2) \frac{d\phi(r)}{dr} - \frac{1 + r^2}{r} m\phi(r) - 3r\phi(r) = 0 , \tag{4.21}$$

where we put $\phi(1) = 1$. We solve this equation and obtain the solution $\phi(r) = ((1+r^2)/2)^{3/2} r^m$. We denote this solution by $\phi_m(r)$. Next, for the negative spinor with the eigenvalue $-(m+3/2)$, we have

$$\sum \sigma_i Z_i \psi^+_{-m(k,l)}(h) = (-m-3)\psi^+_{-m(k,l)}(h). \tag{4.22}$$

So we solve the equation

$$(1+r^2)\frac{d\phi(r)}{dr} - \frac{1+r^2}{r}(-m-3)\phi(r) - 3r\phi(r) = 0 \tag{4.23}$$

and get the solution $\phi_{-m}(r) := ((1+r^2)/2)^{3/2} r^{-m-3}$.

Now, it is useful that we extend the functions $\{v^m_{ij}(h)\}$ on S^3 to the ones on $\mathbf{R}^4$. We recall that, for the representation (ρ_m, V_m),

$$\rho_m(h) z^k = (bz+d)^{m-k}(az+c)^k \quad \text{for } h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2). \tag{4.24}$$

Then we put

$$\begin{aligned} \rho_m(x) z^k &:= (bz+d)^{m-k}(az+c)^k \\ &\quad \text{for } x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} x_4 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_4 - ix_1 \end{pmatrix} \in \mathbf{R}^4, \end{aligned} \tag{4.25}$$

$$v^m_{i,j}(x) := \left(\frac{\rho_m(x) z^i}{\sqrt{i!(m-i)!}}, \frac{z^j}{\sqrt{j!(m-j)!}} \right), \tag{4.26}$$

where $v^m_{ij}(x)$ is homogeneous of order m in x, that is, $v^m_{ij}(x) = |x|^m v^m_{ij}(x/|x|)$. We define the spinor $\psi^+_{m(k,l)}(x)$ on $\mathbf{R}^4$ by

$$\psi^+_{m(k,l)}(x) := \begin{pmatrix} -\sqrt{k} v^m_{l,m+1-k}(x) \\ \sqrt{m+1-k}\, v^m_{l,m-k}(x) \end{pmatrix} \quad \text{for } x \in \mathbf{R}^4. \tag{4.27}$$

The spinor $\psi^+_{-m(k,l)}(x)$ is defined similarly on $\mathbf{R}^4$. Then the desired zero mode spinor corresponding to $\psi^+_{m(k,l)}(h)$ is

$$\begin{aligned} \phi_m(r)\psi^+_{m(k,l)}(h) &= \left(\frac{1+|x|^2}{2}\right)^{3/2} |x|^m \psi^+_{m(k,l)}\left(\frac{x}{|x|}\right) \\ &= \left(\frac{1+|x|^2}{2}\right)^{3/2} \psi^+_{m(k,l)}(x) \quad \text{for } x \in \mathbf{R}, \end{aligned} \tag{4.28}$$

where we remark that

$$\left(\frac{1+|x|^2}{2}\right)^{3/2} \psi^+_{m(k,l)}(x) \sim O(|x|^{m+3}) \quad \text{for } x \to \infty. \tag{4.29}$$

On the other hand, the extension of negative spinor $\phi_{-m}(r)\psi^+_{-m(k,l)}(x/|x|)$ has a pole of order $m+3$ at $x=0$. So we use the coordinate y instead of x. Then

$$\begin{aligned}\phi_{-m}(r)\psi^+_{-m(k,l)}\left(\frac{x}{|x|}\right) &= (1+r^2)^{3/2}r^{-m-3}\psi^+_{-m(k,l)}\left(\frac{x}{|x|}\right)\\ &= (1+r^2)^{3/2}r^{-m-3}\frac{x}{|x|}\psi^+_{m(l,k)}\left(\left(\frac{x}{|x|}\right)^{-1}\right)\\ &= (1+|y|^{-2})^{3/2}|y|^{m+3}\frac{y^*}{|y|}\psi^+_{m(l,k)}\left(\frac{y}{|y|}\right)\\ &= (1+|y|^2)^{3/2}\frac{y^*}{|y|}\psi^+_{m(l,k)}(y)\,.\end{aligned}$$

As $y^*/|y|$ is the transition function of $\mathbf{S}^+(S^4)$, we should think of the above extended spinor as a smooth spinor on $\widehat{\mathbf{R}^4}$. Besides, we show that $D_4^+((1+|y|^2)^{3/2}\psi^+_{m(l,k)}(y))=0$ on $\widehat{\mathbf{R}^4}$ and the extended spinor $(1+|y|^2)^{3/2}\psi^+_{m(l,k)}(y)$ has a pole of order $m+3$ on $y=\infty$, that is, on $x=0$. Thus we have obtained the following proposition.

PROPOSITION 4.1. *Let $\Psi_{m(k,l)}(x)$ be the $\mathbf{C}^2$-valued function $((1+|x|^2)/2)^{3/2}\times \psi^+_{m(k,l)}(x)$ on $\mathbf{R}^4$. Then,*

1. *$\Psi_{m(k,l)}(x)$ is a smooth spinor on $\mathbf{R}^4$ such that $D_4^+(\Psi_{m(k,l)}(x))=0$ and $\Psi_{m(k,l)}(x)|_{S^3}$ is the positive eigenspinor $\psi^+_{m(k,l)}(h)$ of D_3 on S^3.*
2. *$\Psi_{m(k,l)}(y)$ is a smooth spinor on $\widehat{\mathbf{R}^4}$ such that $D_4^+(\Psi_{m(k,l)}(y))=0$ and $(y^*/|y|)\times \Psi_{m(k,l)}(y)|_{S^3}$ is the negative eigenspinor $\psi^+_{-m(l,k)}(h)$ of D_3 on S^3.*

In [9], Kori proved that the space of zero mode spinors on the lower (resp. upper) hemisphere of S^4 with a suitable metric is isomorphic to the space of positive (resp. negative) spinors on its boundary S^3. We can prove that the same assertion holds in our situation where S^4 has the standard metric.

We set the lower hemisphere of S^4 by

$$B^4 := \{x\in\mathbf{R}^4\subset S^4 \mid |x|=r\le 1\}\,. \tag{4.30}$$

where the Riemannian metric on B^4 is induced by the one on S^4. Then we have the trace map b for $s>1/2$,

$$b: H^s(B^4,\mathbf{S}^+(S^4)|_{B^4})\to H^{s-1/2}(S^3,\mathbf{S}^+(S^4)|_{S^3})\,, \tag{4.31}$$

where H^s is Sobolev s-space and S^3 is the boundary of B^4. In the same way as given in [9], we have the following theorem.

THEOREM 4.2. *If we restrict the domain of b to the space of zero mode spinors on B^4, then b gives the isomorphism*

$$b:\{\Psi\in H^s(B^4,\mathbf{S}^+(S^4)|_{B^4}) \mid D_4^+\Psi=0 \text{ on } B^4\setminus S^3\} \xrightarrow{\sim} \overline{\bigoplus_{m\ge 0} E_m}\,, \tag{4.32}$$

where $\overline{\bigoplus_{m\ge0}E_m}$ is the closure of $\bigoplus_{m\ge0}E_m$ in $H^{s-1/2}(S^3,\mathbf{S}^+(S^4)|_{S^3})$.

REMARK 4.2. Proposition 4.1 implies that we have the inverse mapping of b in the above theorem.

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