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On Sectional Curvature of Boggino-Damek-Ricci Type Spaces

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1. Introduction.

Boggino [B] proved that simply connected solvable Lie groups associated to 1 dimensional extensions of Lie algebras of Heisenberg type admit Einstein metrics with non-positive sectional curvature. These spaces contain the class of non-compact symmetric space of rank 1 and are called Damek-Ricci spaces after Damek and Ricci proved that these are harmonic spaces.

We consider a class of solvable Lie groups which includes Damek-Ricci spaces. Let $\{\mathfrak{n}, \langle\,,\rangle_{\mathfrak{n}}\}$ be a 2-step nilpotent metric Lie algebra, $\mathfrak{a}$ a 1 dimensional real vector space and A a non-zero vector in $\mathfrak{a}$. We denote the center of $\mathfrak{n}$ and the orthogonal complement of the center in $\mathfrak{n}$ by $\mathfrak{z}$ and $\mathfrak{v}$ respectively. For $k \in \mathbf{R}^+$ we define a representation f of $\mathfrak{a}$ on $\mathfrak{n}$ by

$$f(A)V = \frac{k}{2}V \quad f(A)Z = kZ \quad \text{for all } V \in \mathfrak{v} \text{ and } Z \in \mathfrak{z}\,.$$

Since $\mathfrak{a}$ acts on $\mathfrak{n}$ as a derivation by f, the semi-direct sum $\mathfrak{s}_k(A;\mathfrak{n}) = \mathfrak{n} \times_f \mathfrak{a}$ of $\mathfrak{n}$ and $\mathfrak{a}$ becomes a solvable Lie algebra. We define an inner product $\langle\,,\rangle_{\mathfrak{a}}$ on $\mathfrak{a}$ by $\langle A, A\rangle_{\mathfrak{a}} = 1$ and an inner product $\langle\,,\rangle$ on $\mathfrak{s}_k(A;\mathfrak{n})$ by the direct sum of $\langle\,,\rangle_{\mathfrak{a}}$ and $\langle\,,\rangle_{\mathfrak{n}}$. We consider the simply connected Lie group with the induced left invariant metric asociated to $\{\mathfrak{s}_k(A;\mathfrak{n}), \langle\,,\rangle\}$. We denote it by $\{S_k(A;\mathfrak{n}), g\}$ and call it a Boggino-Damek-Ricci type space (abbreviated to a BDR-type space).

Mori [M] and Yamada [Y] studied existence of Einstein metrics with non-positive sectional curvature in BDR-type spaces. By the result of Heintze [H], BDR-type spaces have non-positive sectional curvature for sufficient large k, and we can ask the following question:

Can we determine the smallest value of k such that sectional curvature of BDR-type space is non-positive?

In this paper we answer this question in the case that the nilpotent part of BDR-type space is a Lie algebra of echelon type.

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We define a Lie algebra of echelon type. For $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, we consider a natural gradation of $\mathfrak{sl}(n+m+l;\mathbf{K})$. Let

$$\mathfrak{g}_0(\mathbf{K}) = \left\{ \begin{pmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & R \end{pmatrix} \middle| \begin{array}{l} \mathrm{tr}P + \mathrm{tr}Q + \mathrm{tr}R = 0,\ P \text{ is an } n \times n \text{ matrix, } Q \text{ is} \\ \text{an } m \times m \text{ matrix and } R \text{ is an } l \times l \text{ matrix.} \end{array} \right\},$$

$$\mathfrak{g}_1(\mathbf{K}) = \left\{ \begin{pmatrix} 0 & A & 0 \\ 0 & 0 & B \\ 0 & 0 & 0 \end{pmatrix} \middle| A \text{ is an } n \times m \text{ matrix and } B \text{ is an } m \times l \text{ matrix.} \right\},$$

$$\mathfrak{g}_2(\mathbf{K}) = \left\{ \begin{pmatrix} 0 & 0 & C \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| C \text{ is an } n \times l \text{ matrix.} \right\},$$

$$\mathfrak{g}_{-1}(\mathbf{K}) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ D & 0 & 0 \\ 0 & E & 0 \end{pmatrix} \middle| D \text{ is an } m \times n \text{ matrix and } E \text{ is an } l \times m \text{ matrix.} \right\},$$

$$\mathfrak{g}_{-2}(\mathbf{K}) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ F & 0 & 0 \end{pmatrix} \middle| F \text{ is an } l \times n \text{ matrix.} \right\}.$$

Then $\mathfrak{sl}(n+m+l;\mathbf{K}) = \sum_{i=-2}^{2} \mathfrak{g}_i(\mathbf{K})$ is a graded Lie algebra and $\mathfrak{n}(n,m,l;\mathbf{K}) = \mathfrak{g}_1(\mathbf{K}) \oplus \mathfrak{g}_2(\mathbf{K})$ is a 2-step nilpotent Lie algebra. We call $\mathfrak{n}(n,m,l;\mathbf{K})$ a Lie algebra of echelon type.

Our main results are the followings.

THEOREM 3.2. *BDR-type space* $\{S_k(A;\mathfrak{n}(n,m,l;\mathbf{R})), g\}$ *has non-positive (negative) sectional curvature if and only if* $k \geq 1/\sqrt{2}$ $(k > 1/\sqrt{2})$ *respectively.*

THEOREM 4.3. *BDR-type space* $\{S_k(A;\mathfrak{n}(u,m,l;\mathbf{C})), g\}$ *has non-positive (negative) sectional curvature if and only if* $k \geq 1$ $(k > 1)$ *respectively.*

Let $\sum_{k=-2}^{2} \mathfrak{g}_k$ be a second kind simple graded Lie algebra constructed by simple Lie algebras. Then the Lie algebra $\mathfrak{n} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is 2-step nilpotent. Mori [M] has studied Einstein metrics with non-positive sectional curvature in BDR-type spaces whose nilpotent part is $\mathfrak{n} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, in the case that the graded Lie algebras are defined by certain parabolic subalgebras. BDR-type spaces $\{S_k(A;\mathfrak{n}(n,m,l;\mathbf{R})), g\}$ are also obtained by BDR-type spaces induced by type A_n algebras, but normalization of metric in [M] is different from ours.

Note that $\mathfrak{n}(1,m,1;\mathbf{R})$ is a $2m+1$ dimensional Heisenberg algebra. Wolter [W] proved the following.

THEOREM 3.1 (Wolter). *Let* $\mathfrak{n}_1^m$ *be a* $2m+1$ *dimensional Heisenberg algebra. Then* $\{S_k(A;\mathfrak{n}_1^m), g\}$ *has non-positive (negative) sectional curvature if and only if* $k \geq 1/\sqrt{2}$ $(k > 1/\sqrt{2})$ *respectively.*

In section 3 we give a proof of Theorem 3.2. Notice that Wolter's theorem is a special case of Theorem 3.2.

In section 4, as a generalization of the Boggino's theorem (i.e. a Damek-Ricci space has non-positive sectional curvature), we prove the following theorem.

THEOREM 4.2. *If* $\mathfrak{n}$ *satisfies that* $|J_Z V| \le |Z||V|$ *for all* $V \in \mathfrak{v}$ *and* $Z \in \mathfrak{z}$, *then for* $k \ge 1$ *BDR-type spaces* $\{S_k(A; \mathfrak{n}), g\}$ *have non-positive sectional curvature.*

Notice that BDR-type space $\{S_k(A; \mathfrak{n}(n, m, l; \mathbf{C})), g\}$ is an example of satisfying the assumption of Theorem 4.2.

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2. Boggino-Damek-Ricci type spaces.

In this section we discuss properties of sectional curvature of BDR-type spaces.

Let $J : \mathfrak{z} \to \mathrm{End}(\mathfrak{v})$ be a linear mapping defined by

$$\langle J_Z V_1, V_2 \rangle = \langle Z, [V_1, V_2] \rangle \quad \text{for all } V_1, V_2 \in \mathfrak{v} \text{ and } Z \in \mathfrak{z}.$$

Note that J_Z is skew-symmetric and the linear mapping J characterizes the 2-step nilpotent Lie algebra $\mathfrak{n}$.

DEFINITION 2.1. A 2-step nilpotent metric Lie algebra is called Heisenberg type, if ${J_Z}^2 = -|Z|^2 id$ for all Z in $\mathfrak{z}$.

DEFINITION 2.2. If $\mathfrak{n}$ is a Lie algebra of Heisenberg type and $k = 1$, a BDR-type space $\{S_k(A; \mathfrak{n}), g\}$ is called a Dannek-Ricci space.

We summarize basic properties of Lie algebras of Heisenberg type.

LEMMA 2.3. *Let* $V, V' \in \mathfrak{v}$ *and* $Z, Z' \in \mathfrak{z}$. *Then*

(i) $\ker(ad_V)^\perp = J_{\mathfrak{z}} V$,
(ii) *if* $|V| = 1$, *the mapping* ad_V *is a linear isometry from* $\ker(ad_V)^\perp$ *onto* $\mathfrak{z}$,
(iii) $|J_Z V| = |V||Z|$,
(iv) $\langle J_Z V, J_Z V' \rangle = |Z|^2 \langle V, V' \rangle$,
(v) $\langle J_Z V, J_{Z'} V \rangle = |V|^2 \langle Z, Z' \rangle$,
(vi) $[V, J_Z V] = |V|^2 Z$,
(vii) $J_Z J_{Z'} + J_{Z'} J_Z = -2\langle Z, Z' \rangle id$.

PROOF. See [CDKR, 3–4]. □

We compute the Levi-Civita connection ∇ and the sectional curvature κ_k of a BDR-type space $\{S_k(A; \mathfrak{n}), g\}$.

LEMMA 2.4. (i) *Let* $V_1, V_2 \in \mathfrak{v}$, $Z_1, Z_2 \in \mathfrak{z}$ *and* $r_1, r_2 \in \mathbf{R}$. *Then*

$$\nabla_{V_1+Z_1+r_1A}(V_2 + Z_2 + r_2A) = -\frac{1}{2}J_{Z_1}V_2 - \frac{1}{2}J_{Z_2}V_1 - \frac{1}{2}kr_2V_1 + \frac{1}{2}[V_1, V_2] - kr_2Z_1 + \frac{1}{2}k\langle V_1, V_2\rangle A + k\langle Z_1, Z_2\rangle A.$$

(ii) *For a plane π spanned by the orthonormal vectors $X_1 = V_1 + Z_1 + rA$ and $X_2 = V_2 + Z_2$, where $V_1, V_2 \in \mathfrak{v}$, $Z_1, Z_2 \in \mathfrak{z}$ and $r \in \mathbf{R}$, the sectional curvature $\kappa_k(\pi)$ of the BDR-type space is given by*

$$\begin{aligned}\kappa_k(\pi) = &-\frac{3}{4}|[V_1, V_2] + krZ_2|^2 - \frac{1}{4}k^2r^2|V_2|^2 - \frac{1}{4}k^2r^2|Z_2|^2 \\ &+ \frac{1}{4}|J_{Z_1}V_2|^2 + \frac{1}{4}|J_{Z_2}V_1|^2 - \langle J_{Z_1}V_1, J_{Z_2}V_2\rangle + \frac{1}{2}\langle J_{Z_1}V_2, J_{Z_2}V_1\rangle \\ &- k^2\left(\frac{1}{2}|Z_1|^2|V_2|^2 + \frac{1}{2}|Z_2|^2|V_1|^2 + \frac{1}{4}|V_1|^2|V_2|^2 + |Z_1|^2|Z_2|^2\right. \\ &\left. -\langle V_1, V_2\rangle\langle Z_1, Z_2\rangle - \frac{1}{4}\langle V_1, V_2\rangle^2 - \langle Z_1, Z_2\rangle^2\right).\end{aligned}$$

PROOF. (i) By considering $X, Y, W \in \mathfrak{s}_k(A; \mathfrak{n})$ as left invariant vector fields on the BDR-type space $\{S_k(A; \mathfrak{n}), g\}$, we have

$$2\langle \nabla_X Y, W\rangle = \langle [X, Y], W\rangle - \langle [Y, W], X\rangle - \langle [X, W], Y\rangle$$

and hence

$$\begin{aligned}\nabla_{V_1+Z_1+r_1A}(V_2 + Z_2 + r_2A) = &-\frac{1}{2}J_{Z_1}V_2 - \frac{1}{2}J_{Z_2}V_1 - \frac{1}{2}kr_2V_1 + \frac{1}{2}[V_1, V_2] - kr_2Z_1 \\ &+ \frac{1}{2}k\langle V_1, V_2\rangle A + k\langle Z_1, Z_2\rangle A.\end{aligned}$$

(ii) Let R be the Riemannian curvature tensor of the BDR-type space $\{S_k(A; \mathfrak{n}), g\}$. Then we have

$$\begin{aligned}\kappa_k(\pi) &= \langle R(X_1, X_2)(X_2), X_1\rangle \\ &= \langle \nabla_{X_1}\nabla_{X_2}X_2 - \nabla_{X_2}\nabla_{X_1}X_2 - \nabla_{[X_1,X_2]}X_2, X_1\rangle \\ &= |\nabla_{X_1}X_2|^2 - \langle \nabla_{X_1}X_1, \nabla_{X_2}X_2\rangle - \langle \mathrm{ad}_{X_2}{}^2X_1, X_1\rangle - |[X_1, X_2]|^2.\end{aligned}$$

By Lemma 2.4(i) we see that

$$\begin{aligned}\kappa_k(\pi) = &\left|-\frac{1}{2}J_{Z_1}V_2 - \frac{1}{2}J_{Z_2}V_1 + \frac{1}{2}[V_1, V_2] + \left(\frac{1}{2}k\langle V_1, V_2\rangle + k\langle Z_1, Z_2\rangle\right)A\right|^2 \\ &- \left\langle -J_{Z_1}V_1 - \frac{1}{2}krV_1, -J_{Z_2}V_2\right\rangle \\ &- \left\langle\left(\frac{1}{2}k|V_1|^2 + k|Z_1|^2\right)A, \left(\frac{1}{2}k|V_2|^2 + k|Z_2|^2\right)A\right\rangle \\ &- \left|\frac{1}{2}krV_2 + [V_1, V_2] + krZ_2\right|^2\end{aligned}$$

$$= \frac{1}{4}|J_{Z_1}V_2|^2 + \frac{1}{4}|J_{Z_2}V_1|^2 - \langle J_{Z_1}V_1, J_{Z_2}V_2\rangle + \frac{1}{2}\langle J_{Z_1}V_2, J_{Z_2}V_1\rangle$$
$$- k^2\left(\frac{1}{2}|Z_1|^2|V_2|^2 + \frac{1}{2}|Z_2|^2|V_1|^2 + \frac{1}{4}|V_1|^2|V_2|^2 + |Z_1|^2|Z_2|^2\right.$$
$$\left.-\langle V_1, V_2\rangle\langle Z_1, Z_2\rangle - \frac{1}{4}\langle V_1, V_2\rangle^2 - \langle Z_1, Z_2\rangle^2\right)$$
$$- \frac{3}{4}|[V_1, V_2] + krZ_2|^2 - \frac{1}{4}k^2r^2|V_2|^2 - \frac{1}{4}k^2r^2|Z_2|^2 . \qquad \square$$

3. Wolter's theorem and its generalization.

We consider the Lie algebra $\mathfrak{n}(n, m, l; \mathbf{R})$. We define an inner product $\langle\, ,\rangle$ on $\mathfrak{n}(n, m, l; \mathbf{R})$ by

$$\langle X, Y\rangle = \mathrm{tr}\, {}^tXY \quad \text{for all } X, Y \in \mathfrak{n}(n, m, l; \mathbf{R}) . \tag{1}$$

Then $\mathfrak{g}_2(\mathbf{R})$ is the center of $\mathfrak{n}(n, m, l; \mathbf{R})$ and $\mathfrak{g}_1(\mathbf{R})$ is the orthogonal complement of the center in $\mathfrak{n}(n, m, l; \mathbf{R})$.

Wolter [W] prove the following theorem.

THEOREM 3.1 (Wolter). *Let $\mathfrak{n}_1^m$ be the $2m + 1$ dimensional Heisenberg algebra. Then $\{S_k(A; \mathfrak{n}_1^m), g\}$ has non-positive (negative) sectional curvature if and only if $k \geq 1/\sqrt{2}$ ($k > 1/\sqrt{2}$) respectively.*

Note that our class of Lie algebras $\mathfrak{n}(n, m, l; \mathbf{R})$ contains the $2m+1$ dimensional Heisenberg algebra. In fact, $\mathfrak{n}(1, m, 1; \mathbf{R})$ is a $2m + 1$ dimensional Heisenberg algebra.

The following lemma plays an important role in the proof of Theorem 3.3.

LEMMA 3.2.

$$J_ZV = -[Z, {}^tV] \quad \textit{for all } V \in \mathfrak{g}_1(\mathbf{R}) \textit{ and } Z \in \mathfrak{g}_2(\mathbf{R}) . \tag{2}$$

PROOF. For $U \in \mathfrak{g}_1(\mathbf{R})$

$$\begin{aligned}\langle J_ZV, U\rangle &= \langle Z, [V, U]\rangle \\ &= \langle Z, VU - UV\rangle \\ &= \langle Z, VU\rangle - \langle Z, UV\rangle \\ &= \langle {}^tVZ, U\rangle - \langle Z{}^tV, U\rangle \\ &= \langle {}^tVZ - Z{}^tV, U\rangle \\ &= \langle -[Z, {}^tV], U\rangle . \end{aligned} \qquad \square$$

THEOREM 3.3. *BDR-type space $\{S_k(A; \mathfrak{n}(n, m, l; \mathbf{R})), g\}$ has non-positive (negative) sectional curvature if and only if $k \geq 1/\sqrt{2}$ ($k > 1/\sqrt{2}$) respectively.*

PROOF. By Lemma 2.4(ii), for a plane π spanned by the orthonormal vectors $U + X + rA$ and $V + Y$, where $U, V \in \mathfrak{g}_1(\mathbf{R})$, $X, Y \in \mathfrak{g}_2(\mathbf{R})$ and $r \in \mathbf{R}$, the sectional curvature $\kappa_k(\pi)$ is given by

$$
\begin{aligned}
\kappa_k(\pi) = & -\frac{3}{4}|[U, V] + krY|^2 - \frac{1}{4}k^2r^2|V|^2 - \frac{1}{4}k^2r^2|Y|^2 \\
& +\frac{1}{4}|J_Y U|^2 - \frac{1}{4}|U|^2|Y|^2 + \frac{1}{4}|J_X V|^2 - \frac{1}{4}|X|^2|V|^2 && (3) \\
& -\frac{1}{2}\langle J_X U, J_Y V\rangle + \frac{1}{2}\langle U, V\rangle\langle X, Y\rangle && (4) \\
& +\frac{1}{2}(\langle J_X V, J_Y U\rangle - \langle J_X U, J_Y V\rangle) && (5) \\
& -\frac{1}{2}\left(\frac{1}{4}|U|^2|V|^2 - \frac{1}{4}\langle U, V\rangle^2 + |X|^2|Y|^2 - \langle X, Y\rangle^2\right) && (6) \\
& -\left(k^2 - \frac{1}{2}\right)\left(\frac{1}{2}|U|^2|Y|^2 + \frac{1}{2}|V|^2|X|^2 - \langle U, V\rangle\langle X, Y\rangle\right. \\
& \left. +\frac{1}{4}|U|^2|V|^2 + |X|^2|Y|^2 - \frac{1}{4}\langle U, V\rangle^2 - \langle X, Y\rangle^2\right).
\end{aligned}
$$

First, we show that (3) + (4) ≤ 0. We write

$$
X = \begin{pmatrix} 0 & 0 & X_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & Y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
$$
$$
U = \begin{pmatrix} 0 & U_1 & 0 \\ 0 & 0 & U_1' \\ 0 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & V_1 & 0 \\ 0 & 0 & V_1' \\ 0 & 0 & 0 \end{pmatrix}.
$$

We note that X_2 and Y_2 are $n \times l$ real matrices, that U_1 and V_1 are $n \times m$ real matrices and that U_1' and V_1' are $m \times l$ real matrices. By (1) and (2), we have

$$
\begin{aligned}
(3) + (4) = & \frac{1}{4}(\operatorname{tr} {}^tY_2U_1{}^tU_1Y_2 - \operatorname{tr} {}^tU_1U_1 \operatorname{tr} {}^tY_2Y_2) && (7) \\
& +\frac{1}{4}(\operatorname{tr} {}^tX_2V_1{}^tV_1X_2 - \operatorname{tr} {}^tV_1V_1 \operatorname{tr} {}^tX_2X_2) && (8) \\
& +\frac{1}{2}(\operatorname{tr} {}^tU_1V_1 \operatorname{tr} {}^tX_2Y_2 - \operatorname{tr} {}^tX_2U_1{}^tV_1Y_2) && (9) \\
& +\frac{1}{4}(\operatorname{tr} U_1'{}^tY_2Y_2{}^tU_1' - \operatorname{tr} {}^tU_1'U_1' \operatorname{tr} {}^tY_2Y_2) && (10) \\
& +\frac{1}{4}(\operatorname{tr} V_1'{}^tX_2X_2{}^tV_1' - \operatorname{tr} {}^tV_1'V_1' \operatorname{tr} {}^tX_2X_2) && (11) \\
& +\frac{1}{2}(\operatorname{tr} {}^tU_1'V_1' \operatorname{tr} {}^tX_2Y_2 - \operatorname{tr} U_1'{}^tX_2Y_2{}^tV_1'). && (12)
\end{aligned}
$$

We denote (i, j) elements of matrices U_1, U'_1, V_1, V'_1, X_2 and Y_2 by $u_{i,j}$, $u'_{i,j}$, $v_{i,j}$, $v'_{i,j}$, $x_{i,j}$ and $y_{i,j}$ respectively. Then if $n \geq 2$, we have

$$(9) = \frac{1}{2}\left\{\left(\sum_{k=1}^{m}\sum_{i=1}^{n} u_{i,k}v_{i,k}\right)\left(\sum_{h=1}^{l}\sum_{j=1}^{n} x_{j,h}y_{j,h}\right) - \sum_{h=1}^{l}\sum_{k=1}^{m}\left(\sum_{i=1}^{n} x_{i,h}u_{i,k}\right)\left(\sum_{j=1}^{n} v_{j,k}y_{j,h}\right)\right\}$$
$$= \frac{1}{2}\sum_{h=1}^{l}\sum_{k=1}^{m}\sum_{1\leq i<j\leq n}(x_{i,h}v_{j,k} - x_{j,h}v_{i,k})(y_{i,h}u_{j,k} - y_{j,h}u_{i,k})\,.$$

As a special case of (9), we have

$$(7) = -\frac{1}{4}\sum_{h=1}^{l}\sum_{k=1}^{m}\sum_{1\leq i<j\leq n}(y_{i,h}u_{j,k} - y_{j,h}u_{i,k})^2\,,$$
$$(8) = -\frac{1}{4}\sum_{h=1}^{l}\sum_{k=1}^{m}\sum_{1\leq i<j\leq n}(x_{i,h}v_{j,k} - x_{j,h}v_{i,k})^2\,.$$

Therefore if $n \geq 2$, we see that

$$(7) + (8) + (9) = -\frac{1}{4}\sum_{h=1}^{l}\sum_{k=1}^{m}\sum_{1\leq i<j\leq n}\{(y_{i,h}u_{j,k} - y_{j,h}u_{i,k}) - (x_{i,h}v_{j,k} - x_{j,h}v_{i,k})\}^2$$
$$\leq 0\,.$$

If $n = 1$, we have $(7) = 0$, $(8) = 0$ and $(9) = 0$. Thus we see that $(7) + (8) + (9) \leq 0$. Similarly we get $(10) + (11) + (12) \leq 0$. Thus we see that $(3) + (4) \leq 0$.

Next we prove that $(5) + (6) \leq 0$. By (1) and (2), we have

$$\langle J_X V, J_Y U\rangle - \langle J_X U, J_Y V\rangle$$
$$= \operatorname{tr} {}^tX_2 V_1 {}^tU_1 Y_2 - \operatorname{tr} {}^tX_2 U_1 {}^tV_1 Y_2 \tag{13}$$
$$+\operatorname{tr} V_1'{}^tX_2 Y_2 {}^tU_1' - \operatorname{tr} U_1'{}^tX_2 Y_2 {}^tV_1'\,. \tag{14}$$

Noting that $|(A - {}^tA)B| \leq |A||B|$ for $A \in M_{n,n}(\mathbf{R})$ and $B \in M_{n,l}(\mathbf{R})$, we have

$$(13) = \langle {}^tV_1X_2, {}^tU_1Y_2\rangle - \langle {}^tU_1X_2, {}^tV_1Y_2\rangle$$
$$= \langle X_2, V_1{}^tU_1Y_2\rangle - \langle X_2, U_1{}^tV_1Y_2\rangle$$
$$\leq |X_2||(V_1{}^tU_1 - U_1{}^tV_1)Y_2|$$
$$\leq |X_2||V_1{}^tU_1||Y_2|$$
$$\leq |X_2||Y_2||V_1||U_1|\,.$$

Similarly we get

$$(14) \leq |X_2||Y_2||V'_1||U'_1|\,.$$

Thus we obtain

$$\langle J_X V, J_Y U\rangle - \langle J_X U, J_Y V\rangle \leq |X_2||Y_2|(|V_1||U_1| + |V'_1||U'_1|)$$
$$\leq |X_2||Y_2||U||V|\,.$$

We write $V = \alpha U + W$ and $Y = \beta X + Z$, where $\alpha, \beta \in \mathbf{R}$, $W \in \mathfrak{g}_1(\mathbf{R})$ with $\langle U, W\rangle = 0$ and $Z \in \mathfrak{g}_2(\mathbf{R})$ with $\langle X, Z\rangle = 0$. Then we see that

$$\begin{aligned}(5)+(6) &= \frac{1}{2}(\langle J_X W, J_Z U\rangle - \langle J_X U, J_Z W\rangle) - \frac{1}{2}\left(\frac{1}{4}|U|^2|W|^2 + |X|^2|Z|^2\right) \\ &\le \frac{1}{2}|X||Z||U||W| - \frac{1}{8}|U|^2|W|^2 - \frac{1}{2}|X|^2|Z|^2 \\ &= -\frac{1}{2}\left(\frac{1}{2}|U||W| - |X||Z|\right)^2 \\ &\le 0.\end{aligned}$$

Summing up these inequalities, we see that $\kappa_k(\pi) \le 0$ for $k \ge 1/\sqrt{2}$.

If

$$\begin{aligned}0 &= \frac{1}{2}|U|^2|Y|^2 + \frac{1}{2}|V|^2|X|^2 + \frac{1}{4}|U|^2|V|^2 + |X|^2|Y|^2 \\ &\quad - \langle U, V\rangle\langle X, Y\rangle - \frac{1}{4}\langle U, V\rangle^2 - \langle X, Y\rangle^2 \\ &= \frac{1}{2}|U|^2|Y|^2 + \frac{1}{2}|V|^2|X|^2 + \frac{1}{4}|U|^2|V|^2 + \left(|X|^2|Y|^2 - \frac{1}{4}\langle X, Y\rangle^2\right),\end{aligned}$$

then we see that $U = 0$, $X = 0$ and $r = 1$, since $U + X + rA$ and $V + Y$ are orthogonal. Then $r^2|Y|^2 \ne 0$ or $r^2|V|^2 \ne 0$. Therefore $\kappa_k(\pi) < 0$ for $k > 1/\sqrt{2}$.

If we choose a plane σ spanned by the orthonormal vectors $E_{1,n+1}$ and $E_{1,n+m+1}$, where $E_{i,j}$ is an $(n+m+l) \times (n+m+l)$ matrix unit, we have

$$\kappa_k(\sigma) = \frac{1}{4} - \frac{1}{2}k^2 = \frac{1}{2}\left(\frac{1}{2} - k^2\right).$$

This proves our Theorem 3.2. □

4. Boggino's theorem and its generalization.

We can now give a different proof of the Boggino's theorem [B].

THEOREM 4.1 (Boggino). *A Damek-Ricci space has non-positive sectional curvature.*

This theorem is obtained as a special case of the following theorem.

THEOREM 4.2. *If* $\mathfrak{n}$ *satisfies that* $|J_Z V| \le |Z||V|$ *for all* $V \in \mathfrak{v}$ and $Z \in \mathfrak{z}$, *then for* $k \ge 1$ *BDR-type spaces* $\{S_k(A; \mathfrak{n}), g\}$ *have non-positive sectional curvature.*

PROOF. By Lemma 2.4(ii), for a plane π spanned by the orthonormal vectors $U + X + rA$ and $V + Y$, where $U, V \in \mathfrak{v}$, $X, Y \in \mathfrak{z}$ and $r \in \mathbf{R}$, the sectional curvature $\kappa_k(\pi)$ is given by

$$\kappa_k(\pi) = -\frac{3}{4}|[U, V] + krY|^2 - \frac{1}{4}k^2r^2|V|^2 - \frac{1}{4}k^2r^2|Y|^2$$

$$-\frac{1}{4}(|U|^2|Y|^2-|J_YU|^2)-\frac{1}{4}(|V|^2|X|^2-|J_XV|^2) \tag{15}$$

$$-\frac{1}{4}|U|^2|Y|^2-\frac{1}{4}|V|^2|X|^2-\frac{1}{2}\langle J_XU, J_YV\rangle \tag{16}$$

$$-\frac{1}{4}|U|^2|V|^2+\frac{1}{4}\langle U, V\rangle^2-|X|^2|Y|^2+\langle X, Y\rangle^2 \tag{17}$$

$$-\frac{1}{2}\langle J_XU, J_YV\rangle+\frac{1}{2}\langle J_XV, J_YU\rangle \tag{18}$$

$$+\langle U, V\rangle\langle X, Y\rangle \tag{19}$$

$$-(k^2-1)\left(\frac{1}{2}|U|^2|Y|^2+\frac{1}{2}|V|^2|X|^2-\langle U, V\rangle\langle X, Y\rangle\right.$$
$$\left.+\frac{1}{4}|U|^2|V|^2-\frac{1}{4}\langle U, V\rangle^2+|X|^2|Y|^2-\langle X, Y\rangle^2\right).$$

By our assumption that $|J_ZV|\leq|Z||V|$ for all $V\in\mathfrak{v}$ and $Z\in\mathfrak{z}$, we have (15) ≤ 0 and

$$\begin{aligned}(16)&\leq-\frac{1}{4}|U|^2|Y|^2-\frac{1}{4}|V|^2|X|^2+\frac{1}{2}|X||U||Y||V|\\&=-\frac{1}{4}(|U||Y|-|V||X|)^2\\&\leq 0.\end{aligned}$$

We write $V=\alpha U+W$ and $Y=\beta X+Z$, where $\alpha,\beta\in\mathbf{R}$, $W\in\mathfrak{v}$ with $\langle U, W\rangle=0$ and $Z\in\mathfrak{z}$ with $\langle X, Z\rangle=0$. Then we have

$$\begin{aligned}(17)+(18)&=-\frac{1}{4}|U|^2|W|^2-|X|^2|Z|^2-\frac{1}{2}\langle J_XU, J_ZW\rangle+\frac{1}{2}\langle J_XW, J_ZU\rangle\\&\leq-\frac{1}{4}|U|^2|W|^2-|X|^2|Z|^2+|X||Z||U||W|\\&=-\left(\frac{1}{2}|U||W|-|X||Z|\right)^2\\&\leq 0.\end{aligned}$$

Since $U+X+rA$ and $V+Y$ are orthonormal vectors, (19) $=-\langle X, Y\rangle^2\leq 0$. By the above argument we see $\kappa_k(\pi)\leq 0$ for $k\geq 1$. □

We consider the Lie algebra $\mathfrak{n}(n, m, l;\mathbf{C})$. We define an inner product $\langle\,,\rangle$ on $\mathfrak{n}(n, m, l;\mathbf{C})$ by

$$\langle X, Y\rangle=\mathrm{Re}(\mathrm{tr}\,{}^tX\overline{Y})\quad\text{for all } X, Y\in\mathfrak{n}(n, m, l;\mathbf{C}), \tag{20}$$

where $\mathrm{Re}\,z=1/2(z+\bar{z})$ for $z\in\mathbf{C}$. Then $\mathfrak{g}_2(\mathbf{C})$ is the center of $\mathfrak{n}(n, m, l;\mathbf{C})$ and $\mathfrak{g}_1(\mathbf{C})$ is the orthogonal complement of the center in $\mathfrak{n}(n, m, l;\mathbf{C})$.

The following lemma plays an important role in the proof of Theorem 4.4.

LEMMA 4.3.

$$J_ZV=-[Z,{}^t\overline{V}]\quad\textit{for all } V\in\mathfrak{g}_1(\mathbf{C})\textit{ and } Z\in\mathfrak{g}_2(\mathbf{C}). \tag{21}$$

PROOF. For $U \in \mathfrak{g}_1(\mathbf{C})$

$$\begin{aligned}\langle J_Z V, U\rangle &= \langle Z, [V, U]\rangle \\ &= \langle Z, VU - UV\rangle \\ &= \langle Z, VU\rangle - \langle Z, UV\rangle \\ &= \langle \overline{{}^tV} Z, U\rangle - \langle Z\overline{{}^tV}, U\rangle \\ &= \langle \overline{{}^tV} Z - Z\overline{{}^tV}, U\rangle \\ &= \langle -[Z, \overline{{}^tV}], U\rangle .\end{aligned}$$

□

THEOREM 4.4. *BDR-type space* $\{S_k(A; \mathfrak{n}(n, m, l; \mathbf{C})), g\}$ *has non-positive* (*negative*) *sectional curvature if and only if* $k \geq 1$ ($k > 1$) *respectively*.

PROOF. Let $V \in \mathfrak{g}_1(\mathbf{C})$ and $Z \in \mathfrak{g}_2(\mathbf{C})$. We write

$$V = \begin{pmatrix} 0 & V_1 & 0 \\ 0 & 0 & V_1' \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 0 & 0 & Z_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where Z_2 is an $n \times l$ complex matrix, V_1 is an $n \times m$ complex matrix and V_1' is an $m \times l$ complex matrix.

$$J_Z V = -[Z, \overline{{}^tV}] = \begin{pmatrix} 0 & -Z_2\overline{{}^tV'_1} & 0 \\ 0 & 0 & \overline{{}^tV_1}Z_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore we have

$$\begin{aligned}|J_Z V| &= |Z_2\overline{{}^tV'_1}| + |\overline{{}^tV_1}Z_2| \\ &\leq |Z_2|(|\overline{{}^tV_1'}| + |\overline{{}^tV_1}|) \\ &= |Z||V| .\end{aligned}$$

Thus by Theorem 4.2, we see that BDR-type spaces $\{S_k(A; \mathfrak{n}(n, m, l; \mathbf{C})), g\}$ have non-positive sectional curvature for $k \geq 1$.

We can show that, for $k > 1$, BDR-type spaces $\{S_k(A; \mathfrak{n}(n, m, l; \mathbf{C})), g\}$ have negative sectional curvature in the same way as Theorem 3.3's.

For a plane σ spanned by the orthonormal vectors $\sqrt{2}/\sqrt{3}E_{1,n+1} + 1/\sqrt{3}E_{1,n+m+1}$ and $\sqrt{2}/\sqrt{3}F_{1,n+1} + 1/\sqrt{3}F_{1,n+m+1}$, where $E_{i,j}$ is an $(n+m+l) \times (n+m+l)$ matrix unit and $F_{i,j} = \sqrt{-1}E_{i,j}$, we see that the sectional curvature $\kappa_k(\sigma)$ is given by

$$\kappa_k(\sigma) = \frac{4}{9}(1 - k^2).$$

□

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