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Spherical Harmonics on $U(n)/U(n-1)$ and Associated Hilbert Spaces of Analytic Functions

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1. Introduction.

V. Bargmann showed in [1] that a generating function for the system of the Hermite polynomials can be regarded as the integral kernel of a unitary mapping from an L^2 space onto a Hilbert space of analytic functions. Then we have the following problem: is a similar construction possible for any system of classical orthogonal polynomials? That is, for any system of orthogonal polynomials, can we construct its generating function which can be regarded as the integral kernel of a unitary mapping from an L^2 space onto a Hilbert space of analytic functions? This is also indicated in Bargmann's paper.

Let $\mathbf{R}$ or $\mathbf{C}$ be the field of real or complex numbers, $S(\mathbf{R}^n)$ or $S(\mathbf{C}^n)$ the unit sphere in $\mathbf{R}^n$ or $\mathbf{C}^n$ and $x \mapsto \bar{x}$ the usual conjugation in $\mathbf{C}$.

We denote by F the Hilbert space of analytic functions $f(w)$ of n complex variables $w = {}^t(w_1, w_2, \cdots, w_n) \in \mathbf{C}^n$, with the inner product defined by

$$(f, g) = \pi^{-n} \int_{C^n} \overline{f(w)} g(w) \exp(-|w_1|^2 - \cdots - |w_n|^2) dw_1 \cdots dw_n ,$$

where

$$dw_1 \cdots dw_n = du_1 \cdots du_n dv_1 \cdots dv_n , \quad w_j = u_j + iv_j \ (u_j, v_j \in \mathbf{R}) ,$$

and by H the usual Hilbert space $L^2(\mathbf{R}^n)$.

V. Bargmann constructed in [1] a unitary mapping A from H onto F given by an integral operator whose kernel is considered as a generating function of the Hermite polynomials. More precisely, $f = A\phi$ for $\phi \in H$ is defined by

$$f(w) = \int_{R^n} A(w, t) \phi(t) d^n t ,$$

where

$$A(w, t) = \pi^{-n/4} \prod_{j=1}^{n} \exp\left\{ -\frac{1}{2}(w_j^2 + t_j^2) + 2^{1/2} w_j t_j \right\} .$$

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On the other hand, we showed in [8] that this problem can be affirmatively solved for the Gegenbauer polynomials C_m^λ, $m = 0, 1, 2, \cdots$, which essentially give the zonal spherical functions on the homogeneous space $SO(n)/SO(n-1) \cong S(\mathbf{R}^n)$. That is to say, let F_λ be the Hilbert space of analytic functions f of one complex variable on the unit disk B in $\mathbf{C}$, with the inner product given by

$$\langle f, g\rangle_\lambda = \int_B \overline{f(w)}g(w)\rho_\lambda(|w|^2)dudv \quad (w = u + iv,\ u, v \in \mathbf{R}),$$

where

$$\rho_\lambda(t) = \begin{cases} \dfrac{1}{\Gamma(2\lambda-1)}t^{\lambda-1}\displaystyle\int_t^1 s^{-\lambda}(1-s)^{2\lambda-2}ds & (\lambda > 1/2) \\ t^{\lambda-1}\left\{\dfrac{\Gamma(1-\lambda)}{\Gamma(\lambda)} - \dfrac{1}{\Gamma(2\lambda-1)}\displaystyle\int_0^t s^{-\lambda}(1-s)^{2\lambda-2}ds\right\} & (0 < \lambda \le 1/2), \end{cases}$$

and let K_λ be the usual L^2 space on the open interval $(-1, 1)$ with respect to the measure $(1-x^2)^{\lambda-1/2}dx$. Then we have the following proposition (cf. [8]).

PROPOSITION 1. *A unitary operator, $f = A_\lambda\phi$, of K_λ onto F_λ is defined by*

$$f(w) = \int_{-1}^1 A_\lambda(w, t)\phi(t)(1-t^2)^{\lambda-1/2}dt,$$

where

$$\begin{aligned} A_\lambda(w, t) &= \frac{2^{\lambda-1/2}\Gamma(\lambda+1)}{\pi}\frac{1-w^2}{(1-2wt+w^2)^{\lambda+1}} \\ &= \frac{2^{\lambda-1/2}\Gamma(\lambda)}{\pi}\sum_{m=0}^\infty (m+\lambda)C_m^\lambda(t)w^m. \end{aligned}$$

We should remark that $A_\lambda(w, t)$ can be regarded as a generating function of the Gegenbauer polynomials and the following generating function expansion plays an important role in the proof of this proposition.

$$(1-2wt+w^2)^{-\lambda} = \sum_{m=0}^\infty C_m^\lambda(t)w^m, \quad (-1 < t < 1,\ |w| < 1).$$

As stated above, the Gegenbauer polynomials give the spherical functions on the space $SO(n)/SO(n-1) \cong S(\mathbf{R}^n)$, more precisely, for a zonal spherical function ϕ on $SO(n)/SO(n-1) \cong S(\mathbf{R}^n)$, there exists a unique nonnegative integer p such that

$$\phi(b) = C_p^{(n-2)/2}(b_1)/C_p^{(n-2)/2}(1), \quad b = {}^t(b_1, \cdots, b_n) \in S(\mathbf{R}^n).$$

Here the identification $SO(n)/SO(n-1) \cong S(\mathbf{R}^n)$ is given by $kSO(n-1) \mapsto ke_1, k \in SO(n)$ and $e_1 = {}^t(1, 0, \cdots, 0) \in S(\mathbf{R}^n)$.

Let us turn to the analogous geometrical object $U(n)/U(n-1) \cong S(\mathbf{C}^n)$, where we consider that the identification $U(n)/U(n-1) \cong S(\mathbf{C}^n)$ is given by $kU(n-1) \mapsto ke_1$, $k \in U(n)$ and $e_1 = {}^t(1, 0, \cdots, 0) \in S(\mathbf{C}^n)$. Let $H_{pq}^{(n)}$ be the space of restrictions to $S(\mathbf{C}^n)$ of harmonic polynomials $f(\xi, \bar{\xi})$ on $\mathbf{C}^n$ which are homogeneous of degree p in ξ and degree q

in $\bar{\xi}$. Then it is known (cf. [5], [3]) that $H_{pq}^{(n)}$ is $U(n)$-invariant and irreducible, and moreover $L^2(U(n)/U(n-1)) = \bigoplus_{p,q=0}^{\infty} H_{pq}^{(n)}$. In what follows, we denote by $\phi_{pq}^{(n)}$ the zonal spherical function which belongs to $H_{pq}^{(n)}$ (cf. [5]).

The purpose of the present paper is to give a construction similar to that for the Hermite or Gegenbauer case for the functions $\phi_{pq}^{(n)}$. Moreover we shall give some characterization of the spherical function $\phi_{pq}^{(n)}$, which is also a generalization of the usual Poisson integral for $\phi_{pq}^{(n)}$.

Suppose that $n \geq 3$ throughout this paper.

2. Preliminary to main theorem.

In this section we deal with some lemmas preparatory to our main theorem.

Let $\lambda > -1/2$ and we denote by ρ_λ the function on the open set $(0, 1) \times (0, 1)$ in $\mathbf{R}^2$ defined by

$$\rho_\lambda(u, v) = (uv)^{\lambda/2} \int_1^{\min(1/u, 1/v)} \frac{f_\lambda(tu, tv)}{t} dt,$$

where

$$f_\lambda(u, v) = (uv)^{-\lambda/2}\{(1-u)(1-v)\}^\lambda.$$

From the condition $\lambda > -1/2$ it follows that the integral in the definition of ρ_λ converges.

Let $\mathbf{F}_\lambda$ be the Hilbert space of analytic functions $f(\xi, \eta)$ of two complex variables $(\xi, \eta) \in B \times B$, the direct product of the unit disk B in $\mathbf{C}$ with itself, with the inner product defined by

$$\langle f, g \rangle_\lambda = \int_{|\xi|<1} \int_{|\eta|<1} \overline{f(\xi, \eta)} g(\xi, \eta) \rho_\lambda(|\xi|^2, |\eta|^2) d\xi d\eta,$$

where

$$d\xi = d\xi_1 d\xi_2, \quad d\eta = d\eta_1 d\eta_2, \quad \xi = \xi_1 + i\xi_2, \quad \eta = \eta_1 + i\eta_2, \quad \xi_j, \eta_j \in \mathbf{R},$$

and let $\mathbf{K}_\lambda$ be the usual L^2 space on the unit disk B in $\mathbf{C}$ with respect to the measure $(1 - |z|^2)^{\lambda+1} dxdy$, $z = x + iy$, $x, y \in \mathbf{R}$. We consider that the inner product in $\mathbf{K}_\lambda$ is given by

$$(\varphi_1, \varphi_2)_\lambda = \int_B \overline{\varphi_1(z)} \varphi_2(z) (1 - |z|^2)^{\lambda+1} dxdy.$$

We now suppose that $-1/2 < \lambda < 0$ and define the functions $c_\lambda(u, v)$, $h_\lambda(t)$ and $\tilde{\rho}_\lambda(u, v)$ by

$$c_\lambda(u, v) = \int_0^1 f_\lambda(tu, tv) t^{-1} dt$$

$$h_\lambda(t) = \begin{cases} \displaystyle\int_0^1 f_\lambda(st, s) s^{-1} ds & (0 \leq t \leq 1) \\ \displaystyle\int_0^1 f_\lambda(st^{-1}, s) s^{-1} ds & (t > 1) \end{cases}$$

$$\tilde{\rho}_\lambda(u, v) = (uv)^{\lambda/2} (h_\lambda(uv^{-1}) - c_\lambda(u, v)).$$

We should remark that the condition $-1/2 < \lambda < 0$ guarantees the convergences of all the integrals in the above.

Then the functions c_λ, $\tilde{\rho}_\lambda$ satisfy the following partial differential equations and boundary conditions:

LEMMA 1.

$$u\frac{\partial c_\lambda}{\partial u} + v\frac{\partial c_\lambda}{\partial v} = f_\lambda$$

$$u\frac{\partial \tilde{\rho}_\lambda}{\partial u} + v\frac{\partial \tilde{\rho}_\lambda}{\partial v} = \lambda\tilde{\rho}_\lambda - (uv)^{\lambda/2} f_\lambda$$

$$\lim_{u\to 1-0} \tilde{\rho}_\lambda(u,v) = 0, \quad \lim_{u\to +0} u\tilde{\rho}_\lambda(u,v) = 0$$

$$\lim_{v\to 1-0} \tilde{\rho}_\lambda(u,v) = 0, \quad \lim_{v\to +0} v\tilde{\rho}_\lambda(u,v) = 0$$

PROOF. By the definition of c_λ,

$$\begin{aligned} u\frac{\partial c_\lambda}{\partial u}(u,v) + v\frac{\partial c_\lambda}{\partial v}(u,v) &= \int_0^1 \left(u\frac{\partial f_\lambda}{\partial u}(tu,tv) + v\frac{\partial f_\lambda}{\partial v}(tu,tv)\right) dt \\ &= \int_0^1 \left(\frac{\partial}{\partial t} f_\lambda(tu,tv)\right) dt \\ &= [f_\lambda(tu,tv)]_{t=0}^{t=1} \\ &= f_\lambda(u,v), \end{aligned}$$

which means that the first differential equation is true. It follows from this equation that the function $(uv)^{\lambda/2}c_\lambda(u,v)$ satisfies the following differential equation:

$$u\frac{\partial}{\partial u}((uv)^{\lambda/2}c_\lambda) + v\frac{\partial}{\partial v}((uv)^{\lambda/2}c_\lambda) = \lambda(uv)^{\lambda/2}c_\lambda + ((1-u)(1-v))^\lambda .$$

On the other hand, it is easy to see that the function $\tilde{h}_\lambda(u,v) = (uv)^{\lambda/2}h_\lambda(uv^{-1})$ satisfies the following:

$$u\frac{\partial \tilde{h}_\lambda}{\partial u} + v\frac{\partial \tilde{h}_\lambda}{\partial v} = \lambda\tilde{h}_\lambda .$$

So we obtain the second differential equation. It is not difficult to check the boundary conditions. □

From this lemma, we can deduce the following:

LEMMA 2.

$$\int_0^1\int_0^1 u^p v^q \tilde{\rho}_\lambda(u,v)dudv = \frac{p!q![\Gamma(\lambda+1)]^2}{(p+q+\lambda+2)\Gamma(p+\lambda+2)\Gamma(q+\lambda+2)} .$$

PROOF.

$$(p+q+\lambda+2)\int_0^1\int_0^1 u^p v^q \tilde{\rho}_\lambda(u,v)dudv = A + \tilde{A},$$

where

$$A = \left(p + \frac{\lambda+2}{2}\right) \int_0^1 \int_0^1 u^p v^q \tilde{\rho}_\lambda(u,v) du dv$$

$$\tilde{A} = \left(q + \frac{\lambda+2}{2}\right) \int_0^1 \int_0^1 u^p v^q \tilde{\rho}_\lambda(u,v) du dv .$$

If we put

$$A_0(v) = \left(p + \frac{\lambda+2}{2}\right) \int_0^1 u^p \tilde{\rho}_\lambda(u,v) du$$

$$\tilde{A}_0(u) = \left(q + \frac{\lambda+2}{2}\right) \int_0^1 v^q \tilde{\rho}_\lambda(u,v) dv ,$$

then we have

$$A = \int_0^1 A_0(v) v^q dv$$

$$\tilde{A} = \int_0^1 \tilde{A}_0(u) u^p du .$$

Using integrations by parts and the boundary conditions in Lemma 1, we see that

$$\begin{aligned} A_0(v) &= \int_0^1 (u^{p+(\lambda+2)/2})'(u^{-\lambda/2}\tilde{\rho}_\lambda(u,v))du \\ &= [u^{p+1}\tilde{\rho}_\lambda(u,v)]_{u=0}^{u=1} - \int_0^1 u^{p+(\lambda+2)/2}\frac{\partial}{\partial u}(u^{-\lambda/2}\tilde{\rho}_\lambda(u,v))du \\ &= -\int_0^1 u^{p+(\lambda+2)/2}\left(-\frac{\lambda}{2}u^{-\lambda/2-1}\tilde{\rho}_\lambda(u,v) + u^{-\lambda/2}\frac{\partial\tilde{\rho}_\lambda}{\partial u}(u,v)\right)du \\ &= -\int_0^1 u^p\left(-\frac{\lambda}{2}\tilde{\rho}_\lambda(u,v) + u\frac{\partial\tilde{\rho}_\lambda}{\partial u}(u,v)\right)du , \end{aligned}$$

and

$$\tilde{A}_0(u) = -\int_0^1 v^q\left(-\frac{\lambda}{2}\tilde{\rho}_\lambda(u,v) + v\frac{\partial\tilde{\rho}_\lambda}{\partial v}(u,v)\right)dv .$$

And moreover, from the partial differential equations in the same lemma, we obtain that

$$\begin{aligned} A + \tilde{A} = &-\int_0^1\int_0^1 u^p v^q\left(-\frac{\lambda}{2}\tilde{\rho}_\lambda(u,v) + u\frac{\partial\tilde{\rho}_\lambda}{\partial u}(u,v)\right)dudv \\ &-\int_0^1\int_0^1 u^p v^q\left(-\frac{\lambda}{2}\tilde{\rho}_\lambda(u,v) + v\frac{\partial\tilde{\rho}_\lambda}{\partial v}(u,v)\right)dudv \\ = &-\int_0^1\int_0^1 u^p v^q\left(-\lambda\tilde{\rho}_\lambda(u,v) + u\frac{\partial\tilde{\rho}_\lambda}{\partial u}(u,v) + v\frac{\partial\tilde{\rho}_\lambda}{\partial v}(u,v)\right)dudv \end{aligned}$$

$$= \int_0^1 \int_0^1 u^p v^q (uv)^{\lambda/2} f_\lambda(u,v)\,du\,dv$$
$$= \int_0^1 \int_0^1 u^p v^q (1-u)^\lambda (1-v)^\lambda\,du\,dv$$
$$= B(p+1,\lambda+1)B(q+1,\lambda+1)$$
$$= \frac{\Gamma(p+1)\Gamma(\lambda+1)}{\Gamma(p+\lambda+2)} \frac{\Gamma(q+1)\Gamma(\lambda+1)}{\Gamma(q+\lambda+2)}.$$

Thus we can conclude that

$$(p+q+\lambda+2)\int_0^1 \int_0^1 u^p v^q \tilde{\rho}_\lambda(u,v)\,du\,dv = \frac{p!q![\Gamma(\lambda+1)]^2}{\Gamma(p+\lambda+2)\Gamma(q+\lambda+2)},$$

which implies our assertion. □

By simple calculations, we have $\tilde{\rho}_\lambda = \rho_\lambda$ for $-1/2 < \lambda < 0$. Therefore, if we set $\tilde{\rho}_\lambda = \rho_\lambda$ for $\lambda \geq 0$, then this lemma holds also for $\lambda > -1/2$ owing to the analyticity with respect to λ. It follows from this result that the system of functions $u_{pq}^{(\lambda)}$, $p, q = 0, 1, 2, \cdots$, where

$$u_{pq}^{(\lambda)}(\xi,\eta) = \frac{1}{\pi\Gamma(\lambda+1)}\sqrt{\frac{(p+q+\lambda+2)\Gamma(p+\lambda+2)\Gamma(q+\lambda+2)}{p!q!}}\,\xi^p\eta^q,$$

is an orthonormal basis in $\mathbf{F}_\lambda$.

From now on, let $\lambda = n-3$. We showed in [9] the following proposition, which gives a generating function for the functions $\phi_{pq}^{(n)}$.

PROPOSITION 2. *If* $w, z \in \mathbf{C}$, $|w| < 1$, $|z| \leq 1$, *then*

$$(1 - 2\,\mathrm{Re}(wz) + |w|^2)^{1-n} = \sum_{p,q=0}^{\infty} R_{pq}^{(n)}(z) w^p \overline{w}^q,$$

where

$$R_{pq}^{(n)}(b_1) = \binom{n+p-2}{p}\binom{n+q-2}{q}\phi_{pq}^{(n)}(b), \quad b = {}^t(b_1, \cdots, b_n) \in S(\mathbf{C}^n),$$

and the identification $U(n)/U(n-1) \cong S(\mathbf{C}^n)$ *is given by* $kU(n-1) \mapsto ke_1$, $k \in U(n)$ *and* $e_1 = {}^t(1, 0, \cdots, 0) \in S(\mathbf{C}^n)$. *The series on the right hand side converges absolutely and uniformly for* $|z| \leq 1$ *and* $|w| \leq \rho$ *for each* $0 < \rho < 1$.

We should remark that this expansion is equivalent to the following:

$$(1 - \xi z - \eta\bar{z} + \xi\eta)^{1-n} = \sum_{p,q=0}^{\infty} R_{pq}^{(n)}(z)\xi^p\eta^q,$$

where $z, \xi, \eta \in \mathbf{C}$ and $|z| \leq 1$, $|\xi| < 1$, $|\eta| < 1$.

The functions $R_{pq}^{(n)}$ have the following orthogonality relation in $\mathbf{K}_{n-3}$:

$$\int_{|z|\leq 1} R_{pq}^{(n)}(z)\overline{R_{p'q'}^{(n)}(z)}(1-|z|^2)^{n-2}\,dx\,dy = \delta_{pp'}\delta_{qq'}\frac{\pi\Gamma(p+n-1)\Gamma(q+n-1)}{(p+q+n-1)p!q![\Gamma(n-1)]^2}.$$

So the normalization $H_{pq}^{(n)}$ of $R_{pq}^{(n)}$ with respect to the inner product of $\mathbf{K}_{n-3}$ is given by

$$H_{pq}^{(n)}(z) = \sqrt{\frac{(p+q+n-1)p!q![\Gamma(n-1)]^2}{\pi\Gamma(p+n-1)\Gamma(q+n-1)}} R_{pq}^{(n)}(z).$$

Then the system of the functions $H_{pq}^{(n)}$ is an orthonormal basis in $\mathbf{K}_{n-3}$.

3. Main theorem.

From what has been described in the previous section, we can obtain the following main theorem. We denote $u_{pq}^{(n-3)}$ by $U_{pq}^{(n)}$.

THEOREM 1. *A unitary operator, $f = A_n\varphi$, of $\mathbf{K}_{n-3}$ onto $\mathbf{F}_{n-3}$ is defined by*

$$f(\xi, \eta) = \int_{|z|<1} A_n(\xi, \eta; z)\varphi(z)(1-|z|^2)^{n-2}dxdy,$$

where

$$\begin{aligned} A_n(\xi, \eta; z) &= \frac{(n-2)(n-1)}{\pi^{3/2}} \frac{1-\xi\eta}{(1-\xi z-\eta\bar{z}+\xi\eta)^n} \\ &= \frac{n-2}{\pi^{3/2}} \sum_{p,q=0}^{\infty} (p+q+n-1)R_{pq}^{(n)}(z)\xi^p\eta^q \\ &= \sum_{p,q=0}^{\infty} H_{pq}^{(n)}(z)U_{pq}^{(n)}(\xi, \eta). \end{aligned} \tag{1}$$

PROOF. First of all, we notice that $\overline{H_{pq}^{(n)}(z)} = H_{qp}^{(n)}(z)$, in particular

$$A_n(\xi, \eta; z) = \sum_{p,q=0}^{\infty} \overline{H_{qp}^{(n)}(z)}U_{pq}^{(n)}(\xi, \eta).$$

For any $(\xi, \eta) \in B \times B$, we have

$$\sum_{p,q=0}^{\infty} |U_{pq}^{(n)}(\xi, \eta)|^2 < \infty.$$

So we can consider the right hand side of (1) is the Fourier expansion for $A_n(\xi, \eta; z)$ as a function of z. Thus, for $\varphi \in \mathbf{K}_{n-3}$, we have

$$\begin{aligned} (A_n\varphi)(\xi, \eta) &= \left(\sum_{p,q=0}^{\infty} \overline{U_{pq}^{(n)}(\xi, \eta)} H_{qp}^{(n)}, \varphi \right)_{n-3} \\ &= \sum_{p,q=0}^{\infty} (H_{qp}^{(n)}, \varphi)_{n-3}\, U_{pq}^{(n)}(\xi, \eta). \end{aligned}$$

Hence, $\langle A_n\varphi, A_n\varphi\rangle_{n-3} = \sum_{p,q=0}^{\infty} |(H_{pq}^{(n)}, \varphi)_{n-3}|^2$ and $A_n H_{qp}^{(n)} = U_{pq}^{(n)}$, that is to say $\langle A_n\varphi, A_n\varphi\rangle_{n-3} = (\varphi, \varphi)_{n-3}$ and the mapping A_n is onto. This implies that the mapping A_n is a unitary operator of $\mathbf{K}_{n-3}$ onto $\mathbf{F}_{n-3}$. □

4. Some characterization of the spherical function $\phi_{pq}^{(n)}$.

In this section, we shall derive some characterization of the function $\phi_{pq}^{(n)}$ using the preceding main theorem.

Let dk, dm be the normalized Haar measures on $U(n)$, $U(n-1)$ respectively and $d\sigma$ be the normalized measure on $S(\mathbf{C}^n)$ which is invariant under the action of $U(n)$. As is well known (cf. [4]), a continuous function ϕ on $S(\mathbf{C}^n)$ is a zonal spherical function on the homogeneous space $U(n)/U(n-1)$ if and only if

$$\int_{U(n-1)} \phi(k'mke_1)dm = \phi(k'e_1)\phi(ke_1)$$

for all $k', k \in U(n)$.

For a fixed pair of nonnegative integers (p, q), we denote $\phi_{pq}^{(n)}$ by ϕ. From the function equation for ϕ, for a continuous function f on B and $\omega = k'e_1 \in S(\mathbf{C}^n)$ $(k' \in U(n))$ we see that

$$\begin{aligned}
\int_{S(C^n)} f((\omega, \tau))\phi(\tau)d\sigma(\tau) &= \int_{U(n)} f((k'e_1, ke_1))\phi(ke_1)dk \\
&= \int_{U(n)} f((k'e_1, k'mke_1))\phi(k'mke_1)dk \quad (m \in U(n-1)) \\
&= \int_{U(n)} f((e_1, ke_1))\phi(k'mke_1)dk \\
&= \int_{U(n)} f((e_1, ke_1)) \left(\int_{U(n-1)} \phi(k'mke_1)dm \right) dk \\
&= \phi(k'e_1) \int_{U(n)} f((e_1, ke_1))\phi(ke_1)dk \\
&= \phi(\omega) \int_{S(C^n)} f((e_1, \tau))\phi(\tau)d\sigma(\tau) \\
&= C\phi(\omega) \int_{|z|<1} f(z)Q(z)(1-|z|^2)^{n-2}dxdy,
\end{aligned}$$

where $C = (n-1)/\pi$ and $Q(\tau_1) = \phi(\tau)$ $(\tau = {}^t(\tau_1, \tau_2, \cdots, \tau_n))$. This is nothing but another characterization for ϕ, which is well known as the Funk-Hecke (type) theorem (cf. [7]).

Now, for any $(\xi, \eta) \in B \times B$, if we set $f(z) = A_n(\xi, \eta; z)$ and apply the preceding consideration and the main theorem, then we obtain

$$\begin{aligned}
&\int_{S(C^n)} A_n(\xi, \eta; (\omega, \tau))\phi(\tau)d\sigma(\tau) \\
&\qquad = C\phi(\omega) \int_{|z|<1} A_n(\xi, \eta; z)Q(z)(1-|z|^2)^{n-2}dxdy
\end{aligned}$$

$$= CC_{pq}\phi(\omega)\int_{|z|<1} A_n(\xi,\eta;z)H_{pq}^{(n)}(z)(1-|z|^2)^{n-2}dxdy$$
$$= CC_{pq}\phi(\omega)(A_nH_{pq}^{(n)})(\xi,\eta) = CC_{pq}\phi(\omega)U_{qp}^{(n)}(\xi,\eta),$$

where the constant C_{pq} satisfies $C_{pq}H_{pq}^{(n)}(z) = Q(z)$, that is $C_{pq}H_{pq}^{(n)}(1) = Q(1) = 1$.

Thus we can deduce the following:

$$\int_{S(C^n)} A_n(\xi,\eta;(\omega,\tau))\phi(\tau)d\sigma(\tau) = C\frac{n-2}{\sqrt{\pi}}\xi^q\eta^p\phi(\omega).$$

In other words, for $(\xi,\eta)\in B\times B$, if we put

$$P_n(\xi,\eta;z) = \frac{1-\xi\eta}{(1-\xi z-\eta\bar{z}+\xi\eta)^n}, \quad (z\in\mathbf{C},\ |z|\le 1),$$

and define the integral transformation $P_n[\xi,\eta]$ of $L^2(S(\mathbf{C}^n))$ by

$$P_n[\xi,\eta]\psi(\omega) = \int_{S(C^n)} P_n(\xi,\eta;(\omega,\tau))\psi(\tau)d\sigma(\tau), \quad \psi\in L^2(S(\mathbf{C}^n)),$$

then we can conclude that:

THEOREM 2. *Let ϕ be a continuous function on $S(\mathbf{C}^n)$ which is invariant under the action of $U(n-1)$ and satisfies $\phi(e_1)=1$. Then ϕ is a zonal spherical function belonging to $H_{pq}^{(n)}$ on the homogeneous space $U(n)/U(n-1)$ if and only if*

$$P_n[\xi,\eta]\phi = \xi^q\eta^p\phi \tag{2}$$

for all $(\xi,\eta)\in B\times B$.

REMARK. In this theorem, if we set $\xi=\eta=r>0$, then the formula (2) transforms into the following:

$$P_n[r,r]\phi = r^{p+q}\phi,$$

which is the usual Poisson integral of ϕ, i.e. the kernel $P_n(r,r;z)$ is the Euclidean Poisson kernel (cf. [6]). In particular, for two pairs (p,q), (p',q') which satisfy $p+q=p'+q'$, we have

$$P_n[r,r]\phi_{pq}^{(n)} = r^{p+q}\phi_{pq}^{(n)}$$
$$P_n[r,r]\phi_{p'q'}^{(n)} = r^{p'+q'}\phi_{p'q'}^{(n)},$$

where the eigenvalue of the Poisson integral with respect to $\phi_{pq}^{(n)}$ coincides with that for the case of $\phi_{p'q'}^{(n)}$.

On the other hand, from the formula (2), we have

$$P_n[\xi,\eta]\phi_{pq}^{(n)} = \xi^q\eta^p\phi_{pq}^{(n)}$$
$$P_n[\xi,\eta]\phi_{p'q'}^{(n)} = \xi^{q'}\eta^{p'}\phi_{p'q'}^{(n)}.$$

If $(p,q)\ne(p',q')$, the eigenvalues $\xi^q\eta^p$, $\xi^{q'}\eta^{p'}$ do not always coincide. Therefore we can consider that the formula (2) gives not only a generalization of the Poisson integral but a characterization of the spherical function $\phi_{pq}^{(n)}$.

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