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On the Special Functions Higher than the Multiple Gamma-Functions

Koji KATAYAMA

Tsuda College

The present article can be viewed as a continuation of Katayama-Ohtsuki [5] and roughly speaking, concerns with the special functions derived from the multiple Riemann zeta-functions by differentiating twice. Such functions are necessary for studying zeta- or L-function whose functional equation involves $\Gamma(s)^r$, $r \geqq 2$, from the view point, at least, of the theory of Shintani L-functions. Also, for example, study of Kronecker's limit formula for Eisenstein series of Hilbert type, which will be a next task, demands our special functions since the series have $\Gamma(s)^r$, $r \geqq 2$, as Γ-factors in their functional equations.

Our theory will go quite parallel to [5] and its key is Lemma 1 in §2 which is analogous to Lemma 1 in [5]. In §1, first we quote necessary facts on multiple gamma function from Shintani [6] and we introduce new special functions: namely (r, k)-gamma functions ${}_k\Gamma_r(w; \tilde{\omega})$, Stirling (r, k)-modular forms ${}_k\rho_r(\tilde{\omega})$, $L\Gamma_r(w; \tilde{\omega})$ and auxiliary functions $L_2G_m(z)$. But we shall mainly concern with latter two. Their definitions are quite parallel to that of multiple gamma function and $LG(z)$ by Shintani [6], [7]. In §2, we derive asymptotic expansions of them on the basis of the key Lemma 1.

In §3, we construct the above functions by Weierstrass principle. In §4, the special cases for $r = 0, 1$, are considered for supplying our theory.

1. The definition of the function $L\Gamma_r(w; \tilde{\omega})$.

1.1. The multiple Riemann zeta-function. Let $w, \omega_1, \cdots, \omega_r$ be complex numbers with positive real parts. Then r-ple Riemann zeta-function ζ_r is defined by

$$\zeta_r(s; w; \tilde{\omega}) = \sum_{\tilde{m}=\tilde{0}}^{\infty} (w + m_1\omega_1 + \cdots + m_r\omega_r)^{-s}, \quad \mathrm{Re}\, s > r, \tag{1.1.1}$$

where $\tilde{\omega} = (\omega_1, \cdots, \omega_r)$, $\tilde{0} = (0, \cdots, 0)$ and $\tilde{m} = (m_1, \cdots, m_r)$, $m_i \in \mathbf{Z}$, $m_i \geqq 0$. Here

$$w^s = \exp(s \log w),$$
$$\log w = \log |w| + i \arg w, \quad -\pi < \arg w < \pi.$$

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The contour integral representation of ζ_r is well-known:

$$\zeta_r(s; w; \tilde{\omega}) = \frac{\Gamma(1-s)e^{-s\pi i}}{2\pi i} \int_{I(\lambda,\infty)} \frac{e^{-wt}t^{s-1}}{\prod_{i=1}^r (1-e^{-\omega_i t})} dt \tag{1.1.2}$$

where $0 < \lambda < \min_{1 \leqq i \leqq r}(|2\pi/\omega_i|)$ and $I(\lambda, \infty)$ is the path described in Fig. 1:

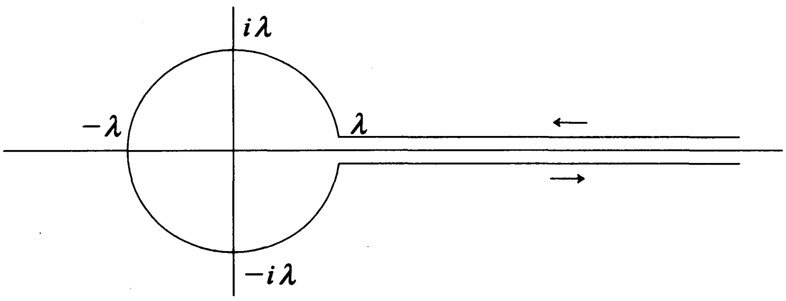

FIGURE 1.

The contour integral in the right hand side of (1.1.2) exists for any s. ζ_r is holomorphic except for simple poles at $s = 1, 2, \cdots, r$.

1.2. The multiple Bernoulli polynomials. We have the expansion of the type

$$\frac{(-1)^r t e^{-wt}}{\prod_{i=1}^r (1-e^{-\omega_i t})} = \sum_{k=1}^{r} \frac{(-1)^k {}_rS_1^{(k+1)}(w; \tilde{\omega})}{t^{k-1}} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} {}_rS'_n(w; \tilde{\omega}) t^n}{n!} \tag{1.2.1}$$

for $|t| < |2\pi/\omega_1|, \cdots, |2\pi/\omega_r|$, with polynomials, ${}_rS_n^{(k)}(w; \tilde{\omega})$ of w and of parameters $\omega_1, \cdots, \omega_r$ [1]. Here ${}_rS_n^{(k)}$ means the k-th derivative of ${}_rS_n$ with respect to w. ${}_rS_n(w; \tilde{\omega})$ is called the r-ple n-th Bernoulli polynomial.

The k-th Bernoulli polynomial $B_k(w)$ is defined by

$$\frac{te^{wt}}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k(w)}{k!} t^k .$$

$B_k = B_k(0)$ is the k-th Bernoulli number. Then

$$B_n(w) = (B + w)^n$$

where we understand that $B = B_1$ and $B^j = B_j$, the k-th Bernoulli number, in the binomial expansion of the right hand side.

From the expansions

$$\begin{aligned} \frac{(-1)^r t e^{-wt}}{\prod_{i=1}^r (1-e^{-\omega_i t})} &= \frac{\prod_{i=1}^r (-\omega_i t)}{\prod_{i=1}^r (e^{-\omega_i t} - 1)} e^{-wt} \frac{1}{t^{r-1}} \frac{(-1)^r}{\prod_{i=1}^r \omega_i} \\ &= \exp(-({}^1B\omega_1 + \cdots + {}^rB\omega_r + w)t) \frac{1}{t^{r-1}} \frac{(-1)^r}{\prod_{i=1}^r \omega_i} \end{aligned}$$

and (1.2.1), we have

$$(1.2.2)\qquad {}_rS'_n(w;\tilde\omega)=\frac{({}^1B\omega_1+\cdots+{}^rB\omega_r+w)^{n+r-1}n!}{\prod_{i=1}^r\omega_i\cdot(n+r-1)!}$$

where ${}^iB=B_1$ for every i and in the multinomial expansion of the numerator,

$$^iB^j=B_j$$

but

$$^iB^j\cdot{}^{i'}B^k\neq B_{j+k}\quad\text{for } i,i'=1,\cdots,r\,.$$

We know ([5], Proposition 1.(i))

$$(1.2.3)\qquad \zeta_r(0;w,\tilde\omega)=(-1)^r{}_rS'_1(w;\tilde\omega)\,.$$

1.3. The functions ${}_k\rho_r(\tilde\omega)$, ${}_k\Gamma_r(w;\tilde\omega)$ and $L\Gamma_r(w;\tilde\omega)$. The multiple gammafunction $\Gamma_r(w;\tilde\omega)$ and $\rho_r(\tilde\omega)$ in the sense of Barnes are defined as follows ([2], [6]):

$$-\rho_r(\tilde\omega)=\lim_{w\to0}\left[\left\{\frac{\partial}{\partial s}\zeta_r(s;w;\tilde\omega)\right\}_{s=0}+\log w\right],$$

$$\left\{\frac{\partial}{\partial s}\zeta_r(s;w;\tilde\omega)\right\}_{s=0}=\log\frac{\Gamma_r(w;\tilde\omega)}{\rho_r(\tilde\omega)}\,.$$

By (1.1.2)

$$\log\frac{\Gamma_r(w;\tilde\omega)}{\rho_r(\tilde\omega)}=\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\frac{e^{-wt}}{\prod_{i=1}^r(1-e^{-\omega_it})}\cdot\frac{\log t}{t}dt+(\gamma-\pi i)\zeta_r(0;w;\tilde\omega)$$

where $\log t$ is real valued on the upper segment of $I(\lambda,\infty)$ and γ is Euler's constant.

In what follows, for brevity, we write

$$\zeta_r^{(k)}(0;w;\tilde\omega)=\frac{\partial^k}{\partial s^k}\zeta_r(s;w;\tilde\omega)\bigg|_{s=0}.$$

Observe that

$$\begin{aligned}\zeta_r(s;w;\tilde\omega)&=w^{-s}+\sum_{j=1}^r\sum_{\tilde m_j=\tilde 0_j}^{\infty}(w+\omega_j+m_j\omega_j+\cdots+m_r\omega_r)^{-s}\\&=w^{-s}+\frac{\Gamma(1-s)e^{-s\pi i}}{2\pi i}\sum_{j=1}^r\int_{I(\lambda,\infty)}\frac{e^{-(w+\omega_j)t}t^{s-1}}{\prod_{i=j}^r(1-e^{-\omega_it})}dt\,.\end{aligned}$$

Hence the singular part at $w=0$ of $\zeta_r^{(k)}(0;w;\tilde\omega)$ is

$$\frac{\partial^kw^{-s}}{\partial s^k}\bigg|_{s=0}=(-1)^k\log^kw\,,$$

where $\log^kw=(\log w)^k$ and so we define ${}_k\rho_r(\tilde\omega)={}_k\rho_r(\omega_1,\omega_2,\cdots,\omega_r)$ by

$$(1.3.1)\qquad -\log_k\rho_r(\tilde\omega)=\lim_{w\to0}[\zeta_r^{(k)}(0;w;\tilde\omega)+(-1)^{k-1}\log^kw]\,.$$

${}_k\rho_r(\tilde\omega)$ is called the Stirling (r,k)-modular form.

Then (r, k)-gamma-function

$$ {}_k\Gamma_r(w; \tilde{\omega}) = {}_k\Gamma_r(w; \omega_1, \omega_2, \cdots, \omega_r) $$

is defined by

$$ \zeta_r^{(k)}(0; w; \tilde{\omega}) = \log \frac{{}_k\Gamma_r(w; \tilde{\omega})}{{}_k\rho_r(\tilde{\omega})}. \tag{1.3.2} $$

The function ${}_1\Gamma_r(w; \tilde{\omega}) = \Gamma_r(w; \tilde{\omega})$ is nothing but the multiple gamma-function and ${}_1\rho_r(\tilde{\omega}) = \rho_r(\tilde{\omega})$ is the Stirling's modular form in the sense of Barnes [5], [3].

We have

$$ \lim_{w \to 0} w^{(-1)^{k-1} \log^{k-1} w} \cdot {}_k\Gamma_r(w; \tilde{\omega}) = 1. $$

Put

$$ g(s) = \Gamma(1-s)e^{-s\pi i} $$

$$ I(s) = \frac{1}{2\pi i} \int_{I(\lambda,\infty)} \frac{e^{-wt}t^{s-1}}{\prod_{i=1}^r (1 - e^{-\omega_i t})} dt. $$

Then we have

$$ \zeta_r^{(k)}(0; w; \tilde{\omega}) = \log \frac{{}_k\Gamma_r(w; \tilde{\omega})}{{}_k\rho_r(\tilde{\omega})} \tag{1.3.3} $$

$$ = \sum_{p=0}^{k} \binom{k}{p} g^{(k-p)}(0) \cdot I^{(p)}(0). $$

We put, for $k = 2$,

$$ L\Gamma_r(w; \tilde{\omega}) = (g''(0) - 2g'^2(0)) \cdot I^{(0)}(0) + g^{(0)}(0) \cdot I''(0) \tag{1.3.4} $$

$$ = \frac{1}{2\pi i} \int_{I(\lambda,\infty)} \frac{e^{-wt}}{\prod_{i=1}^r (1 - e^{-\omega_i t})} \cdot \frac{\log^2 t}{t} dt $$

$$ + \left(\frac{7}{6}\pi^2 + 2\pi i \gamma - \gamma^2\right) \zeta_r(0; w; \tilde{\omega}), $$

Thus

$$ \zeta_r''(0; w; \tilde{\omega}) = L\Gamma_r(w; \tilde{\omega}) + 2(\gamma - \pi i) \log \frac{\Gamma_r(w; \tilde{\omega})}{\rho_r(\tilde{\omega})}. \tag{1.3.5} $$

1.4. Some properties of $L\Gamma_r(w; \tilde{\omega})$. We denote by $\tilde{\omega}(i)$ the $(r-1)$-tuple obtained by omitting ω_i from $\tilde{\omega}$. Then it is known that the difference equation

$$ \frac{\Gamma_r(w; \tilde{\omega})}{\Gamma_r(w + \omega_i; \tilde{\omega})} = \frac{\Gamma_{r-1}(w; \tilde{\omega}(i))}{\rho_{r-1}(\tilde{\omega}(i))} \tag{1.4.1} $$

holds for every $i = 1, 2, \cdots, r$ $(r > 1)$, ([5]). Further, for positive t,

$$ \log \frac{\Gamma_r(tw; t\tilde{\omega})}{\rho_r(t\tilde{\omega})} = -\zeta_r(0; w; \tilde{\omega}) \log t + \log \frac{\Gamma_r(w; \tilde{\omega})}{\rho_r(\tilde{\omega})} \tag{1.4.2} $$

holds ([5], (1.2.8). For $r = 1$, see [7]).

PROPOSITION 1.

$$L\Gamma_r(w+\omega_i;\tilde{\omega}) - L\Gamma_r(w;\tilde{\omega}) = -L\Gamma_{r-1}(w;\tilde{\omega}(i))$$

holds for every $i = 1, 2, \cdots, r$ $(r > 1)$.

PROOF. Our Proposition follows easily from (1.3.5), (1.4.1) and the fact

$$\zeta_r''(0; w+\omega_i; \omega) - \zeta_r''(0; w; \omega) = -\zeta_{r-1}''(0; w; \omega(i)).$$

PROPOSITION 2. *For positive* t,

$$L\Gamma_r(tw; t\tilde{\omega}) = L\Gamma_r(w;\tilde{\omega}) + \log t(\log t + 2\gamma - 2\pi i)\zeta_r(0; w; \tilde{\omega}) - 2\log t \cdot \log\frac{\Gamma_r(w;\tilde{\omega})}{\rho_r(\tilde{\omega})}.$$

PROOF. We have

$$\zeta_r(s; tw; t\tilde{\omega}) = t^{-s}\zeta_r(s; w; \tilde{\omega}),$$

$$\zeta_r''(0; tw; t\tilde{\omega}) = \log^2 t\zeta_r(0; w; \tilde{\omega}) - 2(\log t)\zeta_r''(s; w; \tilde{\omega}) + \zeta_r''(s; w; \tilde{\omega}).$$

From this and (1.3.5), (1.3.3) for $k = 1$, it follows

$$L\Gamma_r(tw; t\tilde{\omega}) + 2(\gamma - \pi i)\log\frac{\Gamma_r(tw; t\tilde{\omega})}{\rho_r(t\tilde{\omega})}$$

$$= (\log^2 t)\zeta_r(0; w; \tilde{\omega}) - 2(\log t)\log\frac{\Gamma_r(w;\tilde{\omega})}{\rho_r(\tilde{\omega})} + 2(\gamma - \pi i)\log\frac{\Gamma_r(w;\tilde{\omega})}{\rho_r(\tilde{\omega})} + L\Gamma_r(w;\tilde{\omega}).$$

Then (1.4.2) shows our Proposition.

2. Asymptotic formulas.

2.1. Key lemma. For the investigation of $L\Gamma_r(w;\tilde{\omega})$, it is necessary to establish the Lemma analogous to Lemma 1 of [3]. It is easily derived from Hankel's representation of $\Gamma(s)$ [9]:

$$\Gamma(s) = \frac{i}{2\sin\pi s}\int_{I(\lambda,\infty)} e^{-t}(-t)^{s-1}dt \tag{2.1.1}$$

$$= \frac{1}{e^{2\pi i s} - 1}\int_{I(\lambda,\infty)} e^{-t}t^{s-1}dt$$

where in $(-t)^{s-1} = e^{(s-1)\log(-t)}$, $\log(-t)$ takes a real value for $t = -\lambda$.

For $z > 0$, we have

$$\Gamma(s)z^{-s}(e^{2\pi i s} - 1) = \int_{I(\lambda,\infty)} e^{-zt}t^{s-1}dt \tag{2.1.2}$$

Put

$$\varphi(s) = \Gamma(s)z^{-s}(e^{2\pi i s} - 1).$$

Define, for $h \geqq 0$, $h \in \mathbf{Z}$,

$$c_h = \sum_{h_1+\cdots+h_m=h}\left(\prod_{i=1}^{m} i^{-h_i}h_i!\right), \quad c_0 = 1, \tag{2.1.3}$$

where the sum runs over all tuples $(h_1, \cdots, h_m)$ of non-negative integers $h_1, \cdots, h_m$ satisfying $h_1 + \cdots + h_m = h$.

Further put

$$\varepsilon_p = 1 \quad \text{for } p \geqq 1 \text{ and } \varepsilon_0 = 0.$$

The Laurent expansion of $\Gamma(s)$ at $s = -m$, $m \in \mathbf{Z}^+$, is given by

$$\text{(2.1.4)} \qquad \Gamma(s) = \frac{(-1)^m}{m!}\frac{1}{s+m} + \frac{(-1)^m}{m!}\sum_{n=0}^{\infty}\left\{c_{n+1} + \sum_{h+l=n} c_h \frac{\Gamma^{(l+1)}(1)}{(l+1)!}\right\}(s+m)^n.$$

In fact, the Taylor expansion of

$$f(s) = \frac{1}{(s-1)(s-2)\cdots(s-m)}$$

at $s = 0$ is

$$f(s) = \frac{(-1)^m}{m!}\sum_{h=0}^{\infty} c_h s^h$$

and the Laurent expansion of $\Gamma(s)$ at $s = 0$ is

$$\Gamma(s) = \frac{1}{s} + \sum_{k=0}^{\infty}\frac{\Gamma^{(k+1)}(1)}{(k+1)!}s^k.$$

Thus, at $s = 0$,

$$\Gamma(s-m) = \Gamma(s)f(s) = \frac{(-1)^m}{m!}\left[\frac{1}{s} + \sum_{n=0}^{\infty}\left\{c_{n+1} + \sum_{h+l=n} c_h\frac{\Gamma^{(l+1)}(1)}{(l+1)!}\right\}s^n\right].$$

Then changing s to $s+m$, we have (2.1.4).

Thus we have the Taylor expansion of $\varphi(s)$ at $s = -m$:

$$\begin{aligned}\varphi(s) = {} & \frac{(-z)^m}{m!}\sum_{n=0}^{\infty}\left(\sum_{p+q=n}\frac{\varepsilon_p(2\pi i)^p}{p!}\cdot\frac{(-\log z)^q}{q!}\right)(s+m)^{n-1} \\ & + \frac{(-z)^m}{m!}\sum_{n=0}^{\infty}\left\{\sum_{h+l=n} c_{h+1}\left(\sum_{p+q=l}\frac{\varepsilon_p(2\pi i)^p}{p!}\cdot\frac{(-\log z)^q}{q!}\right)\right\}(s+m)^n \\ & + \frac{(-z)^m}{m!}\sum_{n=0}^{\infty}\sum_{j+v=n}\left\{\left(\sum_{h+l=j} c_h\frac{\Gamma^{(l+1)}(1)}{(l+1)!}\right)\right. \\ & \left. \times\left(\sum_{p+q=v}\frac{\varepsilon_p(2\pi i)^p}{p!}\cdot\frac{(-\log z)^q}{q!}\right)\right\}(s+m)^n.\end{aligned}$$

Then, putting

$$A_m = \sum_{k=1}^{m}\frac{1}{k} \quad \text{for } m \geqq 1, \quad A_0 = 0,$$

$$A_m^{(2)} = \sum_{k=1}^{m}\frac{1}{k^2} \quad \text{for } m \geqq 1, \quad A_0^{(2)} = 0,$$

we have the following

LEMMA 1. *For* $\operatorname{Re} z > 0$ and an integer $m \geqq 0$

$$(1)\quad \frac{1}{2\pi i}\int_{I(\lambda,\infty)} \frac{e^{-zt}}{t^{m+1}} \log^n t\,dt$$
$$= \frac{1}{2\pi i}\cdot\frac{(-z)^m}{m!} n!\Bigg[\Bigg(\sum_{p+q=n+1} \frac{\varepsilon_p(2\pi i)^p}{p!}\cdot\frac{(-\log z)^q}{q!}\Bigg)$$
$$+\sum_{h+l=n} c_{h+1}\Bigg(\sum_{p+q=l}\frac{\varepsilon_p(2\pi i)^p}{p!}\cdot\frac{(-\log z)^q}{q!}\Bigg)$$
$$+\sum_{j+v=n}\Bigg\{\Bigg(\sum_{h+l=j} c_h\frac{\Gamma^{(l+1)}(1)}{(l+1)!}\Bigg)\Bigg(\sum_{p+q=v}\frac{\varepsilon_p(2\pi i)^p}{p!}\cdot\frac{(-\log z)^q}{q!}\Bigg)\Bigg\}\Bigg].$$

In particular

$$(2)\quad \frac{1}{2\pi i}\int_{I(\lambda,\infty)} \frac{e^{-zt}}{t^{m+1}}\log t\,dt = \frac{(-z)^m}{m!}(A_m - \log z - \gamma + \pi i). \quad (=\textit{Lemma } 1, [5])$$

$$(3)\quad \frac{1}{2\pi i}\int_{I(\lambda,\infty)} \frac{e^{-zt}}{t^{m+1}}\log^2 t\,dt = \frac{(-z)^m}{m!}\Big\{-\frac{7}{6}\pi^2 + 2(\pi i - \log z - \gamma)A_m$$
$$-2\pi i(\log z + \gamma) + \gamma^2 + \log^2 z + 2\gamma\log z + A_m^{(2)} + (A_m)^2\Big\}.$$

Here, note that the left hand side of (1) *is*

$$\frac{d^n}{ds^n}\left(\frac{1}{2\pi i}\int_{I(\lambda,\infty)} e^{-t}t^{s-1}dt\right)\Bigg|_{s=-m}.$$

As in [5], we write

$$\left[\sum_{n=0}^{\infty} a_n x^n\right]_m = \sum_{n=0}^{m} a_n x^n,$$
$$\left\{\sum_{n=0}^{\infty} a_n x^n\right\}_{m+1} = \sum_{n=m+1}^{\infty} a_n x^n,$$
$$\frac{t^r e^{-at}}{\prod_{i=1}^{r}(1-e^{-\omega_i t})} = \sum_{n=0}^{\infty}\frac{{}_rT_n(a;\tilde\omega)t^n}{n!}.$$

Hence

$${}_rT_n(a;\tilde\omega) = \begin{cases} (-1)^n n!\,{}_rS_1^{(r-n+1)}(a;\tilde\omega) & 0 \leqq n \leqq r-1 \\ \dfrac{(-1)^n n!\,{}_rS'_{n+1-r}(a;\tilde\omega)}{(n-r+1)!} & r \leqq n. \end{cases}$$

Then by Lemma 1, we have the following

LEMMA 2. *For* $\operatorname{Re} a \geqq 0$, $\operatorname{Re} z > 0$ *and* $\operatorname{Re} \omega_i > 0$, $i = 1, \cdots, r$,

$$\frac{1}{2\pi i} \int_{I(\lambda,\infty)} \frac{1}{t^{m+1}} \left[\frac{t^r e^{-at}}{\prod_{i=1}^{r}(1 - e^{-\omega_i t})} \right]_m e^{-zt} \log^2 t dt$$
$$= \sum_{n=0}^{m} \frac{{}_rT_n(a;\tilde{\omega})}{n!} \frac{(-z)^{m-n}}{(m-n)!} \cdot \left\{ -\frac{7}{6}\pi^2 + 2(\pi i - \log z - \gamma) A_{m-n} \right.$$
$$\left. - 2\pi i (\log z + \gamma) + \gamma^2 + \log^2 z + 2\gamma \log z + A_{m-n}^{(2)} + (A_{m-n})^2 \right\}.$$

The following is Lemma 3 of [5]:

LEMMA 3. (1) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} {}_rT_k(a;\tilde{\omega})(-z)^{n-k} = {}_rT_n(z + a;\tilde{\omega})$

(2) $\displaystyle\sum_{k=0}^{r} \binom{r}{k} {}_rT_k(a;\tilde{\omega})(-z)^{r-k} = (-1)^r r!{}_rS_1'(z + a;\tilde{\omega})$.

LEMMA 4. *For* $\operatorname{Re} z > 0$, *and* $\operatorname{Re} a \geqq 0$,

$$\lim_{\lambda\to 0} \int_{U(\lambda)} \frac{1}{t^{m+1}} \left\{ \frac{te^{-at}}{1 - e^{-t}} \right\}_{m+1} e^{-zt} \log^2 t dt = 0 .$$

This Lemma corresponds to Lemma 4 *of* [5] *and follows easily from*

$$\lim_{\lambda\to 0} \int_{U(\lambda)} t^m \log^2 t = 0, \quad m \geqq 0 .$$

LEMMA 5. *For* $\operatorname{Re} a \geqq 0$, $\operatorname{Re} z > 0$ *and* $\operatorname{Re} \omega_i > 0$, $i = 1, \cdots, r$,

$$\int_0^\infty \frac{1}{t^{m+1}} \left\{ \frac{t^r e^{-at}}{\prod_{i=1}^{r}(1 - e^{-\omega_i t})} \right\}_{m+1} e^{-zt} \log t dt$$
$$= \sum_{n=m+1}^{\infty} \frac{{}_rT_n(a;\tilde{\omega})(A_{n-m-1} - \gamma - \log z)}{n(n-1)\cdots(n-m)z^{n-m}},$$

("*asymptotically*" *in an extended sence*).

PROOF. First, we consider the integral

$$J_k = \int_0^\infty e^{-zt} t^{k-1} \log t dt , \quad k \geqq 1 .$$

We have

$$J_1 = -\frac{\gamma}{z} - \frac{\log z}{z}$$

because

$$J_1 = \lim_{\lambda\to 0} \int_\lambda^\infty e^{-zt} \log t dt$$

and

$$\begin{aligned}\int_\lambda^\infty e^{-zt}\log t\,dt &= \left[-\frac{1}{z}e^{-zt}\log t\right]_\lambda^\infty + \frac{1}{z}\int_\lambda^\infty \frac{e^{-zt}}{t}dt \\ &= \frac{e^{-zt}}{z}\log\lambda + \frac{1}{z}(-\gamma - \log z - \log\lambda) + \varepsilon(\lambda) \\ &= -\frac{\gamma}{z} - \frac{\log z}{z} + \varepsilon(\lambda)\end{aligned}$$

since from Lemma 1, (2), it follows easily

$$\int_\lambda^\infty \frac{e^{-zt}}{t}dt = -\gamma - \log z - \log\lambda + \varepsilon(\lambda)$$

where $\varepsilon(\lambda)$ is any quantity which goes to 0 when $\lambda \to 0$.

We have the recurrence formula for J_k:

$$J_k = \frac{k-1}{z}J_{k-1} + \frac{\Gamma(k-1)}{z^k}.$$

From this, follow

$$\begin{aligned}J_k &= \frac{(k-1)!}{z^{k-1}}J_1 + \frac{A_{k-1}\Gamma(k)}{z^{k-1}}, \\ J_k &= \frac{(k-1)!}{z^{k-1}}\left(-\frac{\gamma}{z} - \frac{\log z}{z} + \frac{A_{k-1}}{z}\right).\end{aligned}$$

Then from

$$\int_0^\infty \frac{1}{t^{m+1}}\left\{\frac{t^r e^{-at}}{\prod_{i=1}^r(1-e^{-\omega_i t})}\right\}_{m+1} e^{-zt}\log t\,dt = \sum_{n=m+1}^\infty \frac{{}_rT_n(a;\tilde{\omega})}{n!}\int_0^\infty e^{-zt}t^{n-m-1}\log t\,dt,$$

our lemma follows.

We shall quote Lemma 5 of [5] as

LEMMA 6. *For* $\operatorname{Re} a \geqq 0$, $\operatorname{Re} z > 0$ *and* $\operatorname{Re}\omega_i > 0$, $i = 1, \cdots, r$,

$$\begin{aligned}&\int_0^\infty \frac{1}{t^{m+1}}\left\{\frac{t^r e^{-at}}{\prod_{i=1}^r(1-e^{-\omega_i t})}\right\}_{m+1} e^{-zt}dt \\ &= \sum_{n=m+1}^\infty \frac{{}_rT_n(a;\tilde{\omega})}{n(n-1)\cdots(n-m)z^{n-m}}, \quad (\textit{asymptotically}).\end{aligned}$$

2.3. The function $L_2G_m(z)$. For every integer $m \geqq 0$ and for $\operatorname{Re} z > 0$, we define the function $L_2G_m(z)$:

$$\begin{aligned}L_2G_m(z) = &\frac{1}{2\pi i}\int_{I(\lambda,\infty)} \frac{t}{1-e^{-t}}\frac{e^{-zt}}{t^{m+1}}\log^2 t\,dt \\ &+ \left(\frac{7}{6}\pi^2 + 2\pi i\gamma - \gamma^2\right)\frac{(-1)^m}{m!}B_m(z), \quad \operatorname{Re} z > 0.\end{aligned}$$

This is the function analogous to, or in the neighborhood of, $LG_m(z)$:

$$LG_m(z) = \frac{1}{2\pi i}\int_{I(\lambda,\infty)} \frac{t}{1-e^{-t}}\frac{e^{-zt}}{t^{m+1}}\log t\,dt + (\gamma - \pi i)\frac{(-1)^m}{m!}B_m(z)$$

considered in [5]. Shintani is the first who introduced $LG_1(z) = LG(z)$, [7].

PROPOSITION 3. *For* $\operatorname{Re} z > 0$

(1) $$L_2G_m(z) = 2(\pi i - \log z - \gamma)\frac{(-1)^m}{m!}\sum_{k=0}^{m}\binom{m}{k}B_k z^{m-k}A_{m-k}$$
$$+\frac{(-1)^m}{m!}\sum_{k=0}^{m}\binom{m}{k}B_k z^{m-k}(A_{m-k}^{(2)} + (A_{m-k})^2 - \log^2 z + 2\gamma\log z - 2\pi i\log z)$$
$$+2\sum_{n=m+1}^{\infty}\frac{(-1)^n B_n(A_{n-m-1} + \pi i - \gamma - \log z)}{n(n-1)\cdots(n-m)z^{n-m}}, \quad (\text{“}asymptotically\text{”}).$$

(2) $$\frac{dL_2G_m(z)}{dz} = -L_2G_{m-1}(z), \quad (m \geqq 1).$$

(3) $$L_2G_1(z) = L\Gamma_1(z;1),$$

(4) $$\frac{dL_2G_0(z)}{dz} = -(1-\gamma+2\pi i)\zeta(2,z) - 2\zeta'(2,z),$$

where $\zeta(s,z)$ *is the Hurwitz zeta-function*

$$\zeta(s,z) = \sum_{m=0}^{\infty}(z+m)^{-s} \quad \operatorname{Re} s > 1 \tag{2.3.1}$$
$$= \frac{\Gamma(1-s)e^{-s\pi i}}{2\pi i}\int_{I(\lambda,\infty)}\frac{e^{-zt}t^{s-1}}{1-e^{-t}}dt = \zeta_1(s;z,1)$$

and ζ' *means the differentiation of* ζ *with respect to s.*

(5) $$L_2G_m(z+1) - L_2G_m(z) = \frac{-(-z)^{m-1}}{(m-1)!}\{-2\pi i\log z + \log^2 z + 2\gamma\log z$$
$$+A_{m-1}^{(2)} + (A_{m-1})^2 + 2(\pi i - \log z - \gamma)A_{m-1}\}, \quad m \geqq 1.$$

(6) $$L_2G_m(1) = \frac{(-1)^{m-1}}{(m-1)!}(\zeta''(1-m) + 2(\pi i + A_{m-1} - \gamma)\zeta'(1-m))$$
$$+(A_{m-1}^{(2)} + (A_{m-1} - 2\gamma + 2\pi i)A_{m-1})\frac{(-1)^m B_m}{m!}, \quad m \geqq 2,$$
$$L_2G_1(1) = \zeta''(0) + (\gamma - \pi i)\log(2\pi),$$

where $\zeta(s)$ *is the Riemann zeta-function defined by*

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}, \quad \operatorname{Re} s > 1$$

and has the contour integral expression

$$\zeta(s) = \frac{\Gamma(1-s)e^{-s\pi i}}{2\pi i}\int_{I(\lambda,\infty)}\frac{e^{-t}t^{s-1}}{1-e^{-t}}dt \tag{2.3.2}$$

valid for any s.

PROOF.

$$(1)\quad L_2G_m(z) = \frac{1}{2\pi i}\int_{I(\lambda,\infty)} \frac{t}{1-e^{-t}}\frac{e^{-zt}}{t^{m+1}}\log^2 t dt$$
$$+\left(\frac{7}{6}\pi^2 + 2\pi i\gamma - \gamma^2\right)\frac{(-1)^m}{m!}B_m(z)$$
$$= \frac{1}{2\pi i}\int_{I(\lambda,\infty)} \left[\frac{t}{1-e^{-t}}\right]_m \frac{e^{-zt}}{t^{m+1}}\log^2 t dt$$
$$+\frac{1}{2\pi i}\int_{I(\lambda,\infty)} \left\{\frac{t}{1-e^{-t}}\right\}_{m+1} \frac{e^{-zt}}{t^{m+1}}\log^2 t dt$$
$$+\left(\frac{7}{6}\pi^2 + 2\pi i\gamma - \gamma^2\right)\frac{(-1)^m}{m!}B_m(z)\,.$$

The first integral is evaluated in Lemma 2 for $r = 1$. Then the last term of the above is cancelled. The second integral is divided into the sum of the integral on $U(\lambda)$, which is 0 by Lemma 4, and of the integral on the infinite interval $[\lambda, \infty)$, which becomes

$$2\int_0^\infty \left\{\frac{t}{1-e^{-t}}\right\}_{m+1}\frac{e^{-zt}}{t^{m+1}}\log t dt + 2\pi i\int_0^\infty \left\{\frac{t}{1-e^{-t}}\right\}_{m+1}\frac{e^{-zt}}{t^{m+1}}dt\,.$$

These integrals are given in Lemma 5 and 6, both for $r = 1$. Note that

$${}_1T_n(0;1) = (-1)^n B_n\,,\quad n \geqq 1\,,$$
$${}_1T_0(0;1) = 1\,.$$

(2) and (3) are easily shown.

(4) To compute

$$\frac{dL_2G_0(z)}{dz} = -\frac{1}{2\pi i}\int_{I(\lambda,\infty)} \frac{t}{1-e^{-t}}e^{-zt}\log^2 t dt\,,$$

we note first that

$$\int_{I(\lambda,\infty)} \frac{te^{-zt}}{1-e^{-t}}dt = 0 \tag{2.3.3}$$

holds. This is easily shown by putting $s = 2$ in (2.3.1).

Next, we consider

$$\zeta(s,z)\Gamma(s)\sin\pi s = \frac{e^{-s\pi i}}{2i}\int_{I(\lambda,\infty)} \frac{e^{-zt}t^{s-1}}{1-e^{-t}}dt\,, \tag{2.3.4}$$

which is obtained by multiplying $\Gamma(s)$ to both hands of (2.3.1) and by using

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s}\,.$$

Differentiate both hands of (2.3.4) with respect to s and put $s = 2$. Then we have, by using (2.3.3),

$$\text{(2.3.5)} \qquad \pi\zeta(2,z) = \frac{1}{2i}\int_{I(\lambda,\infty)} \frac{te^{-zt}}{1-e^{-t}}\log t\,dt$$
$$= \frac{1}{2i}\int_{U(\lambda)} \frac{te^{-zt}}{1-e^{-t}}\log t\,dt + \pi\int_{\lambda}^{\infty} \frac{t}{1-e^{-t}}e^{-zt}\,dt\,.$$

The integral on $U(\lambda)$ is 0. Hence

$$\text{(2.3.6)} \qquad \zeta(2,z) = \int_0^{\infty} \frac{t}{1-e^{-t}}e^{-zt}\,dt\,.$$

Differentiate both hands of (2.3.4) twice and put $s=2$. Then, by (2.3.6),

$$\pi\Gamma'(2)\zeta(2,z) + 2\pi\zeta'(2,z) = -\frac{\pi^2}{2i}\int_{I(\lambda,\infty)} \frac{te^{-zt}}{1-e^{-t}}dt - \pi\int_{I(\lambda,\infty)} \frac{te^{-zt}}{1-e^{-t}}\log t\,dt$$
$$+ \frac{1}{2i}\int_{I(\lambda,\infty)} \frac{t}{1-e^{-t}}e^{-zt}\log^2 t\,dt\,.$$

By (2.3.3) and (2.3.5), we have

$$\pi\Gamma'(2)\zeta(2,z) + 2\pi\zeta'(2,z) = -2\pi^2 i\zeta(2,z) + \frac{1}{2i}\int_{I(\lambda,\infty)} \frac{t}{1-e^{-t}}e^{-zt}\log^2 t\,dt$$

and (4), since

$$\Gamma'(2) = 1-\gamma\,.$$

(5) Since

$$B_m(z+1) - B_m(z) = mz^{m-1}\,,$$

we have

$$L_2G_m(z+1) - L_2G_m(z) = \frac{-1}{2\pi i}\int_{I(\lambda,\infty)} \frac{e^{-zt}\log^2 t}{t^m}dt$$
$$+ \left(\frac{7}{6}\pi^2 + 2\pi i\gamma - \gamma^2\right)\frac{(-1)^m}{(m-1)!}z^m\,.$$

Then (5) is straightforward by Lemma 1, (3).

(6) Put

$$g(s) = \Gamma(1-s)e^{-s\pi i}\,.$$

Then, for a positive integer m,

$$g(1-m) = (-1)^{m-1}\Gamma(m)\,,$$
$$g'(1-m) = (-1)^m\pi i\Gamma(m) + (-1)^m\Gamma'(m)\,,$$
$$g''(1-m) = (-1)^m\pi^2\Gamma(m) - 2(-1)^m\pi i\Gamma'(m) - (-1)^m\Gamma''(m)\,.$$

Differentiating (2.3.2) twice with respect to s and putting $s=1-m$, we have

$$\text{(2.3.7)} \quad \zeta''(1-m) = \frac{g''(1-m)}{g(1-m)}\zeta(1-m) + 2g'(1-m)\left(LG_m(1) - (\gamma-\pi i)\frac{B_m}{(m-1)!}\right)$$
$$+ g(1-m)\left\{L_2G_m(1) - \left(\frac{7}{6}\pi^2 + 2\pi i\gamma - \gamma^2\right)\frac{B_m}{(m-1)!}\right\}.$$

Here we used

$$B_m(1) = (-1)^m B_m\,, \quad m \geqq 1\,.$$

It is known (c.f. [5]) that

$$\zeta(1-m) = -\frac{B_m}{m}\,, \quad m \geqq 1$$

$$\zeta(0) = -\frac{1}{2}\,,$$

$$\frac{\Gamma'(m)}{\Gamma(m)} = A_{m-1} - \gamma\,, \quad m \geqq 1\,.$$

$$LG_1(1) = -\frac{1}{2}\log(2\pi) = \zeta'(0)\,,$$

$$LG_m(1) = \frac{(-1)^{m-1}}{(m-1)!}\zeta'(1-m) + \frac{(-1)^m B_m}{m!}A_{m-1}\,, \quad m \geqq 2\,.$$

Also, from the well known formula

$$\frac{\Gamma'(s)}{\Gamma(s)} = -\gamma - \frac{1}{s} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+s}\right).$$

we can deduce

$$\frac{\Gamma''(m)}{\Gamma(m)} = \frac{\pi^2}{6} - A_{m-1}^{(2)} + (A_{m-1} - \gamma)^2\,.$$

Hence

$$\Gamma''(1) = \gamma^2 + \frac{\pi^2}{6}\,.$$

Here we used

$$\zeta(2) = \frac{\pi^2}{6}\,.$$

Put these data into (2.3.7). Then a straightforward computation, depending on the case of even and odd m, gives (6) for $m \geqq 2$.

For $m = 1$, we have the value of $L_2G_1(1)$ from (2.3.7).

2.4. Asymptotic formula for $L_2G_m(z+a)$. We know

$$\begin{aligned} L_2G_m(z+a) = {} & \frac{1}{2\pi i}\int_{I(\lambda,\infty)}\left[\frac{te^{-at}}{1-e^{-t}}\right]_m \frac{e^{-zt}}{t^{m+1}}\log^2 t\,dt \\ & + \frac{1}{2\pi i}\int_{I(\lambda,\infty)}\left\{\frac{te^{-at}}{1-e^{-t}}\right\}_{m+1} \frac{e^{-zt}}{t^{m+1}}\log^2 t\,dt \\ & + \left(\frac{7}{6}\pi^2 + 2\pi i\gamma + \gamma^2\right)\frac{(-1)^m}{m!}B_m(z+a)\,. \end{aligned}$$

Now

$${}_1T_n(a;1) = (-1)^n B_n(a)\,.$$

Then Lemma 2 for $r = 1$ shows

$$\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\left[\frac{te^{-at}}{1-e^{-t}}\right]_m \frac{e^{-zt}}{t^{m+1}}\log^2 t\,dt$$
$$=\sum_{n=0}^{m}\frac{(-1)^n B_n(a)}{n!}\frac{(-z)^{m-n}}{(m-n)!}\left\{-\frac{7}{6}\pi^2+2(\pi i-\log z-\gamma)A_{m-n}\right.$$
$$\left.-2\pi i(\log z+\gamma)+\gamma^2+\log^2 z+2\gamma\log z+A^{(2)}_{m-n}+(A_{m-n})^2\right\}.$$

Since

$$\sum_{n=0}^{m}\frac{(-1)^n B_n(a)}{n!}\frac{(-z)^{m-n}}{(m-n)!}=\frac{(-1)^m}{m!}B_m(z+a)$$

we have

$$\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\left[\frac{te^{-at}}{1-e^{-t}}\right]_m \frac{e^{-zt}}{t^{m+1}}\log^2 t\,dt+\left(\frac{7}{6}\pi^2+2\pi i\gamma-\gamma^2\right)\frac{(-1)^m}{m!}B_m(z+a)$$
$$=\sum_{n=0}^{m}\frac{(-1)^n B_n(a)}{n!}\frac{(-z)^{m-n}}{(m-n)!}\{2(\pi i-\log z-\gamma)A_{m-n}+A^{(2)}_{m-n}+(A_{m-n})^2$$
$$-(2\pi i\log z-\log^2 z-2\gamma\log z)\}.$$

Further, Lemma 3, Lemma 4 and Lemma 5 give

$$\frac{1}{2\pi i}\int_{I(\lambda,\infty)}\left\{\frac{te^{-at}}{1-e^{-t}}\right\}_{m+1}\frac{e^{-zt}}{t^{m+1}}\log^2 t\,dt$$
$$=2\int_0^\infty\left\{\frac{te^{-at}}{1-e^{-t}}\right\}_{m+1}\frac{e^{-zt}}{t^{m+1}}\log t\,dt+2\pi i\int_0^\infty\left\{\frac{te^{-at}}{1-e^{-t}}\right\}_{m+1}\frac{e^{-zt}}{t^{m+1}}dt$$
$$=2\sum_{n=m+1}^{\infty}\frac{(-1)^n B_n(a)(A_{n-m-1}+\pi i-\gamma-\log z)}{n(n-1)\cdots(n-m)z^{n-m}}.$$

Summing up, we have the following

PROPOSITION 4. *For* $\operatorname{Re} z > 0$ *and* $\operatorname{Re} a \geqq 0$,

$$L_2G_m(z+a)$$
$$=\frac{(-1)^m}{m!}\sum_{n=0}^{m}\binom{m}{n}B_n(a)z^n\{2(\pi i-\log z-\gamma)A_{m-n}+A^{(2)}_{m-n}+(A_{m-n})^2$$
$$-2\pi i\log z+\log^2 z+2\gamma\log z\}$$
$$+2\sum_{n=m+1}^{\infty}\frac{(-1)^n B_n(a)(A_{n-m-1}+\pi i-\gamma-\log z)}{n(n-1)\cdots(n-m)z^{n-m}}$$

"asymptotically" for large $|z|$.

2.5. Asymptotic formula for $L\Gamma_r(w+a;\tilde{\omega})$.

PROPOSITION 5. *For* $\mathrm{Re}\, w > 0$, $\mathrm{Re}\, a \geqq 0$ *and for* $\tilde{\omega} = (\omega_1, \cdots, \omega_r)$ *with* $\mathrm{Re}\,\omega_i > 0$, $i = 1, \cdots, r$, *we have*

$$\begin{aligned} L\Gamma_r(w+a;\tilde{\omega}) &= (-1)^r \sum_{n=0}^{r} \frac{{}_rS_1^{(r-n+1)}(a;\tilde{\omega}) w^{r-n}}{(r-n)!} \{2(\pi i - \log w - \gamma) A_{r-n} + A_{r-n}^{(2)} + (A_{r-n})^2 \\ &\quad - (2\pi i \log w - \log^2 w - 2\gamma \log w)\} \\ &\quad + 2(-1)^r \sum_{n=1}^{\infty} \frac{(-1)^n {}_rS'_{n+1}(a;\tilde{\omega})(A_{n-1} + \pi i - \gamma - \log w)}{n(n+1) w^n} \end{aligned}$$

"asymptotically" for large $|w|$.

Proof goes on the same way as for Proposition 4, by Lemma 5 of [5], Lemma 2, Lemma 3, Lemma 4 and Lemma 5.

3. Construction of $L\Gamma_r(w;\tilde{\omega})$.

3.1. $L_2G_m(w)$ by Weierstrass principle. First, we introduce, for $k = 0, 1, 2, \cdots$, the constants γ_k by

$$\gamma_k = \lim_{N\to\infty} \left\{ \sum_{n=1}^{N-1} \frac{\log^k n}{n} - \frac{1}{k+1} \log^{k+1} N \right\}.$$

Then $\gamma_0 = \gamma$ is the Euler's constant. It is known, by Stieltjes, that

$$\zeta(s) = \frac{1}{s-1} + \gamma_0 - \gamma_1(s-1) + \frac{\gamma_2}{2!}(s-1)^2 + \cdots + \frac{(-1)^k \gamma_k}{k!}(s-1)^k + \cdots.$$

(B. C. Berndt [3], p. 164).

It is easy to see that

$$\lim_{N\to\infty} \left\{ \sum_{n=1}^{N-1} \log^k n - \int_1^N \log^k x dx - (-1)^k k! + \frac{1}{2} \log^k N \right\}$$

exists. We denote the limit by δ_k. Then it is known that

$$\delta_1 = -\zeta'(0) = -\frac{1}{2}\log 2\pi,$$

$$\frac{d^k}{ds^k}\zeta(s)\bigg|_{s=0} = (-1)^k \delta_k, \quad ([3]). \tag{3.1.1}$$

We define

$$P_2G_m(w) = -\sum_{n=1}^{\infty}\Big[L_2G_m(w+n+1) - L_2G_m(w+n)$$
$$-\frac{(-1)^m}{m!}\sum_{k=1}^{m}\binom{m}{k}(B_k(w+1)-B_k(w))n^{m-k}\{2(\pi i - \log n - \gamma)A_{m-k}$$
$$-2\pi i\log n + \log^2 n + 2\gamma\log n + A_{m-k}^{(2)} + (A_{m-k})^2\}$$
$$-2\frac{(-1)^{m+1}(B_{m+1}(w+1)-B_{m+1}(w))}{(m+1)!}\frac{\pi i - \gamma - \log n}{n}\Big].$$

The right hand side converges by Proposition 4 ($z = n$, $a = w$). The term for $k = 0$ of the inner sum is 0. We single out the term for $k = m$ from the inner sum, change k to $k-1$ and apply Proposition 3 (5) to the part { }. Using

$$B_m(z+1) - B_m(z) = mz^{m-1}$$

we have

$$P_2G_m(w)$$
$$= -\sum_{n=1}^{\infty}\Big[L_2G_m(w+n+1) - L_2G_m(w+n)$$
$$-\sum_{k=0}^{m-2}\frac{(-1)^k w^k}{k!}(L_2G_{m-k}(n+1) - L_2G_{m-k}(n))$$
$$+2\frac{(-1)^m w^m}{m!}\frac{\pi i - \gamma - \log n}{n} - \frac{(-1)^m}{(m-1)!}w^{m-1}(-2\pi i\log n + \log^2 n + 2\gamma\log n)\Big].$$

Now the first $(N-1)$-sum of $-P_2G_m(w)$ is

$$\sum_{n=1}^{N-1}\Big[L_2G_m(w+n+1) - L_2G_m(w+n) \tag{3.1.2}$$
$$-\sum_{k=0}^{m-2}\frac{(-1)^k w^k}{k!}(L_2G_{m-k}(n+1) - L_2G_{m-k}(n))$$
$$+2\frac{(-1)^m w^m}{m!}\frac{\pi i - \gamma - \log n}{n}$$
$$-\frac{(-1)^m}{(m-1)!}w^{m-1}(-2\pi i\log n + \log^2 n + 2\gamma\log n)\Big]$$
$$= L_2G_m(w+N) - L_2G_m(w+1)$$
$$-\sum_{k=0}^{m-2}\frac{(-1)^k w^k}{k!}(L_2G_{m-k}(N) - L_2G_{m-k}(1))$$
$$+2\frac{(-1)^m w^m}{m!}(\pi i - \gamma)\sum_{n=1}^{N-1}\frac{1}{n} - 2\frac{(-1)^m w^m}{m!}\sum_{n=1}^{N-1}\frac{\log n}{n}$$

$$+2(\pi i-\gamma)\frac{(-1)^m w^{m-1}}{(m-1)!}\log\Gamma(N)-\frac{(-1)^m w^{m-1}}{(m-1)!}\sum_{n=1}^{N-1}\log^2 n\,.$$

Replace

$$L_2G_m(w+N)\,,\quad L_2G_m(N)\,,\quad \sum_{n=1}^{N-1}\frac{1}{n}\,,\quad \sum_{n=1}^{N-1}\frac{\log n}{n}\,,\quad \log\Gamma(N)\,,\quad\text{and}\quad \sum_{n=1}^{N-1}\log^2 n$$

by

$$\begin{aligned}&L_2G_m(w+N)\\&\quad=\sum_{n=0}^{m}\frac{(-1)^n B_n(w)}{n!}\frac{(-N)^{m-n}}{(m-n)!}\{2(\pi i-\log N-\gamma)A_{m-n}+A^{(2)}_{m-n}+(A_{m-n})^2\}\\&\quad-(2\pi i\log N-\log^2 N-2\gamma\log N)\frac{(-1)^m}{m!}B_m(w+N)+O(N^{-1})\end{aligned}$$

(3.1.3) $$\begin{aligned}&L_2G_m(N)\\&\quad=\sum_{n=0}^{m}\frac{(-1)^n B_n}{n!}\frac{(-N)^{m-n}}{(m-n)!}\{2(\pi i-\log N-\gamma)A_{m-n}+A^{(2)}_{m-n}+(A_{m-n})^2\}\\&\quad-(2\pi i\log N-\log^2 N-2\gamma\log N)\frac{(-1)^m}{m!}B_m(N)+O(N^{-1})\end{aligned}$$

(3.1.4) $$\sum_{n=1}^{N-1}\frac{1}{n}=\gamma+\log N+O(N^{-1})\,,$$

(3.1.5) $$\sum_{n=1}^{N-1}\frac{\log n}{n}=\gamma_1+\frac{1}{2}\log^2 N+\varepsilon(N)\,,$$

(3.1.6) $$\log\Gamma(N)=\left(N-\frac{1}{2}\right)\log N-N+\frac{1}{2}\log(2\pi)+O(N^{-1})\,,$$

(3.1.7) $$\sum_{n=1}^{N-1}\log^2 n=\delta_2+\left(N-\frac{1}{2}\right)\log^2 N-2N\log N+2N+\varepsilon(N)\,.$$

Then terms going to infinity for $N\to\infty$, in the formula so obtained are of the types N^j ($j\geqq 1$), $N^j\log N$ ($j\geqq 1$), $\log N$, $N^j\log^2 N$ ($j\geqq 1$), $\log^2 N$ and (3.1.2) converges to $-P_2G_m(w)$ for $N\to\infty$. Hence, diverging terms must be cancelled with each other. Finally, replace $L_2G_m(w+1)$ by

$$\begin{aligned}L_2G_m(w)-\frac{(-w)^{m-1}}{(m-1)!}\{&-2\pi i\log w+\log^2 w+2\gamma\log w\\&+A^{(2)}_{m-1}+(A_{m-1})^2+2(\pi i-\log w-\gamma)A_{m-1}\}\,.\end{aligned}$$

Thus we obtain the following

THEOREM 1. *For* $m \geqq 1$,

$$L_2G_m(w) = \frac{(-w)^{m-1}}{(m-1)!}\{-2\pi i \log w + \log^2 w + 2\gamma \log w + A^{(2)}_{m-1} + (A_{m-1})^2 + 2(\pi i - \log w - \gamma)A_{m-1}\} + \sum_{k=0}^{m-2} \frac{(-1)^k w^k}{k!} L_2G_{m-k}(1) + 2\frac{(-1)^m w^m}{m!}\gamma(\pi i - \gamma) - 2\frac{(-1)^m w^m}{m!}\gamma_1 + (\pi i - \gamma)\frac{(-1)^m w^{m-1}}{(m-1)!}\log(2\pi) - \frac{(-1)^m w^{m-1}}{(m-1)!}\delta_2 + P_2G_m(w).$$

3.2. On the value $\zeta''(0)$. By a straightforward computation, it can be shown that

$$P_2G_1(1) = 2\gamma(\pi i - \gamma)2\gamma_1,$$
$$L_2G_1(1) = -2\gamma(\pi i - \gamma) + 2\gamma_1 - (\pi i - \gamma)\log(2\pi) + \delta_2 + P_2G_1(1) = \delta_2 - (\pi i - \gamma)\log(2\pi).$$

Then incidentally, by Proposition 3 (6), we have

PROPOSITION 6.

$$\zeta''(0) = \delta_2.$$

This is a special case of (3.1.3).

Further, comparing coefficients of the Laurent expansion at $s = 0$ of the both hands of the functional equation of Riemann zeta-function

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s),$$

we have, noting $\Gamma''(1/2) = \pi^{3/2}/2 + (\log 4 - \gamma)^2\pi^{1/2}$, the following

COROLLARY.

$$\delta_2 - \gamma_1 = -\log 2 \cdot \log \pi - \frac{1}{2}\log^2 \pi - \frac{\pi^2}{24} + \gamma \log 2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\log^2 2.$$

3.3. $L\Gamma_r(w; \tilde{\omega})$ by Weierstrass principle. Recall that $\tilde{\omega} = (\omega_1, \omega_2, \cdots, \omega_r)$. Put $\tilde{\omega}^* = (\omega_1, \omega_2, \cdots, \omega_r, \omega_{r+1})$. Assume throughout that w, ω_i, $i = 1, 2, \cdots, r+1$, are all real positive.

For short, we put

$$K(w, n, \tilde{\omega}^*) = \sum_{m=0}^{r} \frac{(-1)^m {}_rS_1^{(r-m+1)}(w; \tilde{\omega})(-n\omega_{r+1})^{r-m}}{(r-m)!}\{A^{(2)}_{r-m} + (A_{r-m})^2 + 2(\pi i - \log(n\omega_{r+1}) - \gamma)A_{r-m} - 2\pi i \log(n\omega_{r+1}) + \log^2(n\omega_{r+1}) + 2\gamma \log(n\omega_{r+1})\} - (-1)^r \frac{{}_rS_2'(w; \tilde{\omega})(\pi i - \gamma - \log(n\omega_{r+1}))}{n\omega_{r+1}}.$$

Define

$$LP_{r+1}(w;\tilde{\omega}^*) = -\sum_{n=1}^{\infty}\{L\Gamma_r(w+n\omega_{r+1};\tilde{\omega}) - K(w;n;\tilde{\omega}^*)\}.$$

The right hand side converges by Proposition 5. By Proposition 1,

$$L\Gamma_{r+1}(w;\tilde{\omega}^*) - L\Gamma_{r+1}(w+\omega_{r+1};\tilde{\omega}^*) = L\Gamma_r(w;\tilde{\omega}).$$

Hence,

$$L\Gamma_{r+1}(w;\tilde{\omega}^*) - L\Gamma_{r+1}(w+N\omega_{r+1};\tilde{\omega}^*) = \sum_{n=0}^{N-1} L\Gamma_r(w+n\omega_{r+1};\tilde{\omega}),$$

and

$$\begin{aligned}(3.3.1)\quad L\Gamma_{r+1}(w;\tilde{\omega}^*) &= L\Gamma_{r+1}(w+N\omega_{r+1};\tilde{\omega}^*) + L\Gamma_r(w;\tilde{\omega})\\ &\quad+\sum_{n=1}^{N-1}\{L\Gamma_r(w+n\omega_{r+1};\tilde{\omega}) - K(w;n;\tilde{\omega}^*\} + \sum_{n=1}^{N-1} K(w;n;\tilde{\omega}^*).\end{aligned}$$

Now

$$\begin{aligned}(3.3.2)\quad &\sum_{n=1}^{N-1} K(w;n;\tilde{\omega}^*)\\
&= \sum_{n=1}^{N-1}\Bigg[\sum_{m=0}^{r-1}(-1)^m{}_rS_1^{(r-m+1)}(w;\tilde{\omega})(\omega_{r+1})^{r-m}\\
&\qquad\qquad\times\{L_2G_{r-m+1}(n) - L_2G_{r-m+1}(n+1)\}\\
&\quad+(-1)^r{}_rS_1'(w;\tilde{\omega})\{2(\gamma-\pi i)\log(n\omega_{r+1}) + \log^2(n\omega_{r+1})\}\\
&\quad+\sum_{m=0}^{r-1}\frac{(-1)^m{}_rS_1^{(r-m+1)}(w;\tilde{\omega})(-n\omega_{r+1})^{r-m}}{(r-m)!}\{-2\log\omega_{r+1}\cdot A_{r-m}\\
&\qquad\qquad+2(\gamma-\pi i)\log\omega_{r+1} + 2\log n\log\omega_{r+1} + \log^2\omega_{r+1}\}\\
&\quad-\frac{(-1)^r{}_rS_2'(w;\tilde{\omega})}{n\omega_{r+1}}(\pi i - \gamma - \log n - \log\omega_{r+1})\Bigg]\\
&= \sum_{m=0}^{r-1}(-1)^m{}_rS_1^{(r-m+1)}(w;\tilde{\omega})(\omega_{r+1})^{r-m}(L_2G_{r-m+1}(1) - L_2G_{r-m+1}(N))\\
&\quad+2(-1)^r{}_rS_1'(w;\tilde{\omega})(\gamma-\pi i+\log\omega_{r+1})\log\Gamma(N)\\
&\quad+N(-1)^r{}_rS_1'(w;\tilde{\omega})(2(\gamma-\pi i)\log\omega_{r+1} + \log^2\omega_{r+1})\\
&\quad-(-1)^r{}_rS_1'(w;\tilde{\omega})(2(\gamma-\pi i)\log\omega_{r+1} + \log^2\omega_{r+1})\\
&\quad+(-1)^r{}_rS_1'(w;\tilde{\omega})\sum_{n=1}^{N-1}\log^2 n\\
&\quad+\sum_{m=0}^{r-1}\Bigg[\frac{(-1)^m{}_rS_1^{(r-m+1)}(w;\tilde{\omega})(-\omega_{r+1})^{r-m}}{(r-m)!}\{-2\log\omega_{r+1}\cdot A_{r-m}\end{aligned}$$

$$+2(\gamma - \pi i)\log \omega_{r+1} + \log^2 \omega_{r+1}\} \cdot \sum_{n=1}^{N-1} n^{r-m} \Bigg]$$

$$+\sum_{m=0}^{r-1}\left[\frac{(-1)^m {}_rS_1^{(r-m+1)}(w;\tilde{\omega})(-\omega_{r+1})^{r-m}}{(r-m)!} 2\log \omega_{r+1} \sum_{n=1}^{N-1} n^{r-m}\log n\right]$$

$$-\frac{(-1)^r {}_rS_2'(w;\tilde{\omega})}{\omega_{r+1}}(\pi i - \gamma - \log \omega_{r+1})\sum_{n=1}^{N-1}\frac{1}{n} + \frac{(-1)^r {}_rS_2'(w;\tilde{\omega})}{\omega_{r+1}}\sum_{n=1}^{N-1}\frac{\log n}{n}.$$

It is easy to see that for every $k = 0, 1, 2, \cdots$,

$$\sum_{n=1}^{N-1} n^k \log n - \left(N - \frac{1}{2}\right) N^k \log N + \frac{1}{(k+1)^2} N^{k+1}$$

converges. Denote the limit by η_k (hence $\eta_0 = \delta_1 = (1/2)\log(2\pi)$): namely, we can write

$$\sum_{n=1}^{N-1} n^k \log n = \left(N - \frac{1}{2}\right) N^k \log N + \frac{1}{(k+1)^2} N^{k+1} + \eta_k + \varepsilon(N). \tag{3.3.3}$$

We replace, in (3.3.2), $L\Gamma_{r+1}(w + N\omega_{r+1}; \tilde{\omega}^*)$ by

$$(-1)^{r+1}\sum_{n=0}^{r+1}\frac{{}_{r+1}S_1^{(r+1-n+1)}(w;\tilde{\omega}^*)(N\omega_{r+1})^{r+1-n}}{(r+1-n)!}$$
$$\times \{2(\pi i - \log(N\omega_{r+1}) - \gamma)A_{r+1-n} + (A_{r+1-n})^2 + A_{r+1-n}^{(2)}$$
$$- 2\pi i \log(N\omega_{r+1}) + \log^2(N\omega_{r+1}) + 2\log(N\omega_{r+1})\} + O(N^{-1})$$

$\sum_{n=1}^{N-1} n^{r-m}\log n$ by (3.3.3) with $k = r - m$,

$L_2G_{r-m+1}(N)$ by (3.1.3) with $r - m + 1$ instead of m,

$\log \Gamma(N)$ by (3.1.6),

$\sum_{n=1}^{N-1} \log^2 n$ by (3.1.7),

$\sum_{n=1}^{N-1} n^k$ by $\frac{1}{k+1}(B_{k+1}(N) - B_{k+1})$ with $k = r - m \geqq 1$,

$\sum_{n=1}^{N-1} \frac{1}{n}$ by (3.1.4),

$\sum_{n=1}^{N-1} \frac{\log n}{n}$ by (3.1.5).

(3.3.2) must converge when $N \to \infty$. The terms of (3.3.2) going to infinity for $N \to \infty$ are of the types N^p $(p \geqq 1)$, $N^p \log N$ $(p \geqq 0)$, $N^p \log^2 N$ $(p \geqq 0)$. Hence diverging terms must be cancelled with each other. Thus, letting N tend to ∞ in (3.3.1), we have the infinite series representation (of Theorem 2 below) of $L\Gamma_{r+1}(w;\tilde{\omega}^*)$.

Now we define

$$\mathbf{D}_{r+1}=\left\{(w,\tilde{\omega}^*);\,\omega_i\in\mathbf{C}-(-\infty,0],\,i=1,2,\cdots,r+1,\,w\neq-\sum_{i=1}^{r+1}m_iw_i,\right.$$
$$\left. m_i\in\mathbf{Z},\,i=1,2,\cdots,r+1\right\}.$$

Then, (4.2.1) in the next section shows that $L\Gamma_1(w;\omega)$ can be continued holomorphically to the domain $\mathbf{D}_1$. Therefore inductively, $LP_{r+1}(w;\tilde{\omega}^*)$ and $L\Gamma_{r+1}(w;\tilde{\omega}^*)$ can be continued holomorphically to the domain $\mathbf{D}_{r+1}$

THEOREM 2. *Let $w>0$ and $\omega_i>0,\,i=1,2,\cdots,r+1$. Put $\tilde{\omega}=(\omega_1,\cdots,\omega_r)$ and $\tilde{\omega}^*=(\omega_1,\cdots,\omega_{r+1})$. Then*

$$\begin{aligned}L\Gamma_{r+1}(w;\tilde{\omega}^*)&=L\Gamma_r(w;\tilde{\omega})+(-1)^{r+1}{}_{r+1}S_1'(w;\tilde{\omega}^*)(2(\gamma-\pi i)\log\omega_{r+1}+\log^2\omega_{r+1})\\&+\sum_{m=0}^{r-1}(-1)^m{}_rS_1^{(r-m+1)}(w;\tilde{\omega})(\omega_{r+1})^{r-m}L_2G_{r-m+1}(1)\\&+(-1)^r{}_rS_1'(w;\tilde{\omega})\{(\gamma-\pi i+\log\omega_{r+1})\log(2\pi)\\&\qquad-2(\gamma-\pi i)\log\omega_{r+1}-\log^2\omega_{r+1}+\delta_2\}\\&+2\log\omega_{r+1}\sum_{m=0}^{r-1}\frac{(-1)^m{}_rS_1^{(r-m+1)}(w;\tilde{\omega})(-\omega_{r+1})^{r-m}}{(r-m)!}\eta_{r-m}\\&-\frac{(-1)^r{}_rS_2'(w;\tilde{\omega})}{\omega_{r+1}}\{(\pi i-\gamma-\log\omega_{r+1})\gamma-\gamma_1\}-LP_{r+1}(w;\tilde{\omega}^*).\end{aligned}$$

and this can be continued holomorphically to the domain $\mathbf{D}_{r+1}$.

Further, combining this Theorem with Theorem 2 of [5], we have, by a straightforward but long calculation using Proposition 3 (6) of [5] and the present Propositon 3 (6),

THEOREM 3. *Let $w>0$ and $\omega_i>0,\,i=1,2,\cdots,r+1$, and $\tilde{\omega}=(\omega_1,\cdots,\omega_r)$ $\tilde{\omega}^*=(\omega_1,\cdots,\omega_r,\omega_{r+1})$,*

$$\begin{aligned}&\zeta_{r+1}''(0;w;\tilde{\omega}^*)\\&=\zeta_r''(0;w;\tilde{\omega})+\sum_{n=1}^{\infty}\Bigg[\zeta_r''(0;w+n\omega_{r+1};\tilde{\omega})\\&\quad-\sum_{m=0}^{r}\frac{(-1)^m{}_rS_1^{(r-m+1)}(w;\tilde{\omega})(-n\omega_{r+1})^{r-m}}{(r-m)!}(A_{r-m}^{(2)}+(A_{r-m})^2\\&\quad-2A_{r-m}\log(n\omega_{r+1})+\log^2(n\omega_{r+1}))-(-1)^r\frac{{}_rS_2'(w;\tilde{\omega})\log(n\omega_{r+1})}{n\omega_{r+1}}\Bigg]\end{aligned}$$

$$+ (-1)^{r+1}{}_{r+1}S_1'(w;\tilde{\omega}^*)\log^2\omega_{r+1}$$
$$+ (-1)^r{}_rS_1'(w;\tilde{\omega})\left(\log\omega_{r+1}\cdot\frac{1}{2}\log(2\pi) - \log^2\omega_{r+1} + \delta_2\right)$$
$$+ \sum_{m=0}^{r-1}(-1)^m{}_rS_1^{(r-m+1)}(w;\tilde{\omega})\omega_{r+1}^{r-m}\left\{\frac{(-1)^{r-m}}{(r-m)!}(\zeta''(-r+m) + 2A_{r-m}\zeta'(-r+m))\right.$$
$$\left. + (A_{r-m}^{(2)} + (A_{r-m})^2)\frac{(-1)^{r-m+1}B_{r-m+1}}{(r-m+1)!}\right\} + \frac{(-1)^r{}_rS_2'(w;\tilde{\omega})(\gamma\log\omega_{r+1} + \gamma_1)}{\omega_{r+1}}$$

and this can be continued holomorphically to the domain $(w;\tilde{\omega}^*)\in \mathbf{D}_{r+1}$.

4. The cases $r = 0$ and 1.

4.1. In this section, we shall supply the above theory by considering the case $r = 0$ and 1.

First we define $L\Gamma_0(w;[\ ]) = L\Gamma_0(w)$ so as to satisfy the difference equation of Proposition 1 for $r = 1$ ($[\ ]$ means the empty). Thus $-L\Gamma_0(w)$ must be equal to

$$\begin{aligned}
&L\Gamma_1(w+\omega;\omega) - L\Gamma_1(w;\omega)\\
&= \frac{1}{2\pi i}\int_{I(\lambda,\infty)}\frac{e^{-(w+\omega)t} - e^{-wt}}{1-e^{-\omega t}}\frac{\log^2 t}{t}dt\\
&\quad + \left(\frac{7}{6}\pi^2 + 2\pi i\gamma - \gamma^2\right)(\zeta_1(0, w+\omega;\omega) - \zeta_1(0;w;\omega))\\
&= \frac{-1}{2\pi i}\int_{I(\lambda,\infty)}\frac{e^{-wt}\log^2 t}{t}dt - \left(\frac{7}{6}\pi^2 + 2\pi i\gamma - \gamma^2\right)\\
&= 2\pi i\log w - \log^2 w - 2\gamma\log w\,.
\end{aligned}$$

Here, we have used the key Lemma 1.

Hence we define

$$L\Gamma_0(w) = -2(\pi i - \gamma)\log w + \log^2 w\,. \tag{4.1.1}$$

From Proposition 5, the asymptotic expansion for $L\Gamma_1(w+a;\omega)$ is given by, for $\operatorname{Re} w > 0$, $\operatorname{Re} a \geqq 0$ and for large $|w|$,

$$\begin{aligned}
L\Gamma_1(w+a;\omega) &= B_1(a/\omega)(2(\pi i - \gamma)\log w - \log^2 w)\\
&\quad - \frac{w}{\omega}(2(\pi i - \log w - \gamma) + 2 - (2(\pi i - \gamma)\log w - \log^2 w))\\
&\quad + \frac{\omega B_2(a/\omega)(\pi i - \gamma - \log w)}{w} + \cdots\,.
\end{aligned} \tag{4.1.2}$$

4.2. The construction of $L\Gamma_1(w;\omega)$ by Weierstrass principle is as follows.

From the difference equation

$$L\Gamma_1(w;\omega) - L\Gamma_1(w+\omega;\omega) = L\Gamma_0(w)\,,$$

We have

$$\begin{aligned}
L\Gamma_1(w;\omega) &= L\Gamma_1(w+N\omega;\omega) + L\Gamma_0(w) + \sum_{n=1}^{N-1} L\Gamma_0(w+n\omega) \\
&= L\Gamma_1(w+N\omega;\omega) + L\Gamma_0(w) \\
&\quad - \sum_{n=1}^{N-1}\{2(\pi i-\gamma)\log(w+n\omega) - \log^2(w+n\omega)\} \\
&= L\Gamma_1(w+N\omega;\omega) + L\Gamma_0(w) \\
&\quad - \sum_{n=1}^{N-1}\left\{2(\pi i-\gamma)\left(\log(w+n\omega)-\log(n\omega)-\frac{w}{n\omega}\right)\right. \\
&\qquad\qquad \left. - \left(\log^2(w+n\omega)-\log^2(n\omega)-\frac{2\log(n\omega)}{n\omega}w\right)\right\} \\
&\quad - \sum_{n=1}^{N-1}\left\{2(\pi i-\gamma)\left(\log(n\omega)+\frac{w}{n\omega}\right)-\log^2(n\omega)-\frac{2\log(n\omega)}{n\omega}w\right\} \\
&= L\Gamma_1(w+N\omega;\omega) + L\Gamma_0(w) \\
&\quad - \sum_{n=1}^{N-1}\left\{2(\pi i-\gamma)\left(\log(w+n\omega)-\log(n\omega)-\frac{w}{n\omega}\right)\right. \\
&\qquad\qquad \left. - \left(\log^2(w+n\omega)-\log^2(n\omega)-\frac{2\log n\omega}{n\omega}w\right)\right\} \\
&\quad - 2(\pi i-\gamma)\sum_{n=1}^{N-1}\log n - 2(\pi i-\gamma)\frac{w}{\omega}\sum_{n=1}^{N-1}\frac{1}{n} - 2(\pi i-\gamma)\sum_{n=1}^{N-1}\log\omega \\
&\quad + \sum_{n=1}^{N-1}\log^2 n + 2\log\omega\sum_{n=1}^{N-1}\log n + \sum_{n=1}^{N-1}\log^2\omega + 2\frac{w}{\omega}\sum_{n=1}^{N-1}\frac{\log n}{n} \\
&\quad + 2\frac{w\log\omega}{\omega}\sum_{n=1}^{N-1}\frac{1}{n}\,.
\end{aligned}$$

In the above, replace $L\Gamma_0(w)$, $L\Gamma_1(w+N\omega;\omega)$, $\sum_{n=1}^{N-1}\log n$, $\sum_{n=1}^{N-1}\frac{1}{n}$, $\sum_{n=1}^{N-1}\log^2 n$, and $\sum_{n=1}^{N-1}\frac{\log n}{n}$ by

$$\begin{aligned}
&L\Gamma_0(w) = -2(\pi i-\gamma)\log w + \log^2 w \\
&L\Gamma_1(w+N\omega;\omega) = B_1(w/\omega)(2(\pi i-\gamma)\log(N\omega) - \log^2(N\omega)) \\
&\qquad - N(2(\pi i - \log(N\omega) - \gamma) + 2 - (2(\pi i-\gamma)\log(N\omega) - \log^2(N\omega))) + \varepsilon(N) \\
&\sum_{n=1}^{N-1}\log n = \left(N-\frac{1}{2}\right)\log N - N + \frac{1}{2}\log(2\pi) + O(N^{-1})\,,
\end{aligned}$$

$$\sum_{n=1}^{N-1} \frac{1}{n} = \gamma + \log N + O(N^{-1}),$$

$$\sum_{n=1}^{N-1} \log^2 n = \delta_2 + \left(N - \frac{1}{2}\right) \log^2 N - 2N \log N + 2N + \varepsilon(N),$$

$$\sum_{n=1}^{N-1} \frac{\log n}{n} = \gamma_1 + \frac{1}{2} \log^2 N + \varepsilon(N),$$

respectively, where $\varepsilon(N)$ is any quantity which goes to 0 when N tends to ∞.

Then for $N \to \infty$, the right hand side of $L\Gamma_1(w; 1)$ in the above converges. Hence, terms going to ∞ when $N \to \infty$ must be cancelled out and we have

$$\begin{aligned}(4.2.1)\quad L\Gamma_1(w; \omega) = 2(\pi i - \gamma)\Bigg[& B_1(w/\omega) \log \omega - \log w - \frac{1}{2} \log(2\pi) - \frac{w}{\omega}\gamma \\ & + \log \omega - \sum_{n=1}^{\infty} \left(\log(w + n\omega) - \log(n\omega) - \frac{w}{n\omega} \right)\Bigg] \\ & + \log^2 w + \delta_2 + 2\frac{w}{\omega}\gamma_1 + \log \omega \cdot \log(2\pi) - \log^2 \omega + \frac{2w \log \omega}{\omega}\gamma \\ & - B_1(w/\omega) \log^2 \omega + \sum_{n=1}^{\infty} \left(\log^2(w + n\omega) - \log^2(n\omega) - \frac{2 \log(n\omega)}{n\omega} w \right).\end{aligned}$$

Now the part [ ] in the above is

$$-\log \frac{\Gamma_1(w; \omega)}{\sqrt{2\pi/\omega}} = -\log \frac{\Gamma(w/\omega)}{\sqrt{2\pi}} - \log \omega \cdot B_1(w/\omega).$$

Comparing (4.2.1) with (1.3.5) for $r = 1$, we have

PROPOSITION 7. *For* $(w, \omega) \in \mathbf{D}_1$,

$$\begin{aligned}\zeta_1''(0; w; \omega) = \delta_2 &+ \frac{2w\gamma_1}{\omega} + \log^2 w - \log^2 \omega - \log^2 \omega \cdot B_1(w/\omega) \\ &+ \log \omega \cdot \log(2\pi) + \frac{2w \log \omega}{\omega}\gamma \\ &+ \sum_{n=1}^{\infty} \left(\log^2(w + n\omega) - \log^2(n\omega) - \frac{2 \log(n\omega)}{n\omega} w \right).\end{aligned}$$

In Proposition 7, put $\omega = 1$. Then

COROLLARY.

$$\zeta_1''(0; w) = \delta_2 + 2w\gamma_1 + \log^2 w + \sum_{n=1}^{\infty} \left(\log^2(w + n) - \log^2 n - \frac{2 \log n}{n} w \right).$$

This is already known [3], [4] (Deninger's γ_1 is our $2\gamma_1$). Note that for $w = 1$.

$$\sum_{n=1}^{\infty}\left(\log^2(w+n) - \log^2 n - \frac{2\log n}{n}w\right) = \lim_{N\to\infty}\sum_{n=1}^{N-1}\left(\log^2(1+n) - \log^2 n - \frac{2\log n}{n}\right)$$

$$= \lim_{N\to\infty}\left(\log^2 N - 2\sum_{n=1}^{N-1}\frac{\log n}{n}\right) = -2\gamma_1\,.$$

So from this too, $\zeta''(0) = \delta_2$ is obtained by Proposition 7.

Theorem 3 is a generalization of Proposition 7: namely, the Theorem, for $r = 0$ $\tilde{\omega}^* = (\omega)$, gives Proposition 7 if we reasonably understand that

$$\zeta_0(s; w; [\ ]) = w^{-s}\,,$$

$$_0S_r'(w; [\ ]) = n! \cdot w^{n-1}\,,$$

$$\sum_m^{-1} \cdots = 0\,.$$

Here [ ] means empty.

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Present Address:
DEPARTMENT OF MATHEMATICS, TSUDA COLLEGE,
TSUDA-MACHI, KODAIRA-SHI, TOKYO, 187–8577 JAPAN.