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Expansivity, Pseudoleaf Tracing Property and Semistability of Foliations

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1. Introduction.

A foliation is viewed as a generalization of a nonsingular flow, and it is natural to expect that various notions and results in the theory of dynamical systems might also be valid for foliations. In fact, several important results have been obtained from this viewpoint: e.g., the Poincaré-Bendixson theory for C^2 foliations (by Cantwell-Conlon, Hector etc.), the theory of transverse invariant measures and foliation cycles (by Plante, Sullivan etc.). One of the recent works on these lines was done by Ghys, Langevin and Walczak [7, 12], who defined an entropy for a foliation and showed that the notion is effective to measure the qualitative complexity of foliations (see also [9] and [13]). In [10] we considered the expansivity for foliations and obtained a characterization of an expansive codimension-one foliation in terms of local minimal sets (see also [11, Section D]).

In this paper we introduce a notion called the pseudoleaf tracing property (PLTP for short) for foliations. As is well known, the pseudo-orbit tracing property (POTP) plays an important role in topological dynamics to prove stability results. Our PLTP is its natural analogue in foliation theory. We give a few examples of foliations with the PLTP, and generalize a fundamental stability theorem for dynamical systems to foliations:

THEOREM A. *Let M be a closed Riemannian manifold and $\mathcal{F}$ a smooth foliation on M. If $\mathcal{F}$ is expansive and has the PLTP, then $\mathcal{F}$ is semistable.*

In dynamical systems, the corresponding results are obtained by Walters [17] for homeomorphisms and by Thomas [14] for flows.

We also prove that the PLTP, the expansivity and the semistability are preserved under the suspension procedure:

THEOREM B. *Let $\Phi : \Gamma \to \mathrm{Diff}(X)$ be a smooth action of a finitely presented group Γ on a compact Riemannian manifold X, and let $\mathcal{F}_\Phi$ be a suspension foliation of Φ. Then,*

(1) *Φ has the POTP if and only if $\mathcal{F}_\Phi$ has the PLTP.*

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(2) *Φ is semistable if and only if $\mathcal{F}_\Phi$ is semistable.*
(3) *Φ is expansive if and only if $\mathcal{F}_\Phi$ is expansive.*

In dynamical systems, the corresponding results to Theorem B(1), B(2) and B(3) are obtained by Thomas [14, Theorem 2], Thomas [15, Theorems 2 and 3] and Bowen-Walters [3, Theorem 6] respectively.

Throughout this paper we work in the smooth category. The author does not know how to define the PLTP in a purely topological setting as is usually done in dynamical systems.

We note that very recently Biś and Walczak [1] and Walczak [16] have found new applications of the notion of pseudoleaves.

Several parts of this paper were improved through conversations with P. Walczak during the author's short stay in Łódź. In particular, the idea of using the notion of the center of mass in the proof of Theorem B(1) was suggested by him. The author would like to thank him for all of these and also for his hospitality during the stay. The author would also be grateful to S. Matsumoto for indicating the PLTP of Anosov foliations.

2. Notations.

Notations introduced in this section will be used throughout this paper:

Let (M, g) be a closed Riemannian manifold and $\mathcal{F}$ a foliation on M. We denote by F_x the leaf of $\mathcal{F}$ passing through a point $x \in M$ and by d_M (respectively, $d_{\mathcal{F}}$) the distance function on M (respectively, on each leaf of $\mathcal{F}$) induced from g (respectively, $g \mid T\mathcal{F}$). For $r \geq 0$ we put $D_M(x, r) = \{y \in M \mid d_M(x, y) \leq r\}$, $D_{\mathcal{F}}(x, r) = \{y \in F_x \mid d_{\mathcal{F}}(x, y) \leq r\}$, $P_{\mathcal{F}}(x, r) = \{\gamma : [0, 1] \to F_x \mid \gamma$ is a piecewise smooth curve with length $\leq r$ and $\gamma(0) = x\}$, $P_{\mathcal{F}}(x) = \bigcup_{r \geq 0} P_{\mathcal{F}}(x, r)$, $(T_x^{\perp}\mathcal{F})(r) = \{v \in T_x^{\perp}\mathcal{F} \mid \|v\| \leq r\}$ and $D_{\perp}(x, r) = \exp(T_x^{\perp}\mathcal{F})(r)$, where $T_x^{\perp}\mathcal{F}$ is the orthogonal complement of $T_x\mathcal{F}$ in T_xM.

Given a foliation $\mathcal{F}$ on M, we fix once and for all a constant $\kappa > 0$ which is small enough to satisfy the following: for every $x \in M$, $\exp \mid (T_x^{\perp}\mathcal{F})(\kappa)$ is transverse to $\mathcal{F}$ and $\exp : \bigcup\{(T_u^{\perp}\mathcal{F})(\kappa) \mid u \in D_{\mathcal{F}}(x, \kappa)\} \to M$ is an embedding. Then, for every $x \in M$, we can define the local orthogonal projection $p : \bigcup\{D_{\perp}(u, \kappa) \mid u \in D_{\mathcal{F}}(x, \kappa)\} \to D_{\mathcal{F}}(x, \kappa)$ by $p(y) = u$ for $y \in D_{\perp}(u, \kappa)$ (see [12, p. 555]).

Let $x \in M$ and $y \in D_{\perp}(x, \kappa)$. The local orthogonal projection gives us a one-to-one correspondence between the local curves in F_x at x and those in F_y at y. For $\gamma \in P_{\mathcal{F}}(x)$, we can determine a piecewise smooth curve $\bar{\gamma}$ in F_y of origin y by requiring $\bar{\gamma}(t) \in F_y \cap D_{\perp}(\gamma(t), \kappa)$ (that is, $p(\bar{\gamma}(t)) = \gamma(t)$), as long as $\bar{\gamma}(t)$ stays on $D_{\perp}(\gamma(t), \kappa)$. If $\bar{\gamma}(t)$ is defined for all $0 \leq t \leq 1$, we say that γ can be *lifted* to $\bar{\gamma} \in P_{\mathcal{F}}(y)$. Conversely, we can begin with $\bar{\gamma}$ and *project* it to obtain γ.

We also fix a constant $\kappa' > 0$ sufficiently small relative to κ that the following holds: if x and y are points of M such that $d_M(x, y) < \kappa'$, then the leaf through y of the restricted foliation $\mathcal{F} \mid \bigcup\{D_{\perp}(u, \kappa) \mid u \in D_{\mathcal{F}}(x, \kappa)\}$ intersects $D_{\perp}(x, \kappa)$ at exactly one point, and, moreover, the intersection point depends on x and y smoothly. We denote the point by $w(x, y)$.

For two nontrivial subspaces W_1, W_2 of $T_x M$ we put

$$\rho(W_1, W_2) = \max\{\min\{\|w_1 - w_2\| \mid w_2 \in W_2, \|w_2\| = 1\} \mid w_1 \in W_1, \|w_1\| = 1\}.$$

3. Expansivity.

DEFINITION 3.1 ([10], [11, Section D]). A foliation $\mathcal{F}$ is *expansive* if there exists a constant $0 < e_{\mathcal{F}} < \kappa$ with the property that for any $x \in M$ and any $y \in D_\perp(x, e_{\mathcal{F}}) - \{x\}$, one can find a curve $\gamma \in P_{\mathcal{F}}(x)$ which is lifted to a curve $\bar{\gamma} \in P_{\mathcal{F}}(y)$ such that $\bar{\gamma}(1) \notin D_\perp(\gamma(1), e_{\mathcal{F}})$. The constant $e_{\mathcal{F}}$ is called an *expansive constant* for $\mathcal{F}$.

Expansivity is independent of the Riemannian metric of M since we are assuming that M is compact.

EXAMPLE 3.2. A 1-dimensional foliation given by the orbits of an expansive flow ([3]) is expansive.

EXAMPLE 3.3 ([10]). A codimension 1 foliation $\mathcal{F}$ on M is expansive if and only if $\mathcal{F}$ has finitely many nontrivial open local minimal sets whose union is dense in M.

EXAMPLE 3.4. An Anosov foliation (i.e., the weak stable foliation of an Anosov flow) is expansive.

4. Pseudoleaf tracing property.

DEFINITION 4.1. Let $\delta > 0$. Let L be a connected complete Riemannian manifold such that $\dim L = \dim \mathcal{F}$ and $\varphi : L \to M$ an isometric immersion. We say that $\varphi(L)$ is a δ*-pseudoleaf* of $\mathcal{F}$ if for every $x \in L$ we have $\rho(T_{\varphi(x)}\mathcal{F}, \varphi_*(T_x L)) < \delta$.

Hereafter we will usually omit φ and confuse $\varphi(L)$ with L.

We fix a constant $\delta_0 > 0$ sufficiently small that if L is a δ_0-pseudoleaf then $\exp|(T_x^\perp \mathcal{F})(\kappa)$ is transverse to L for all $x \in M$.

DEFINITION 4.2. Let $0 < \delta < \delta_0$ and $0 < \varepsilon < \kappa$. A δ-pseudoleaf L of $\mathcal{F}$ with base point x_0 is ε*-traced* by an $\mathcal{F}$-leaf F with base point y_0 if $x_0 \in D_\perp(y_0, \kappa) \cap L$ and if for every $\gamma \in P_{\mathcal{F}}(y_0)$ there exists a curve $\bar{\gamma} : [0, 1] \to L$ of origin x_0 such that $\bar{\gamma}(t) \in D_\perp(\gamma(t), \varepsilon)$ for all $0 \leq t \leq 1$.

By the same reason as stated in §2, the lift $\bar{\gamma}$ is uniquely determined by γ, and vice versa.

When we simply say that L is ε-traced by F, it means that we can choose points $x_0 \in L$ and $y_0 \in F$ such that (L, x_0) is ε*-traced* by (F, y_0).

REMARK. If (L, x_0) is ε-traced by (F, y_0), we can define a covering map $\phi : \tilde{F} \to L$ by $\phi([\gamma]) = \bar{\gamma}(1)$, where $\tilde{F} = P_{\mathcal{F}}(y_0)$/(homotopy relative to the endpoints) is the universal covering space of F.

DEFINITION 4.3. A foliation $\mathcal{F}$ has the *pseudoleaf tracing property* (PLTP for short) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that any δ-pseudoleaf of $\mathcal{F}$ is ε-traced by some $\mathcal{F}$-leaf.

This definition is independent of the Riemannian metric of M since we are assuming that M is compact.

EXAMPLE 4.4. The Reeb foliation $\mathcal{F}_R$ (see e.g. [8] for definition) of S^3 has the PLTP.

PROOF. Suppose $\varepsilon > 0$ is given. Replacing ε by a smaller one if necessary, we may assume that $\varepsilon < \kappa$ and that $\exp : \bigcup_{x\in T}(T_x^{\perp}\mathcal{F})(\varepsilon) \to S^3$ is an embedding, where T is the toral leaf of $\mathcal{F}_R$. Then we have a well-defined orthogonal projection $p : \bigcup_{x\in T} D_{\perp}(x, \varepsilon) \to T$ which maps each disk $D_{\perp}(x, \varepsilon)$ to a point x. Let N be a compact neighborhood of T contained in $\bigcup_{x\in T} D_{\perp}(x, \varepsilon/2)$ such that ∂N is transverse both to $\mathcal{F}$ and to $D_{\perp}(x, \varepsilon/2)$ for all $x \in T$. Denote by R_1 and R_2 the two connected components of $S^3 - \operatorname{Int} N$. Then, by the choice of N, $\mathcal{F} \mid R_i$ $(i = 1, 2)$ are product foliations by disks. Choose $\delta > 0$ sufficiently small that the following conditions are satisfied:

(1) $2\delta < \rho(T_x\partial N, T_x\mathcal{F})$ for all $x \in \partial N$,

(2) $2\delta < \rho(T_y D_{\mathcal{F}^{\perp}}(x, \varepsilon), T_y\mathcal{F})$ for all $x \in T$ and $y \in N \cap D_{\mathcal{F}^{\perp}}(x, \varepsilon)$, and

(3) if L' is an immersed complete 2-dimensional submanifold of R_i with $\partial L' \subset \partial R_i$ such that $\rho(T_z L', T_z\mathcal{F}) < \delta$ for all $z \in L'$, then L' is diffeomorphic to a disk.

Now let L be any δ-pseudoleaf of $\mathcal{F}_R$. If L is entirely contained in N, then clearly T ε-traces L. So, assume that L intersects $S^3 - N$. Then, by the condition (3) above, $L \cap R_i$ is a union of disks for each $i = 1, 2$. In particular, $L \cap N$ is connected. Since, by (2), $L \cap N$ is transverse to the orthogonal projection p, we can extend $p \mid L \cap N$ to a covering map $\hat{p} : \hat{L} \to T$ by attaching copies of $S^1 \times [0, \infty)$ to $L \cap N$ along all connected components of $L \cap \partial N$. Then $\hat{L}$ is noncompact and hence diffeomorphic to a cylinder or a plane. We see that L intersects only one of R_i $(i = 1, 2)$. In fact, suppose L intersected both. Then, a pair of loops, one chosen from $L \cap \partial R_1$ and the other from $L \cap \partial R_2$, would be projected by $\hat{p}$ to form a meridian-longitude system on T, which is impossible. Without loss of generality we may assume that L only intersects R_1. Next we see that $L \cap \partial R_1$ is a single circle. In fact, suppose $L \cap \partial R_1$ consisted of more than one component. Consider a foliation $\mathcal{C}$ on ∂R_1 by circles such that every leaf meets each component of $L \cap \partial R_1$ transversely exactly once. Then, since $L \cap N$ is transverse to p, it follows from the connectedness of $L \cap N$ that there is a leaf c of $\mathcal{C}$ such that $p^{-1}(p(c)) \cap (L \cap N)$ is a smooth curve joining distinct components of $L \cap \partial R_1$. But this contradicts the conditions (1) and (2). Consequently, L is ε-traced by some leaf in one of the Reeb components of $\mathcal{F}_R$. This proves the PLTP for $\mathcal{F}_R$.

In this paper a *pseudo-orbit* of a flow ϕ means a pseudoleaf of the 1-dimensional foliation by the orbits of ϕ. The following observation is essentially due to Matsumoto.

EXAMPLE 4.5. An Anosov foliation has the PLTP.

PROOF. Let $\mathcal{F}^s$ be an Anosov foliation, (i.e., the weakly stable foliation of a smooth Anosov flow ϕ) on a compact manifold M. Let $\varepsilon > 0$ be given. Since it is well-known that an Anosov flow has the POTP, one can find $\eta > 0$ so that every η-pseudo-orbit of ϕ is ε-traced by some orbit of ϕ. We choose $\delta > 0$ so small that if L is a δ-pseudoleaf of $\mathcal{F}^s$ and if $\mathcal{F}^u$ is the weakly unstable foliation of ϕ, then each connected component of the intersection of L and a leaf of $\mathcal{F}^u$ is an η-pseudo-orbit of ϕ.

Now let L be a δ-pseudoleaf of $\mathcal{F}^s$. The intersections of L and leaves of $\mathcal{F}^u$ yield a one dimensional oriented foliation, say $\mathcal{L}$, on L. Since each leaf ℓ in $\mathcal{L}$ is an η-pseudo-orbit of ϕ, there uniquely exists an orbit o_ℓ of ϕ which ε-traces ℓ (the uniqueness follows from the expansivity of ϕ). We claim that there exists a leaf F of $\mathcal{F}^s$ such that for every $\ell \in \mathcal{L}$, the orbit o_ℓ lies in F. In fact, suppose this claim were not true. Then one can choose a small local cross section (i.e., a codimension one embedded open disk transverse to ϕ) D through a point $x \in L$ satisfying the following property: there are points x', y and y' on D such that

(1) the point y lies on L,

(2) the leaf $\ell(x)$ (respectively, $\ell(y)$) of $\mathcal{L}$ through x (respectively, y) is ε-traced by the orbit, say $o_{\ell(x)}$ (respectively, $o_{\ell(y)}$), of ϕ through x' (respectively, y'), and

(3) the orbits $o_{\ell(x)}$ and $o_{\ell(y)}$ are contained in distinct leaves of $\mathcal{F}^s$.

Let $W^s(x)$ (respectively, $W^u(x)$) be the leaf of $\mathcal{F}^s$ (respectively, $\mathcal{F}^u$) through x. If we go forward along $o_{\ell(x)}$, we of course remain on $W^s(x') \cap W^u(x')$ forever. On the other hand, if we go forward along $o_{\ell(y)}$, we go away from $W^s(x')$ while coming nearer and nearer to $W^u(x')$. Since $\ell(x)$ (respectively, $\ell(y)$) is ε-traced by $o_{\ell(x)}$ (respectively, $o_{\ell(y)}$), $\ell(x)$ (respectively, $\ell(y)$) behaves similarly as $o_{\ell(x)}$ (respectively, $o_{\ell(y)}$). Namely, if we go forward along $\ell(x)$, we must remain near $W^s(x) \cap W^u(x)$, while if we go forward along $\ell(y)$, we must go away from $W^s(x)$ with keeping near $W^u(x)$. But, such a phenomenon cannot occur because L is a δ-pseudoleaf of $\mathcal{F}^s$ and $\ell(x)$ and $\ell(y)$ are contained in L. Thus the claim is shown. The leaf F of $\mathcal{F}^s$ just found in the claim clearly ε-traces L, which proves the PLTP for $\mathcal{F}^s$.

EXAMPLE 4.6. Given a transversely orientable codimension one foliation $\mathcal{F}$ on a closed n-manifold M ($n \geq 4$) satisfying the PLTP, by turbulizing $\mathcal{F}$ along a closed transversal to $\mathcal{F}$ (see e.g. [8, p. 50]) one can produce another foliation $\mathcal{F}'$ possessing the PLTP. Here the turbulization must be done so that the resulting compact leaf has topologically hyperbolic holonomy.

5. Semistability.

We denote by $\mathrm{Fol}_q(M)$ the set of all codimension q smooth foliations on M and define a distance ρ on $\mathrm{Fol}_q(M)$ by $\rho(\mathcal{F}, \mathcal{G}) = \max_{x \in M} \rho(T_x\mathcal{F}, T_x\mathcal{G})$.

DEFINITION 5.1. A foliation $\mathcal{F}$ is *semistable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\mathcal{G} \in \mathrm{Fol}_q(M)$ with $\rho(\mathcal{F}, \mathcal{G}) < \delta$ there exists a continuous map $f : M \to M$ satisfying the following properties:

(1) every $\mathcal{G}$-leaf is mapped by f onto some $\mathcal{F}$-leaf, and

(2) $d_M(f, \mathrm{id}_M)(= \max_{x \in M} d_M(f(x), x)) < \varepsilon$.

Semistability is independent of the Riemannian metric of M since we are assuming that M is compact.

EXAMPLE 5.2 (folklore). The Reeb foliation $\mathcal{F}_R$ of S^3 is semistable. In fact, the only nontrivial perturbation is to cut S^3 open along the toral leaf and then to insert there a foliated interval bundle over T^2.

EXAMPLE 5.3. Similarly to the case of the PLTP, given a transversely orientable codimension one semistable foliation on a closed n-manifold ($n \geq 4$), by a turbulization one can produce another semistable foliation.

6. Expansivity with PLTP implies semistability.

The purpose of this section is to prove Theorem A stated in the introduction. The proof goes parallel with that in [17].

Let $\mathcal{F} \in \mathrm{Fol}_q(M)$ be an expansive foliation with expansive constant $e_{\mathcal{F}}$ ($< \kappa$).

By replacing a smaller one if necessary, we assume that the constant κ' (see §2) is sufficiently small relative to $e_{\mathcal{F}}$ and that the point $w(x, y)$ lies in $D_{\perp}(x, e_{\mathcal{F}})$.

First we prove two preliminary lemmas.

LEMMA 6.1. *Let $0 < \varepsilon < \kappa'/2$ and suppose that a pseudoleaf L of $\mathcal{F}$ is ε-traced by some leaf of $\mathcal{F}$. Then a leaf F of $\mathcal{F}$ which ε-traces L is unique. Moreover, if we fix a base point $x_0 \in L$, then the corresponding base point $y_0 \in F$ is uniquely determined. Equivalently, if $\phi : \tilde{F} \to L$ is the covering map given in the remark after 4.2, there uniquely exists a covering map $\psi : L \to F$ such that $\psi \circ \phi = p$, where $\tilde{F} = P_{\mathcal{F}}(y_0)/\simeq$ is the universal covering space of F and $p : \tilde{F} \to F$ is given by $p([\gamma]) = \gamma(1)$.*

PROOF. Fix a point x_0 of L and suppose that there are two pairs (F_1, y_1), (F_2, y_2) of pointed $\mathcal{F}$-leaves such that $x_0 \in D_{\mathcal{F}^\perp}(y_1, \varepsilon) \cap D_{\mathcal{F}^\perp}(y_2, \varepsilon)$ and that both pairs ε-trace (L, x_0). By the definition of ε-tracing, any $\gamma \in P_{\mathcal{F}}(y_1)$ can be lifted to a curve $\bar{\gamma} : [0, 1] \to L$ of origin x_0 such that $\bar{\gamma}(t) \in D_{\mathcal{F}^\perp}(\gamma(t), \varepsilon)$ for all t. Since (F_2, y_2) also ε-traces (L, x_0), we can project $\bar{\gamma}$ to (F_2, y_2) and obtain a curve $\bar{\bar{\gamma}} \in P_{\mathcal{F}}(y_2)$ such that $\bar{\gamma}(t) \in D_{\mathcal{F}^\perp}(\bar{\bar{\gamma}}(t), \varepsilon)$ for all t. Then we have $d_M(\gamma(t), \bar{\bar{\gamma}}(t)) \leq d_M(\gamma(t), \bar{\gamma}(t)) + d_M(\bar{\gamma}(t), \bar{\bar{\gamma}}(t)) < \varepsilon + \varepsilon = 2\varepsilon < \kappa'$. From this and the choice of κ' we see that γ is lifted to the curve $\tilde{\gamma} \in P_{\mathcal{F}}(w(y_1, y_2))$ defined by $\tilde{\gamma}(t) = w(\gamma(t), \bar{\bar{\gamma}}(t))$, which satisfies that $\tilde{\gamma}(t) \in D_{\mathcal{F}^\perp}(\gamma(t), e_{\mathcal{F}})$ for all $0 \leq t \leq 1$. The expansivity of $\mathcal{F}$ then implies that $(F_1, y_1) = (F_2, w(y_1, y_2))$. This completes the proof since it is obvious that under the condition $D_{\mathcal{F}^\perp}(y_1, \varepsilon) \cap D_{\mathcal{F}^\perp}(y_2, \varepsilon) \neq \emptyset$ the equality $y_1 = w(y_1, y_2)$ holds if and only if $y_1 = y_2$.

LEMMA 6.2. *For every $\nu > 0$ there exists $r > 0$ satisfying the following property: Let* $x \in M$ and $y \in D_{\mathcal{F}^\perp}(x, e_{\mathcal{F}})$. *Suppose any $\gamma \in P_{\mathcal{F}}(x, r)$ is liftable to some curve $\bar{\gamma} \in P_{\mathcal{F}}(y)$ such that $\bar{\gamma}(t) \in D_{\mathcal{F}^\perp}(\gamma(t), e_{\mathcal{F}})$ for all $0 \leq t \leq 1$. Then it holds that $y \in D_{\mathcal{F}^\perp}(x, \nu)$.*

PROOF. Take any $\nu > 0$. We may assume $\nu < e_{\mathcal{F}}$. Suppose by contradiction that there does not exist such r. Then there exist a sequence $\{r_n\}$ of positive numbers diverging to ∞ and sequences $\{x_n\}$, $\{y_n\}$ of points of M with $y_n \in D_{\mathcal{F}^\perp}(x_n, e_{\mathcal{F}}) - D_{\mathcal{F}^\perp}(x_n, \nu)$ which satisfy the following property: Every curve $\gamma \in P_{\mathcal{F}}(x_n, r_n)$ is liftable to some curve $\bar{\gamma} \in P_{\mathcal{F}}(y_n)$ such that $\bar{\gamma}(t) \in D_{\mathcal{F}^\perp}(\gamma(t), e_{\mathcal{F}})$ for all $0 \le t \le 1$. Since M is compact, by taking subsequences if necessary, we may assume that the sequences $\{x_n\}$ and $\{y_n\}$ converge to some points x_∞ and y_∞ of M respectively. Then one can check that these two points satisfy the following properties: firstly, $y_\infty \in D_{\mathcal{F}^\perp}(x_\infty, e_{\mathcal{F}}) - \operatorname{Int} D_{\mathcal{F}^\perp}(x_\infty, \nu)$, and secondly every curve $\gamma \in P_{\mathcal{F}}(x_\infty)$ is liftable to some curve $\bar{\gamma} \in P_{\mathcal{F}}(y_\infty)$ such that $\bar{\gamma}(t) \in D_{\mathcal{F}^\perp}(\gamma(t), e_{\mathcal{F}})$ for all $0 \le t \le 1$. This contradicts the expansivity of $\mathcal{F}$, which completes the proof.

PROOF OF THEOREM A. Suppose that $\mathcal{F} \in \mathrm{Fol}_q(M)$ is expansive with expansive constant $e_{\mathcal{F}}$ and has the PLTP. Let $\varepsilon < \kappa'/3$. Let $\delta > 0$ be such that any δ-pseudoleaf is ε-traced by some $\mathcal{F}$-leaf. Let $\mathcal{G} \in \mathrm{Fol}_q(M)$ be such that $\rho(\mathcal{F}, \mathcal{G}) < \delta$. Then since every leaf of $\mathcal{G}$ is a δ-pseudoleaf of $\mathcal{F}$, by Lemma 6.1 for every $G \in \mathcal{G}$ there exists a unique $\mathcal{F}$-leaf F_G such that F_G ε-traces G. More precisely, there uniquely exists a covering map $\psi_G : G \to F_G$ such that $y \in D_{\mathcal{F}^\perp}(\psi_G(y), \varepsilon)$ for all $y \in G$. Now define a map $f : M \to M$ by $f(x) = \psi_{G_x}(x)$, where G_x is the leaf of $\mathcal{G}$ passing through x. Then obviously $d_M(f, \mathrm{id}_M) < \varepsilon$ and f sends each $\mathcal{G}$-leaf onto some $\mathcal{F}$-leaf.

So, the only thing we have to check is the continuity of f. Let $\lambda > 0$ be given. We will prove that there exists $\mu > 0$ such that $d_M(x, y) < \mu$ implies $d_M(f(x), f(y)) < \lambda$.

First, choose $\nu > 0$ sufficiently small to satisfy the following property: For any $u, v \in M$ with $d_M(u, v) < \kappa'$ if $d_M(u, w(u, v)) < \nu$ and $d_M(D_{\mathcal{F}^\perp}(u, \varepsilon), D_{\mathcal{F}^\perp}(v, \varepsilon)) < \nu$, then $d_M(u, v) < \lambda$. Next, for this ν, take $r > 0$ as in Lemma 6.2. Put $R = \sup\{\mathrm{Length}\, \bar{\gamma} \mid x \in M, \gamma \in P_{\mathcal{F}}(f(x), r), \bar{\gamma} \in P_{\mathcal{G}}(x), f \circ \bar{\gamma} = \gamma\} < \infty$. Then, take $0 < \mu < \nu$ sufficiently small that if $d_M(x, y) < \mu$, then for any $\bar{\gamma} \in P_{\mathcal{G}}(x, R)$ there exists $\hat{\gamma} \in P_{\mathcal{G}}(y)$ such that $d_M(\bar{\gamma}(t), \hat{\gamma}(t)) < \kappa'/3$ for all $0 \le t \le 1$. We will show that this μ is the desired one. Let x and y be two points of M with $d_M(x, y) < \mu$ (Note that this implies that $d_M(D_{\mathcal{F}^\perp}(f(x), \varepsilon), D_{\mathcal{F}^\perp}(f(y), \varepsilon)) < \mu$ since $x \in D_{\mathcal{F}^\perp}(f(x), \varepsilon)$ and $y \in D_{\mathcal{F}^\perp}(f(y), \varepsilon)$). Take any $\gamma \in P_{\mathcal{F}}(f(x), r)$. Then by Lemma 6.1 and the definition of f, γ can be lifted to $\bar{\gamma} \in P_{\mathcal{G}}(x)$ such that $\bar{\gamma}(t) \in D_{\mathcal{F}^\perp}(\gamma(t), \varepsilon)$ for all $0 \le t \le 1$. By the choice of R. Length $\bar{\gamma} \le R$. Hence by the choice of μ, there exists $\hat{\gamma} \in P_{\mathcal{G}}(y)$ such that $d_M(\bar{\gamma}(t), \hat{\gamma}(t)) < \kappa'/3$ for all $0 \le t \le 1$. Again by Lemma 6.1 and the definition of f, $\hat{\gamma}$ can be projected to $\check{\gamma} \in P_{\mathcal{F}}(f(y))$ such that $\hat{\gamma}(t) \in D_{\mathcal{F}^\perp}(\check{\gamma}(t), \varepsilon)$ for all $0 \le t \le 1$. We have

$$\begin{aligned} d_M(\gamma(t), \check{\gamma}(t)) &\le d_M(\gamma(t), \bar{\gamma}(t)) + d_M(\bar{\gamma}(t), \hat{\gamma}(t)) + d_M(\hat{\gamma}(t), \check{\gamma}(t)) \\ &< \varepsilon + \kappa'/3 + \varepsilon < \kappa' \end{aligned}$$

for all $0 \le t \le 1$. This with the choice of κ' implies that γ can be lifted to the curve $\tilde{\gamma} \in P_{\mathcal{F}}(w(f(x), f(y)))$ defined by $\tilde{\gamma}(t) = w(\gamma(t), \check{\gamma}(t))$. Since $\tilde{\gamma}$ satisfies $\tilde{\gamma} \in D_\perp(\gamma(t), e_{\mathcal{F}})$ for all $0 \le t \le 1$ (see the top of this section), by Lemma 6.2 we have $w(f(x), f(y)) \in D_\perp(f(x), \nu)$. By the choice of ν, this with $d_M(x, y) < \nu$ implies that $d_M(f(x), f(y)) < \lambda$, which shows the continuity of f. The theorem is proven.

7. Suspension and pseudoleaf tracing property.

The purpose of this section is to prove the part (1) of Theorem B stated in the introduction. We begin with the definition of the pseudo-orbit tracing property for a group action.

Let X be a compact Riemannian manifold. We denote by Diff(X) the group of diffeomorphisms of X. Let $\Phi : \Gamma \to \mathrm{Diff}(X)$ be an action of a finitely presented group Γ on X. The pseudo-orbit tracing property (POTP) for Φ is defined as follows: Fix an arbitrary finite generating system $\{g_1, g_2, \cdots, g_r\}$ of Γ. Then, a family $\{x(g)\}_{g\in\Gamma}$ of points of X is a δ*-pseudo-oribt* of Φ if $d(\Phi(g_i)(x(g)), x(g_i g)) < \delta$ for all $g \in \Gamma$ and $1 \le i \le r$. A δ-pseudo-orbit $\{x(g)\}_{g\in\Gamma}$ is ε*-traced* by the orbit $\{\Phi(g)(y)\}_{g\in\Gamma}$ of Φ passing through $y \in X$ if $d(x(g), \Phi(g)(y)) < \varepsilon$ for all $g \in \Gamma$. Finally, an action Φ has the *POTP* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that any δ-pseudo-orbit is ε-traced by some orbit of Φ. As is easily seen, the POTP of an action does not depend on the choice of a generating system.

To fix the notation, let us briefly review about the suspension: A *suspension* $\mathcal{F}_\Phi$ of Φ is a foliation which is constructed in the following way: Let B be a compact Riemannian manifold such that $\pi_1(B)$ is isomorphic to Γ. We fix a base point $b_0 \in B$ and identify Γ with $\pi_1(B, b_0)$. Γ acts on the universal covering $\tilde{B}$ of B as covering transformations. Define the action of Γ on $\tilde{B} \times X$ by $g \cdot (b, x) = (gb, \Phi(g)x)$ for $g \in \Gamma$, $b \in \tilde{B}$ and $x \in X$. This action preserves the product foliation $\tilde{\mathcal{F}} = \{\tilde{B} \times \{x\}\}_{x\in X}$ and hence induces a foliation $\mathcal{F}_\Phi$ on the quotient manifold $M_\Phi = (\tilde{B} \times X)/\Gamma$. We denote by q the quotient map $\tilde{B} \times X \to M_\Phi$. We fix a base point $b(e)$ in $\tilde{B}$ such that $\pi(b(e)) = b_0$, where $\pi : \tilde{B} \to B$ is the covering projection, and for $g \in \Gamma$ we put $b(g) = gb(e)$. We identify $p^{-1}(b_0)$ with X by $p_2 \circ (q \mid \{b(e)\} \times X)^{-1}$, where p_i $(i = 1, 2)$ is the projection of $\tilde{B} \times X$ to the i-th factor and $p : M_\Phi \to B$ is the bundle projection such that $p \circ q = \pi \circ p_1$. We fix on M_Φ a Riemannian metric such that its restriction to $p^{-1}(b_0)$ coincides with that on X and that $p^{-1}(b_0)$ is orthogonal to $\mathcal{F}_\Phi$. We regard $\tilde{B} \times X$ as a Riemannian manifold by pulling back the metric on M_Φ by q.

Throughout §§7, 8 and 9 we use the above notations.

Now we show the following

THEOREM B(1). *Φ has the POTP if and only if $\mathcal{F}_\Phi$ has the PLTP.*

PROOF. Put $V_g = \{b \in \tilde{B} \mid d_{\tilde{B}}(b, b(g)) \le d_{\tilde{B}}(b, b(h))$ for all $h \in \Gamma\}$, where $d_{\tilde{B}}$ is the distance on $\tilde{B}$ induced from the pull-back of the Riemannian metric of B by π. Take $\eta > 0$ and set U_g to be an open η-neighborhood of V_g. Then $\{U_g\}_{g\in\Gamma}$ is a locally finite open cover of $\tilde{B}$. Here we choose η so small that $b_e \in U_g$ if and only if $g = e$. Let $\{\lambda(g)\}_{g\in\Gamma}$ be any partition of unity subordinate to $\{U_g\}$.

First we show the "if" part. So assume that $\mathcal{F}_\Phi$ has the PLTP. Let $\varepsilon > 0$ be given. Then there is $\delta' > 0$ such that any δ'-pseudoleaf of $\mathcal{F}_\Phi$ is ε-traced by some real leaf of $\mathcal{F}_\Phi$. Choose $\delta > 0$ sufficiently small (which will be adjusted later) and take a δ-pseudo-orbit $\{x(g)\}_{g\in\Gamma}$ of Φ. We will construct a smooth section $s : \tilde{B} \to \tilde{B} \times X$ as follows: Define a function $\mu : \tilde{B} \times X \to \mathbf{R}$ by $\mu(b, x) = \sum_{g\in\Gamma} \lambda(g^{-1})(b)\{d_b((b, x), (b, \Phi(g^{-1})x(g)))\}^2$, where d_b is the distance on $\{b\} \times X$ induced from the Riemannian metric. If δ is small, for each $b \in \tilde{B}$, $\mu(b, x)$ is strictly convex in x, and attains a minimum at a unique point $s(b)$ (Such a point is called the *center of mass* of the points $\{(b, \Phi(g^{-1})x(g))\}_{g\in\Gamma}$ with respect to the mass distribution $\{\lambda(g^{-1})(b)\}_{g\in\Gamma}$ [4, p. 131]). Moreover, $s(b)$ depends smoothly on b. By the construction, $s(\tilde{B})$ contains the points $(b(g^{-1}), \Phi(g^{-1})x(g))$ for all $g \in \Gamma$. By replacing δ with a smaller one if necessary, we can further assume that the image $qs(\tilde{B})$ is an immersed submanifold of M and that if $\{x(g)\}$ is a δ-pseudo-orbit of Φ, then $qs(\tilde{B})$ is a δ'-pseudoleaf of $\mathcal{F}_\Phi$. Now, by the PLTP of $\mathcal{F}_\Phi$ and the choice of δ', there is a leaf F of $\mathcal{F}_\Phi$ which ε-traces $qs(\tilde{B})$, and, it follows that the Φ-orbit $F \cap p^{-1}(b_0)$ ε-traces $qs(\tilde{B}) \cap p^{-1}(b_0) = \{q(b(e), x(g))\}_{g\in\Gamma} = \{x(g)\}_{g\in\Gamma}$, as desired. This proves the "if" part.

Next we show the "only if" part. Suppose that Φ has the POTP and let ε be given. Choose $0 < \varepsilon' < \varepsilon$ so small that the following property holds: For every $x, y \in X$ with $d(x, y) < 2\varepsilon'$ and for every $b \in U_e$, the geodesic arc in $\tilde{B} \times X$ starting from (b, x), perpendicular to $U_e \times \{x\}$ at (b, x) and arriving at a point of $\tilde{B} \times \{y\}$ has length less than $\varepsilon/2$. (We can find such ε' because X and $\overline{U_e}$ are compact.) By the POTP of Φ there is $\delta' > 0$ such that any δ'-pseudo-orbit of Φ is ε'-traced by some real orbit of Φ. We can take $\delta > 0$ sufficiently small so that it satisfies the following properties:

(1) If L is a δ-pseudoleaf of $\mathcal{F}_\Phi$, then L is transverse to the fibers of p and $L \cap p^{-1}(b_0)$ is a δ'-pseudo-orbit of Φ.

(2) Let L be a δ-pseudoleaf and $\tilde{L}$ a connected component of $q^{-1}(L)$. Then, for every $x \in X$ with $d((b(e), x), (\{b(e)\} \times X) \cap \tilde{L}) < \varepsilon'$ and for every $b \in U_e$, the geodesic arc starting from (b, x), perpendicular to $U_e \times \{x\}$ at (b, x) and arriving at a point of $\tilde{L}$ has length less than ε.

In fact, the property (2) can be established from the choice of ε' because $\tilde{L} \cap (U_e \times X)$ becomes arbitrarily C^1-close to $U_e \times \{w\}$ for some $w \in X$ as δ tends to 0.

Now, let L be a δ-pseudoleaf of $\mathcal{F}_\Phi$. Then, since L is a covering space over B by the property (1) above, each point of $L \cap p^{-1}(b_0)$ can naturally be written as $x(g)$ for some $g \in \Gamma$ as soon as we take an arbitrary point of $L \cap p^{-1}(b_0)$ as $x(e)$. (That is, if γ is a loop in B based at b_0 representing $g \in \Gamma$ and if $\tilde{\gamma}$ is a path in L starting from $x(e)$ such that $q \circ \tilde{\gamma} = \gamma$, then we name the terminal point of $\tilde{\gamma}$ as $x(g)$.) Also, by the choice of δ' we can find an orbit $O = \{\Phi(g)x\}_{g\in\Gamma}$ of Φ such that $d(\Phi(g)x, x(g)) < \varepsilon'$ for all $g \in \Gamma$. Let F be the leaf of $\mathcal{F}_\Phi$ corresponding with O. Denote by $\tilde{L}$ the connected component of $q^{-1}(L)$ passing through $(b(e), x(e))$. By the property (2) and the fact that $(\{b(e)\} \times X) \cap g(\tilde{L}) = (b(e), x(g))$ for each $g \in \Gamma$, we see that, for every $g \in \Gamma$ and $b \in U_e$, the geodesic arc starting from $(b, \Phi(g)x)$, perpendicular to $U_e \times \{\Phi(g)x\}$ at $(b, \Phi(g)x)$ and arriving at a point of $g(\tilde{L})$ has length less than ε. Since the Γ-action on $\tilde{B} \times X$ is isometric and $\{g^{-1}(U_e \times \{\Phi(g)x\})\}_{g\in\Gamma}$ is an open cover of $\tilde{B} \times \{x\}$, it follows that L is ε-traced by F. This proves the "only if" part, and the proof of the theorem is complete.

Let $\mathrm{PO}^+(n, 1)$ be the group of orientation preserving Möbius transformations of S^{n-1}.

COROLLARY 7.1. *Let Γ be a finitely generated torsionless discrete cocompact subgroup of* $\mathrm{PO}^+(n,1)$ *and let* $\Phi : \Gamma \to \mathrm{Diff}(S^{n-1})$ *be the inclusion homomorphism. Then* Φ *has the POTP.*

PROOF. Let Γ and Φ be as in the hypothesis. Then, as is well-known, the geodesic flow φ of $K \setminus \mathrm{PO}^+(n,1)/\Gamma$ is an Anosov flow on the unit tangent bundle $T_1(K \setminus \mathrm{PO}^+(n,1)/\Gamma)$ of $K \setminus \mathrm{PO}^+(n,1)/\Gamma$, where K is a maximal compact subgroup of $\mathrm{PO}^+(n,1)$. The weakly stable foliation $\mathcal{F}_\varphi^s$ of φ is smooth and transverse to the fibers of $T_1(K \setminus \mathrm{PO}^+(n,1)/\Gamma)$ and is obtained from Φ by suspension. Since $\mathcal{F}_\varphi^s$ has the PLTP (Example 4.5), it follows from Theorem B(1) that Φ has the POTP, as desired.

8. Suspension and semistability.

In this section we prove the part (2) of Theorem B.

An action $\Phi : \Gamma \to \mathrm{Diff}(X)$ is *semistable* if for every $\varepsilon > 0$ there exist neighborhoods U_i of $\Phi(g_i)$ for all $1 \le i \le r$ with the property that for any action $\Psi : \Gamma \to \mathrm{Diff}(X)$ with $\Psi(g_i) \in U_i$ for all $1 \le i \le r$ there is a continuous map $h : X \to X$ satisfying $h\Psi(g) = \Phi(g)h$ for all $g \in \Gamma$ and $d_X(h, \mathrm{id}_X) < \varepsilon$.

For the definition of semistability of foliations, see §5.

LEMMA 8.1. *Let* $\Phi : \Gamma \to \mathrm{Diff}(X)$ *be a smooth action. Then for any neighborhood* V *of* $\mathcal{F}_\Phi$ *in* $\mathrm{Fol}_q(M_\Phi)$ *there exist neighborhoods* U_i *of* $\Phi(g_i)$ *in* $\mathrm{Diff}(X)$ *for all* $1 \le i \le r$ *with the property that for any action* $\Psi : \Gamma \to \mathrm{Diff}(X)$ *with* $\Psi(g_i) \in U_i$ *for all* $1 \le i \le r$ *there is a fiber preserving diffeomorphism* $\varphi : M_\Phi \to M_\Psi$ *such that* $\varphi^*\mathcal{F}_\Psi \in V$.

PROOF. It is well-known that $\mathrm{Diff}(X)$ is locally contractible. Hence there exists no obstruction in constructing a desired diffeomorphism if Ψ is sufficiently close to Φ.

Now we will prove

THEOREM B(2). Φ *is semistable if and only if* $\mathcal{F}_\Phi$ *is semistable.*

PROOF. First we prove the "if" part. Suppose that Φ is semistable and consider a perturbation $\mathcal{G}$ of its suspension foliation $\mathcal{F}_\Phi$. We may assume that $\mathcal{G}$ is transverse to the fibers. Thus $\mathcal{G}$ determines uniquely an action Ψ of Γ on $p^{-1}(b_0)(= X)$. Since Ψ is close to Φ, it follows from the semistability of Φ that there exists a semiconjugacy $h : p^{-1}(b_0) \to p^{-1}(b_0)$ close to the identity such that $h\Psi(g) = \Phi(g)h$ for all $g \in \Gamma$. Now, for each $b \in B$ take any path $\gamma : I \to B$ such that $\gamma(0) = b_0$ and $\gamma(1) = b$. For each $z \in p^{-1}(b)$ lift γ to a $\mathcal{G}$-curve γ_1 with $\gamma_1(1) = z$ and also to a $\mathcal{F}_\Phi$-curve γ_2 with $\gamma_2(0) = h(\gamma_1(0))$. Then define H by $H(z) = \gamma_2(1)$. One can easily check that H becomes a well-defined continuous map from M_Φ to itself close to the identity such that $H(\mathcal{G}) = \mathcal{F}_\Phi$. This proves the "if" part.

Next we prove the "only if" part. Suppose that $\mathcal{F}_\Phi$ is semistable and consider a perturbation Ψ of Φ. By Lemma 8.1 there exists a fiber preserving diffeomorphism $\varphi : M_\Phi \to M_\Psi$ such that $\varphi^*\mathcal{F}_\Psi$ is close to $\mathcal{F}_\Phi$. It follows from the semistability of $\mathcal{F}_\Phi$ that there exists a continuous map $H : M_\Phi \to M_\Phi$ close to the identity such that $H(\varphi^*\mathcal{F}_\Psi) = \mathcal{F}_\Phi$. Take a neighborhood U of b_0 so that $\varphi^* \mathcal{F}_\Psi \mid p^{-1}(U)$ and $\mathcal{F}_\Phi \mid p^{-1}(U)$ are product foliations and that if $x \in p^{-1}(b_0)(= X)$ then $H(x) \in p^{-1}(U)$. Then for every $x \in p^{-1}(b_0)$ there exists a unique point, say $h(x)$, in $p^{-1}(b_0)$ such that $H(x)$ and $h(x)$ are on the same leaf of $\mathcal{F}_\Phi \mid p^{-1}(U)$. Now it is easy to check that the map h thus defined is a desired semiconjugacy from Ψ to Φ. This proves the "only if" part, and the proof of the theorem is complete.

REMARK. By [5], the group Homeo(X) of homeomorphisms of X is also locally contractible. So, the topological version of Theorem B(2) would also be true once the space of all foliations could be topologized suitably in a purely topological manner.

9. Suspension and expansivity.

The purpose of this section is to prove the part (3) of Theorem B.

An action $\Phi : \Gamma \to \mathrm{Diff}(X)$ is said to be *expansive* of there exists a constant $e_\Phi > 0$ with the property that for any two distinct points $x, y \in X$, there exists $g \in \Gamma$ such that $d(\Phi(g)(x), \Phi(g)(y)) > e_\Phi$.

THEOREM B(3). *Φ is expansive if and only if $\mathcal{F}_\Phi$ is expansive.*

PROOF. First we show the "if" part. So assume that the suspension foliation $\mathcal{F} = \mathcal{F}_\Phi$ is expansive with expansive constant $e(< \kappa)$. Let x and y be any two distinct points of $X = p^{-1}(b_0)$. In suffices to consider the case where $d_X(x, y) \leq e$. By the expansivity of $\mathcal{F}$ there exists $\gamma \in P_{\mathcal{F}}(x)$ such that the lift $\bar{\gamma} \in P_{\mathcal{F}}(y)$ of γ satisfies $\bar{\gamma}(1) \notin D_{\mathcal{F}^\perp}(\gamma(1), e)$. Put $T = \sup\{t_0 \mid \bar{\gamma}(t) \in D_{\mathcal{F}^\perp}(\gamma(t), e) \text{ for all } 0 \leq t \leq t_0\}$. Then $0 \leq T < 1$. Fix a compact fundamental domain F in $\tilde{B}$ for the action of Γ such that $b(e) \in F$ and that the diameter of F is not greater than the diameter of B. We lift γ to a curve $(\tilde{\gamma}, x)$ in $\tilde{B} \times X$ so that $\tilde{\gamma}(0) = b(e)$.

The following is obvious from the compactness of F.

LEMMA 9.1. *There exists a constant $\eta > 0$ such that for any $g \in \Gamma$, $x, y \in X$ and $b \in g(F)$ if $\tilde{B} \times \{y\}$ intersects $\partial D_\perp((b, x), e)$, then $d_{\{b(g)\}\times X}((b(g), x), (b(g), y)) > \eta$.*

By this lemma if $\tilde{\gamma}(T) \in g(F)$, then $d_{\{b(g)\}\times X}((b(g), x), (b(g), y)) > \eta$. Since by definition the action of Γ on $\tilde{B} \times X$ is isometric, we have

$$d_{\{b(e)\}\times X}((b(e), \Phi(g^{-1})(x)), (b(e), \Phi(g^{-1})(y))) > \eta\,.$$

By the identification of $\{b_e\} \times X$ with X this may be rewritten as

$$d_X(\Phi(g^{-1})(x)), \Phi(g^{-1})(y)) > \eta\,.$$

This proves the expansivity of Φ with expansive constant η.

Now we will prove the "only if" part. Assume that Φ is expansive with expansive constant $e(< \kappa)$. We will see that $\mathcal{F}_\Phi$ is expansive with the same expansive constant e. Let x and y be any two distinct points of M_Φ such that $y \in D_{\mathcal{F}^\perp}(x, e)$. Take an arbitrary path $\gamma \in P_{\mathcal{F}}(x)$ such that $\gamma(1) \in X = p^{-1}(b_0)$. If $\bar{\gamma}(t) \notin D_{\mathcal{F}^\perp}(\gamma(t), e)$ for some $0 \leq t \leq 1$, where $\bar{\gamma}$ is the lift of γ to $P_{\mathcal{F}}(y)$, then we are done. Otherwise, put $x' = \gamma(1) \in X$ and $y' = \bar{\gamma}(1) \in X$. By the expansivity of Φ, there exists $g \in \Gamma$ such that $d_X(\Phi(g)x', \Phi(g)y') > e$. Let τ_g be a loop in B based at b_0 representing g and $\gamma_g \in P_{\mathcal{F}}(x')$ the curve such that $p \circ \gamma_g = \tau_g$. Then $\gamma_g(1) = \Phi(g)x'$ and $\bar{\gamma}_g(1) = \Phi(g)y'$ where $\bar{\gamma}_g$ is the lift of γ_g to $P_{\mathcal{F}}(y')$, thus it must occur that $\bar{\gamma}_g(t) \notin D_{\mathcal{F}^\perp}(\gamma_g(t), e)$ for some $0 \leq t \leq 1$. This means that the composite curve $\gamma * \gamma_g \in P_{\mathcal{F}}(x)$ cannot be lifted to a curve in $P_{\mathcal{F}}(y)$ within the orthogonal distance e, which completes the proof of "only if" part, and, hence, of Theorem B(3).

10. Open Problems.

PROBLEM 10.1 Does the semistability imply the PLTP?

This has been affirmatively solved for homeomorphisms and flows ([17, 14]) by using a technique of local perturbation. But in the case of a foliation of leaf-dimension greater than one, the answer is likely to be in the negative, because the nontriviality of the integrability condition prevents us from making such a local modification.

PROBLEM 10.2. Consider an expansive foliation and denote by $g(r)$ the number of its compact leaves of diameter not greater than r. Is the exponential growth rate of $g(r)$ estimated by the geometric entropy ([7]) from above?

For the corresponding results in dynamical systems, see [6], [2, Proposition 2.8] and [3, Theorem 5].

PROBLEM 10.3. Define the PLTP for topological foliations.

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