On Some Maximal Galois Coverings over Affine and Projective Planes II

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1. Introduction.

This note is the second part of the preceding paper [NT] which we refer as Part I in the following. We recall that in Part I the finite maximal Galois coverings over a complex affine plane $\mathbf{A}^2$ with branch locus $B_q := \{(v, w) \in \mathbf{A}^2 | w^2 = v^q\}$ *with q odd* were studied in some detail, and further, the existence of maximal Galois coverings over a complex projective plane $\mathbf{P}^2$ with branch locus $\overline{B_q} \cup l_\infty$ was discussed, where $\overline{B_q}$ is the projective closure of B_q and l_∞ is the infinite line.

In this note, we study the finite maximal Galois coverings over $\mathbf{A}^2$ (resp. $\mathbf{P}^2$) with branch locus B_q (resp. $\overline{B_q} \cup l_\infty$) *with even q*, which remained untouched in Part I.

Our main results are: (i) the maximal Galois covering of $\mathbf{A}^2$ with branch locus B_q exists and is isomorphic to $\mathbf{A}^2$ if the corresponding maximal Galois group is finite (Theorem 4) and (ii) a criterion for the existence of maximal Galois coverings over $\mathbf{P}^2$ with branch locus $\overline{B_q} \cup l_\infty$ (Theorem 7).

We note that the finite maximal Galois coverings of $\mathbf{P}^2$ thus obtained are all rational so that they will hopefully be good examples of rational normal projective surfaces which are finite maximal Galois coverings over $\mathbf{P}^2$ with simple branch loci and also with explicitly calculated Galois groups.

Our main results mentioned above are easy consequences of the explicit calculations of the maximal Galois groups with some simple presentation. We thus spent many pages on elementary combinatorial group-theoretic computation, which may be boring. The reason for our doing this is twofold: (i) these explicit descriptions of the maximal Galois groups are essentially used in our main results (ii) it will be convenient for the readers who are not accustomed to combinatorial group theory.

In order to calculate the order of finite groups with given presentation, we used Cayley/Magma system, which performs the Todd-Coxeter process on computers.

2. Maximal Galois coverings over A^2 with branch locus B_q.

We recall that *the maximal Galois group* $G(M, D)$ of the pair (M, D), where M is a complex manifold and $D = \sum_{i=1}^{s} e_i D_i$ $(e_i > 0, D_i$: irreducible component) is an effective divisor on M, is defined as follows:

$$G(M, D) := \pi_1(M - \text{Supp}(D), P_0)/N\langle \gamma_1^{e_1}, \cdots, \gamma_s^{e_s} \rangle,$$

where $\pi_1(M - \text{Supp}(D), P_0)$ is the fundamental group of the complement of support of D in M with respect to a base point P_0, γ_i is the homotopy class of a closed path which starts from P_0, goes around D_i once, and returns to P_0, and $N\langle \gamma_1^{e_1}, \cdots, \gamma_s^{e_s} \rangle$ is the smallest normal subgroup which contains $\{\gamma_1^{e_1}, \cdots, \gamma_s^{e_s}\}$.

We note that $G(M, D)$ is a Galois group which controls the branched coverings over M with branch locus at most at D. See [NT, Section 1] or [Na, Chapter 1] for the details on the theory of branched coverings.

Now we specialize to the situation where $M = A^2$. As the branch locus D, we consider $B := \{(v, w) \in A^2 \mid w^2 = v^q\}$, where $q = 2r > 0$ is a positive even integer. Decompose B_q as $B_q = B_q^1 \cup B_q^2$, where $B_q^1 := \{(v, w) \in A^2 \mid w = v^r\}$, $B_q^2 := \{(v, w) \in A^2 \mid w = -v^r\}$. Thus B_q has two nonsingular branches which contact at the origin r-times if $r \geq 2$. In this section, we will first calculate the *finite* maximal Galois groups $G(A^2, e_1 B_q^1 + e_2 B_q^2)$ explicitly. In the following, we set $G(e_1, e_2; q) := G(A^2, e_1 B_q^1 + e_2 B_q^2)$ for short.

PROPOSITION 1. $G(e_1, e_2; q)$ *has a following presentation*:

$$G(e_1, e_2; q) \cong \langle a, b \mid a^{e_1} = b^{e_2} = 1, (ab)^r = (ba)^r \rangle.$$

PROOF.   See [NT, Proposition 2.1].   □

PROPOSITION 2.   *Let $e_1, e_2 \geq 2$, $q = 2r > 0$ be integers. Then $G(e_1, e_2; q)$ is a finite group if and only if $(e_1, e_2, q) = (m_1, m_2, 2), (2, 2, 2n_1), (2, n_2, 4), (2, 3, 6), (3, 3, 4),$ $(2, 3, 8), (2, 4, 6), (3, 4, 4), (2, 3, 10), (2, 5, 6), (3, 5, 4),$ up to permutation in (e_1, e_2). Here $m_i, n_i \geq 2$ are arbitrary integers.*

PROOF.   We show only if part first. Assume $q \geq 4$. We recall that the polyhedral group $P(x, y, z)$ is defined as $P(x, y, z) := \langle c, d \mid c^x = d^y = (cd)^z = 1 \rangle$ $(x, y, z \geq 2)$. Since $P(x, y, z)$ is a homomorphic image of $G(e_1, e_2; q)$, the only if part follows from the well-known finiteness condition for $P(x, y, z)$: $P(x, y, z)$ is finite if and only if $(x, y, z) = (2, 2, z)$ where $z \geq 2$, $(2, 3, 3), (2, 3, 4), (2, 3, 5)$ up to permutation.

The if part follows from calculating orders of these groups explicitly by the Todd-Coxeter process.   □

In the following proposition, we use the notations below.

NOTATION.

(1) $D_{2n} := \langle x, y \mid x^n = y^2 = 1, y^{-1}xy = x^{-1} \rangle$: the dihedral group of order $2n$.

(2) $Q_8 := \langle x, y \mid x^2 = y^2, x^4 = 1, y^{-1}xy = x^{-1} \rangle$: the quaternion group of order 8.

(3) $SL(2, \mathbf{Z}_n) :=$ the special linear group of degree 2 with coefficients in $\mathbf{Z}_n = \mathbf{Z}/n\mathbf{Z}$.

(4) Given two groups H, N with a homomorphism $\alpha : H \to \text{Aut}(N)$, we denote by $H \underset{\alpha}{\bowtie} N$ the semi-direct product of H with N. The product is defined as $(h_1, n_1)(h_2, n_2) :=$ $(h_1 h_2, n_1^{h_2} n_2)$, where $h_i \in H, n_i \in N$ and $n_1^{h_2} = \alpha(h_2)(n_1)$.

PROPOSITION 3. *Suppose that $G(e_1, e_2; q)$ is a finite group. Then we have the following isomorphisms. The description of the isomorphisms φ_i and the actions α_j, β_k is given in the proof.*

(0) $\varphi_0 : G(m_1, m_2; 2) \cong \mathbf{Z}_{m_1} \times \mathbf{Z}_{m_2}$.

(1) $\varphi_1 : G(2, 2; 2n_1) \cong D_{4n_1}$.

(2) $\varphi_2 : G(2, n_2; 4) \cong \mathbf{Z}_2 \underset{\alpha_2}{\bowtie} (\mathbf{Z}_{n_2} \times \mathbf{Z}_{n_2})$.

(3) $\varphi_3 : G(2, 3; 6) \cong \mathbf{Z}_2 \underset{\beta_3}{\bowtie} (\mathbf{Z}_3 \underset{\alpha_3}{\bowtie} Q_8)$.

(4) $\varphi_4 : G(3, 3; 4) \cong (\mathbf{Z}_3 \times \mathbf{Z}_3) \underset{\alpha_4}{\bowtie} Q_8$.

(5) $\varphi_5 : G(2, 3; 8) \cong \mathbf{Z}_2 \underset{\beta_5}{\bowtie} (\mathbf{Z}_3 \underset{\alpha_5}{\bowtie} SL(2, \mathbf{Z}_3))$.

(6) $\varphi_6 : G(2, 4; 6) \cong \mathbf{Z}_2 \underset{\beta_6}{\bowtie} (\mathbf{Z}_4 \underset{\alpha_6}{\bowtie} SL(2, \mathbf{Z}_3))$.

(7) $\varphi_7 : G(3, 4; 4) \cong \mathbf{Z}_{12} \underset{\alpha_7}{\bowtie} SL(2, \mathbf{Z}_3)$.

(8) $\varphi_8 : G(2, 3; 10) \cong \mathbf{Z}_6 \underset{\alpha_8}{\bowtie} SL(2, \mathbf{Z}_5)$.

(9) $\varphi_9 : G(2, 5; 6) \cong \mathbf{Z}_{10} \underset{\alpha_9}{\bowtie} SL(2, \mathbf{Z}_5)$.

(10) $\varphi_{10} : G(3, 5; 4) \cong \mathbf{Z}_{15} \underset{\alpha_{10}}{\bowtie} SL(2, \mathbf{Z}_5)$.

PROOF. In this proof, we write $G := G(e_1, e_2; q)$ for short.

(0), (1) These follow immediately from the presentation given in Proposition 1. The isomorphism φ_0 is given by

$$\varphi_0(a) = (1, 0), \quad \varphi_0(b) = (0, 1),$$

where $(1, 0)$ means $(1 \bmod m_1, 0 \bmod m_2) \in \mathbf{Z}_{m_1} \times \mathbf{Z}_{m_2}$. The isomorphism φ_1 is given by

$$\varphi_1(a) = y, \quad \varphi_1(b) = y^{-1}x.$$

(2) We first form a semi-direct product $\mathbf{Z}_2 \underset{\alpha_2}{\bowtie} (\mathbf{Z}_{n_2} \times \mathbf{Z}_{n_2})$, where the action $\alpha_2 : \mathbf{Z}_2 \to \text{Aut}(\mathbf{Z}_{n_2} \times \mathbf{Z}_{n_2})$ is defined by

$$\alpha_2(1)((1, 0)) := (1, -1), \quad \alpha_2(1)((0, 1)) := (0, -1).$$

Define a map $\varphi_2 : G \to \mathbf{Z}_2 \underset{\alpha_2}{\bowtie} (\mathbf{Z}_{n_2} \times \mathbf{Z}_{n_2})$ by

$$\varphi_2(a) := (1, (0, 0)), \quad \varphi_2(b) := (0, (1, 0)).$$

A direct calculation shows that $\text{ord}(1, (0, 0)) = 2$, $\text{ord}(0, (1, 0)) = n_2$ and $(1, (0, 0))^2 = (0, (1, 0))^2$ so that the map φ_2 is a well-defined group homomorphism. Since a set $\{(1, (0, 0)), (0, (0, 1)), (0, (1, -1))\} = \{\varphi_2(a), \varphi_2(b), \varphi_2(a^{-1}ba)\}$ generates $\mathbf{Z}_2 \underset{\alpha_2}{\bowtie} (\mathbf{Z}_{n_2} \times \mathbf{Z}_{n_2})$, φ_2 is surjective.

In order to show φ_2 is an isomorphism, it is enough to show $\text{ord}(G) = 2n_2^2$.

We calculate $\mathrm{ord}(G)$ after [J, Chapter 9]. Consider the following exact sequence

$$1 \to G' \to G \underset{f}{\to} G^{ab} \to 1 \,,$$

where G' is the commutator subgroup of G, G^{ab} the abelianization of G and f the natural surjection. Since $G^{ab} \cong \mathbf{Z}_2 \times \mathbf{Z}_{n_2}$, we may identify f with $f : G \to \mathbf{Z}_2 \times \mathbf{Z}_{n_2}$, $f(a) = (1, 0)$, $f(b) = (0, 1)$.

We take a Schreier transversal U of G' in G as follows:

$$U := \{1, b, b^2, \cdots, b^{n_2-3}, b^{n_2-2}, b^{n_2-1}, a, ab, ab^2, \cdots, ab^{n_2-3}, ab^{n_2-2}, ab^{n_2-1}\} \,.$$

Then the $B\hat{R}$-table and $B\hat{S}$-table are given as follows:

$B\hat{R}$-TABLE

	a	b	a^2	b^{n_2}	$ababa^{-1}b^{-1}a^{-1}b^{-1}$
1	1	1	a^2	b^{n_2}	$ababa^{-1}b^{-1}a^{-1}b^{-1}$
b	$bab^{-1}a^{-1}$	1	ba^2b^{-1}	b^{n_2}	$bababa^{-1}b^{-1}a^{-1}b^{-2}$
b^2	$b^2ab^{-2}a^{-1}$	1	$b^2a^2b^{-2}$	b^{n_2}	$b^2ababa^{-1}b^{-1}a^{-1}b^{-3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
b^{n_2-3}	$b^{n_2-3}ab^{-n_2+3}a^{-1}$	1	$b^{n_2-3}a^2b^{-n_2+3}$	b^{n_2}	$b^{n_2-3}ababa^{-1}b^{-1}a^{-1}b^{-n_2+2}$
b^{n_2-2}	$b^{n_2-2}ab^{-n_2+2}a^{-1}$	1	$b^{n_2-2}a^2b^{-n_2+2}$	b^{n_2}	$b^{n_2-2}ababa^{-1}b^{-1}a^{-1}b^{-n_2+1}$
b^{n_2-1}	$b^{n_2-1}ab^{-n_2+1}a^{-1}$	b^{n_2}	$b^{n_2-1}a^2b^{-n_2+1}$	b^{n_2}	$b^{n_2-1}ababa^{-1}b^{-1}a^{-1}b^{-n_2}$
a	a^2	1	a^2	$ab^{n_2}a^{-1}$	$a^2bababa^{-1}b^{-1}a^{-1}b^{-1}a^{-1}$
ab	$abab^{-1}$	1	$aba^2b^{-1}a^{-1}$	$ab^{n_2}a^{-1}$	$ababab a^{-1}b^{-1}a^{-1}b^{-2}a^{-1}$
ab^2	ab^2ab^{-2}	1	$ab^2a^2b^{-2}a^{-1}$	$ab^{n_2}a^{-1}$	$ab^2ababa^{-1}b^{-1}a^{-1}b^{-3}a^{-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
ab^{n_2-3}	$ab^{n_2-3}ab^{-n_2+3}$	1	$ab^{n_2-3}a^2b^{-n_2+3}a^{-1}$	$ab^{n_2}a^{-1}$	$ab^{n_2-3}ababa^{-1}b^{-1}a^{-1}b^{-n_2+2}a^{-1}$
ab^{n_2-2}	$ab^{n_2-2}ab^{-n_2+2}$	1	$ab^{n_2-2}a^2b^{-n_2+2}a^{-1}$	$ab^{n_2}a^{-1}$	$ab^{n_2-2}ababa^{-1}b^{-1}a^{-1}b^{-n_2+1}a^{-1}$
ab^{n_2-1}	$ab^{n_2-1}ab^{-n_2+1}$	$ab^{n_2}a^{-1}$	$ab^{n_2-1}a^2b^{-n_2+1}a^{-1}$	$ab^{n_2}a^{-1}$	$ab^{n_2-1}ababa^{-1}b^{-1}a^{-1}b^{-n_2}a^{-1}$

$B\hat{S}$-TABLE

		c_{n_2}	c_{2n_2}	$c_{n_2+1}c_{n_2+2}^{-1}c_1^{-1}$
c_1	—	$c_1c_{n_2+1}$	c_{2n_2}	$c_1c_{n_2+2}c_{n_2+3}^{-1}c_2^{-1}$
c_2	—	$c_2c_{n_2+2}$	c_{2n_2}	$c_2c_{n_2+3}c_{n_2+4}^{-1}c_3^{-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
c_{n_2-3}	—	$c_{n_2-3}c_{2n_2-3}$	c_{2n_2}	$c_{n_2-3}c_{2n_2-2}c_{2n_2-1}^{-1}c_{n_2-2}^{-1}$
c_{n_2-2}	—	$c_{n_2-2}c_{2n_2-2}$	c_{2n_2}	$c_{n_2-2}c_{2n_2-1}c_{2n_2}^{-1}c_{2n_2+1}^{-1}c_{n_2-1}^{-1}$
c_{n_2-1}	c_{2n_2}	$c_{n_2-1}c_{2n_2-1}$	c_{2n_2}	$c_{n_2-1}c_{2n_2+1}c_{n_2}c_{n_2+1}^{-1}c_{2n_2}^{-1}$
c_{n_2}	—	c_{n_2}	c_{2n_2+1}	$c_1c_2^{-1}c_{n_2+1}^{-1}$
c_{n_2+1}	—	$c_{n_2+1}c_1$	c_{2n_2+1}	$c_{n_2+1}c_2c_3^{-1}c_{n_2+2}^{-1}$
c_{n_2+2}	—	$c_{n_2+2}c_2$	c_{2n_2+1}	$c_{n_2+2}c_3c_4^{-1}c_{n_2+3}^{-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
c_{2n_2-3}	—	$c_{2n_2-3}c_{n_2-3}$	c_{2n_2+1}	$c_{2n_2-3}c_{n_2-2}c_{n_2-1}^{-1}c_{2n_2-2}^{-1}$
c_{2n_2-2}	—	$c_{2n_2-2}c_{n_2-2}$	c_{2n_2+1}	$c_{2n_2-2}c_{n_2-1}c_{2n_2+1}c_{2n_2}^{-1}c_{2n_2-1}^{-1}$
c_{2n_2-1}	c_{2n_2+1}	$c_{2n_2-1}c_{n_2-1}$	c_{2n_2+1}	$c_{2n_2-1}c_{2n_2}c_1^{-1}c_{n_2}^{-1}c_{2n_2+1}^{-1}$

Therefore we have

$$G' \cong \langle c_i (1 \le i \le n_2) \mid 6n_2 \text{ relations in the } B\hat{S}\text{-table} \rangle$$

$$\cong \langle c_1 \mid c_1^{n_2} = 1 \rangle$$

$$\cong \mathbf{Z}_{n_2}.$$

Hence $\text{ord}(G) = \text{ord}(G')\text{ord}(G^{ab}) = 2n_2^2$ as desired.

(3) Consider a group $\mathbf{Z}_2 \underset{\beta_3}{\rhd\!\!\!<} (\mathbf{Z}_3 \underset{\alpha_3}{\rhd\!\!\!<} Q_8)$, where the actions β_3, α_3 are defined as follows: The action $\alpha_3 : \mathbf{Z}_3 \to \text{Aut}(Q_8)$ is defined by

$$\alpha_3(1)(x) := yx^2, \quad \alpha_3(1)(y) := yx$$

and the action $\beta_3 : \mathbf{Z}_3 \to \text{Aut}(\mathbf{Z}_3 \underset{\alpha_3}{\rhd\!\!\!<} Q_8)$ is defined by

$$\beta_3(1)((1,1)) := (1, y), \quad \beta_3(1)((0, x)) := (0, x^3), \quad \beta_3(1)((0, y)) := (0, yx^2).$$

Now define a map $\varphi_3 : G \to \mathbf{Z}_2 \underset{\beta_3}{\rhd\!\!\!<} (\mathbf{Z}_3 \underset{\alpha_3}{\rhd\!\!\!<} Q_8)$ by

$$\varphi_3(a) := (1, (0, e)), \quad \varphi_3(b) := (0, (1, e)),$$

where $e \in Q_8$ is the identity element. We note that, since $\text{ord}((1, (0, e))) = 2$, $\text{ord}((0, (1, e))) = 3$, $\{(1, (0, e)) \cdot (0, (1, e))\}^3 = \{(0, (1, e)) \cdot (1, (0, e))\}^3$ by a direct calculation, φ_3 is a well-defined group homomorphism.

The map φ_3 is surjective since a set $\{(1, (0, e)), (0, (1, e)), (0, (0, y)), (0, (0, yx))\} = \{\varphi_3(a), \varphi_3(b), \varphi_3(b^{-1}a^{-1}ba), \varphi_3(b^{-2}a^{-1}bab)\}$ generates $\mathbf{Z}_2 \underset{\beta_3}{\rhd\!\!\!<} (\mathbf{Z}_3 \underset{\alpha_3}{\rhd\!\!\!<} Q_8)$.

On the other hand, we have $\text{ord}(G) = 48$ by the Todd-Coxeter process performed with Cayly/Magma system. Since $\text{ord}(\mathbf{Z}_2 \underset{\beta_3}{\rhd\!\!\!<} (\mathbf{Z}_3 \underset{\alpha_3}{\rhd\!\!\!<} Q_8)) = 48$, we conclude φ_3 is an isomorphism.

Since the proofs of (4)–(10) below are similar to that of (3), we only indicate explicitly the isomorphism φ_i and the actions α_j, β_k, omitting the details.

(4) The isomorphism $\varphi_4 : G \cong (\mathbf{Z}_3 \times \mathbf{Z}_3) \underset{\alpha_4}{\rhd\!\!\!<} Q_8$ by

$$\varphi_4(a) := ((1, 0), e), \quad \varphi_4(b) := ((0, 1), xy).$$

The action $\alpha_4 : \mathbf{Z}_3 \times \mathbf{Z}_3 \to \text{Aut}(Q_8)$ is given by

$$\alpha_3((1, 0))(x) := yx, \quad \alpha_3((0, 1))(x) := y, \quad \alpha_3((1, 0))(y) := x, \quad \alpha_3((0, 1))(y) := yx.$$

(5) Set $X := \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$, $Y := \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \in \text{SL}(2, \mathbf{Z}_3)$.

The isomorphism $\varphi_4 : G \cong \mathbf{Z}_2 \underset{\beta_5}{\rhd\!\!\!<} (\mathbf{Z}_3 \underset{\alpha_5}{\rhd\!\!\!<} \text{SL}(2, \mathbf{Z}_3))$ is given by

$$\varphi_5(a) := (1, (0, I_2)), \quad \varphi_5(b) := (0, (1, I_2)),$$

where $I_2 \in \text{SL}(2, \mathbf{Z}_3)$ is the identity matrix.

The action $\alpha_5 : \mathbf{Z}_3 \to \text{Aut}(\text{SL}(2, \mathbf{Z}_3))$ is given by

$$\alpha_5(1)(X) := Y^{-1}, \quad \alpha_5(1)(Y) := XY^{-1}.$$

The action $\beta_5 : \mathbf{Z}_3 \to \mathrm{Aut}(\mathbf{Z}_3 \underset{\alpha_5}{\triangleright\!\!\triangleleft} \mathrm{SL}(2, \mathbf{Z}_3))$ is given by

$$\beta_5(1)((1, I_2)) := (1, Y), \quad \beta_5(1)((0, X)) := (0, X^{-1}), \quad \beta_5(1)((0, Y)) := (0, Y^{-1}).$$

(6) Set $X := \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}, Y := \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \in \mathrm{SL}(2, \mathbf{Z}_3).$

The isomorphism $\varphi_6 : G \cong \mathbf{Z}_2 \underset{\beta_6}{\triangleright\!\!\triangleleft} (\mathbf{Z}_4 \underset{\alpha_6}{\triangleright\!\!\triangleleft} \mathrm{SL}(2, \mathbf{Z}_3))$ is given by

$$\varphi_6(a) := (1, (0, I_2)), \quad \varphi_6(b) := (0, (1, I_2)).$$

The action $\alpha_6 : \mathbf{Z}_4 \to \mathrm{Aut}(\mathrm{SL}(2, \mathbf{Z}_3))$ is given by

$$\alpha_6(1)(X) := Y^{-1}, \quad \alpha_6(1)(Y) := YX.$$

The action $\beta_6 : \mathbf{Z}_3 \to \mathrm{Aut}(\mathbf{Z}_3 \underset{\alpha_6}{\triangleright\!\!\triangleleft} \mathrm{SL}(2, \mathbf{Z}_3))$ is given by

$$\beta_6(1)((1, I_2)) := (1, Y), \quad \beta_6(1)((0, X)) := (0, X^{-1}), \quad \beta_6(1)((0, Y)) := (0, Y^{-1}).$$

(7) Set $X := \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}, Y := \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \in \mathrm{SL}(2, \mathbf{Z}_3).$

The isomorphism $\varphi_7 : G \cong \mathbf{Z}_{12} \underset{\alpha_7}{\triangleright\!\!\triangleleft} \mathrm{SL}(2, \mathbf{Z}_3)$ is given by

$$\varphi_7(a) := (4, Y^{-1}XY^{-1}) = \left(4, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right), \quad \varphi_7(b) := (9, I_2).$$

The action $\alpha_7 : \mathbf{Z}_4 \to \mathrm{Aut}(\mathrm{SL}(2, \mathbf{Z}_3))$ is given by

$$\alpha_7(1)(X) = (XY)^{-1}, \quad \alpha_7(1)(Y) = Y.$$

(8) Set $X = \begin{bmatrix} 4 & 2 \\ 0 & 4 \end{bmatrix}, Y = \begin{bmatrix} 4 & 0 \\ 2 & 4 \end{bmatrix} \in \mathrm{SL}(2, \mathbf{Z}_5).$

The isomorphism $\varphi_8 : G \cong \mathbf{Z}_6 \underset{\alpha_8}{\triangleright\!\!\triangleleft} \mathrm{SL}(2, \mathbf{Z}_5)$ is given by

$$\varphi_8(a) := (3, I_2), \quad \varphi_8(b) := (4, Y^2 X^{-2} Y^{-6}) = \left(4, \begin{bmatrix} 4 & 4 \\ 1 & 0 \end{bmatrix}\right).$$

The action $\alpha_8 : \mathbf{Z}_6 \to \mathrm{Aut}(\mathrm{SL}(2, \mathbf{Z}_5))$ is given by

$$\alpha_8(1)(X) = X^{-1}, \quad \alpha_8(1)(Y) = Y^{-1}.$$

(9) Set $X = \begin{bmatrix} 4 & 2 \\ 0 & 4 \end{bmatrix}, Y = \begin{bmatrix} 4 & 0 \\ 3 & 4 \end{bmatrix} \in \mathrm{SL}(2, \mathbf{Z}_5).$

The isomorphism $\varphi_9 : G \cong \mathbf{Z}_{10} \underset{\alpha_9}{\triangleright\!\!\triangleleft} \mathrm{SL}(2, \mathbf{Z}_5)$ is given by

$$\varphi_9(a) := (5, I_2), \quad \varphi_{10}(b) := (6, Y^4 X^{-4} Y^{-4}) = \left(6, \begin{bmatrix} 2 & 3 \\ 3 & 0 \end{bmatrix}\right).$$

The action $\alpha_9 : \mathbf{Z}_{10} \to \mathrm{Aut}(\mathrm{SL}(2, \mathbf{Z}_5))$ is given by

$$\alpha_9(1)(X) = X^{-1}, \quad \alpha_9(1)(Y) = Y^{-1}.$$

(10)   Set $X = \begin{bmatrix} 4 & 0 \\ 2 & 4 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix} \in \mathbf{SL}(2, \mathbf{Z}_5)$.

The isomorphism $\varphi_{10} : G \cong \mathbf{Z}_{15} \underset{\alpha_{10}}{\rtimes} \mathbf{SL}(2, \mathbf{Z}_5)$ is given by

$$\varphi_{10}(a) := (10, Y^2 X Y^{-1} X^2) = \left(10, \begin{bmatrix} 3 & 4 \\ 3 & 1 \end{bmatrix} \right), \quad \varphi_{10}(b) := (6, I_2).$$

The action $\alpha_{10} : \mathbf{Z}_{10} \to \mathrm{Aut}(\mathbf{SL}(2, \mathbf{Z}_5))$ is given by

$$\alpha_{10}(1)(X) = X Y^{-1} X = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}, \quad \alpha_{10}(1)(Y) = X^2 Y^{-1} X = \begin{bmatrix} 4 & 3 \\ 4 & 2 \end{bmatrix}.$$

$\square$

Now we are ready for deducing a geometric result from Proposition 3.

Let (M, D) be a pair of a complex manifold and an effective divisor on M. We recall that a Galois covering over M which branches at D is called *the maximal Galois covering of* (M, D) if it is universal among the branched coverings over M with branch locus at most at D (cf. [NT, Definition 1.2]). Our first main result is the following:

THEOREM 4.   *For* (e_1, e_2, q) *in Proposition* 2, *the maximal Galois covering of* $(\mathbf{A}^2, e_1 B_q^1 + e_2 B_q^2)$ *exists and is all isomorphic to* $\mathbf{A}^2$.

PROOF.   The existence of the maximal Galois covering X follows from Proposition 3 and [NT, Corollary 1.5] by calculating explicitly the orders of a and b in $G(e_1, e_2; q)$.

We show $X \cong \mathbf{A}^2$ in the case of $(e_1, e_2, q) = (3, 5, 4)$. The other cases are checked similarly. Let $H \subset \mathbf{SL}(2, \mathbf{C})$ be a binary icosahedral group, which is isomorphic to $\mathbf{SL}(2, \mathbf{Z}_5)$, and $f : \mathbf{A}^2 \to \mathbf{A}^2/H \cong \{(x, y, z) \in \mathbf{C}^3 \mid x^2 + y^3 + z^5 = 0\}$ the quotient map. By [NT, Proposition 2.3], we have a $\mathbf{Z}_{15}$-cyclic covering $g : \mathbf{A}^2/H \cong \{(u_1, u_2, v) \in \mathbf{C}^3 \mid u_1^3 + 2v^2 - u_2^5 = 0\} \to \mathbf{A}^2, (u_1, u_2, v) \mapsto (v, u_1^3 + v^2)$. Then we get a composite covering $g \circ f : \mathbf{A}^2 \to \mathbf{A}^2$ which branches at $3B_q^1 + 5B_q^2$.

Let $\pi : X \to \mathbf{A}^2$ be the maximal Galois covering of $(\mathbf{A}^2, 3B_q^1 + 5B_q^2)$. Then, by maximality, $g \circ f$ factors through π. Since the covering degrees of $g \circ f$ and π are equal, we conclude that $g \circ f \cong \pi$. $\square$

REMARK.   (1)   In Part 1 [NT, Remark 2.7], we showed that also for *odd* q the maximal Galois covering of $(\mathbf{A}^2, eB_q)$ exists and is all isomorphic to $\mathbf{A}^2$ if the corresponding maximal Galois group is finite.

(2)   The finite maximal Galois groups in Proposition 3, when considered as subgroups of $\mathbf{GL}(2, \mathbf{C})$, are all *complex reflection groups* (cf. [ST]). Indeed, by Theorem 4, we have $\mathbf{A}^2/G \cong \mathbf{A}^2$ where $G := G(e_1, e_2; q)$. Since G is a finite group, we may assume $G \subset \mathbf{GL}(2, \mathbf{C})$ by a polynomial change of coordinates (cf. [K]). Hence G is a complex reflection group by [ST, 5.1].

3. Maximal Galois coverings over $\mathbf{P}^2$ with branch locus $\overline{B_q} \cup l_\infty$.

In this section, we give a criterion for the existence of the maximal Galois coverings over $\mathbf{P}^2$ with branch locus $\overline{B_q^1} \cup \overline{B_q^2} \cup l_\infty$, where $\overline{B_q^i}$ is the projective closure of B_q^i and l_∞ the infinite line. We recall the following proposition in Part I.

PROPOSITION 5. *Let $q := 2r > 0$ be an even positive integer. Set* $\overline{B_q} := \{(x_0 : x_1 : x_2) \in \mathbf{P}^2 \mid x_0^q = x_1^2 x_2^{q-2}\} = \overline{B_q^1} \cup \overline{B_q^2}$, *where* $\overline{B_q^1} := \{(x_0 : x_1 : x_2) \in \mathbf{P}^2 \mid x_0^r = x_1 x_2^{r-1}\}$, $\overline{B_q^2} := \{(x_0 : x_1 : x_2) \in \mathbf{P}^2 \mid x_0^r = -x_1 x_2^{r-1}\}$ *and* $l_\infty := \{(x_0 : x_1 : x_2) \in \mathbf{P}^2 \mid x_2 = 0\}$ *the infinite line. Then the presentation of the maximal Galois group $G(\mathbf{P}^2, e_1\overline{B_q^1} + e_2\overline{B_q^2} + ml_\infty)$ is given as follows*:

$$G(\mathbf{P}^2, e_1\overline{B_q^1} + e_2\overline{B_q^2} + ml_\infty) \cong \langle a, b, c \mid a^{e_1} = b^{e_2} = c^m = 1, (ab)^r = (ba)^r = c^{-1} \rangle.$$

PROOF. See [NT, Section 3]. ☐

Unfortunately, we do not know precisely when $G(\mathbf{P}^2, e_1\overline{B_q^1} + e_2\overline{B_q^2} + ml_\infty)$ is finite. We therefore restrict ourselves to the easiest cases where this group is clearly finite, namely, the cases where $G(\mathbf{P}^2, e_1\overline{B_q^1} + e_2\overline{B_q^2} + ml_\infty)$ is a quotient of the *finite* group $G(\mathbf{A}^2, e_1 B_q^1 + e_2 B_q^2)$ in Proposition 3.

In the following, we also assume $q \geq 4$ so that we exclude the case (0) in Proposition 3, since we are mainly interested in *non-abelian* maximal Galois groups. We set $G[e_1, e_2, m; q] := G(\mathbf{P}^2, e_1\overline{B_q^1} + e_2\overline{B_q^2} + ml_\infty)$ for short.

PROPOSITION 6. *Suppose that $G(e_1, e_2; q)$ $(q \geq 4)$ is a finite group (so that (e_1, e_2, q) is one of the triples in Proposition 3 except the case (0)). Then the explicit structure of $G[e_1, e_2, m; q]$ is given as follows*:

1. $G[2, 2, m; 2n_1] \cong \begin{cases} D_{2n_1} & \text{if } m : \text{odd} \\ G(2, 2; 2n_1) & \text{if } m : \text{even} \end{cases}$

2. $G[2, n_2, m; 4] \cong \mathbf{Z}_2 \underset{\alpha_{21}}{\bowtie} (\mathbf{Z}_l \times \mathbf{Z}_{n_2})$ $(l := \mathrm{GCD}(m, n_2))$

3. $G[2, 3, m; 6] \cong \begin{cases} A_4 & \text{if } m \equiv 1, 3 \pmod 4 \\ \mathbf{Z}_2 \underset{\beta_{31}}{\bowtie} (\mathbf{Z}_3 \underset{\alpha_{31}}{\bowtie} (\mathbf{Z}_2 \times \mathbf{Z}_2)) & \text{if } m \equiv 2 \pmod 4 \\ G(2, 3; 6) & \text{if } m \equiv 0 \pmod 4 \end{cases}$

4. $G[3, 3, m; 4] \cong \begin{cases} A_4 & \text{if } m \equiv 1, 5 \pmod 6 \\ \mathbf{SL}(2, \mathbf{Z}_3) & \text{if } m \equiv 2, 4 \pmod 6 \\ (\mathbf{Z}_3 \times \mathbf{Z}_3) \underset{\alpha_{41}}{\bowtie} (\mathbf{Z}_2 \times \mathbf{Z}_2) & \text{if } m \equiv 3 \pmod 6 \\ G(3, 3; 4) & \text{if } m \equiv 0 \pmod 6 \end{cases}$

5. $\quad G[2, 3, m; 8] \cong$
$$
\begin{cases}
S_4 & \text{if } m \equiv 1, 5 \pmod 6 \\
\mathbf{Z}_2 \underset{\alpha_{51}}{\bowtie} \mathbf{SL}(2, \mathbf{Z}_3) & \text{if } m \equiv 2, 4 \pmod 6 \\
\mathbf{Z}_2 \underset{\beta_{52}}{\bowtie} (\mathbf{Z}_3 \underset{\alpha_{52}}{\bowtie} A_4) & \text{if } m \equiv 3 \pmod 6 \\
G(2, 3; 8) & \text{if } m \equiv 0 \pmod 6
\end{cases}
$$

6. $\quad G[2, 4, m; 6] \cong$
$$
\begin{cases}
S_4 & \text{if } m \equiv 1, 3, 5, 7 \pmod 8 \\
(\mathbf{Z}_2 \times \mathbf{Z}_2) \underset{\alpha_{61}}{\bowtie} A_4 & \text{if } m \equiv 2, 6 \pmod 8 \\
\mathbf{Z}_2 \underset{\beta_{62}}{\bowtie} (\mathbf{Z}_4 \underset{\alpha_{62}}{\bowtie} A_4) & \text{if } m \equiv 4 \pmod 8 \\
G(2, 4; 6) & \text{if } m \equiv 0 \pmod 8
\end{cases}
$$

7. $\quad G[3, 4, m; 4] \cong$
$$
\begin{cases}
S_4 & \text{if } m \equiv 1, 5, 7, 11 \pmod{12} \\
\mathbf{Z}_4 \underset{\alpha_{71}}{\bowtie} A_4 & \text{if } m \equiv 2, 10 \pmod{12} \\
\mathbf{Z}_6 \underset{\alpha_{72}}{\bowtie} A_4 & \text{if } m \equiv 3, 9 \pmod{12} \\
\mathbf{Z}_4 \underset{\alpha_{73}}{\bowtie} \mathbf{SL}(2, \mathbf{Z}_3) & \text{if } m \equiv 4, 8 \pmod{12} \\
\mathbf{Z}_{12} \underset{\alpha_{74}}{\bowtie} A_4 & \text{if } m \equiv 6 \pmod{12} \\
G(3, 4; 4) & \text{if } m \equiv 0 \pmod{12}
\end{cases}
$$

8. $\quad G[2, 3, m; 10] \cong$
$$
\begin{cases}
A_5 & \text{if } m \equiv 1, 5, 7, 11 \pmod{12} \\
\mathbf{Z}_2 \underset{\alpha_{81}}{\bowtie} \mathbf{PSL}(2, \mathbf{Z}_5) & \text{if } m \equiv 2, 10 \pmod{12} \\
\mathbf{Z}_3 \underset{\alpha_{82}}{\bowtie} \mathbf{PSL}(2, \mathbf{Z}_5) & \text{if } m \equiv 3, 9 \pmod{12} \\
\mathbf{Z}_2 \underset{\alpha_{83}}{\bowtie} \mathbf{SL}(2, \mathbf{Z}_5) & \text{if } m \equiv 4, 8 \pmod{12} \\
\mathbf{Z}_6 \times \mathbf{PSL}(2, \mathbf{Z}_5) & \text{if } m \equiv 6 \pmod{12} \\
G(2, 3; 10) & \text{if } m \equiv 0 \pmod{12}
\end{cases}
$$

9. $\quad G[2, 5, m; 6] \cong$
$$
\begin{cases}
A_5 & \text{if } m \equiv 1, 3, 7, 9, 11, 13, 17, 19 \pmod{20} \\
\mathbf{Z}_2 \underset{\alpha_{91}}{\bowtie} \mathbf{PSL}(2, \mathbf{Z}_5) & \text{if } m \equiv 2, 6, 14, 18 \pmod{20} \\
\mathbf{Z}_2 \underset{\alpha_{92}}{\bowtie} \mathbf{SL}(2, \mathbf{Z}_5) & \text{if } m \equiv 4, 8, 12, 16 \pmod{20} \\
\mathbf{Z}_5 \underset{\alpha_{93}}{\bowtie} \mathbf{PSL}(2, \mathbf{Z}_5) & \text{if } m \equiv 5, 15 \pmod{20} \\
\mathbf{Z}_{10} \times \mathbf{PSL}(2, \mathbf{Z}_5) & \text{if } m \equiv 10 \pmod{20} \\
G(2, 5; 6) & \text{if } m \equiv 0 \pmod{20}
\end{cases}
$$

$$
10. \quad G[3, 5, m; 4] \cong
\begin{cases}
A_5 & \text{if } m \equiv 1, 7, 11, 13, 17, 19, 23, 29 \pmod{30} \\[4pt]
SL(2, \mathbf{Z}_5) & \text{if } m \equiv 2, 4, 8, 14, 16, 22, 26, 28 \pmod{30} \\[4pt]
\mathbf{Z}_3 \underset{\alpha_{10,1}}{\rtimes} PSL(2, \mathbf{Z}_5) & \text{if } m \equiv 3, 9, 21, 27 \pmod{30} \\[4pt]
\mathbf{Z}_5 \underset{\alpha_{10,2}}{\rtimes} PSL(2, \mathbf{Z}_5) & \text{if } m \equiv 5, 25 \pmod{30} \\[4pt]
\mathbf{Z}_3 \underset{\alpha_{10,3}}{\rtimes} SL(2, \mathbf{Z}_5) & \text{if } m \equiv 6, 12, 18, 24 \pmod{30} \\[4pt]
\mathbf{Z}_5 \underset{\alpha_{10,4}}{\rtimes} SL(2, \mathbf{Z}_5) & \text{if } m \equiv 10, 20 \pmod{30} \\[4pt]
\mathbf{Z}_{15} \times PSL(2, \mathbf{Z}_5) & \text{if } m \equiv 15 \pmod{30} \\[4pt]
G(3, 5; 4) & \text{if } m \equiv 0 \pmod{30}
\end{cases}
$$

PROOF. Since the proof of this proposition is similar to that of Proposition 3, we omit the details of the proof. However, we give the explicit descriptions of isomorphisms and actions so that we can calculate explicitly the orders of a, b, c in $G[e_1, e_2, m; q]$ (see Theorem 7 below). In the following, we write $G := G[e_1, e_2, m; q]$ for short.

(1) (i) Case where m is odd. The isomorphism $G \cong D_{2n_1}$ is given by $a \mapsto y$, $b \mapsto y^{-1}x, c \mapsto e$.

(ii) Case where m is even. The isomorphism $G \cong D_{4n_1}$ is given by $a \mapsto y, b \mapsto y^{-1}x$, $c \mapsto x^{n_1}$.

(2) The isomorphism $G \cong \mathbf{Z}_2 \underset{\alpha_{21}}{\rtimes} (\mathbf{Z}_l \times \mathbf{Z}_{n_2})$ is given by $a \mapsto (1, (0, 0))$, $b \mapsto (0, (0, 1)), c \mapsto (0, (1, 0))$. The action $\alpha_{21} : \mathbf{Z}_2 \to \mathrm{Aut}(\mathbf{Z}_l \times \mathbf{Z}_{n_2})$ is given by $\alpha_{21}(1)((1, 0)) = (1, 0), \alpha_{21}(1)((0, 1)) = (1, -1)$.

(3) (i) Case where $m \equiv 1, 3 \pmod 4$. The isomorphism $G \cong A_4$ is given by $a \mapsto (12)(34), b \mapsto (234), c \mapsto (1)$.

(ii) Case where $m \equiv 2 \pmod 4$. The isomorphism $G \cong \mathbf{Z}_2 \underset{\beta_{31}}{\rtimes} (\mathbf{Z}_3 \underset{\alpha_{31}}{\rtimes} (\mathbf{Z}_2 \times \mathbf{Z}_2))$ is given by $a \mapsto (1, (0, (0, 0))), b \mapsto (0, (1, (0, 0))), c \mapsto (1, (0, (1, 1)))$. The action $\alpha_{31} : \mathbf{Z}_3 \to \mathrm{Aut}(\mathbf{Z}_2 \times \mathbf{Z}_2)$ is given by $\alpha_{31}(1)((1, 0)) = (0, 1), \alpha_{31}(1)((0, 1)) = (1, 1)$ and the action $\beta_{31} : \mathbf{Z}_2 \to \mathrm{Aut}(\mathbf{Z}_3 \underset{\alpha_{31}}{\rtimes} (\mathbf{Z}_2 \times \mathbf{Z}_2))$ is given by $\beta_{31}(1)((1, (0, 0))) = (1, (0, 1))$, $\beta_{31}(1)((0, (1, 0))) = (0, (1, 0)), \beta_{31}(1)((0, (0, 1))) = (0, (0, 1))$.

(iii) Case where $m \equiv 0 \pmod 4$. The isomorphism $G \cong \mathbf{Z}_2 \underset{\beta_3}{\rtimes} (\mathbf{Z}_3 \underset{\alpha_3}{\rtimes} Q_8)$ is given by φ_3 in Proposition 3 since $\mathrm{ord}((ab)^3) = 4$ in $G(2, 3; 6)$.

(4) (i) Case where $m \equiv 1, 5 \pmod 6$. The isomorphism $G \cong A_4$ is given by $a \mapsto (123), b \mapsto (234), c \mapsto (1)$.

(ii) Case where $m \equiv 2, 4 \pmod 6$. The isomorphism $G \cong SL(2, \mathbf{Z}_3)$ is given by $a \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, b \mapsto \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, c \mapsto 2I_2$.

(iii) Case where $m \equiv 3 \pmod 6$. The isomorphism $G \cong (\mathbf{Z}_3 \times \mathbf{Z}_3) \underset{\alpha_{41}}{\rtimes} (\mathbf{Z}_2 \times \mathbf{Z}_2)$ is given by $a \mapsto ((1, 0), (0, 0)), b \mapsto ((0, 1), (1, 1)), c \mapsto ((1, 1), (0, 0))$. The action α_{41} is given

by $\alpha_{41}((1,0))((1,0)) = (1,1)$, $\alpha_{41}((0,1))((1,0)) = (0,1)$, $\alpha_{41}((1,0))((0,1)) = (1,0)$, $\alpha_{41}((0,1))((0,1)) = (1,1)$.

(iv)   Case where $m \equiv 0 \pmod 6$. The isomorphism $G \cong (\mathbf{Z}_3 \times \mathbf{Z}_3) \underset{\alpha_4}{\rhd\!\!\!<} Q_8$ is given by φ_4 in Proposition 3 since $\mathrm{ord}((ab)^2) = 6$ in $G(3,3;4)$.

(5)   (i)   Case where $m \equiv 1,5 \pmod 6$. The isomorphism $G \cong S_4$ is given by $a \mapsto (34), b \mapsto (123), c \mapsto (1)$.

(ii)   Case where $m \equiv 2,4 \pmod 6$. Set $X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. The isomorphism $G \cong \mathbf{Z}_2 \underset{\alpha_{51}}{\rhd\!\!\!<} \mathrm{SL}(2,\mathbf{Z}_3)$ is given by $a \mapsto (1, I_2), b \mapsto (0, Y), c \mapsto (0, 2I_2)$. The action α_{51} is given by $\alpha_{51}(1)(X) = Y, \alpha_{51}(1)(Y) = X$.

(iii)   Case where $m \equiv 3 \pmod 6$. Set $s = (234), t = (142) \in S_4$. The isomorphism $G \cong \mathbf{Z}_2 \underset{\beta_{52}}{\rhd\!\!\!<} (\mathbf{Z}_3 \underset{\alpha_{52}}{\rhd\!\!\!<} A_4)$ is given by $a \mapsto (1, (0, (1))), b \mapsto (0, (1, (1))), c \mapsto (0, (2, t^2 s)) = (0, (2, (123)))$. The action $\alpha_{52} : \mathbf{Z}_3 \to \mathrm{Aut}(A_4)$ is given by $\alpha_{52}(1)(s) = t^2 = (124)$, $\alpha_{52}(1)(t) = st^2 = (134)$. The action $\beta_{52} : \mathbf{Z}_2 \to \mathrm{Aut}(\mathbf{Z}_3 \underset{\alpha_{52}}{\rhd\!\!\!<} A_4)$ is given by $\beta_{52}(1)(1, (1)) = (1, t), \beta_{52}(1)((0, s)) = (0, s^2) = (0, (243)), \beta_{52}(1)((0, t)) = (0, t^2) = (0, (124))$.

(iv)   Case where $m \equiv 0 \pmod 6$. The isomorphism $G \cong \mathbf{Z}_2 \underset{\beta_5}{\rhd\!\!\!<} (\mathbf{Z}_3 \underset{\alpha_5}{\rhd\!\!\!<} \mathrm{SL}(2, \mathbf{Z}_3))$ is given by φ_5 in Proposition 3 since $\mathrm{ord}((ab)^4) = 6$ in $G(2,3;8)$.

(6)   (i)   Case where $m \equiv 1,3,5,7 \pmod 8$. The isomorphism $G \cong S_4$ is given by $a \mapsto (34), b \mapsto (123), c \mapsto (1)$.

(ii)   Case where $m \equiv 2,6 \pmod 8$. Set $s = (234), t = (12)(34) \in S_4$. The isomorphism $G \cong (\mathbf{Z}_2 \times \mathbf{Z}_2) \underset{\alpha_{61}}{\rhd\!\!\!<} A_4$ is given by $a \mapsto ((1,0), (1)), b \mapsto ((0,1), s^2 t s^2) = ((0,1), (143)), c \mapsto ((1,1), (1))$. The action $\alpha_{61} : \mathbf{Z}_2 \times \mathbf{Z}_2 \to \mathrm{Aut}(A_4)$ is given by $\alpha_{61}((1,0))(s) = s^2 = (243), \alpha_{61}((0,1))(s) = s^2 = (243), \alpha_{61}((1,0))(t) = s^2 t s = (14)(23), \alpha_{61}((0,1))(t) = s^2 t s = (14)(23)$.

(iii)   Case where $m \equiv 4 \pmod 8$. The isomorphism $G \cong \mathbf{Z}_2 \underset{\beta_{62}}{\rhd\!\!\!<} (\mathbf{Z}_4 \underset{\alpha_{62}}{\rhd\!\!\!<} A_4)$ is given by $a \mapsto (1, (0, (1))), b \mapsto (0, (1, (1))), c \mapsto (1, (1, (st)^2)) = (1, (1, (132)))$. The action $\alpha_{62} : \mathbf{Z}_4 \to \mathrm{Aut}(A_4)$ is given by $\alpha_{62}(1)(s) = sts = (134), \alpha_{62}(1)(t) = s^2 t s = (14)(23)$ and the action $\beta_{62} : \mathbf{Z}_2 \to \mathrm{Aut}(\mathbf{Z}_4 \underset{\alpha_{62}}{\rhd\!\!\!<} A_4)$ is given by $\beta_{62}(1)(1, (1)) = (1, s^2 t s^2) = (1, (143)), \beta_{62}(1)((0, s)) = (0, s^2) = (0, (243)), \beta_{62}(1)((0, t)) = (0, t)$.

(iv)   Case where $m \equiv 0 \pmod 8$. The isomorphism $G \cong \mathbf{Z}_2 \underset{\beta_6}{\rhd\!\!\!<} (\mathbf{Z}_4 \underset{\alpha_6}{\rhd\!\!\!<} \mathrm{SL}(2, \mathbf{Z}_3))$ is given by φ_6 in Proposition 3 since $\mathrm{ord}((ab)^3) = 8$ in $G(2,4;6)$.

(7)   (i)   Case where $m \equiv 1,5,7,11 \pmod{12}$. The isomorphism $G \cong S_4$ is given by $a \mapsto (123), b \mapsto (4321), c \mapsto (1)$.

(ii)   Case where $m \equiv 2,10 \pmod{12}$. Set $s = (234), t = (142) \in S_4$. The isomorphism $G \cong \mathbf{Z}_4 \underset{\alpha_{71}}{\rhd\!\!\!<} A_4$ is given by $a \mapsto (0, s), b \mapsto (1, (1)), c \mapsto (2, st) = (2, (12)(34))$. The action $\alpha_{71} : \mathbf{Z}_4 \to \mathrm{Aut}(A_4)$ is given by $\alpha_{71}(1)(s) = t^2 s = (123), \alpha_{71}(1)(t) = s$.

(iii) Case where $m \equiv 3, 9 \pmod{12}$. Set $s = (234)$, $t = (12)(34) \in S_4$. The isomorphism $G \cong \mathbf{Z}_6 \underset{\alpha_{72}}{\rtimes} A_4$ is given by $a \mapsto (4, st) = (4, (132))$, $b \mapsto (3, tst) = (3, (143))$, $c \mapsto (4, (1))$. The action $\alpha_{72} : \mathbf{Z}_6 \to \text{Aut}(A_4)$ is given by $\alpha_{72}(1)(s) = s^2 = (243)$, $\alpha_{72}(1)(t) = s^2 ts = (14)(23)$.

(iv) Case where $m \equiv 4, 8 \pmod{12}$. Sex $X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in \text{SL}(2, \mathbf{Z}_3)$.

The isomorphism $G \cong \mathbf{Z}_4 \underset{\alpha_{73}}{\rtimes} \text{SL}(2, \mathbf{Z}_3)$ is given by $a \mapsto (0, X)$, $b \mapsto (1, I_2), c \mapsto (2, (XY)^3)$

$= \left(2, \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}\right)$. The action $\alpha_{73} : \mathbf{Z}_4 \to \text{Aut}(\text{SL}(2, \mathbf{Z}_3))$ is given by $\alpha_{73}(1)(X) = XYX^{-1}$

$= \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}$, $\alpha_{73}(1)(Y) = X$.

(v) Case where $m \equiv 6 \pmod{12}$. Set $s = (234)$, $t = (12)(34) \in S_4$. The isomorphism $G \cong \mathbf{Z}_{12} \underset{\alpha_{74}}{\rtimes} A_4$ is given by $a \mapsto (4, ts) = (4, (124))$, $b \mapsto (9, st) = (9, (132))$, $c \mapsto (10, (1))$. The action $\alpha_{74} : \mathbf{Z}_{12} \to \text{Aut}(A_4)$ is given by $\alpha_{74}(1)(s) = s^2 = (243)$, $\alpha_{74}(1)(t) = t$.

(vi) Case where $m \equiv 0 \pmod{12}$. The isomorphism $G \cong \mathbf{Z}_{12} \underset{\alpha_7}{\rtimes} \text{SL}(2, \mathbf{Z}_3)$ is given by φ_7 in Proposition 3 since $\text{ord}((ab)^2) = 12$ in $G(3, 4; 4)$.

(8) (i) Case where $m \equiv 1, 5, 7, 11 \pmod{12}$. The isomorphism $G \cong A_5$ is given by $a \mapsto (12)(45)$, $b \mapsto (134)$, $c \mapsto (1)$.

(ii) Case where $m \equiv 2, 10 \pmod{12}$. Set $X = \begin{bmatrix} 0 & 1 \\ 4 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & 1 \\ 4 & 4 \end{bmatrix} \in \text{PSL}(2, \mathbf{Z}_5)$. The isomorphism $G \cong \mathbf{Z}_2 \underset{\alpha_{81}}{\rtimes} \text{PSL}(2, \mathbf{Z}_5)$ is given by $a \mapsto (1, I_2)$, $b \mapsto (0, X)$, $c \mapsto (2, YXYXY) = \left(2, \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\right)$. The action $\alpha_{81} : \mathbf{Z}_2 \to \text{Aut}(\text{PSL}(2, \mathbf{Z}_5))$ is given by $\alpha_{81}(1)(X) = Y, \alpha_{81}(1)(Y) = X$.

(iii) Case where $m \equiv 3, 9 \pmod{12}$. Set $X = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$, $Z = \begin{bmatrix} 1 & 2 \\ 4 & 4 \end{bmatrix} \in \text{PSL}(2, \mathbf{Z}_5)$. The isomorphism $G \cong \mathbf{Z}_3 \underset{\alpha_{82}}{\rtimes} \text{PSL}(2, \mathbf{Z}_5)$ is given by $a \mapsto (0, X)$, $b \mapsto (1, I_2)$, $c \mapsto (1, ZXYZXYZXYZ) = \left(1, \begin{bmatrix} 1 & 1 \\ 4 & 0 \end{bmatrix}\right)$. The action $\alpha_{82} : \mathbf{Z}_3 \to \text{Aut}(\text{PSL}(2, \mathbf{Z}_5))$ is given by $\alpha_{82}(1)(X) = Z, \alpha_{82}(1)(Y) = X, \alpha_{82}(1)(Z) = Y$.

(iv) Case where $m \equiv 4, 8 \pmod{12}$. Set $X = \begin{bmatrix} 0 & 1 \\ 4 & 4 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & 4 \\ 1 & 4 \end{bmatrix} \in \text{SL}(2, \mathbf{Z}_5)$. The isomorphism $G \cong \mathbf{Z}_2 \underset{\alpha_{83}}{\rtimes} \text{SL}(2, \mathbf{Z}_5)$ is given by $a \mapsto (1, I_2)$, $b \mapsto (0, X)$, $c \mapsto (1, (XY)^7 X)$

$= \left(1, \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right)$. The action $\alpha_{83} : \mathbf{Z}_2 \to \mathrm{Aut}(\mathbf{SL}(2, \mathbf{Z}_5))$ is given by $\alpha_{83}(1)(X) = Y$, $\alpha_{83}(1)(Y) = X$.

(v) Case when $m \equiv 6 \pmod{12}$. Set $X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \in \mathbf{PSL}(2, \mathbf{Z}_5)$.

The isomorphism $G \cong \mathbf{Z}_6 \times \mathbf{PSL}(2, \mathbf{Z}_5)$ is given by $a \mapsto (3, XY^3X^3Y) = \left(3, \begin{bmatrix} 2 & 0 \\ 3 & 3 \end{bmatrix}\right)$,

$b \mapsto (4, Y^3X^2) = \left(4, \begin{bmatrix} 4 & 3 \\ 3 & 0 \end{bmatrix}\right)$, $c \mapsto (5, I_2)$.

(vi) Case where $m \equiv 0 \pmod{12}$. The isomorphism $G \cong \mathbf{Z}_6 \underset{\alpha_8}{\ltimes} \mathbf{SL}(2, \mathbf{Z}_5)$ is given by φ_8 is Proposition 3 since $\mathrm{ord}((ab)^5) = 12$ in $G(2, 3; 10)$.

(9)   (i)   Case where $m \equiv 1, 3, 5, 7, 9, 11, 13, 17, 19 \pmod{20}$. The isomorphism $G \cong A_5$ is given by $a \mapsto (12)(45)$, $b \mapsto (13542)$, $c \mapsto (1)$.

(ii)   Case where $m \equiv 2, 6, 14, 18 \pmod{20}$. Set $X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & 4 \\ 1 & 2 \end{bmatrix} \in \mathbf{PSL}(2, \mathbf{Z}_5)$. The isomorphism $G \cong \mathbf{Z}_2 \underset{\alpha_{91}}{\ltimes} \mathbf{PSL}(2, \mathbf{Z}_5)$ is given by $a \mapsto (1, I_2)$, $b \mapsto (0, X)$,

$c \mapsto (1, XYX) = \left(1, \begin{bmatrix} 4 & 4 \\ 2 & 1 \end{bmatrix}\right)$. The action $\alpha_{91} : \mathbf{Z}_2 \to \mathrm{Aut}(\mathbf{PSL}(2, \mathbf{Z}_5))$ is given by $\alpha_{91}(1)(X) = Y, \alpha_{91}(1)(Y) = X$.

(iii)   Case where $m \equiv 4, 8, 12, 16 \pmod{20}$. Set $X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \in \mathbf{SL}(2, \mathbf{Z}_5)$. The isomorphism $G \cong \mathbf{Z}_2 \underset{\alpha_{92}}{\ltimes} \mathbf{SL}(2, \mathbf{Z}_5)$ is given by $a \mapsto (1, I_2)$, $b \mapsto (0, X)$,

$c \mapsto (1, (XYX)^3) = \left(1, \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}\right)$. The action $\alpha_{92} : \mathbf{Z}_2 \to \mathrm{Aut}(\mathbf{SL}(2, \mathbf{Z}_5))$ is given by $\alpha_{92}(1)(X) = Y, \alpha_{92}(1)(Y) = X$.

(iv)   Case where $m \equiv 5, 15 \pmod{20}$. Set $X = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$, $Z = \begin{bmatrix} 1 & 4 \\ 2 & 4 \end{bmatrix} \in \mathbf{PSL}(2, \mathbf{Z}_5)$. This isomorphism $G \cong \mathbf{Z}_5 \underset{\alpha_{93}}{\ltimes} \mathbf{PSL}(2, \mathbf{Z}_5)$ is given by $a \mapsto (0, X), b \mapsto (1, I_2)$,

$c \mapsto (2, (XYZ)^4) = \left(1, \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\right)$. The action $\alpha_{93} : \mathbf{Z}_5 \to \mathrm{Aut}(\mathbf{PSL}(2, \mathbf{Z}_5))$ is given by $\alpha_{93}(1)(X) = XYZYX = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$, $\alpha_{93}(1)(Y) = X, \alpha_{93}(1)(Z) = Y$.

(v)   Case where $m \equiv 10 \pmod{20}$. Set $X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathbf{PSL}(2, \mathbf{Z}_5)$.

The isomorphism $G \cong \mathbf{Z}_{10} \times \mathbf{PSL}(2, \mathbf{Z}_5)$ is given by $a \mapsto (5, Y^2 XY X^2) = \left(5, \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\right)$,

$b \mapsto (6, X^{-1} Y^{-1} X) = \left(6, \begin{bmatrix} 0 & 4 \\ 1 & 2 \end{bmatrix}\right)$, $c \mapsto (7, I_2)$.

(vi)   Case where $m \equiv 0 \pmod{20}$. The isomorphism $G \cong \mathbf{Z}_{10} \underset{\alpha_9}{\rtimes} \mathbf{SL}(2, \mathbf{Z}_5)$ is given by

φ_9 in Theorem 9 since $\mathrm{ord}((ab)^3) = 20$ in $G(2, 5; 6)$.

(10)   (i)   Case where $m \equiv 1, 7, 11, 13, 17, 19, 23, 29 \pmod{30}$. The isomorphism $G \cong A_5$ is given by $a \mapsto (134)$, $b \mapsto (12)(45)$, $c \mapsto (1)$.

(ii)   Case where $m \equiv 2, 4, 8, 14, 16, 22, 26, 28 \pmod{30}$. Set $X = \begin{bmatrix} 0 & 1 \\ 4 & 4 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in \mathbf{SL}(2, \mathbf{Z}_5)$. The isomorphism $G \cong \mathbf{SL}(2, \mathbf{Z}_5)$ is given by $a \mapsto X$, $b \mapsto Y$, $c \mapsto 4I_2$.

(iii)   Case where $m \equiv 3, 9, 21, 27 \pmod{30}$. Set $X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & 4 \\ 1 & 2 \end{bmatrix} \in \mathbf{PSL}(2, \mathbf{Z}_5)$. The isomorphism $G \cong \mathbf{Z}_2 \underset{\alpha_{10,1}}{\rtimes} \mathbf{PSL}(2, \mathbf{Z}_5)$ is given by $a \mapsto (1, I_2)$, $b \mapsto (10, X)$, $c \mapsto (1, (XY)^2)$. The action $\alpha_{10,1} : \mathbf{Z}_3 \to \mathrm{Aut}(\mathbf{PSL}(2, \mathbf{Z}_5))$ is given by $\alpha_{10,1}(1)(X) = XYX^{-1} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$, $\alpha_{10,1}(1)(Y) = X$.

(iv)   Case where $m \equiv 5, 25 \pmod{30}$. Set $X = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 1 \\ 4 & 0 \end{bmatrix} \in \mathbf{PSL}(2, \mathbf{Z}_5)$. The isomorphism $G \cong \mathbf{Z}_5 \underset{\alpha_{10,2}}{\rtimes} \mathbf{PSL}(2, \mathbf{Z}_5)$ is given by $a \mapsto (0, X)$, $b \mapsto (1, I_2)$, $c \mapsto (3, (XY)^4) = \left(3, \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix}\right)$. The action $\alpha_{10,2} : \mathbf{Z}_5 \to \mathrm{Aut}(\mathbf{PSL}(2, \mathbf{Z}_5))$ is given by $\alpha_{10,2}(1)(X) = XYX^{-1} = \begin{bmatrix} 3 & 1 \\ 3 & 3 \end{bmatrix}$, $\alpha_{10,2}(1)(Y) = X$.

(v)   Case where $m \equiv 6, 12, 18, 24 \pmod{30}$. Set $X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \in \mathbf{SL}(2, \mathbf{Z}_5)$. The isomorphism $G \cong \mathbf{Z}_3 \underset{\alpha_{10,3}}{\rtimes} \mathbf{SL}(2, \mathbf{Z}_5)$ is given by $a \mapsto (1, I_2)$, $b \mapsto (0, X)$, $c \mapsto (1, (XY)^5) = \left(1, \begin{bmatrix} 0 & 1 \\ 4 & 1 \end{bmatrix}\right)$. The action $\alpha_{10,3} : \mathbf{Z}_3 \to \mathrm{Aut}(\mathbf{SL}(2, \mathbf{Z}_5))$ is given by $\alpha_{10,3}(1)(X) = XYX^{-1} = \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix}$, $\alpha_{10,3}(1)(Y) = X$.

(vi)   Case where $m \equiv 10, 20 \pmod{30}$. Set $X = \begin{bmatrix} 0 & 1 \\ 4 & 4 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & 4 \\ 1 & 4 \end{bmatrix} \in \mathbf{SL}(2, \mathbf{Z}_5)$.

The isomorphism $G \cong \mathbf{Z}_5 \underset{\alpha_{10,4}}{\bowtie} \mathbf{SL}(2, \mathbf{Z}_5)$ is given by $a \mapsto (0, X)$, $b \mapsto (1, I_2)$,

$c \mapsto (3, (XY)^9) = \left(3, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \right)$. The action $\alpha_{10,4} : \mathbf{Z}_5 \to \mathrm{Aut}(\mathbf{SL}(2, \mathbf{Z}_5))$ is given by

$\alpha_{10,4}(1)(X) = XYX^{-1} = \begin{bmatrix} 3 & 4 \\ 3 & 1 \end{bmatrix}$, $\alpha_{10,4}(1)(Y) = X$.

(vii)   Case where $m \equiv 15 \pmod{30}$. Set $X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathbf{PSL}(2, \mathbf{Z}_5)$.

The isomorphism $G \cong \mathbf{Z}_{15} \times \mathbf{PSL}(2, \mathbf{Z}_5)$ is given by $a \mapsto (10, Y^{-1}X^{-1}Y^2X^2Y^{-2}X^{-2}) = $

$\left(10, \begin{bmatrix} 4 & 2 \\ 2 & 0 \end{bmatrix} \right)$, $b \mapsto (6, X^{-1}Y^{-1}XY) = \left(6, \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix} \right)$, $c \mapsto (13, I_2)$.

(viii)   Case where $m \equiv 0 \pmod{30}$. The isomorphism $G \cong \mathbf{Z}_{15} \underset{\alpha_{10}}{\bowtie} \mathbf{SL}(2, \mathbf{Z}_5)$ is given

by φ_{10} in Proposition 3 since $\mathrm{ord}((ab)^2) = 30$ in $G(3, 5; 4)$. $\square$

From Proposition 6, we get the following geometric result, which is our second main result.

THEOREM 7.   (1)   *Let the triple* (e_1, e_2, q) *be as in Proposition* 6. *Then the necessary and sufficient condition for the existence of the maximal Galois covering of* $(\mathbf{P}^2, e_1 \overline{B_q^1} + e_2 \overline{B_q^2} + ml_\infty)$ *is as follows*:

   1.   *In the case* $(e_1, e_2, q) = (2, 2, 2n_1)$, $m = 1, 2$.
   2.   *In the case* $(e_1, e_2, q) = (2, n_2, 4)$, $m = $ *any positive divisor of* n_2.
   3.   *In the case* $(e_1, e_2, q) = (2, 3, 6)$, $m = 1, 2, 4$.
   4.   *In the case* $(e_1, e_2, q) = (3, 3, 4)$, $m = 1, 2, 3, 6$.
   5.   *In the case* $(e_1, e_2, q) = (2, 3, 8)$, $m = 1, 2, 3, 6$.
   6.   *In the case* $(e_1, e_2, q) = (2, 4, 6)$, $m = 1, 2, 4, 8$.
   7.   *In the case* $(e_1, e_2, q) = (3, 4, 4)$, $m = 1, 2, 3, 4, 6, 12$.
   8.   *In the case* $(e_1, e_2, q) = (2, 3, 10)$, $m = 1, 2, 3, 4, 6, 12$.
   9.   *In the case* $(e_1, e_2, q) = (2, 5, 6)$, $m = 1, 2, 4, 5, 10, 20$.
   10.   *In the case* $(e_1, e_2, q) = (3, 5, 4)$, $m = 1, 2, 3, 5, 6, 10, 15, 30$.

   (2)   *The maximal Galois coverings in* (1) *are all normal projective rational surfaces.*

   (3)   *Assume* m *is the maximal value in each of the above* 10 *cases (for instance,* $m = 30$ *in the case* 10). *Then the corresponding maximal Galois covering contains* $\mathbf{A}^2$ *as an open dense subset.*

PROOF.   (1)   We check the case 1, the other cases being treated similarly. Suppose m is odd. Then we have $\mathrm{ord}(a) = \mathrm{ord}(b) = 2$, $\mathrm{ord}(c) = 1$ by Proposition 6. Hence the maximal Galois covering exists if and only if $m = 1$ by [NT, Corollary 1.5].

Suppose m is even. Then we have $\mathrm{ord}(a) = \mathrm{ord}(b) = 2$, $\mathrm{ord}(c) = 2$ and the maximal Galois covering exists if and only if $m = 2$ similarly.

(2) From the general construction of maximal Galois coverings (cf. [Na, Theorem 1.3.8]), our maximal Galois covering X is a projectivization of a quotient of $\mathbf{A}^2$ by a finite group so that it is a unirational surface. Since X is rational (cf. [B, Corollaire V.5]), and normal by definition, the assertion (2) is done.

(3) If m is maximal, then X is a projectiviazation of the quotient of $\mathbf{A}^2$ by the trivial group. $\square$

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