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Sheaves on Local Ringed Spaces Associated to Hilbert Rings

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Introduction.

Let $(X, \mathcal{O}_X)$ be a local ringed space, X_{cl} the set of closed points of X and $i : X_{cl} \to X$ the inclusion mapping. For a sheaf $\mathcal{F}$ of rings or modules over X we denote by

$$i^{\sharp} : \mathcal{F} \to i_* i^{-1} \mathcal{F}$$

the natural morphism of sheaves, and introduce the following conditions for X:
(s_1) $i^{\sharp} : \mathcal{F} \to i_* i^{-1}\mathcal{F}$ is an isomorphism for any sheaf $\mathcal{F}$ over X,
(s_2) $i^{\sharp} : \mathcal{O}_X \to i_* i^{-1}\mathcal{O}_X$ is an isomorphism.
Moreover we consider the conditions:
(c_1) $\mathcal{O}_X(X)$ is a Hilbert ring (see §1),
(c_2) $ti : t(X_{cl}) \to tX$ is a homeomorphism.
For the functor t on topological spaces, see [4, II, Proposition 2.6] or [10, §1].

Using the morphism $\pi_X : X \to Spec\, \mathcal{O}_X(X)$ of local ringed spaces defined in [8, §1], we put

$$I_X(E) = \bigcap_{x \in E} \pi_X(x), \quad Z_X(\mathfrak{a}) = \{x \in X \mid \mathfrak{a} \subset \pi_X(x)\} = \pi_X^{-1}(V(\mathfrak{a})),$$

for $E \subset X$ and for an ideal $\mathfrak{a}$ of $\mathcal{O}_X(X)$. Then we introduce the following condition for X_{cl}:
(c_3) $\sqrt{\mathfrak{a}} = I_{X_{cl}}(Z_{X_{cl}}(\mathfrak{a}))$ for any ideal $\mathfrak{a}$ of $\mathcal{O}_{X_{cl}}(X_{cl})$, where $\mathcal{O}_{X_{cl}} = i^{-1}\mathcal{O}_X$.

In this paper we shall study the relationship among these conditions and consider an abstract form of Hilbert Nullstellensatz. The main results are as follows:

THEOREM 1. *For a ring A, we put $X = Spec\, A$ and introduce the condition*:
(s_1') $i^{\sharp} : \mathcal{F} \to i_* i^{-1}\mathcal{F}$ *is an isomorphism for any sheaf $\mathcal{F}$ of quasi-coherent $\mathcal{O}_X$-modules over X.*

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Then

$$A \text{ is a Hilbert ring} \Leftrightarrow (c_1) \Leftrightarrow (c_2) \Leftrightarrow (c_3) \Leftrightarrow (s_1) \Leftrightarrow (s_1') \Rightarrow (s_2).$$

THEOREM 2. *For a field K and a subring A of K, we put $X = Zar(K|A)$ and introduce the condition*:
(s_1'') $i^{\sharp} : \mathcal{F} \to i_* i^{-1}\mathcal{F}$ *is an isomorphism for any intersection sheaf $\mathcal{F}$ of $\mathcal{O}_X$-algebras over X (for intersection sheaves, see* [9, §0]).
Then

$$A \text{ is a Hilbert ring} \Leftrightarrow (c_1) \Leftrightarrow (c_2) \Leftrightarrow (c_3) \Leftrightarrow (s_1) \Leftrightarrow (s_1'') \Rightarrow (s_2).$$

REMARK 1. Suppose that $\mathcal{O}_X$ is a sheaf of mappings to a field k, in other words, $(X, \mathcal{O}_X)$ and k satisfy the condition (a) in [8, Proposition 1]. Then $\pi_X(x) = \{f \in \mathcal{O}_X(X) \mid f(x) = 0\}$ for any $x \in X$. Thus $I_X(E) = \{f \in \mathcal{O}_X(X) \mid x \in E \Rightarrow f(x) = 0\}$ for any $E \subset X$ and $Z_X(\mathfrak{a}) = \{x \in X \mid f \in \mathfrak{a} \Rightarrow f(x) = 0\}$ for any ideal $\mathfrak{a}$ of $\mathcal{O}_X(X)$. Therefore Theorem 1 is one of the generalizations of Hilbert Nullstellensatz. See also Theorem 1″ and its corollary in §2.

REMARK 2. For general local ringed spaces, even if we assume that any irreducible closed subset has a unique generic point, the conditions (c_1), (c_2) and (c_3) are independent. See Example 1, $\cdots$, Example 6 in §4. For generalizations of other parts of Theorems 1 and 2, see Theorem 3 and Theorem 4 in §1.

REMARK 3. In general, the implication "$(s_2) \Rightarrow (s_1)$" does not hold in Theorems 1 and 2. See Example 7 in §4.

1. Here we prove Theorem 3 and Theorem 4.

LEMMA 1.1. *For a continuous mapping $f : X \to Y$, we obtain*:

(i) *The topology of X is the induced topology of Y with respect to f*
$\Leftrightarrow E = f^{-1}(\overline{f(E)})$ *for any closed subset E of X*
$\Rightarrow E = f^{-1}(\overline{f(E)})$ *for any $E \in tX$*
$\Rightarrow tf : tX \to tY$ *is an injection.*

(ii) $tf : tX \to tY$ *is a surjection*
$\Rightarrow \overline{f(f^{-1}(F))} = F$ *for any $F \in tY$*
$\Leftrightarrow \overline{f(f^{-1}(F))} = F$ *for any closed subset F of Y.*

(iii) $tf : tX \to tY$ *is a homeomorphism*
$\Leftrightarrow tf$ *is a surjection and the topology of X is the induced topology of Y with respect to f*
$\Leftrightarrow E = f^{-1}(\overline{f(E)})$ *for any closed subset E of X and $\overline{f(f^{-1}(F))} = F$ for any closed subset F of Y.*

Here "overline" means the closure of topological spaces.

The proof is easy.

For a topological space X and a subset E of X, we put

$$E^* = \{x \in X \mid \overline{\{x\}} \cap E \neq \emptyset\}.$$

A ring A is said to be Hilbert if any prime ideals of A are intersections of maximal ideals.

LEMMA 1.2. *For a ring A, the following conditions are equivalent*:

(c_0) *A is a Hilbert ring.*

(1) $\overline{F \cap m.Spec\, A} = F$ *for any closed subset F of Spec A.*

(2) $(V \cap m.Spec\, A)^* = V$ *for any open subset V of Spec A.*

(2′) $(D(f) \cap m.Spec\, A)^* = D(f)$ *for any $f \in A$.*

The proof is induced from [10, Lemma 8].

LEMMA 1.3. *Let $(X, \mathcal{O}_X)$ be a local ringed space. Then $\mathcal{O}_{X,E}$ is a local ring for any irreducible subset E of X.*

The proof is easy.

COROLLARY 1. *We obtain a functor $t : (L.R.S.) \to (L.R.S.)$, where $(L.R.S.)$ denotes the category of local ringed spaces.*

COROLLARY 2. *The condition (c_2) is equivalent to the following one*:

(c_2') $t(i, i^\sharp) : t(X_{cl}, \mathcal{O}_{X_{cl}}) \to t(X, \mathcal{O}_X)$ *is an isomorphism of local ringed spaces.*

THEOREM 3. *Let $(X, \mathcal{O}_X)$ be a local ringed space.*

(i) *The following conditions are equivalent*:

(a) $\bar{E} = Z_X(I_X(E))$ *for any $E \subset X$.*

(b) $E = \pi_X^{-1}(\overline{\pi_X(E)})$ *for any closed subset E of X.*

(ii) *The following conditions are equivalent*:

(c) $\sqrt{\mathfrak{a}} = I_X(Z_X(\mathfrak{a}))$ *for any ideal $\mathfrak{a}$ of $\mathcal{O}_X(X)$.*

(d) $\overline{\pi_X(\pi_X^{-1}(F))} = F$ *for any closed subset F of Spec $\mathcal{O}_X(X)$.*

(iii) *E is irreducible $\Rightarrow I_X(E) \in$ Spec $\mathcal{O}_X(X)$. If X satisfies* (a), *then the converse holds.*

PROOF. (i): Verified from Lemma 1.1, (i).

(ii): Induced from [8, Lemma 2].

(iii): Easy to prove.

COROLLARY 1. (i) *The mapping $I_X : tX \to$ Spec $\mathcal{O}_X(X)$ defined by restriction satisfies $\pi_X = I_X \circ \alpha_X$ and $\alpha_{\mathrm{Spec}\,\mathcal{O}_X(X)} \circ I_X = t(\pi_X)$. Therefore I_X gives rise to a morphism of local ringed spaces. Moreover $I_X = \pi_{tX}$.*

(ii) *X satisfies the conditions* (a) *and* (c) $\Leftrightarrow$ *π_{tX} is a homeomorphism.*

COROLLARY 2. *The condition (c_3) is equivalent to the following one*:

(c_3') $\overline{\pi_{X_{cl}}(\pi_{X_{cl}}^{-1}(F))} = F$ *for any closed subset F of Spec $\mathcal{O}_{X_{cl}}(X_{cl})$.*

THEOREM 4. *Let $(X, \mathcal{O}_X)$ be a local ringed space.*

(i) *If any irreducible closed subset of X has a unique generic point, then*

$$(c_2) \Leftrightarrow (s_1)\,.$$

(ii) *If X satisfies (s_2) and $\pi_X(X_{cl}) \supset m.Spec\, \mathcal{O}_X(X)$, then*

$$(c_1) \Rightarrow (c_3')\,.$$

(iii) *If X satisfies*

$$Spec\, \mathcal{O}_X(X) = \bigcup_{x \in X_{cl}} \mathrm{Im}(Spec\, \rho_{X,x})$$

and $\pi_X(X_{cl}) \subset m.Spec\, \mathcal{O}_X(X)$, then

$$(c_3') \Rightarrow (c_1)\,.$$

(iii′) *If X satisfies $\pi_X(X) = Spec\, \mathcal{O}_X(X)$, then*

$$(X_{cl})^* = X \Rightarrow Spec\, \mathcal{O}_X(X) = \bigcup_{x \in X_{cl}} \mathrm{Im}(Spec\, \rho_{X,x})\,.$$

(iv) *If $\pi_X(X) = Spec\, \mathcal{O}_X(X)$ and $\pi_X(X_{cl}) \subset m.Spec\, \mathcal{O}_X(X)$, then*

$$(c_2) \Rightarrow (c_1)\,.$$

PROOF. We put $W = X_{cl}$. (i) $(c_2) \Rightarrow (s_1)$: Easy from [10, Lemma 3]. $(s_1) \Rightarrow (c_2)$: From Lemma 1.1, it is sufficient to prove that $\overline{F \cap W} = F$ for any closed subset F of X. We put $i_F : F \hookrightarrow X$, $\mathcal{O}_F = i_F^{-1}\mathcal{O}_X$ and $\mathcal{F}_X = i_{F*}i_F^{-1}\mathcal{O}_X$. Then $\mathcal{F}_X$ is a sheaf of $\mathcal{O}_X$-algebras. Thus $i^\sharp : \mathcal{F}_X \to i_*i^{-1}\mathcal{F}_X$ is an isomorphism of sheaves from (s_1), and hence $i^\sharp(V) : \mathcal{F}_X(V) \to (i^{-1}\mathcal{F}_X)(V \cap W)$ is an isomorphism of rings for any open subsets V of X. Since there exists a homomorphism: $\mathcal{O}_F(V \cap W \cap F) \to \mathcal{O}_F(V \cap F)$ of rings, we obtain that $V \cap F \neq \emptyset \Rightarrow V \cap W \cap F \neq \emptyset$. Therefore $\overline{F \cap W} = F$.

(ii) By (s_2), $i^\sharp(X) : \mathcal{O}_X(X) \to \mathcal{O}_W(W)$ is an isomorphism of rings, and hence we put $A = \mathcal{O}_X(X) = \mathcal{O}_W(W)$. Then $\pi_X \circ i = \pi_W$. From $m.Spec\, A \subset \pi_X(W) = \pi_W(W)$, we obtain $F = \overline{F \cap m.Spec\, A} \subset \overline{F \cap \pi_W(W)} = \overline{\pi_W(\pi_W^{-1}(F))} \subset F$ for any closed subset F of $Spec\, A$. Therefore $\overline{\pi_W(\pi_W^{-1}(F))} = F$.

(iii) We put $\varphi = i^\sharp(X)$ and $f = Spec\, \varphi$. Then $\pi_X \circ i = f \circ \pi_W$. Since f is a surjection, we obtain $F = f(f^{-1}(F)) = f(\overline{f^{-1}(F) \cap \pi_W(W)}) \subset \overline{f(f^{-1}(F) \cap \pi_W(W))} \subset \overline{F \cap \pi_X(W)} \subset \overline{F \cap m.Spec\, \mathcal{O}_X(X)} \subset F$ for any closed subset F of $Spec\, \mathcal{O}_X(X)$. Therefore $\overline{F \cap m.Spec\, \mathcal{O}_X(X)} = F$ and hence $\mathcal{O}_X(X)$ is a Hilbert ring.

(iii′): Easy to prove.

(iv) From Lemma 1.1, we obtain $F = \pi_X(\pi_X^{-1}(F)) = \pi_X(\overline{\pi_X^{-1}(F) \cap W}) \subset \overline{\pi_X(\pi_X^{-1}(F) \cap W)} \subset \overline{F \cap \pi_X(W)} \subset \overline{F \cap m.Spec\, \mathcal{O}_X(X)} \subset F$ for any closed subset F of $Spec\, \mathcal{O}_X(X)$. Therefore $\overline{F \cap m.Spec\, \mathcal{O}_X(X)} = F$ and hence $\mathcal{O}_X(X)$ is a Hilbert ring.

EXAMPLE 0. Let $(X, \mathcal{O}_X)$ be a local ringed space. Suppose that the topology of X is discrete.

(i) X satisfies (c_2) and $\pi_X(X) \subset m.Spec\, \mathcal{O}_X(X)$.

(ii) X satisfies $(c_1) \Rightarrow \dim \mathcal{O}_{X,x} = 0$ for any $x \in X$.
(iii) $\mathcal{O}_{X,x}$ is a field for any $x \in X$
$\Rightarrow \dim \mathcal{O}_X(X) = 0$ and $\pi_X(X)$ is open in $Spec\, \mathcal{O}_X(X)$.
(iv) X is a finite set $\Leftrightarrow \pi_X(X) = m.Spec\, \mathcal{O}_X(X)$.
(v) $(X, \mathcal{O}_X)$ is an affine scheme $\Leftrightarrow \pi_X$ is a surjection
$\Leftrightarrow X$ is a finite set and $\dim \mathcal{O}_X(X) = 0$.
(vi) If X is a finite set, then

$$(c_1) \Leftrightarrow (X, \mathcal{O}_X) \text{ is an affine scheme} \Leftrightarrow (c_3)\,.$$

The proof is easy.

2. Here we prove Theorem 1 and Theorem 2.

Let A be a ring, $i : m.Spec\, A \to Spec\, A$ the inclusion mapping and M an A-module. Then we consider the homomorphism of modules

$$i^\sharp(D(f)) : M_f \to (i^{-1}\tilde{M})(D(f) \cap m.Spec\, A)$$

for any $f \in A$, induced from $i^\sharp : \tilde{M} \to i_* i^{-1}\tilde{M}$. Here we write $\Psi_f^M = i^\sharp(D(f))$. Then the following three lemmas are shown.

LEMMA 2.1. *Let A be a ring and M an A-module.*
(i) *Ψ_f^M is an injection for any $f \in A$*
$\Leftrightarrow$ *$\overline{V(Ann_M(\alpha)) \cap m.Spec\, A} = V(Ann_M(\alpha))$ for any $\alpha \in M$.*
(ii) *Ψ_f^A is an injection for any $f \in A$*
$\Leftrightarrow$ *$\Psi_f^{\mathfrak{a}}$ is an injection for any ideal $\mathfrak{a}$ of A and $f \in A$*
$\Leftrightarrow$ *$\Psi_f^{\mathfrak{p}}$ is an injection for any $\mathfrak{p} \in Spec\, A$ and $f \in A$*
$\Leftrightarrow$ *$\Psi_f^{\mathfrak{m}}$ is an injection for any $\mathfrak{m} \in m.Spec\, A$ and $f \in A$.*

LEMMA 2.2. *Let A be a ring. If $\Psi_f^{A/\mathfrak{p}}$ is an injection for any $\mathfrak{p} \in Spec\, A$ and $f \in A$, then A is a Hilbert ring.*

LEMMA 2.3. *Let A be a ring and $i : m.Spec\, A \hookrightarrow Spec\, A$ the inclusion mapping. If $i^\sharp : \tilde{\mathfrak{p}} \to i_* i^{-1}\tilde{\mathfrak{p}}$ is an isomorphism for any $\mathfrak{p} \in Spec\, A$, then $(D(f) \cap m.Spec\, A)^* = D(f)$ for any $f \in A$.*

The next result is induced from Lemma 1.2, Theorem 4, (i), Lemma 2.2 and Lemma 2.3.

THEOREM 1′. *For a ring A, we put $X = Spec\, A$ and $i : X_{cl} \hookrightarrow X$. Then*
A is a Hilbert ring
$\Leftrightarrow$ *$i^\sharp : \tilde{\mathfrak{p}} \to i_* i^{-1}\tilde{\mathfrak{p}}$ is an isomorphism for any $\mathfrak{p} \in Spec\, A$*
$\Leftrightarrow$ *$i^\sharp : \widetilde{A/\mathfrak{p}} \to i_* i^{-1}\widetilde{A/\mathfrak{p}}$ is an isomorphism for any $\mathfrak{p} \in Spec\, A$.*

PROOF OF THEOREM 1. $(c_2) \Leftrightarrow (s_1)$: Already proved in Theorem 4, (i).
$(c_0) \Leftrightarrow (c_1)$: Obvious from $A = \mathcal{O}_X(X)$.
$(c_0) \Leftrightarrow (c_2)$: Easy from Lemma 1.1 and Lemma 1.2.
$(c_1) \Leftrightarrow (c_3)$: Induced from Corollary 2 to Theorem 3, Theorem 4, (ii), (iii) and (iii′).

$(s_1) \Rightarrow (s_1')$, $(s_1) \Rightarrow (s_2)$: Trivial.
$(s_1') \Rightarrow (c_0)$: Obvious from Theorem 1′.

From Theorem 1 and Corollary 1 to Theorem 3, we have:

THEOREM 1″. *For a ring A, we put $X = m.Spec\, A$ and $\mathcal{O}_X = \tilde{A}|_X$. Then*
A is a Hilbert ring $\Leftrightarrow \sqrt{\mathfrak{a}} = I_X(Z_X(\mathfrak{a}))$ for any ideal $\mathfrak{a}$ of $\mathcal{O}_X(X)$
$\Leftrightarrow tX$ is an affine scheme.

COROLLARY. *Suppose that A is a reduced ring of finite type over an algebraically closed field k.*
(i) *$\mathcal{O}_X$ is a sheaf of mappings to k.*
(ii) *For any ideal $\mathfrak{a}$ of A and $f \in A$, we obtain*

$$f|_{Z_X(\mathfrak{a})} = 0 \Rightarrow f \in \sqrt{\mathfrak{a}}.$$

PROOF OF THEOREM 2. $(c_2) \Leftrightarrow (s_1)$: Already proved in Theorem 4, (i).
$(c_0) \Leftrightarrow (c_1)$: Easy from the fact that $A \subset \mathcal{O}_X(X)$ is an integral extension.
$(c_0) \Leftrightarrow (c_2)$: Already proved in [10, Theorem 3].
$(c_1) \Leftrightarrow (c_3)$: Induced from Corollary 2 to Theorem 3, Theorem 4, (ii), (iii) and (iii′).
$(s_1) \Rightarrow (s_1'')$, $(s_1) \Rightarrow (s_2)$: Trivial.
$(s_1'') \Rightarrow (c_2)$: We put $W = X_{cl}$. From Lemma 1.1, it is sufficient to prove that $\overline{F \cap W} = F$ for any closed subset F of X. The mapping $s : X \to Loc(K|A)$ defined by $s(R) = R$ for $R \in F$ and $s(R) = K$ for $R \notin F$ is continuous. Let $\mathcal{F}_X$ denote the intersection sheaf over X with respect to s. Then $\mathcal{F}_X$ is a sheaf of $\mathcal{O}_X$-algebras. Thus $i^\sharp : \mathcal{F}_X \to i_*i^{-1}\mathcal{F}_X$ is an isomorphism of sheaves from (s_1''), and hence $i^\sharp(V) : \mathcal{F}_X(V) \to (i^{-1}\mathcal{F}_X)(V \cap W)$ is an isomorphism of rings for any open subsets V of X. By [9, Lemma 2] and that W is irreducible, $i^{-1}\mathcal{F}_X$ is an intersection sheaf over W. Thus we obtain

$$V \cap F \neq \emptyset \Rightarrow V \cap W \neq \emptyset \Rightarrow \bigcap_{R \in V} s(R) = \bigcap_{R \in V \cap W} s(R) \Rightarrow V \cap F \cap W \neq \emptyset.$$

Therefore $\overline{F \cap W} = F$.

3. Here we consider the sheaves of real-valued continuous functions.

Let C_X^0 denote the sheaf of real-valued continuous functions over a topological space X. Then we obtain a local ringed space (X, C_X^0).

LEMMA 3.1. *Let X be a topological space.*
(i) *X is completely regular*
$\Leftrightarrow \pi_X : X \to Spec\, C_X^0(X)$ is an into-homeomorphism.
(ii) *$\pi_X : X \to Spec\, C_X^0(X)$ is dominant and*

$$\pi_X(X) = \{\mathfrak{m} \in m.Spec\, C_X^0(X) \mid Z_X(\mathfrak{m}) \neq \emptyset\}.$$

Moreover if X is compact, then $\pi_X(X) = m.Spec\, C_X^0(X)$.

(iii) *If X is normal, then $(\pi_X^\sharp)_x : C_X^0(X)_{\pi_X(x)} \to C_{X,x}^0$ is an isomorphism of rings for any $x \in X$.*

The proof is easy.

COROLLARY 1. *X is a compact T_2 space*
$\Leftrightarrow$ *$\pi_X : X \to m.Spec\, C_X^0(X)$ is an homeomorphism*
$\Leftrightarrow$ *$\pi_X : X \to m.Spec\, C_X^0(X)$ is an isomorphism of local ringed spaces.*

COROLLARY 2. (i) *(X, C_X^0) is an affine scheme* $\Rightarrow \dim C_X^0(X) = 0$
$\Rightarrow$ *$C_X^0(X)$ is a Hilbert ring.*

(ii) *For a compact T_2 space X, all the conditions in* (i) *are equivalent.*

REMARK. In general, for any topological spaces X, we obtain

$$C_X^0(X) \text{ is a Hilbert ring} \Leftrightarrow \dim C_X^0(X) = 0$$

from [2, 2.11].

LEMMA 3.2. *Let X be a topological space.*

(i) *If X is a T_1 space, then X satisfies (c_2).*

(ii) *If X is a compact T_2 space, then*

$$(c_1) \Leftrightarrow (X, C_X^0) \text{ is an affine scheme} \Leftrightarrow (c_3)\,.$$

The proof is induced from Lemma 1.2, Lemma 3.1, (ii) and Corollary 2 to Lemma 3.1.

LEMMA 3.3. *Suppose that X is a compact T_2 space. Then*
$C_X^0(X)$ is a noetherian Hilbert ring
$\Leftrightarrow$ *$C_X^0(X)$ is an Artin ring*
$\Leftrightarrow$ *$C_X^0(X)$ is a Hilbert ring and $\pi_X(x)$ is a principal ideal for any $x \in X$*
$\Leftrightarrow$ *X is a finite set.*

The proof is easy.

COROLLARY. *For a compact metric space X, we obtain*

$$C_X^0(X) \text{ is a Hilbert ring} \Leftrightarrow X \text{ is a finite set}.$$

4. Here we give some examples related to Theorems 1, 2 and 4. Note that all topological spaces X appeared in the following examples satisfy the property that any irreducible closed subset of X has a unique generic point.

First we show six examples described in Remark 2.

EXAMPLE 1. Let $(X, \mathcal{O}_X)$ be a local ringed space. Suppose that X is an infinite discrete space and $\mathcal{O}_{X,x}$ is a field for any $x \in X$. Then X satisfies (c_1) and (c_2) but does not satisfy (c_3).

The proof is easy from Example 0.

EXAMPLE 2. For a field K and a subring A of K, we put $X = Loc(K|A)$. Suppose that A is a Hilbert ring but is not a Prüfer ring. Then X satisfies (c_1) and (c_3) but does not satisfy (c_2).

For a proof see [10, Theorem 2].

EXAMPLE 3. For a three points set $X = \{x_1, x_2, y\}$, we introduce a topology by defining $\emptyset$, $\{y\}$, $\{x_1, y\}$, $\{x_2, y\}$ and X to be open subsets. Taking a field k and an indeterminate T over k, we define a mapping $s : X \to Loc(k(T)|k)$ by $s(x_1) = k$, $s(x_2) = k[T]_{(T)}$, $s(y) = k(T)$. Then s is continuous. Let $\mathcal{O}_X$ denote the intersection sheaf over X with respect to s. Then the local ringed space $(X, \mathcal{O}_X)$ satisfies (c_1) but does not satisfy (c_2) and (c_3).

The proof is easy.

EXAMPLE 4. Let X be the set of 0 and all primes. We introduce a topology for X by defining X and finite subsets of $\{2, 3, 5, \cdots\}$ to be closed subsets. Let $\mathcal{O}_X$ denote the intersection sheaf over X with respect to the continuous mapping $s : X \to Loc(\mathbf{Q}|\mathbf{Z})$ defined by $s(2) = \mathbf{Z}_{(2)}$, $s(x) = \mathbf{Q}$ $(x \neq 2)$. Then the local ringed space $(X, \mathcal{O}_X)$ satisfies (c_2) and (c_3) but does not satisfy (c_1).

The proof is easy.

EXAMPLE 5. (i) Let $(X, \mathcal{O}_X)$ be a local ringed space. Suppose that X is a finite discrete space and $\dim \mathcal{O}_{X,x} \geqq 1$ for some $x \in X$. Then X satisfies (c_2) but does not satisfy (c_1) and (c_3).

(ii) If a compact metric space X is not a finite set, then (X, C^0_X) satisfies (c_2) but does not satisfy (c_1) and (c_3).

PROOF. (i) is easy from Example 0.

(ii) is verified from Lemma 3.2 and Corollary to Lemma 3.3.

EXAMPLE 6. Let X be the set of -1, 0 and all primes. We introduce a topology for X by defining X, $\{0, 2, 3, 5, \cdots\}$ and finite subsets of $\{2, 3, 5, \cdots\}$ to be closed subsets. Let $\mathcal{O}_X$ denote the intersection sheaf over X with respect to the continuous mapping $s : X \to Loc(\mathbf{Q}|\mathbf{Z})$ defined by $s(2) = \mathbf{Z}_{(2)}$, $s(x) = \mathbf{Q}$ $(x \neq 2)$. Then the local ringed space $(X, \mathcal{O}_X)$ satisfies (c_3) but does not satisfy (c_1) and (c_2).

The proof is easy.

Next we show an example described in Remark 3.

EXAMPLE 7. For an algebraically closed field k, we put $\Lambda = k \cup \{\infty\}$. Take a family $(T_\lambda)_{\lambda\in\Lambda}$ of indeterminates over k, and consider the polynomial ring $A = k[T_\lambda \mid \lambda \in \Lambda]$ of infinite indeterminates. Then

(i) $X = Spec\, A$ satisfies (s_2) but does not satisfy (s_1).

(ii) $X = Zar(QA|A)$ satisfies (s_2) but does not satisfy (s_1).

PROOF. Since A is a polynomial ring over an algebraically closed field, X satisfies (s_2). For any $R \in Zar(k(T)|k)$, there exists $\mathfrak{p} \in Spec\, A$ such that $A/\mathfrak{p} \cong R$ (k-isomorphism). Thus A is not a Hilbert ring, and hence X does not satisfy (s_1).

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