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Lévy Processes with Negative Drift Conditioned to Stay Positive

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Abstract. Let X be a Lévy process with negative drift starting from $x > 0$, and let τ and τ_s be the first passage times to $(-\infty, 0]$ and (s, ∞), respectively. Under appropriate exponential moment conditions of X, we show that, for every $A \in \mathcal{F}_t$, the conditional laws $P_x(X \in A \mid \tau > s)$ and $P_x(X \in A \mid \tau > \tau_s)$ converge to different distributions as $s \to \infty$. Both of them can be regarded as the laws of X conditioned to stay positive. We characterize these limit laws in terms of h-transforms, by the renewal functions, of some Lévy processes killed at the entrance time into $(-\infty, 0]$.

1. Introduction.

Let $X = (X_t, t \geq 0)$ be a Lévy process with negative drift starting from $x > 0$, and let τ be the first passage time to $(-\infty, 0]$. One of our main aims in this paper is to investigate properties of the process when a conditioning event tends to the event $(\tau = \infty)$, which has probability zero. We make conditioning of X on $(\tau = \infty)$ by approximating this event in two ways. One natural definition for the conditional law of X given $(\tau = \infty)$ is

$$\lim_{s \to \infty} P_x(X \in A \mid \tau > s), \tag{1}$$

where $A \in \mathcal{F}_t$. Another natural way is to consider

$$\lim_{s \to \infty} P_x(X \in A \mid \tau > \tau_s), \tag{2}$$

where τ_s is the first passage time to (s, ∞). We are interested in studying the limit distributions in these approximations.

To clarify our purpose, we give typical examples obtained by Martinez-San Martin [16]. Suppose $X_t = B_t - \alpha t$ where B is a one-dimensional Brownian motion and $\alpha > 0$. According to [16], the first approximation yields

$$\lim_{s \to \infty} P_x(X \in A \mid \tau > s) = Q_x^1(A), \tag{3}$$

where Q_x^1 is the law of the 3-dimensional Bessel process, whose transition function is

$$q^1(t, x, dy) = \frac{y}{x} P_x(B_t \in dy, \tau^B > t),$$

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with $\tau^B = \inf\{t > 0 : B_t = 0\}$. On the other hand the second approximation implies

$$\lim_{s\to\infty} P_x(X \in A \mid \tau > \tau_s) = Q_x^2(A)\,. \tag{4}$$

Here Q_x^2 is the law of the diffusion process with generator $(1/2)\Delta + \alpha\coth(\alpha x)(d/dx)$, whose transition function is given by

$$q^2(t, x, dy) = \frac{1 - e^{-2\alpha y}}{1 - e^{-2\alpha x}} P_x(X_t^* \in dy, \tau^{X^*} > t)\,,$$

where $X_t^* = B_t + \alpha t$ and $\tau^{X^*} = \inf\{t > 0 : X_t^* = 0\}$. A qualitative difference between two limits Q_x^1 and Q_x^2 is that the former does not rely on $\alpha > 0$, and the latter does. However we note that both of q^1 and q^2 are expressed as h-transforms of some diffusions. More precisely, q^1 (resp. q^2) corresponds to the h-transform of B (resp. X^*) killed at the entrance time into $(-\infty, 0]$ by the harmonic function $h_1(y) = y$ (resp. $h_2(y) = 1 - e^{-2\alpha y}$). Hence Q_x^1 and Q_x^2 are thought of the laws of B and X^* conditioned to stay positive.

It is anticipated from (3) and (4) that the two approximations give different limits for a Lévy process X. Indeed we shall show that, under an appropriate exponential moment condition of X, the first approximation leads to the law of a certain oscillating Lévy process conditioned to stay positive, while the second leads to the law of a Lévy process drifting to ∞ conditioned to stay positive.

Our motivation comes from plenty of limit theorems for random walks with negative drifts conditioned by events of the above types, investigated by a number of other authors. Keener [15] obtained a result similar to (3) for integer-valued random walks. Bertoin-Doney [2] gave a simple proof for his result, and showed that it also holds for non-lattice random walks. Moreover they considered random walk analogue of (4). Some of the results in [2] and [15] are expanded by the author [11]. In continuous time, (3) was extended to diffusions by Collet-Martinez-San Martin [6].

This paper is organized as follows. In Section 2 we introduce notation and recall some fundamental facts which we shall use later. Section 3 includes several limit theorems concerned with the first approximation. In particular we get results which elucidate the limit (1). The limit distribution in (2) is determined in Section 4. In Appendix we prove certain asymptotic results which play crucial roles in Section 3.

2. Notation and auxiliary results.

Let $(X_t, t \geq 0, P_x)$ be a Lévy process, which is a process with stationary independent increments whose sample paths belong to the space of right continuous real-valued functions on $[0, \infty)$ with left limits on $(0, \infty)$. For $x \in \mathbf{R}$, P_x stands for the law of X starting from x, and set $P = P_0$. Let $\mathcal{F}_t$ be the sigma field generated by $(X_s, s \leq t)$. Let $M = (M_t, t \geq 0)$ be the supremum process, i.e., $M_t = \sup_{0\leq s\leq t} X_s$. We write $M_\infty = \sup_{t\geq 0} X_t$. The first passage times to $(-\infty, 0]$ and to (x, ∞) are denoted by τ and τ_x, respectively, i.e.,

$$\tau = \inf\{t > 0 : X_t \leq 0\}, \quad \tau_x = \inf\{t > 0 : X_t > x\}\,.$$

The dual process of X is denoted by $\bar{X}$; that is $\bar{X} = -X$ when the starting point is 0. For the quantities introduced to X, the corresponding ones for $\bar{X}$ are denoted by bars, for instance, $\bar{M}$ and $\bar{\tau}$. Let L be a local time process of $M - X$ at 0. If 0 is regular for $\{0\}$, i.e., $P(\inf\{t > 0 : M_t = X_t\} = 0) = 1$, then L is constructed as in Blumenthal-Getoor [5]. If 0 is not regular for $\{0\}$, then L is defined in the following manner as in Fristedt [9]. In this case $\{t \geq 0 : M_t = X_t\}$ is a.s. discrete. Let $\{\mathcal{L}_n\}_{n\geq 1}$ be a sequence of independent, identically and exponentially distributed random variables independent of X. We set $L(t) = \sum[\mathcal{L}_j; j \leq \#\{s < t : M_s = X_s\}]$. In both cases, L^{-1} denotes the right continuous inverse of L, i.e., $L^{-1}(\cdot) = \inf\{t > 0 : L(t) > \cdot\}$. The ladder height process H is given by $H_t = M(L_t^{-1})$ on $L_t^{-1} < \infty$ and $H_t = \infty$ on $L_t^{-1} = \infty$. Fristedt [9] showed that (L^{-1}, H) is a bivariate subordinator and obtained a formula of the Laplace exponent κ of (L^{-1}, H) defined by $E(e^{-aL_t^{-1} - bH_t}) = e^{-t\kappa(a,b)}$, $a, b \geq 0$:

$$\kappa(a,b) = k \exp\left[\int_0^\infty \frac{dt}{t} \int_{0-}^\infty (e^{-t} - e^{-at-bx}) P(X_t \in dx)\right], \tag{5}$$

where k depends on the normalization of L. *Throughout this paper we take* $k = 1$. The above identity is connected with the Wiener-Hopf factorization identity, see for instance Bertoin [1], Bingham [4] and Fristedt [9]. Among many relations we use the following. If $z > 0$ and $u, v \geq 0$,

$$z \int_0^\infty e^{-zt} E(e^{-uM_t - v(M_t - X_t)})\, dt = \frac{\kappa(z,0)}{\kappa(z,u)} \cdot \frac{\bar{\kappa}(z,0)}{\bar{\kappa}(z,v)}. \tag{6}$$

The renewal function of the ladder height process H is defined by

$$V(x) = \int_0^\infty P(H_t \leq x)\, dt, \qquad x \geq 0.$$

The left limit of V is denoted by $V(\cdot -)$. Recall that, if $\limsup_{t\to\infty} X_t = \infty$ a.s., then $(\bar{V}(X_t -) 1_{\{\tau > t\}}, t \geq 0)$ is a P_x martingale for every $x > 0$, see e.g. [1, p.184]. Thus

$$p(t, x, dy) = \frac{\bar{V}(y-)}{\bar{V}(x-)} P_x(X_t \in dy, \tau > t)$$

is a strict Markovian transition function on $(0, \infty)$. One may say that its law is the h-transform, by the harmonic function $\bar{V}(\cdot -)$, of the law of X killed at time τ.

In this paper we classify Lévy processes into three types according to the form of the characteristic exponent ξ of X defined by $E(e^{i\theta X_t}) = e^{t\xi(\theta)}$. It is well known that ξ is expressed as

$$\xi(\theta) = ia\theta - \frac{d^2}{2}\theta^2 + \int_{\mathbf{R}} (e^{i\theta x} - 1 - i\theta x 1_{\{|x|<1\}}) \Lambda(dx),$$

where $a \in \mathbf{R}$, $d \geq 0$ and Λ is a measure on $\mathbf{R} - \{0\}$ with $\int_{\mathbf{R}} (1 \wedge x^2) \Lambda(dx) < \infty$. We omit the trivial case where $d = 0$ and $\Lambda \equiv 0$. We say that X is in Class I if X satisfies the following:

Class I. *X_1 has a non-lattice distribution, i.e.,* $|E(e^{i\theta X_1})| < 1$ *if* $\theta \neq 0$.

By the form of ξ, we see that

$$
\begin{aligned}
X \text{ is in Class I} \quad &\Leftrightarrow \quad \Re\xi(\theta) < 0 \text{ if } \theta \neq 0, \\
&\Leftrightarrow \quad \frac{d^2}{2}\theta^2 + \int_{-\infty}^{\infty} (1 - \cos\theta x)\Lambda(dx) > 0 \text{ if } \theta \neq 0, \\
&\Leftrightarrow \quad d > 0 \text{ or } \Lambda(\mathbf{R} - r\mathbf{Z}) > 0 \text{ for } \forall r > 0.
\end{aligned}
$$

If X is not in Class I, then, by the above argument, $d = 0$ and Λ is supported by $\{\pm rn : n \in \mathbf{N}\}$ with maximal span $r > 0$, so that $\xi(\theta) = ib\theta + \sum_{j\neq 0}(1 - e^{i\theta rj})\Lambda(\{rj\})$ with some $b \in \mathbf{R}$. Hence we define the following two classes.

Class II. *ξ is expressed as $\xi(\theta) = ib\theta + \sum_{j\neq 0}(1 - e^{i\theta rj})\Lambda(\{rj\})$ where $b \neq 0$ and $r > 0$ is the maximal span of Λ.*

Class III. *ξ is expressed as $\xi(\theta) = \sum_{j\neq 0}(1 - e^{i\theta rj})\Lambda(\{rj\})$ where $r > 0$ is the maximal span of Λ.*

3. Limit theorems related to the first approximation.

We introduce the Laplace exponent ϕ of X defined by

$$E(e^{\theta X_t}) = e^{t\phi(\theta)}, \quad t > 0,\ \theta \in \mathbf{R}.$$

The Laplace exponent serves to state conditions for our results. Throughout this section we assume that the following conditions are satisfied.

(A) *There exists $\alpha > 0$ such that $\phi < \infty$ in a neighbourhood of α, and $\phi'(\alpha) = 0$.*

(B) *X is in either Class* I *or Class* III.

Similar conditions have appeared as key hypotheses for the conditional limit theorems of random walks in [2], [7], [14], [15] and many other papers. We point out that those limit theorems of random walks do not deal with the random walks of which the one step distribution is concentrated in non-centered lattice set of the form $\{a + c\mathbf{Z}\}$ with $0 < a < c$. This restriction is corresponding to removing Lévy processes in Class II from our limit theorems in this section. Since ϕ is convex, (A) implies $EX_1 < 0$, so that X drifts to $-\infty$ and M_∞ is a.s. finite. If $\phi(\theta) < \infty$, the exponential martingale transform P^θ of P is defined by

$$P^\theta = e^{\theta X(t) - t\phi(\theta)} \cdot P \quad \text{on } \mathcal{F}_t.$$

Plainly, this relation also holds if the fixed time $t > 0$ is replaced by an $\mathcal{F}_t$-stopping time. Under P^θ, X is a Lévy process with Laplace exponent $\phi(\theta) - \phi(\cdot + \theta)$. To simplify the notation we write $\hat{P}$ and γ instead of P^α and $e^{\phi(\alpha)}$, respectively. We note that an exponential martingale transform does not vary the class of Lévy process, see e.g.[18]. This fact is frequently used in the arguments in this paper. Set

$$U(x) = \int_0^\infty \hat{P}(H_t < x)dt, \qquad \bar{U}(x) = \int_0^\infty \hat{P}(\bar{H}_t < x)dt.$$

Namely U is the left limit of the renewal function associated to the ladder height process of the Lévy process with Laplace exponent $\phi(\alpha) - \phi(\cdot + \alpha)$, and $\bar{U}$ is its dual.

In our proofs, we mainly deal with the case where X is in Class I since essentially the same methods work for X in Class III. Our results in this section depend on the following lemma.

LEMMA 1. *Let $x > 0$ and $\theta > 0$.*

(a) *If X is in Class* I, *then*

$$\lim_{t\to\infty} t^{3/2}\hat{E}_x(e^{-\theta X(t)}, \tau > t) = \frac{c_1}{\sqrt{2\pi\phi''(\alpha)}}\bar{U}(x)\int_0^\infty e^{-\theta z}U(z)dz.$$

(b) *If X is in Class* III, *then*

$$\lim_{t\to\infty} t^{3/2}\hat{E}_x(e^{-\theta X(t)}, \tau > t) = \frac{c_1 r}{\sqrt{2\pi\phi''(\alpha)}}\bar{U}(x)\sum_{j\in r\mathbf{N}} e^{-\theta(j+l(x))}U(j).$$

Here $c_1 = \exp\{\int_0^\infty (e^{-t}-1)t^{-1}\hat{P}(X_t = 0)dt\}$ and $l(x) = x - rk$ if $r(k-1) < x \le rk$, $k \in \mathbf{N}$. In addition, we can replace τ, X, U and $\bar{U}$ by their duals.

Since $\hat{E}X_1 = 0$ and $0 < \hat{E}|X_1|^2 = \phi''(\alpha) < \infty$, Lemma 1 follows from Lemma A in Appendix. See Appendix for the proof. Note that $l(x) = l(y)$ if $x \equiv y \pmod r$. This is a key point of the proof for X in Class III. Using Lemma 1 we get

LEMMA 2. *Let $x, y > 0$. Suppose that $x \equiv y$* (mod r) *if X is in Class* III. *Then we have, for $s > 0$,*

$$\lim_{t\to\infty}\frac{P_y(\tau > t-s)}{P_x(\tau > t)} = \gamma^{-s}\frac{\bar{U}(y)}{\bar{U}(x)}e^{\alpha(y-x)}.$$

PROOF. Let $\theta \ge 0$. From the definition of $\hat{P}$ and (a) of Lemma 1,

$$\begin{aligned} E_x(e^{-\theta X(t)}, \tau > t) &= \gamma^t e^{\alpha x}\hat{E}_x(e^{-(\theta+\alpha)X(t)}, \tau > t) \\ &\sim \gamma^t t^{-3/2} c_2 e^{\alpha x}\bar{U}(x)\int_0^\infty e^{-(\theta+\alpha)z}U(z)dz, \end{aligned} \tag{7}$$

as $t \to \infty$ with a constant $c_2 > 0$. Set $\theta = 0$. Then

$$P_x(\tau > t) \sim \gamma^t t^{-3/2} c_3 e^{\alpha x}\bar{U}(x), \quad \text{as } t \to \infty, \tag{8}$$

where $c_3 = c_2\int_0^\infty e^{-\alpha z}U(z)dz$. This shows the lemma. □

The following proposition is a Lévy process analogue of [6, Theorem C] and [15, Theorem 1.3].

PROPOSITION 1. *Assume the conditions* (A) *and* (B).

(a) *If X is in Class* I, *then, for all $x > 0$ and $y \ge 0$,*

$$\lim_{t\to\infty} P_x(X_t \le y \mid \tau > t) = \frac{\int_0^y e^{-\alpha z}U(z)dz}{\int_0^\infty e^{-\alpha z}U(z)dz}.$$

(b) *If X is in Class* III, *then, for all $x, y \in r\mathbf{N}$,*

$$\lim_{t\to\infty} P_x(X_t = y \mid \tau > t) = \frac{e^{-\alpha y}U(y)}{\sum_{j\in r\mathbf{N}} e^{-\alpha j}U(j)}.$$

PROOF. We prove only (a). Dividing (7) by (8), we have

$$\lim_{t\to\infty} E_x(e^{-\theta X(t)} \mid \tau > t) = \int_0^\infty e^{-(\theta+\alpha)z}U(z)dz \Big/ \int_0^\infty e^{-\alpha z}U(z)dz.$$

By the continuity theorem for Laplace transform, we get (a). □

REMARK. The convergence of $E_x(e^{-\theta X(t)} \mid \tau > t)$ has been considered by Pechinkin [17] if X has no negative jumps. However there is something unclear in his proof.

To state our results, we introduce time homogeneous Markov processes Y and $\bar{Y}$ on $(0, \infty)$ whose transition functions are given by the following q and $\bar{q}$, respectively.

$$q(t, x, dy) = \frac{\bar{U}(y)}{\bar{U}(x)}\hat{P}_x(X_t \in dy, \tau > t), \qquad x > 0,$$
$$\bar{q}(t, x, dy) = \frac{U(y)}{U(x)}\hat{P}_x(\bar{X}_t \in dy, \bar{\tau} > t), \qquad x > 0.$$

As was mentioned in Section 2, q and $\bar{q}$ are strict Markovian transition function on $(0, \infty)$ since X oscillates under $\hat{P}$. In other words, Y and $\bar{Y}$ are conservative. The next theorem gives the limit distribution of (1).

THEOREM 1. *Assume the conditions* (A) *and* (B), *and let $x > 0$. Then, for $A \in \mathcal{F}_s$,*

$$\lim_{t\to\infty} P_x(X \in A \mid \tau > t) = P_x(Y \in A).$$

PROOF. By the Markov property,

$$P_x(X \in A \mid \tau > t) = E_x\left(\frac{P_{X(s)}(\tau > t - s)}{P_x(\tau > t)}, \tau > s, X \in A\right).$$

Using Fatou's lemma and Lemma 2, we have

$$\begin{aligned}\liminf_{t\to\infty} P_x(X \in A \mid \tau > t) &\geq E_x\left(\gamma^{-s}\frac{\bar{U}(X_s)}{\bar{U}(x)}e^{\alpha(X_s - x)}, \tau > s, X \in A\right) \\ &= \hat{E}_x\left(\frac{\bar{U}(X_s)}{\bar{U}(x)}, \tau > s, X \in A\right) \equiv P_x(Y \in A).\end{aligned}$$

Replacing A by A^c, we have

$$\begin{aligned}\limsup_{t\to\infty} P_x(X \in A \mid \tau > t) &= 1 - \liminf_{t\to\infty} P_x(X \in A^c \mid \tau > t) \\ &\leq 1 - P_x(Y \in A^c) = P_x(Y \in A).\end{aligned}$$

The proof of the theorem is complete. □

Denote by $D[0,t]$ the set of real-valued functions on $[0,t]$ which are right continuous and have left limits on $(0,t]$. $D[0,t]$ is endowed with Skorohod's topology. The following lemma is a direct consequence of the well known duality between X and $\bar{X}$ with respect to the Lebesgue measure, see e.g. [1, p.46].

LEMMA 3. *If $f, g : \mathbf{R} \to [0,\infty)$ and $H : D[0,t] \to [0,\infty)$, it holds that*

$$\int_{\mathbf{R}} dx f(x) E_x(H(X_{(t-s)-}, s \le t) g(X_t)) = \int_{\mathbf{R}} dx\, g(x) E_x(H(\bar{X}_s, s \le t) f(\bar{X}_t)) .$$

The next theorem is an extention of Theorem 1.

THEOREM 2. *Assume the conditions* (A) *and* (B). *Let $F, G : D[0,s] \to \mathbf{R}$ be bounded and measurable.*

(a) *If X is in Class* I *and G is continuous, then, for $x > 0$,*

$$\begin{aligned} &E_x(F(X_u, u \le s) G(X_{(t-u)-}, u \le s) \mid \tau > t) \\ &\to E_x(F(Y_u, u \le s)) \cdot c^{-1} \int_0^\infty dz\, e^{-\alpha z} U(z) E_z(G(\bar{Y}_u, u \le s)) \end{aligned}$$

as $t \to \infty$, where $c = \int_0^\infty e^{-\alpha z} U(z) dz$.

(b) *If X is in Class* III, *then, for $x \in r\mathbf{N}$,*

$$\begin{aligned} &E_x(F(X_u, u \le s) G(X_{(t-u)-}, u \le s) \mid \tau > t) \\ &\to E_x(F(Y_u, u \le s)) \cdot c^{-1} \sum_{j \in r\mathbf{N}} e^{-\alpha j} U(j) E_j(G(\bar{Y}_u, u \le s)) \end{aligned}$$

as $t \to \infty$, where $c = \sum_{j \in r\mathbf{N}} e^{-\alpha j} U(j)$.

PROOF. We prove only (a) because the same argument is valid for (b). Without loss of generality we may assume that $0 \le F, G \le 1$. For ease of notation we write $F(X) = F(X_u, u \le s)$ and $g(y) = E_y[G(X_{(s-u)-}, u \le s), \tau > s]$. Conditioning on $\mathcal{F}_{t-s}$ and then on $\mathcal{F}_s$, we have by the Markov property

$$\begin{aligned} E_x(F(X) G(X_{(t-u)-}, u \le s), \tau > t) &= E_x(F(X) g(X_{t-s}), \tau > t-s) \\ &= E_x(F(X) E_{X(s)}[g(X_{t-2s}), \tau > t-2s], \tau > s) \\ &= E_x(F(X) P_{X(s)}(\tau > t-2s) h_{t-2s}(X_s), \tau > s) , \end{aligned}$$

where $h_t(y) = E_y(g(X_t) \mid \tau > t)$. Thus

$$E_x(F(X) G(X_{(t-u)-}, u \le s) \mid \tau > t) = E_x\left(F(X) \frac{P_{X(s)}(\tau > t-2s)}{P_x(\tau > t)} h_{t-2s}(X_s), \tau > s \right) .$$

It is easy to see that the function g is bounded and left continuous. Therefore (a) of Proposition 1 shows that, if $y > 0$,

$$\lim_{t \to \infty} h_t(y) = c^{-1} \int_0^\infty e^{-\alpha z} g(z) U(z) dz .$$

Using the definitions of g, $\hat{P}$ and Lemma 3 in turn, we have

$$\begin{aligned}\int_0^\infty e^{-\alpha z} g(z) U(z) dz &= \int_0^\infty dz\, e^{-\alpha z} U(z) E_z(G(X_{(s-u)-}, u \le s), \tau > s) \\ &= \gamma^s \int_0^\infty dz\, U(z) \hat{E}_z(G(X_{(s-u)-}, u \le s) e^{-\alpha X_s}, \tau > s) \\ &= \gamma^s \int_0^\infty dz\, e^{-\alpha z} \hat{E}_z(G(\bar{X}_u, u \le s) U(\bar{X}_s), \bar{\tau} > s) \\ &= \gamma^s \int_0^\infty dz\, e^{-\alpha z} U(z) E_z(G(\bar{Y})) .\end{aligned}$$

The last equality follows from the definition of $\bar{Y}$. In view of Lemma 2 and the above, it follows from Fatou's lemma that

$$\begin{aligned}&\liminf_{t\to\infty} E_x(F(X) G(X_{(t-u)-}, u \le s) \mid \tau > t) \\ &\ge E_x\left(F(X) \gamma^{-s} \frac{\bar{U}(X_s)}{\bar{U}(x)} e^{\alpha(X_s - x)}, \tau > s \right) \cdot \frac{\gamma^{-s}}{c} \int_0^\infty e^{-\alpha z} g(z) U(z) dz \\ &= E_x(F(Y)) \cdot c^{-1} \int_0^\infty dz\, e^{-\alpha z} U(z) E_z(G(\bar{Y})) . \qquad (9)\end{aligned}$$

Putting $G = 1$ and replacing F by $1 - F$ in (9), we see

$$\limsup_{t\to\infty} E_x(F(X) \mid \tau > t) \le E_x(F(Y)) .$$

Thus we have

$$\lim_{t\to\infty} E_x(F(X) \mid \tau > t) = E_x(F(Y)) .$$

Replacing G by $1 - G$ in (9) and using the above, we have

$$\limsup_{t\to\infty} E_x(F(X) G(X_{(t-u)-}, u \le s) \mid \tau > t) \le E_x(F(Y)) \cdot c^{-1} \int_0^\infty dz\, e^{-\alpha z} U(z) E_z(G(\bar{Y})) .$$

This combined with (9) concludes the proof of the theorem. □

Theorem 2 tells us that, for fixed $s > 0$, the conditional processes $(X_u, u \le s \mid \tau > t)$ and $(X_{(t-u)-}, u \le s \mid \tau > t)$ are asymptotically independent as $t \to \infty$.

A limit theorem of this type for random walks has been obtained by the author [11]. It was applied to a certain problem of a random walk in a random medium. One of our main aims of this investigation is to get Theorem 2. The application of Theorem 2 to a similar problem for a diffusion process in a random Lévy environment will be discussed in the forthcoming paper [12].

The finiteness of $E(e^{\theta M_\infty})$ will be used. That is, we need the following.

LEMMA 4. *If $\phi(\theta) < 0$ for some $\theta > 0$, then $E(e^{\theta M_\infty}) < \infty$.*

PROOF. Denote by κ_θ the Laplace exponent of the ladder process under P^θ. Let $\beta = -\phi(\theta)$. Using (5), we see that, for any $u > 0$,

$$\begin{aligned}\frac{\kappa_\theta(\beta, \theta+u)}{\kappa_\theta(1+\beta, \theta)} &= \exp\left[\int_0^\infty \frac{dt}{t}\int_{0-}^\infty (e^{-t} - e^{-ux})e^{-\beta t - \theta x} P^\theta(X_t \in dx)\right] \\ &= \exp\left[\int_0^\infty \frac{dt}{t}\int_{0-}^\infty (e^{-t} - e^{-ux}) P(X_t \in dx)\right] \\ &= \kappa(0, u).\end{aligned}$$

Recall that $M_\infty < \infty$ a.s. Setting $v = 0$ in (6) and then letting $z \to 0$, we have

$$E(e^{-uM_\infty}) = \frac{\kappa(0,0)}{\kappa(0,u)} = \frac{c_4}{\kappa_\theta(\beta, \theta+u)} = c_4 \int_0^\infty e^{-(u+\theta)x} dV_\theta(x),$$

where $V_\theta(x) = \int_0^\infty E^\theta(e^{-\beta L_t^{-1}}, H_t \le x)dt$. This shows $P(M_\infty \in dx) = c_4 e^{-\theta x} V_\theta(dx)$. Hence we get $E(e^{\theta M_\infty}) = c_4 \kappa_\theta(\beta, 0)^{-1} < \infty$. The proof of the lemma is complete. □

By the condition (A) we can pick $\varepsilon > 0$ such that $\phi(\alpha + \varepsilon) < 0$. Thus Chebyshev's inequality combined with Lemma 4 shows the following.

LEMMA 5. *Let $y > 0$. Then, for some $\varepsilon > 0$,*

$$e^{\alpha y} P(M_\infty > y) \le \text{const.} e^{-\varepsilon y}.$$

We turn our attention to the conditional limit theorems related to $\hat{P}$ and the dual process $\bar{X}$. We identify $\hat{P}$ as the law of X conditioned to oscillate, i.e., we get

PROPOSITION 2. *Assume the conditions* (A) *and* (B). *Then, for $A \in \mathcal{F}_s$,*

$$\lim_{t\to\infty} P(A \mid \exists u > t, X_u > 0) = \hat{P}(A).$$

PROOF. Firstly we show the following. For fixed $x \in \mathbf{R}$ ($x \in r\mathbf{Z}$ if X is in Class III),

$$P(\exists s > t, X_s > x) \sim \gamma^t t^{-1/2} c_5 e^{-\alpha x} E(e^{\alpha M_\infty})/(2\pi\phi''(\alpha))^{1/2} \quad \text{as } t \to \infty, \tag{10}$$

with some $c_5 > 0$. To see this we put $M^{(t)} = \sup_{s \ge t}\{X_s - X_t\}$ and $h(y) = P(M_\infty > y)$. Since $M^{(t)}$ is independent of X_t and has the same law as M_∞,

$$\begin{aligned}P(\exists s > t, X_s > x) &= P(M^{(t)} + X_t > x) \\ &= E(h(x - X_t)) \\ &= \gamma^t \hat{E}(e^{-\alpha X_t} h(x - X_t)) \\ &= \gamma^t\left[\hat{E}(e^{-\alpha X_t}, X_t > x) + \hat{E}(e^{-\alpha X_t} h(x - X_t), X_t \le x)\right].\end{aligned}$$

From Lemma 5, $e^{-\alpha y} h(x - y) \le \text{const.} e^{\varepsilon y}$ if $y \le x$. Let X be in Class I. Use (a.3) in Appendix and the Kolmogorov-Rogozin inequality, $P(0 \le X_t \le x) \le \text{const.}(x+1)t^{-1/2}$, see e.g. [10] or [18]. Then we get

$$\lim_{t\to\infty} \gamma^{-t} t^{1/2} P(\exists s > t, X_s > x) = \frac{1}{\sqrt{2\pi\phi''(\alpha)}} \left[\int_x^\infty e^{-\alpha y} dy + \int_{-\infty}^x e^{-\alpha y} h(x-y) dy \right]$$
$$= \frac{1}{\sqrt{2\pi\phi''(\alpha)}} \cdot \frac{e^{-\alpha x}}{\alpha} E(e^{\alpha M_\infty}), \quad \text{(Fubini's Theorem)}.$$

That is, (10) holds with $c_5 = \alpha^{-1}$. If X is in Class III, similar calculation gives (10) with $c_5 = r/(e^{\alpha r} - 1)$. We deduce from the Markov property, Fatou's lemma and (10) that

$$\liminf_{t\to\infty} P(A \mid \exists u > t, X_u > 0) \geq E\left(\lim_{t\to\infty} \frac{P_{X(s)}(\exists u > t-s, X_u > 0)}{P(\exists u > t, X_u > 0)}, A \right)$$
$$= \gamma^{-s} E(e^{\alpha X(s)}, A) \equiv \hat{P}(A).$$

Replacing A by A^c, we finish the proof of the proposition. □

The law of $\bar{Y}$ appears in the following conditional limit theorem for $\bar{X}$ which is similar to Theorem 1.

PROPOSITION 3. *Assume the conditions* (A) *and* (B). *Then, for* $x > 0$ *and* $A \in \mathcal{F}_s$,

$$\lim_{t\to\infty} P_x(\bar{X} \in A \mid \infty > \bar{\tau} > t) = P_x(\bar{Y} \in A).$$

PROOF. As before we assume that X is in Class I. Let $h(y) = P(M_\infty > y)$. By the Markov property

$$P_x(\infty > \bar{\tau} > t) = E_x(h(\bar{X}_t), \bar{\tau} > t)$$
$$= \gamma^t e^{-\alpha x} \hat{E}_x(e^{\alpha \bar{X}(t)} h(\bar{X}_t), \bar{\tau} > t).$$

By Lemma 5, we have $e^{\alpha z} h(z) \leq \text{const.} e^{-\varepsilon z}$ if $z \geq 0$. Therefore (a) of Lemma 1 applied to $\bar{X}$ shows

$$\lim_{t\to\infty} \gamma^{-t} t^{3/2} P_x(\infty > \bar{\tau} > t) = c_2 U(x) e^{-\alpha x} \int_0^\infty e^{\alpha z} h(z) \bar{U}(z) dz.$$

Therefore we have, for all $x, y > 0$ and $s > 0$,

$$\lim_{t\to\infty} \frac{P_y(\infty > \bar{\tau} > t-s)}{P_x(\infty > \bar{\tau} > t)} = \gamma^{-s} \frac{U(y)}{U(x)} e^{\alpha(x-y)}.$$

If X is in Class III, then the above is valid for $x \equiv y \pmod r$. The proposition follows from the above and arguments based on Fatou's lemma and the Markov property similar to those in Theorem 1. □

4. Limit theorems by the second approximation.

Our aim of this section is to determine the limit distribution in (2) for suitable Lévy processes. All the results in this section are a continuous time analogue of the corresponding results for random walks, see [2] and [13]. Here we are out of the conditions (A) and (B). Throughout this section we assume the following condition.

(C) *There exists* $\omega > 0$ *such that* $\phi(\omega) = 0$ *and* $E(X_1 e^{\omega X_1}) < \infty$.

This condition is known as Cramér estimate in random walk theory. Since ϕ is convex, (C) leads to (A). When (C) holds, we can introduce the new probability P^* as in the previous section. That is,

$$P^* = e^{\omega X_t} P \quad \text{on} \quad \mathcal{F}_t .$$

Under P^*, we have $0 < E^*(X_1) < \infty$. Set

$$\bar{U}_*(x) = \int_0^\infty P^*(\bar{H}_t < x) dt .$$

The starting point is the following lemma obtained by Bertoin-Doney; see [1, p.153] and [3].

LEMMA 6. *If* X *is not in Class* III, *then*

$$\lim_{y \to \infty} e^{\omega y} P(M_\infty > y) = k_1 .$$

If X *is in Class* III, *then*

$$\lim_{y \to \infty} e^{\omega r [y/r]} P(M_\infty > y) = k_2 ,$$

where k_1 *and* k_2 *are positive and finite.*

As a result of Lemma 6, we have the following.

LEMMA 7. *Let* $x, z \in \mathbf{R}$. *Assume that* $x \equiv z \pmod r$ *if* X *is in Class* III. *Then*

$$\lim_{y \to \infty} P_x(M_\infty > y)/P_z(M_\infty > y) = e^{\omega(x-z)} .$$

We can think of P^* as the law of X conditioned to drift to ∞. More precisely we have

PROPOSITION 4. *Under the condition* (C), *we have, for any* $A \in \mathcal{F}_s$,

$$\lim_{y \to \infty} P(A \mid M_\infty > y) = P^*(A) .$$

PROOF. The equivalence $(M_\infty > y) = (\tau_y < \infty)$ and the Markov property imply

$$\begin{aligned} P(A \mid M_\infty > y) &\geq P(A, \infty > \tau_y > s)/P(M_\infty > y) \\ &= E\left(\frac{P_{X(s)}(M_\infty > y)}{P(M_\infty > y)}, \tau_y > s, A \right) . \end{aligned}$$

We deduce from Fatou's lemma and Lemma 7 that

$$\liminf_{y \to \infty} P(A \mid M_\infty > y) \geq E(e^{\omega X(s)}, A) \equiv P^*(A) .$$

The limsup estimate follows from replacing A by A^c. The proof is complete. □

REMARK. Proposition 4 has been originally demonstrated by Williams [20] in case of Brownian motion with negative drift. His result was extended to the spectrally negative Lévy processes, see for instance [1].

LEMMA 8. *Let $x, z > 0$. Suppose that $x \equiv z \pmod r$ if X is in Class* III. *Then we have*

$$\lim_{y\to\infty} \frac{P_z(\tau > \tau_y)}{P_x(\tau > \tau_y)} = \frac{\bar{U}_*(z)}{\bar{U}_*(x)} e^{\omega(z-x)} .$$

PROOF. Decomposition of the event $(M_\infty > y)$ and the strong Markov property yield

$$\begin{aligned} P_x(M_\infty > y) &= P_x(\tau > \tau_y) + P_x(\infty > \tau_y > \tau) \\ &= P_x(\tau > \tau_y) + E_x(P_{X(\tau)}(M_\infty > y), \tau_y > \tau) . \end{aligned}$$

It is clear that $P_{X(\tau)}(M_\infty > y)/P_x(M_\infty > y)$ is less than one and converges as $y \to \infty$ to $e^{\omega(X(\tau)-x)}$ a.s. by Lemma 7. Hence, by the dominated convergence theorem we have

$$\begin{aligned} \lim_{y\to\infty} P_x(\tau > \tau_y)/P_x(M_\infty > y) &= 1 - E_x(e^{\omega(X(\tau)-x)}) \\ &= 1 - P_x^*(\tau < \infty) \\ &= P_x^*(\tau = \infty) = P^*(\bar{M}_\infty < x) . \end{aligned}$$

Recall that $P^*(\bar{M}_\infty < x) = k_3\bar{U}_*(x)$ where $k_3 \in (0, \infty)$. Combining this identity and Lemma 7 with the above equality, we get the lemma. □

The following theorem gives the limit distribution in (2).

THEOREM 3. *Under the condition* (C), *we have, for any $x > 0$ and $A \in \mathcal{F}_s$,*

$$\lim_{y\to\infty} P_x(X \in A \mid \tau > \tau_y) = R_x(A) ,$$

where R_x is the law of the homogeneous Markov process with transition function

$$r(t, x, dy) = \frac{\bar{U}_*(y)}{\bar{U}_*(x)} P_x^*(X_t \in dy, \tau > t) .$$

PROOF. Applying the Markov property, we see

$$\begin{aligned} P_x(X \in A \mid \tau > \tau_y) &\geq P_x(X \in A, \tau > \tau_y > s)/P_x(\tau > \tau_y) \\ &= E_x\left(\frac{P_{X(s)}(\tau > \tau_y)}{P_x(\tau > \tau_y)}, (\tau_y \wedge \tau) > s, X \in A \right) . \end{aligned}$$

Using Fatou's lemma and Lemma 8, we get

$$\begin{aligned} \liminf_{y\to\infty} P_x(X \in A \mid \tau > \tau_y) &\geq E_x\left(\frac{\bar{U}_*(X_s)}{\bar{U}_*(x)} e^{\omega(X(s)-x)}, \tau > s, X \in A \right) \\ &= E_x^*\left(\frac{\bar{U}_*(X_s)}{\bar{U}_*(x)}, \tau > s, X \in A \right) \equiv R_x(A) . \end{aligned}$$

The limsup estimate is derived from replacing A by A^c. Our theorem is proved. □

REMARK. If X is spectrally negative (i.e., has no positive jumps), Theorem 3 is well-known and the transition function is expressed as

$$r(t,x,dy) = \frac{W(y)}{W(x)} P_x(X_t \in dy, \tau > t)\,,$$

where W is the scale function of X. It is easy to see that the two expressions of $r(t,x,dy)$ are equivalent in that case because we can show $W(x) = \text{const.}e^{\omega x}\bar{U}_*(x)$, see for instance [1, Chapter VII].

Appendix : Lemma A and its proof.

We use the same notation as in Section 2. We show the following.

LEMMA A. *Suppose that* $EX_1 = 0$ *and* $0 < \sigma^2 := E|X_1|^2 < \infty$. *Let* $x > 0$ *and* $\theta > 0$.

(a) *If* X *is in Class* I, *then*

$$\lim_{t\to\infty} t^{3/2} E_x(e^{-\theta X(t)}, \tau > t) = \frac{a_1}{\sqrt{2\pi}\sigma}\,\bar{V}(x-)\int_0^\infty e^{-\theta z}V(z)dz\,.$$

(b) *If* X *is in Class* III, *then*

$$\lim_{t\to\infty} t^{3/2} E_x(e^{-\theta X(t)}, \tau > t) = \frac{a_1 r}{\sqrt{2\pi}\sigma}\,\bar{V}(x-)\sum_{k\in r\mathbf{Z}_+} e^{-\theta(k+j(x))}V(k)\,.$$

Here $a_1 = \exp\{\int_0^\infty (e^{-t}-1)t^{-1}P(X_t = 0)dt\}$ *and* $j(x) = x - rk$ *if* $rk < x \le r(k+1)$, $k \in \mathbf{Z}_+$.

For the proof of Lemma A, we need a series of lemmas. The following one is close to Lemma 2.2 in [14].

LEMMA 9. *Let* $t \ge 0$ *and* $\mu_t,\ \eta_t : [0,\infty)^d \to [0,\infty)$ *be right continuous and non-decreasing functions. Suppose that, for fixed* $x \in [0,\infty)^d$, $\mu(x) := \int_0^\infty \mu_t(x)dt < \infty$, $\mu_t(x) = O(t^{-1})$ *as* $t\to\infty$ *and* $\eta(x) := \lim_{t\to\infty}\sqrt{t}\,\eta_t(x) < \infty$. *Then we have*

$$\lim_{t\to\infty}\sqrt{t}\int_0^t \mu_s * \eta_{t-s}(x)ds = \mu * \eta(x)\,,$$

where $*$ *denotes convolution in* $[0,\infty)^d$.

PROOF. Fix $x \in [0,\infty)^d$ and $\varepsilon \in (0,1)$. By assumptions, there exists $a = a(x)$ such that $\mu_t(x) \le at^{-1}$ and $\eta_t(x) \le at^{-1/2}$. Using these estimates, we have

$$\begin{aligned}t^{1/2}\int_{(1-\varepsilon)t}^{t}\mu_s*\eta_{t-s}(x)ds &= t^{1/2}\int_{(1-\varepsilon)t}^{t}ds\int_{0\le y\le x}\eta_{t-s}(x-y)d\mu_s(y)\\ &\le a^2t^{1/2}\int_{(1-\varepsilon)t}^{t}(t-s)^{-1/2}s^{-1}ds\\ &\le a^2(1-\varepsilon)^{-1}t^{-1/2}\int_{(1-\varepsilon)t}^{t}(t-s)^{-1/2}ds\\ &= 2a^2(1-\varepsilon)^{-1}\varepsilon^{1/2}.\end{aligned}$$

If $s\in[0,(1-\varepsilon)t]$ and $y\le x$, we have $t^{1/2}\eta_{t-s}(x-y)\le a\varepsilon^{-1/2}$. Hence, by the dominated convergence theorem,

$$\begin{aligned}\lim_{t\to\infty}t^{1/2}\int_0^{(1-\varepsilon)t}\mu_s*\eta_{t-s}(x)ds &= \int_0^\infty ds\int_{0\le y\le x}\eta(x-y)d\mu_s(y)\\ &= \int_{0\le y\le x}\eta(x-y)d\mu(y)=\mu*\eta(x),\end{aligned}$$

which shows the lemma. □

For ease of notation, we set

$$\begin{aligned}\mu_t(x,y) &= P(M_t\le x, M_t-X_t\le y),\\ \eta_t(x,y) &= P(0\le X_t\le x)1_{\{y\ge 0\}}+1_{\{x\ge 0\}}P(0<\bar{X}_t\le y).\end{aligned}$$

Let $F_{u,v}(t)$ and $G_{u,v}(t)$ denote their double Laplace transforms, i.e., for $u, v>0$,

$$\begin{aligned}F_{u,v}(t) &= E(e^{-uM_t-v(M_t-X_t)}),\\ G_{u,v}(t) &= E(e^{-uX_t}, X_t\ge 0)+E(e^{vX_t}, X_t<0).\end{aligned}$$

The following results are immediate consequences of the formulae (5) and (6).

LEMMA 10. *Let $x\ge 0$ and $y\ge 0$.*

(1) $t\mu_t(x,y)=\int_0^t\mu_s*\eta_{t-s}(x,y)ds, \quad \forall t\ge 0.$

(2) $\int_0^\infty\mu_t(x,y)dt=a_1V(x)\bar{V}(y)$.

PROOF. Using (5), $F_{u,v}$ and $G_{u,v}$, we rewrite (6) as

$$z\int_0^\infty e^{-zt}F_{u,v}(t)dt=\exp\left[\int_0^\infty\frac{e^{-zt}}{t}(G_{u,v}(t)-1)dt\right].$$

Differentiating the above with respect to z, we have

$$\begin{aligned}\int_0^\infty e^{-zt}tF_{u,v}(t)dt &= \int_0^\infty e^{-zt}F_{u,v}(t)dt\int_0^\infty e^{-zt}G_{u,v}(t)dt\\ &= \int_0^\infty e^{-zt}\Big(\int_0^t F_{u,v}(s)G_{u,v}(t-s)ds\Big)dt.\end{aligned}$$

By the uniqueness theorem for Laplace transform we have

$$tF_{u,v}(t) = \int_0^t F_{u,v}(s)G_{u,v}(t-s)ds\,, \qquad \forall t \geq 0\,, \tag{a.1}$$

since both sides are right continuous in $t \geq 0$. Thus inverting the Laplace transform in the variables (u, v) shows (1). Let $z > 0$. By (5), we have

$$\begin{aligned}\kappa(z,0)\bar{\kappa}(z,0) &= \exp\left[\int_0^\infty \frac{e^{-t}-e^{-zt}}{t}(1+P(X_t=0))dt\right] \\ &= z\exp\left[\int_0^\infty \frac{e^{-t}-e^{-zt}}{t}P(X_t=0)dt\right].\end{aligned}$$

Thus we have

$$\lim_{z\to 0}\kappa(z,0)\bar{\kappa}(z,0)/z = a_1\,.$$

Dividing both sides in (6) by z and then letting $z \to 0$, we have

$$\int_0^\infty F_{u,v}(t)dt = \frac{a_1}{\kappa(0,u)\bar{\kappa}(0,v)} = a_1\int_0^\infty\int_0^\infty e^{-ux-vy}dV(x)d\bar{V}(y)\,. \tag{a.2}$$

The second equality follows from the identity $\int_0^\infty e^{-ux}dV(x) = \kappa(0,u)^{-1}$. Inverting the Laplace transform in the variables (u, v), we get (2). □

Henceforth we assume $EX_1 = 0$ and $0 < \sigma^2 < \infty$. The next lemma is an analogue of the well-known result of random walks. For this lemma, the reader had better refer to Bertoin [1, Theorem 18, p. 173], Feller [8, Theorem 1, p. 612] and their proofs.

LEMMA 11. *If $x > 0$ and $u > 0$, then, as $t \to \infty$,*

$$P(M_t \leq x) \sim a_2V(x)t^{-1/2} \quad \textit{and} \quad E(e^{-uM_t}) \sim a_2\kappa(0,u)^{-1}t^{-1/2}\,,$$

where $a_2 = (2/\pi\sigma^2)^{1/2}EH_1$.

The asymptotic behavior of η_t is given by the following lemma.

LEMMA 12. (a) *If X is in Class* I, *then, for all $x, y \geq 0$,*

$$\lim_{t\to\infty}\sqrt{t}\,\eta_t(x,y) = (x+y)/\sqrt{2\pi}\sigma\,.$$

(b) *If X is in Class* III, *then, for all $x, y \in r\mathbf{Z}_+$,*

$$\lim_{t\to\infty}\sqrt{t}\,\eta_t(x,y) = (x+y+r)/\sqrt{2\pi}\sigma\,.$$

We omit the proof of this lemma because we can show the following by applying the techniques used in Shepp [19] to Lévy processes. If X is in Class I and $x \leq y$, then

$$\lim_{t\to\infty}\sqrt{t}P(x \leq X_t \leq y) = (y-x)/\sqrt{2\pi}\sigma\,. \tag{a.3}$$

This comes from the non-lattice property of X_1. If X is in Class III and $k \in \mathbf{Z}$, then

$$\lim_{t\to\infty} \sqrt{t}P(X_t = rk) = r/\sqrt{2\pi}\sigma . \tag{a.4}$$

This corresponds to a local limit theorem of random walks. By these facts the lemma is proved. Moreover, if X is in Class II with $r = 1$, we see

$$(2\pi\sigma^2 t)^{1/2} P(x \le X_t \le y) = \#[x - \{bt\}, y - \{bt\}] + o(1) ,$$

where $\{x\}$ denotes the fractional part of x and $\#[x, y]$ is the number of integers in $[x, y]$. Since $b \neq 0$, the right hand side does not converge as $t \to \infty$, and hence nor does $\sqrt{t}\,\eta_t$. This is a reason why we removed Lévy processes in Class II from Lemma A.

For the later convenience we give the following. The Kolmogorov-Rogozin inequality $P(0 \le X_t \le x) \le \text{const.}(x+1)t^{-1/2}$, see [10] or [18], combined with (a.3) and (a.4) shows that for each $u > 0$ and $v > 0$,

$$\lim_{t\to\infty} \sqrt{t}G_{u,v}(t) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\sigma}\left(\dfrac{1}{u} + \dfrac{1}{v}\right), & \text{if } X \text{ is in Class I}, \\ \dfrac{r}{\sqrt{2\pi}\sigma}\left(\dfrac{1}{1-e^{-ur}} + \dfrac{1}{e^{vr}-1}\right), & \text{if } X \text{ is in Class III}. \end{cases} \tag{a.5}$$

In the proofs below we treat the case where X is in Class I. The proofs for X in Class III are similar. Collecting Lemmas 9 through 12, we get the following.

LEMMA 13. (a) *If X is in Class* I, *then, for all $x, y \ge 0$,*

$$\lim_{t\to\infty} t^{3/2}\mu_t(x, y) = \frac{a_1}{\sqrt{2\pi}\sigma}\left\{V(x)\int_0^y \bar{V}(s)ds + \int_0^x V(s)ds \cdot \bar{V}(y)\right\}. \tag{a.6}$$

(b) *If X is in Class* III, *then, for all $x, y \in r\mathbf{Z}_+$,*

$$\lim_{t\to\infty} t^{3/2}\mu_t(x, y) = \frac{a_1 r}{\sqrt{2\pi}\sigma}\left\{V(x)\sum_{j=0}^{y/r}\bar{V}(rj) + \sum_{j=0}^{x/r} V(rj)\cdot\bar{V}(y) - V(x)\bar{V}(y)\right\}.$$

PROOF. Let $x, y \ge 0$ be fixed. Recalling (1) of Lemma 10 and then using Lemmas 11 and 12 for the third inequality below, we have

$$\begin{aligned} t\mu_t(x, y) &\le \int_0^t \eta_{t-s}(x, y)\mu_s(x, y)ds \\ &\le \int_0^t \eta_{t-s}(x, y)P(M_s \le x)ds \\ &\le \text{const.}\int_0^t \frac{ds}{\sqrt{(t-s)s}} = \text{const.} \end{aligned}$$

According to (2) of Lemma 10, Lemma 12 and the above, μ_t and η_t satisfy the assumptions in Lemma 9. Hence by Lemma 9

$$\lim_{t\to\infty} t^{3/2}\mu_t(x, y) = \frac{a_1}{\sqrt{2\pi}\sigma}\int_0^x\int_0^y \{(x-z) + (y-w)\}dV(z)d\bar{V}(w) .$$

Integration by parts applied to the right hand side shows the lemma. □

Now we are in a final part of the proof of Lemma A. We first remark the following. If we use (a.1), (a.2) and (a.5) instead of Lemmas 10 and 12, calculations similar to the proof of Lemma 13 show that, for $u, v > 0$,

$$\lim_{t\to\infty} t^{3/2} F_{u,v}(t) = \frac{a_1}{\sqrt{2\pi}\sigma\,\kappa(0,u)\bar{\kappa}(0,v)}\left(\frac{1}{u}+\frac{1}{v}\right). \tag{a.7}$$

Let $z > 0$ and $\theta > 0$. Using duality in the second equality below, we see

$$\begin{aligned} E_z(e^{-\theta X(t)}, \tau > t) &= e^{-\theta z} E(e^{-\theta X(t)}, \inf_{0\le s\le t} X_s > -z) \\ &= e^{-\theta z} E(e^{-\theta X(t)}, M_t - X_t < z) \\ &= e^{-\theta z} \int_0^{\infty}\int_0^{z-} e^{-\theta(x-y)} d\mu_t(x,y). \end{aligned}$$

Set $\mu(x,y) = \lim_{t\to\infty} t^{3/2}\mu_t(x,y)$. Then, for $k > 0$,

$$\lim_{t\to\infty} t^{3/2} \int_0^{k}\int_0^{z-} e^{-\theta(x-y)} d\mu_t(x,y) = \int_0^{k}\int_0^{z-} e^{-\theta(x-y)} d\mu(x,y).$$

If $\lambda > 0$ and $y \le z$, then $e^{\theta y} \le e^{(\lambda+\theta)z-\lambda y}$ and $\int_k^{\infty}\int_0^{z-} e^{-\theta(x-y)} d\mu_t(x,y)$ is less than const.$e^{-\theta k/2} F_{\theta/2,\lambda}(t)$. Taking this and (a.7) into account, we have

$$\limsup_{k\to\infty} \limsup_{t\to\infty} t^{3/2} \int_k^{\infty}\int_0^{z-} e^{-\theta(x-y)} d\mu_t(x,y) = 0.$$

Combining the three results above, we get

$$\lim_{t\to\infty} t^{3/2} E_z(e^{-\theta X(t)}, \tau > t) = e^{-\theta z} \int_0^{\infty}\int_0^{z-} e^{-\theta(x-y)} d\mu(x,y).$$

Putting the right hand side of (a.6) in the place of $\mu(x,y)$ and then applying the integration by parts, we establish Lemma A.

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