TOKYO J. MATH.
VOL. 24, NO. 1, 2001

A Diffusion Process with a One-Sided Brownian Potential

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(Communicated by Y. Maeda)

Introduction.

Let $\mathbf{W}$ be the space of continuous functions w defined in $\mathbf{R}$ and vanishing identically on $[0, \infty)$. We denote by P the Wiener measure on $\mathbf{W}$, namely, P is the probability measure on $\mathbf{W}$ such that $\{w(-x), x \geq 0, P\}$ is a Brownian motion with time parameter x. Let $\Omega = \mathbf{C}([0, \infty); \mathbf{R})$ and write $X(t) = X(t, \omega) = \omega(t)$, where $\omega(t)$ is the value of $\omega(\in \Omega)$ at time t. Given $w \in \mathbf{W}$ and $x_0 \in \mathbf{R}$ we denote by $P_w^{x_0}$ the probability measure on Ω such that $\{X(t), t \geq 0, P_w^{x_0}\}$ is a diffusion process with generator

$$\mathcal{L}_w = \frac{1}{2} e^{w(x)} \frac{d}{dx} \left(e^{-w(x)} \frac{d}{dx} \right)$$

starting from x_0. Let $\mathcal{P}^{x_0}$ be the probability measure on $\mathbf{W} \times \Omega$ defined by

$$\mathcal{P}^{x_0}(dwd\omega) = P(dw) P_w^{x_0}(d\omega) .$$

The process $\{X(t), t \geq 0, \mathcal{P}^{x_0}\}$ is regarded as defined on the probability space $(\mathbf{W} \times \Omega, \mathcal{P}^{x_0})$, which we call a diffusion process with a one-sided Brownian potential. We are interested in the limiting behavior of $\{X(t), t \geq 0, \mathcal{P}^0\}$ as $t \to \infty$.

Our present model is a variant of the Brox-Schumacher diffusion ([1], [9]) that was introduced as a diffusion analogue of Sinai's random walk ([10]). When $w(x)$ does not vanish identically for $x \geq 0$, or more precisely speaking, when $\{w(x), x \geq 0, P\}$ and $\{w(-x), x \geq 0, P\}$ are independent Brownian motions, Brox [1] and Schumacher [9] proved that $\{(\log t)^{-2} X(t), t \geq 0, \mathcal{P}^0\}$ has a nondegenerate limit distribution. This result was extended to the case of a considerably wider class of (asymptotically) self-similar random environments by Kawazu, Tamura and Tanaka ([6], [7]). See [12] for a survey of results concerning diffusion processes in random environments. In our present model the random environment is self-similar but does not belong to the class of random environments of [6] because

Received January 4, 2000
Revised June 2, 2000
[1,2,3] Partially supported by Grant-in-Aid for Scientific Research No. 11640122([1]), 10740058([2]) and 11640137([3]), the Ministry of Education, Science and Culture, Japan.

of the non-existence of valleys containing 0. The result for the present model is much different from those of [1], [6], [7]. In fact, for the diffusion $\{X(t), t \geq 0, \mathcal{P}^0\}$ with a one-sided Brownian potential the limit distribution of $t^{-1/2}X(t)$ as $t \to \infty$ exists and is given by

$$\frac{1}{2}\sqrt{\frac{2}{\pi}}e^{-x^2/2}dx + \frac{1}{2}\delta_0(dx),$$

the support being $[0, \infty)$. This result shows that the long-term behavior of $X(t)$ is *diffusive* (in the sense that a limit distribution exists under the Brownian scaling) with probability $1/2$ and *subdiffusive* with the rest probability $1/2$.

Our model may look simpler than those studied previously but the result obtained will suggest the difficulty of anticipating the due result for the long-term behavior of a diffusion process with a random potential consisting of two independent strictly stable processes with different exponents for the right and the left hand sides of the origin.

We state our result in a more precise form. We put

$$X_\lambda(t) = \lambda^{-1/2}X(\lambda t), \quad t \geq 0,$$

for a constant $\lambda > 0$ and introduce two probability laws P_N and P_R on Ω as follows:

$P_N =$ the probability law of the process vanishing identically (the probability measure in Ω concentrated at the null path),

$P_R =$ the probability law of the reflecting Brownian motion on $[0, \infty)$ starting from 0.

Denote by $\mathcal{M}$ the space of probability laws on Ω and let ρ be the Prokhorov metric on $\mathcal{M}$. We also denote by $P_\lambda(w)$ the probability law of the process $\{X_\lambda(t), t \geq 0, P_w^0\}$. Thus P_N, P_R and $P_\lambda(w)$ are elements of $\mathcal{M}$. Our main result is then stated as follows.

THEOREM 1. *For any ε such that $0 < \varepsilon < \rho(P_N, P_R)/2$*

$$\lim_{\lambda\to\infty} P\{\rho(P_\lambda(w), P_N) < \varepsilon\} = \frac{1}{2}, \tag{0.1a}$$

$$\lim_{\lambda\to\infty} P\{\rho(P_\lambda(w), P_R) < \varepsilon\} = \frac{1}{2}. \tag{0.1b}$$

In particular, the following (0.2) *and* (0.3) *hold*:

$$\lim_{t\to\infty} \mathcal{P}^0\{-\varepsilon < t^{-1/2}X(t) \leq x\} = \frac{1}{2} + \frac{1}{2}\int_0^x \sqrt{\frac{2}{\pi}}e^{-y^2/2}dy, \quad x > 0, \ \varepsilon > 0. \tag{0.2}$$

$$\begin{aligned} \lim_{t\to\infty} \mathcal{P}^0\left\{0 \leq t^{-1/2}\max_{0\leq s\leq t} X(s) \leq x\right\} &= \frac{1}{2} + \frac{1}{2}P_R\left\{\max_{0\leq s\leq 1} X(s) \leq x\right\} \\ &= \frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\exp\left\{-\frac{(2n+1)^2\pi^2}{8x^2}\right\}, \quad x > 0. \end{aligned} \tag{0.3}$$

As for the minimum process of $X(t)$ we have the following result (Theorem 2), which is quite different from the result (0.3) for the maximum process. For $w \in \mathbf{W}$ and $a \in \mathbf{R}$ we put

$$\sigma(a) = \sigma(a, w) = \sup\{x < 0 : w(x) = a\}, \tag{0.4}$$

$$\zeta = \zeta(w) = \sup\left\{x < 0 : w(x) - \min_{x \le y \le 0} w(y) = 1\right\}, \tag{0.5}$$

$$M = M(w) = \begin{cases} \sigma(1/2), & \text{if } \sigma(-1/2) < \sigma(1/2), \\ \zeta(w), & \text{if } \sigma(1/2) < \sigma(-1/2). \end{cases} \tag{0.6}$$

Here we may take w from a suitable subset of $\mathbf{W}$ that has a full P-measure to avoid unpleasant cases such as $\sigma(1/2) = -\infty$, $\zeta(w) = -\infty$, etc.

THEOREM 2. (i) $\{(\log t)^{-2} \min_{0 \le s \le t} X(s), \mathcal{P}^0\}$ *converges in law to* $\{M, P\}$ *as* $t \to \infty$. (ii) $-M$ *is identical in law to the exit time from the interval* $[0, 2]$ *of a standard* 1-*dimensional Brownian motion starting from* $1/2$, *namely,*

$$P\{M < x\} = \begin{cases} \dfrac{4}{\pi} \displaystyle\sum_{n=0}^{\infty} \frac{1}{2n+1} \exp\left\{\frac{(2n+1)^2}{8} \pi^2 x\right\} \sin \frac{(2n+1)\pi}{4}, & x < 0, \\ 1, & x \ge 0. \end{cases} \tag{0.7}$$

1. Preliminaries.

Let $\lambda > 0$ be fixed. For $w \in \mathbf{W}$ and $x_0 \in \mathbf{R}$ let $P_{\lambda w}^{x_0}$ be the probability measure on Ω such that $\{X(t), t \ge 0, P_{\lambda w}^{x_0}\}$ is a diffusion process with generator

$$\mathcal{L}_{\lambda w} = \frac{1}{2} e^{\lambda w(x)} \frac{d}{dx} \left(e^{-\lambda w(x)} \frac{d}{dx} \right)$$

starting from x_0. Denote by $E_{\lambda w}^{x_0}$ the expectation with respect to $P_{\lambda w}^{x_0}$. Such a diffusion process can be constructed as follows ([3]). Let $(\tilde{\Omega}, \tilde{P})$ be a probability space, and let $B(t)$, $t \ge 0$, be a 1-dimensional Brownian motion starting from 0 defined on $(\tilde{\Omega}, \tilde{P})$. Put

$$L(t, x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{[x, x+\varepsilon)}(B(s)) ds \quad \text{(local time)},$$

$$S_\lambda(x) = \int_0^x e^{\lambda w(y)} dy, \quad x \in \mathbf{R},$$

$$A_\lambda(t) = \int_0^t e^{-2\lambda w(S_\lambda^{-1}(B(s)))} ds = \int_{\mathbf{R}} e^{-2\lambda w(S_\lambda^{-1}(x))} L(t, x) dx, \quad t \ge 0,$$

$$X(t; 0, \lambda w) = S_\lambda^{-1}(B(A_\lambda^{-1}(t))), \quad t \ge 0, \tag{1.1}$$

where S_λ^{-1} and A_λ^{-1} denote the inverse functions. Then the process $X(t; 0, \lambda w)$, $t \ge 0$, defined on $(\tilde{\Omega}, \tilde{P})$ is a diffusion process with generator $\mathcal{L}_{\lambda w}$ starting from 0. Given $x_0 \in \mathbf{R}$ we define $w^{x_0} \in \mathbf{C}(\mathbf{R})$ by $w^{x_0}(\cdot) = w(\cdot + x_0)$, and put $X(t; x_0, \lambda w) = x_0 + X(t; 0, \lambda w^{x_0})$. Then $X(t; x_0, \lambda w)$, $t \ge 0$, is a diffusion process with generator $\mathcal{L}_{\lambda w}$ starting from x_0.

REMARK. (i) The notation $X(t; x_0, \lambda w)$ should not be confused with $X(t) = X(t, \omega)$; the former is defined on $\tilde{\Omega}$ and the latter on Ω or on $\mathbf{W} \times \Omega$. (ii) If $S_\lambda(x) \to -\infty (x \to -\infty)$, then the diffusion process $X(t; x_0, \lambda w)$ is recurrent and hence conservative. From now on we reduce the whole space $\mathbf{W}$ so that it equals the set of w satisfying $S_\lambda(x) \to -\infty (x \to -\infty)$ for all $\lambda > 0$, which has still a full P-measure. Thus $X(t; x_0, \lambda w)$ is always recurrent.

For $w \in \mathbf{W}$ define $w_\lambda \in \mathbf{W}$ by

$$w_\lambda(x) = \lambda^{-1} w(\lambda^2 x), \quad x \in \mathbf{R}.$$

Then we have

$$\{w_\lambda, P\} \stackrel{\mathrm{d}}{=} \{w, P\}, \tag{1.2}$$

where $\stackrel{\mathrm{d}}{=}$ means the equality in distribution. The following lemma is proved in [1].

LEMMA 1.1 ([1]). *For any $\lambda > 0$ and $w \in \mathbf{W}$,*

$$\{X(t; 0, \lambda w_\lambda), t \geq 0, \tilde{P}\} \stackrel{\mathrm{d}}{=} \{\lambda^{-2} X(\lambda^4 t; 0, w), t \geq 0, \tilde{P}\},$$

or equivalently,

$$\{X(t), t \geq 0, P^0_{\lambda w_\lambda}\} \stackrel{\mathrm{d}}{=} \{\lambda^{-2} X(\lambda^4 t), t \geq 0, P^0_w\}.$$

Note that $P^0_{\lambda w}$ is an element of $\mathcal{M}$. By (1.2) and Lemma 1.1, for the proof of Theorem 1 it is enough to show the following theorem.

THEOREM 1′. *For any ε such that* $0 < \varepsilon < \rho(P_N, P_R)/2$

$$\lim_{\lambda \to \infty} P\{\rho(P^0_{\lambda w}, P_N) < \varepsilon\} = \frac{1}{2},$$

$$\lim_{\lambda \to \infty} P\{\rho(P^0_{\lambda w}, P_R) < \varepsilon\} = \frac{1}{2}.$$

For the proof of Theorem 1′ we first calculate concretely the limit, as $\lambda \to \infty$, of the Laplace transform of the distribution of the hitting time to $a > 0$ for the process $\{X(t), t \geq 0, P \otimes P^0_{\lambda w}\}$ in Section 2. As a result we see the limit is half of the corresponding quantity for the reflecting Brownian motion on $[0, \infty)$ starting from 0. By looking into carefully what this fact means, we shall arrive at Theorem 1′ after all. In Section 3 we explain a coupling method which is needed for clarifying our argument. In Section 4 we prove Theorem 1′ and in Section 5 we prove Theorem 2.

2. The Laplace transform of the distribution of a hitting time.

In this section we examine the limit, as $\lambda \to \infty$, of the Laplace transform of the distribution of the hitting time to $a > 0$ for the process $\{X(t), t \geq 0, P \otimes P^0_{\lambda w}\}$, and compare the limit with the corresponding quantity for the reflecting Brownian motion on $[0, \infty)$ starting from 0.

Let $\Omega^+ = \mathbf{C}([0, \infty); [0, \infty))$ and write $X^+(t) = X^+(t, \omega^+) = \omega^+(t)$, where $\omega^+(t)$ is the value of $\omega^+ (\in \Omega^+)$ at time t. Given $x \geq 0$ we denote by P^x_R the probability measure on

Ω^+ such that $\{X^+(t), t \geq 0, P_R^x\}$ is a reflecting Brownian motion on $[0, \infty)$ starting from x, and by E_R^x the expectation with respect to P_R^x. For $\omega^+ \in \Omega^+$ we put

$$\tau^+(a) = \tau^+(a, \omega^+) = \inf\{t > 0 : X^+(t) = a\}, \quad a > 0. \tag{2.1}$$

The following lemma is well-known.

LEMMA 2.1. *For $\xi > 0$ and $a > 0$,*

$$E_R^x\{e^{-\xi\tau^+(a)}\} = \frac{e^{\sqrt{2\xi}x} + e^{-\sqrt{2\xi}x}}{e^{\sqrt{2\xi}a} + e^{-\sqrt{2\xi}a}}, \quad 0 \leq x \leq a.$$

We also put, for $\omega \in \Omega$,

$$\tau(a) = \tau(a, \omega) = \inf\{t > 0 : X(t) = a\}, \quad a \in \mathbf{R}. \tag{2.2}$$

The main result in this section is the following.

PROPOSITION 2.2. *For $\xi > 0$ and $a > 0$,*

$$\lim_{\lambda\to\infty} E[E_{\lambda w}^0\{e^{-\xi\tau(a)}\}] = \frac{1}{e^{\sqrt{2\xi}a} + e^{-\sqrt{2\xi}a}} = \frac{1}{2}E_R^0\{e^{-\xi\tau^+(a)}\}. \tag{2.3}$$

To prove Proposition 2.2, we prepare some lemmas. First we derive Kotani's formula (see [5]) in our case.

LEMMA 2.3. *For $\xi > 0$ and $a > 0$,*

$$E_{\lambda w}^0\{e^{-\xi\tau(a)}\} = \exp\left\{-\int_0^a U_\xi^\lambda(x)dx\right\}, \quad P\text{-a.s.}, \tag{2.4}$$

where $U_\xi^\lambda(x)$ is a positive solution of

$$dU_\xi^\lambda(x) = (2\xi - U_\xi^\lambda(x)^2)dx, \quad x > 0, \tag{2.5}$$

$$dU_\xi^\lambda(x) = \lambda U_\xi^\lambda(x)dw(x) + \{2\xi + (\lambda^2/2)U_\xi^\lambda(x) - U_\xi^\lambda(x)^2\}dx, \quad x < 0. \tag{2.6}$$

Moreover $U_\xi^\lambda(x), x < 0$, is a unique stationary positive solution of (2.6) *and $U_\xi^\lambda(x)$ is continuous at $x = 0$.*

PROOF. We follow the proof of Kotani's formula in [5]. For $b \leq 0$ and $c > 0$ put

$$u(x) = 1/E_{\lambda w}^b\{e^{-\xi\tau(x)}\}, \quad b < x,$$
$$v(x) = E_{\lambda w}^x\{e^{-\xi\tau(c)}\}, \quad b < x < c.$$

Then we have

$$\begin{aligned} E_{\lambda w}^b\{e^{-\xi\tau(c)}\} &= E_{\lambda w}^b\{e^{-\xi\tau(x)}\}E_{\lambda w}^x\{e^{-\xi\tau(c)}\} \\ &= v(x)/u(x), \quad b < x < c. \end{aligned}$$

Since $\mathcal{L}_{\lambda w}v(x) = \xi v(x), b < x < c$, $u(x)$ also satisfies

$$\mathcal{L}_{\lambda w}u(x) = \xi u(x), \quad b < x. \tag{2.7}$$

Namely $u(x)$ satisfies

$$u''(x) = 2\xi u(x), \quad x > 0, \tag{2.8}$$

$$\frac{d}{dx}\{e^{-\lambda w(x)}u'(x)\} = 2\xi e^{-\lambda w(x)}u(x), \quad b < x < 0. \tag{2.9}$$

If we put $U_\xi^\lambda(x) = (\log u(x))' = u'(x)/u(x)$, then $U_\xi^\lambda(x) > 0$. Let us compute the stochastic differential $dU_\xi^\lambda(x)$. Using (2.8), we have

$$dU_\xi^\lambda(x) = (2\xi - U_\xi^\lambda(x)^2)dx, \quad x > 0.$$

On the other hand, for $b < x < 0$ we have, by (2.9),

$$\begin{aligned} dU_\xi^\lambda(x) &= d(e^{-\lambda w(x)}u'(x)e^{\lambda w(x)}u(x)^{-1}) \\ &= e^{\lambda w(x)}u(x)^{-1}d(e^{-\lambda w(x)}u'(x)) + e^{-\lambda w(x)}u'(x)d(e^{\lambda w(x)}u(x)^{-1}) \\ &= \lambda U_\xi^\lambda(x)dw(x) + \{2\xi + (\lambda^2/2)U_\xi^\lambda(x) - U_\xi^\lambda(x)^2\}dx. \end{aligned}$$

In the above we fixed $b \le 0$. But we see $U_\xi^\lambda(x)$ does not depend on b. Therefore (2.6) holds for all $x < 0$. Since $u'(x)$ is continuous at $x = 0$ by (2.7), $U_\xi^\lambda(x)$ is also continuous at $x = 0$. The last assertion for $U_\xi^\lambda(x)$, $x < 0$, can be proved in the same way as in [5].

LEMMA 2.4. *For the solution $U_\xi^\lambda(x)$ of (2.5) under $U_\xi^\lambda(0) = \eta$,*

$$\exp\left\{-\int_0^a U_\xi^\lambda(x)dx\right\} = \frac{2\sqrt{2\xi}e^{\sqrt{2\xi}a}}{(e^{2\sqrt{2\xi}a} - 1)\eta + \sqrt{2\xi}(e^{2\sqrt{2\xi}a} + 1)}. \tag{2.10}$$

PROOF. The solution of the differential equation

$$\begin{cases} \dfrac{dU_\xi^\lambda(x)}{dx} = 2\xi - U_\xi^\lambda(x)^2, & x > 0, \\ U_\xi^\lambda(0) = \eta, \end{cases}$$

is

$$U_\xi^\lambda(x) = \sqrt{2\xi}\,\frac{\eta + \sqrt{2\xi} + (\eta - \sqrt{2\xi})e^{-2\sqrt{2\xi}x}}{\eta + \sqrt{2\xi} - (\eta - \sqrt{2\xi})e^{-2\sqrt{2\xi}x}}. \tag{2.11}$$

Calculating the left-hand side of (2.10) by using (2.11), we obtain the lemma.

The generator of the diffusion process $U_\xi^\lambda(x)$, $x < 0$, appearing in Lemma 2.3 is $\mathcal{L}_\xi^\lambda = d/m_\xi^\lambda(d\eta) \cdot d/dS_\xi^\lambda(\eta)$, where

$$m_\xi^\lambda(d\eta) = \frac{2}{\lambda^2}\exp\left\{-\frac{4\xi}{\lambda^2\eta} - \frac{2\eta}{\lambda^2}\right\}\frac{d\eta}{\eta}, \tag{2.12}$$

$$S_\xi^\lambda(\eta) = \int_1^\eta \exp\left\{\frac{4\xi}{\lambda^2\zeta} + \frac{2\zeta}{\lambda^2}\right\}\frac{d\eta}{\zeta}, \quad \lambda > 0, \; \xi > 0. \tag{2.13}$$

It is guaranteed that $U_\xi^\lambda(0) \equiv \lim_{x\uparrow 0} U_\xi^\lambda(x)$ is a Borel function of w by Theorem 18 of [4]. The distribution of $U_\xi^\lambda(0)$ is

$$\mu_\xi^\lambda(d\eta) = m_\xi^\lambda(d\eta)/m_\xi^\lambda([0,\infty)).$$

LEMMA 2.5. *Let $\xi > 0$ be fixed. If μ_ξ^λ is regarded as a probability measure on* $[0,\infty]$, *then*

$$\mu_\xi^\lambda \to \frac{1}{2}\delta_0 + \frac{1}{2}\delta_\infty \quad \textit{weakly as } \lambda \to \infty.$$

PROOF. For $\varepsilon > 0$ we have, as $\lambda \to \infty$,

$$\int_\varepsilon^\infty \exp\left\{-\frac{4\xi}{\lambda^2\eta} - \frac{2\eta}{\lambda^2}\right\}\frac{d\eta}{\eta} \sim \int_\varepsilon^\infty \exp\left\{-\frac{2\eta}{\lambda^2}\right\}\frac{d\eta}{\eta} = \int_{2\varepsilon/\lambda^2}^\infty \exp\{-\eta\}\frac{d\eta}{\eta} \sim 2\log\lambda, \tag{2.14}$$

$$\int_0^\varepsilon \exp\left\{-\frac{4\xi}{\lambda^2\eta} - \frac{2\eta}{\lambda^2}\right\}\frac{d\eta}{\eta} \sim \int_0^\varepsilon \exp\left\{-\frac{4\xi}{\lambda^2\eta}\right\}\frac{d\eta}{\eta} = \int_{4\xi/(\lambda^2\varepsilon)}^\infty \exp\{-\eta\}\frac{d\eta}{\eta} \sim 2\log\lambda, \tag{2.15}$$

because for any $a > 0$

$$\int_{a/\lambda^2}^\infty \exp\{-\eta\}\frac{d\eta}{\eta} \sim 2\log\lambda \quad \text{as } \lambda \to \infty.$$

Moreover, for any $0 < \varepsilon < A < \infty$

$$\int_\varepsilon^A \exp\left\{-\frac{4\xi}{\lambda^2\eta} - \frac{2\eta}{\lambda^2}\right\}\frac{d\eta}{\eta} < \log\frac{A}{\varepsilon}, \tag{2.16}$$

which remains bounded as $\lambda \to \infty$. By (2.14) $\sim$ (2.16), we obtain for any $0 < \varepsilon < A < \infty$

$$\lim_{\lambda\to\infty} \mu_\xi^\lambda([0,\varepsilon]) = \lim_{\lambda\to\infty} \mu_\xi^\lambda([A,\infty]) = \frac{1}{2}.$$

Hence the lemma is proved.

PROOF OF PROPOSITION 2.2. By Lemma 2.3, Lemma 2.4 and recalling that the distribution of $U_\xi^\lambda(0)$ is μ_ξ^λ, we have

$$E[E_{\lambda w}^0\{e^{-\xi\tau(a)}\}] = 2\sqrt{2\xi}e^{\sqrt{2\xi}a}\int_0^\infty \frac{1}{(e^{2\sqrt{2\xi}a}-1)\eta + \sqrt{2\xi}(e^{2\sqrt{2\xi}a}+1)}\mu_\xi^\lambda(d\eta). \tag{2.17}$$

Put

$$f(\eta) = \frac{1}{(e^{2\sqrt{2\xi}a}-1)\eta + \sqrt{2\xi}(e^{2\sqrt{2\xi}a}+1)}, \quad \eta \in [0,\infty].$$

Then f is a continuous function on $[0, \infty]$. Therefore, by (2.17) and Lemma 2.5, we obtain

$$\lim_{\lambda\to\infty} E[E^0_{\lambda w}\{e^{-\xi\tau(a)}\}] = 2\sqrt{2\xi}e^{\sqrt{2\xi}a}\{f(0)/2 + f(\infty)/2\}$$
$$= \frac{1}{e^{\sqrt{2\xi}a} + e^{-\sqrt{2\xi}a}}.$$

3. A coupling method.

In this section we explain a coupling method which plays an important role for the proof of Theorem 1′.

First we prepare a lemma. Given $x \in \mathbf{R}$, a continuous function $\beta(t)$ defined in $[0, \infty)$ with $\beta(0) = 0$ and a Lipschitz continuous function $b(x)$ defined in $\mathbf{R}$, we denote by $Y(t, x)$ the solution of

$$Y(t) = x + \beta(t) + \int_0^t b(Y(s))ds\,, \quad t \geq 0\,. \tag{3.1}$$

Given $y \geq 0$ we also consider the solution $Y^+(t, y)$ of the Skorohod problem

$$Y^+(t) = y + \beta(t) + \int_0^t b(Y^+(s))ds + \varphi(t)\,, \quad t \geq 0\,, \tag{3.2}$$

in which $Y^+(t)$ is to be found under the following conditions:

(3.3) $Y^+(t) \geq 0$.

(3.4) $\varphi(t)$ is continuous, nondecreasing and $\varphi(0) = 0$.

(3.5) $\varphi(t)$ is constant on each connected component of $\{t > 0 : Y^+(t) > 0\}$.

It is well-known that the Skorohod problem (3.2) has a unique solution (e.g. see [11] or [8]). The following lemma is essentially the same as Lemma 3 of [8].

LEMMA 3.1. *If $x \leq y$, then*

$$Y(t, x) \leq Y^+(t, y) \quad \textit{for all } t \geq 0\,. \tag{3.6}$$

PROOF. Let $\varepsilon > 0$ be an arbitrary constant. Let $Y^+_\varepsilon(t, y)$ be the solution of the Skorohod problem

$$Y^+(t) = y + \beta(t) + \int_0^t \{b(Y^+(s)) + \varepsilon\}ds + \psi(t)\,, \tag{3.7}$$

and let us prove

$$Y(t, x) \leq Y^+_\varepsilon(t, y) \quad \text{for all } t \geq 0\,, \tag{3.8}$$

from which (3.6) follows by letting $\varepsilon \downarrow 0$. Define

$$T = \inf\{t > 0 : Y(t, x) > Y^+_\varepsilon(t, y)\}$$

with convention $\inf \phi = \infty$ and suppose $T < \infty$. We put

$$\xi_1(t) = Y(T+t, x), \quad \varphi_1(t) = \varphi(T+t) - \varphi(T),$$
$$\xi_2(t) = Y_\varepsilon^+(T+t, y), \quad \varphi_2(t) = \psi(T+t) - \psi(T).$$

Then

(3.9) $\xi_1(0) = \xi_2(0) \geq 0$,

(3.10) there exist $t_n > 0, n = 1, 2, \cdots$, such that t_n tends to 0 and $\xi_1(t_n) > \xi_2(t_n)$.

Moreover, $\xi_1(t)$ is the solution of

$$\xi_1(t) = \xi_1(0) + \beta(T+t) - \beta(T) + \int_0^t b(\xi_1(s))ds, \tag{3.11}$$

and $\xi_2(t)$ is the solution of the Skorohod problem

$$\xi_2(t) = \xi_2(0) + \beta(T+t) - \beta(T) + \int_0^t \{b(\xi_2(s)) + \varepsilon\}ds + \varphi_2(t). \tag{3.12}$$

From (3.9), (3.11) and (3.12) it follows that $\xi_1(t) \leq \xi_2(t)$ for all sufficiently small $t > 0$. But this contradicts (3.10). Therefore $T = \infty$ and this proves (3.8).

Let $\lambda > 0$ and $w \in \mathbf{W}$ be fixed. We consider a sequence $\{w_n, n = 1, 2, \cdots\}$ such that

(3.13) $w_n, n \geq 1$, are C^2-functions in $\mathbf{R}$ with the bounded second derivatives and satisfying $w_n(x) = 0$ for $x \geq 0$,

(3.14) w_n converges to w as $n \to \infty$ uniformly on each finite inerval.

From the way of the construction of the process $X(t; x, \lambda w)$, $t \geq 0$, described in Section 1, it follows that

(3.15) for each x, $X(t; x, \lambda w_n)$ converges to $X(t; x, \lambda w)$ as $n \to \infty$ uniformly on each finite t-interval.

Hence

$$P_{\lambda w_n}^x \quad \text{converges to} \quad P_{\lambda w}^x \quad \text{as } n \to \infty. \tag{3.16}$$

Now we construct a coupled process of a diffusion process with generator $\mathcal{L}_{\lambda w}$ starting from $x \in \mathbf{R}$ and a reflecting Brownian motion on $[0, \infty)$ starting from $y \geq 0$ on a suitable probability space $(\bar{\Omega}, \bar{P})$ by using stochastic differential equations. Let $\beta(t), t \geq 0$, be a 1-dimensional Brownian motion starting from 0 defined on $(\bar{\Omega}, \bar{P})$. For $x \in \mathbf{R}$ consider the stochastic differential equation

$$Y_n(t) = x + \beta(t) - \frac{1}{2}\int_0^t \lambda w_n'(Y_n(s))ds, \quad t \geq 0. \tag{3.17}$$

Since w_n' is a Lipschitz continuous function, the equation (3.17) has a unique strong solution. The solution $Y_n(t), t \geq 0$, of (3.17) is a diffusion process with generator $\mathcal{L}_{\lambda w_n}$ starting from x so its probability law is $P_{\lambda w_n}^x$. Notice that $w_n'(x) = 0$ for $x \geq 0$ and consider, for $y \geq 0$, the Skorohod problem

$$Y^+(t) = y + \beta(t) + \varphi(t), \quad t \geq 0. \tag{3.18}$$

The solution $Y^+(t)$, $t \geq 0$, of (3.18) is a reflecting Brownian motion on $[0, \infty)$ starting from y. We see that

(3.19) the coupled process $(Y_n(t), Y^+(t))$, $t \geq 0$, is a 2-dimensional diffusion process such that each component process is also a 1-dimensional diffusion.

If $x \leq y$, then we get, by Lemma 3.1,

$$Y_n(t) \leq Y^+(t) \quad \text{for all } t \geq 0. \tag{3.20}$$

Let $\hat{\Omega} = \mathbf{C}([0, \infty); \mathbf{R} \times [0, \infty))$. An element $\hat{\omega}$ of $\hat{\Omega}$ can be expressed as $\hat{\omega} = (\omega, \omega^+)$, where $\omega \in \Omega = \mathbf{C}([0, \infty); \mathbf{R})$ and $\omega^+ \in \Omega^+ = \mathbf{C}([0, \infty); [0, \infty))$. We denote by $\hat{\omega}(t) = (\omega(t), \omega^+(t))$ the value of $\hat{\omega}$ at time t and write

$$X(t) = \omega(t), \quad X^+(t) = \omega^+(t),$$
$$\hat{X}(t) = \hat{\omega}(t) = (X(t), X^+(t)).$$

We introduce a right continuous filtration $\{\hat{\mathfrak{F}}_t\}$ on $\hat{\Omega}$ by

$$\hat{\mathfrak{F}}_t = \bigcap_{\varepsilon > 0} \sigma\{\hat{X}(s); 0 \leq s \leq t + \varepsilon\}.$$

LEMMA 3.2. *Let $\lambda > 0$ and $w \in \mathbf{W}$ be fixed, and assume $x \in \mathbf{R}$ and $y \geq 0$ satisfy $x \leq y$. Then there exists a probability measure $Q_{\lambda w}$ on $\hat{\Omega}$ with the following properties* (3.21), (3.22) *and* (3.23).

(3.21) *The projections (marginal distributions) of $Q_{\lambda w}$ on the subspaces Ω and Ω^+ are $P^x_{\lambda w}$ and P^y_R, respectively.*

(3.22) *$X(t) \leq X^+(t)$ for all $t \geq 0$, $Q_{\lambda w}$-a.s.*

(3.23) *Each of $\{X(t), t \geq 0, Q_{\lambda w}\}$ and $\{X^+(t), t \geq 0, Q_{\lambda w}\}$ has the strong Markov property with respect to the filtration $\{\hat{\mathfrak{F}}_t\}$; more precisely, for any bounded continuous f and g defined in $\mathbf{R}$ and in $[0, \infty)$ respectively and for any $\{\hat{\mathfrak{F}}_t\}$-stopping time τ*

(i) $E_{Q_{\lambda w}}\{f(X(\tau + s)) \,|\, \hat{\mathfrak{F}}_\tau\} = F(X(\tau))$, *$Q_{\lambda w}$-a.s. on* $\{\tau < \infty\}$,

(ii) $E_{Q_{\lambda w}}\{g(X^+(\tau + s)) \,|\, \hat{\mathfrak{F}}_\tau\} = G(X^+(\tau))$, *$Q_{\lambda w}$-a.s. on* $\{\tau < \infty\}$, *where*

$$F(\cdot) = E^{\cdot}_{\lambda w}\{f(X(s))\}, \quad G(\cdot) = E^{\cdot}_R\{g(X^+(s))\},$$

and $E_{Q_{\lambda w}}$ denotes the expectation with respect to $Q_{\lambda w}$.

PROOF. Taking a sequence $\{w_n, n = 1, 2, \cdots\}$ with the properties (3.13) and (3.14), we consider the probability measure $Q_{\lambda w_n}$ on $\hat{\Omega}$ such that the process $\{\hat{X}(t), t \geq 0, Q_{\lambda w_n}\}$ is identical in law to the process $\{(Y_n(t), Y^+(t)), t \geq 0, \bar{P}\}$. Then (3.16) implies that the sequence of probability measures $Q_{\lambda w_n}$, $n = 1, 2, \cdots$, is tight. Therefore there exist $n_1 < n_2 < \cdots$ such that $Q_{\lambda w_n}$ converges to some probability measure $Q_{\lambda w}$ on $\hat{\Omega}$ as $n \to \infty$ via the sequence $\{n_k\}$. It is clear that $Q_{\lambda w}$ satisfies (3.21). Since the property (3.20) is inherited by the process $\{\hat{X}(t), t \geq 0, Q_{\lambda w_n}\}$, it is also inherited by the process $\{\hat{X}(t), t \geq 0, Q_{\lambda w}\}$; namely we have (3.22). By (3.19), we see that each of the processes $\{X(t), t \geq 0, Q_{\lambda w_n}\}$ and $\{X^+(t), t \geq 0, Q_{\lambda w_n}\}$ has the Markov property with respect to $\{\hat{\mathfrak{F}}_t\}$. By virtue of (3.16) it is easy to see that this property is also shared by each of the processes $\{X(t), t \geq 0, Q_{\lambda w}\}$ and

$\{X^+(t), t \geq 0, Q_{\lambda w}\}$. By a routine argument we can also prove the strong Markov property with respect to $\{\hat{\mathfrak{F}}_t\}$ as stated in (3.23).

4. Proof of Theorem 1′.

First we state a lemma due to Brox ([1]). Let $w \in \mathbf{W}$, $a < c < 0$ and denote by $w_{[a,c]}$ the restriction of w on $[a, c]$. We call $w_{[a,c]}$ a valley if there exists $b \in (a, c)$ and if the following conditions (i) and (ii) are satisfied.

(i) $$w(a) > w(x) > w(b) \quad \text{for any } x \in (a, b),$$
$$w(c) > w(x) > w(b) \quad \text{for any } x \in (b, c).$$

(ii) $$w(a) - w(b) > \sup\{w(y) - w(x) : b < y < x < c\},$$
$$w(c) - w(b) > \sup\{w(y) - w(x) : a < x < y < b\}.$$

We call $D = \{w(a) - w(b)\} \wedge \{w(c) - w(b)\}$ the depth of the valley $w_{[a,c]}$.

LEMMA 4.1 (Brox [1]: Lemma 3.1). *If $w_{[a,c]}$ is a valley with depth D, then, for each closed interval $I \subset (a, c)$ and for any $\varepsilon > 0$,*

$$\lim_{\lambda \to \infty} P^x_{\lambda w}\{e^{\lambda(D-\varepsilon)} < \tau(a, c) < e^{\lambda(D+\varepsilon)}\} = 1,$$

uniformly in $x \in I$, where $\tau(a, c)$ is the exit time of $X(t)$ from (a, c).

If we put

$$\mathbf{W}_0 = \left\{ w \in \mathbf{W} : \begin{array}{l} \min_{-\varepsilon \leq x \leq 0} w(x) < 0 < \max_{-\varepsilon \leq x \leq 0} w(x) \text{ for any } \varepsilon > 0, \text{ and} \\ w \text{ can not have the same value of local minimum} \\ \text{at distinct points in } (-\infty, 0) \end{array} \right\},$$

then $P\{\mathbf{W}_0\} = 1$. Let $w \in \mathbf{W}_0$ be fixed. Then for any $\varepsilon > 0$ there exist a and c such that $-\varepsilon < a < c < 0$ and $w(a) = w(c) > w(x)$ for any $x \in (a, c)$. From Lemma 4.1 we have

(4.1) $$\lim_{\lambda \to \infty} P^x_{\lambda w}\{\tau(a, c) > e^{\lambda D/2}\} = 1, \quad x \in (a, c),$$

where D is the depth of the valley $w_{[a,c]}$, and from (4.1) we easily obtain the following lemma.

LEMMA 4.2. *Suppose $w \in \mathbf{W}_0$. Then for any $\varepsilon > 0$ and $T > 0$*

$$\lim_{\lambda \to \infty} P^x_{\lambda w}\left\{\min_{0 \leq t \leq T} X(t) > -\varepsilon\right\} = 1, \quad x \in [0, \infty).$$

By Lemma 4.2, for $w \in \mathbf{W}_0$ and $n \in \mathbf{N}$, we have

$$\lim_{\lambda \to \infty} P^0_{\lambda w}\left\{\min_{0 \leq t \leq n} X(t) > -\frac{1}{n}\right\} = 1.$$

Define

$$\Lambda_n(w) = \inf\left\{\Lambda > 0 : P^0_{\lambda w}\left(\min_{0 \leq t \leq n} X(t) > -\frac{1}{n}\right) > 1 - \frac{1}{n} \text{ for all } \lambda > \Lambda\right\},$$

with convention $\inf \phi = \infty$, and

$$\Gamma_{\lambda,n} = \{w \in \mathbf{W}_0 : \Lambda_n(w) < \lambda\}.$$

For each $n \in \mathbf{N}$ we have

$$\lim_{\lambda\to\infty} P\{\Gamma_{\lambda,n}\} = 1.$$

Choose a positive sequence $\{\lambda_n, n = 1, 2, \cdots\}$ satisfying $\lambda_n \uparrow \infty$ as $n \to \infty$ and

$$P\{\Gamma_{\lambda_n,n}\} > 1 - \frac{1}{n}.$$

For $\lambda \geq \lambda_1$ define

$$n(\lambda) = \max\{n \geq 1 : \lambda_n \leq \lambda\}, \quad \Gamma_\lambda = \Gamma_{\lambda_{n(\lambda)},n(\lambda)}.$$

Since $n(\lambda) \to \infty$ as $\lambda \to \infty$, we have

$$\lim_{\lambda\to\infty} P\{\Gamma_\lambda\} = 1.$$

We easily obtain the following lemma.

LEMMA 4.3. *Suppose $w \in \Gamma_\lambda$. Then*

$$P^0_{\lambda w}\left\{\min_{0\leq t\leq n(\lambda)} X(t) > -\frac{1}{n(\lambda)}\right\} > 1 - \frac{1}{n(\lambda)}.$$

To proceed we introduce two subsets A_λ and B_λ of Γ_λ by

$$A_\lambda = \left\{w \in \Gamma_\lambda : U^\lambda_1(0) < \frac{1}{\log\lambda}\right\}, \quad B_\lambda = \{w \in \Gamma_\lambda : U^\lambda_1(0) > \log\lambda\}.$$

Then the following lemma can be obtained in the same way as in the proof of Lemma 2.5.

LEMMA 4.4. $\lim_{\lambda\to\infty} P\{A_\lambda\} = \lim_{\lambda\to\infty} P\{B_\lambda\} = \frac{1}{2}$.

In the following $Q_{\lambda w}$ denotes the probability measure on $\hat{\Omega}$ which corresponds to $x = y = 0$ in Lemma 3.2. Namely we consider the coupled process of the diffusion process with generator $\mathcal{L}_{\lambda w}$ starting from 0 and the reflecting Brownian motion on $[0, \infty)$ starting from 0. For $a > 0$ and $x \geq 0$, we put

$$f(a,x) = \frac{e^{\sqrt{2}x} + e^{-\sqrt{2}x}}{e^{\sqrt{2}a} + e^{-\sqrt{2}a}}, \quad g(a,x) = \frac{2\sqrt{2}e^{\sqrt{2}a}}{(e^{2\sqrt{2}a} - 1)x + \sqrt{2}(e^{2\sqrt{2}a} + 1)}.$$

Then

$$f(a,0) = g(a,0). \tag{4.2}$$

By Lemma 2.1, Lemma 2.3 and Lemma 2.4, we have for $a > 0$

$$E^x_R\{e^{-\tau^+(a)}\} = f(a,x), \quad 0 \leq x \leq a, \tag{4.3}$$

$$E^0_{\lambda w}\{e^{-\tau(a)}\} = g(a, U^\lambda_1(0)). \tag{4.4}$$

PROPOSITION 4.5. *For any $a > 0$ and $\varepsilon > 0$ there exists $K_{a,\varepsilon} > 0$ such that*

$$Q_{\lambda w}\left\{\max_{0\le t\le T}|X^+(t) - X(t)| > \varepsilon\right\} \le e^T\left\{\frac{K_{a,\varepsilon}}{\log\lambda} + \frac{1}{n(\lambda)} + 2e^{-\sqrt{2}a}\right\} \tag{4.5}$$

holds provided that $w \in A_\lambda$, $n(\lambda) > T$ and $n(\lambda) > 2/\varepsilon$.

PROPOSITION 4.6. *Suppose $w \in B_\lambda$, $n(\lambda) > T$ and $n(\lambda) > 1/\varepsilon$. Then*

$$P^0_{\lambda w}\left\{\max_{0\le t\le T}|X(t)| > \varepsilon\right\} \le e^T g(\varepsilon, \log\lambda) + \frac{1}{n(\lambda)}. \tag{4.6}$$

Combining Lemma 4.4 with Propositions 4.5 and 4.6, we obtain Theorem 1′. For the proof of Proposition 4.5, we prepare two lemmas. We use the notation $\tau(a)$ defined in (2.2) both for $\{X(t), t \ge 0, P^x_{\lambda w}\}$ and $\{X(t), t \ge 0, Q_{\lambda w}\}$. Similarly we use the notation $\tau^+(a)$ defined in (2.1) both for $\{X^+(t), t \ge 0, P^x_R\}$ and $\{X^+(t), t \ge 0, Q_{\lambda w}\}$.

LEMMA 4.7. *For any $a > 0$ there exists $C_a > 0$ such that*

$$E_{Q_{\lambda w}}\{e^{-\tau^+(a)}\} - E_{Q_{\lambda w}}\{e^{-\tau(a)}\} \le \frac{C_a}{\log\lambda} \quad \text{for all } w \in A_\lambda. \tag{4.7}$$

PROOF. By Lemma 3.2 and (4.2)~(4.4), the left-hand side of (4.7) is equal to

$$E^0_R\{e^{-\tau^+(a)}\} - E^0_{\lambda w}\{e^{-\tau(a)}\} = g(a, 0) - g(a, U^\lambda_1(0)). \tag{4.8}$$

Since $w \in A_\lambda$, the right-hand side of (4.8) is dominated by

$$\begin{aligned} g(a,0) - g(a, 1/\log\lambda) &= \frac{2e^{\sqrt{2}a}(e^{2\sqrt{2}a} - 1)}{(e^{2\sqrt{2}a} + 1)\{(e^{2\sqrt{2}a} - 1) + \sqrt{2}(e^{2\sqrt{2}a} + 1)\log\lambda\}} \\ &\le \frac{\sqrt{2}e^{\sqrt{2}a}(e^{2\sqrt{2}a} - 1)}{(e^{2\sqrt{2}a} + 1)^2\log\lambda}, \end{aligned}$$

which completes the proof.

For $\varepsilon > 0$ we put

$$\tau = \inf\{t > 0 : X^+(t) - X(t) = \varepsilon\}. \tag{4.9}$$

LEMMA 4.8. *For any $w \in \mathbf{W}$ and $a > 0$,*

$$\begin{aligned} &E_{Q_{\lambda w}}[e^{-\tau}\{f(a, X^+(\tau)) - f(a, X(\tau)\vee 0)\}; \tau < \tau^+(a)] \\ &\quad\le E_{Q_{\lambda w}}\{e^{-\tau^+(a)}\} - E_{Q_{\lambda w}}\{e^{-\tau(a)}\}. \end{aligned} \tag{4.10}$$

PROOF. First of all, by (3.22), we notice that for any $a > 0$

$$\tau^+(a) \le \tau(a) \quad Q_{\lambda w}\text{-a.s.}$$

Using (3.23) (i), we have

$$\begin{aligned} E_{Q_{\lambda w}}\{e^{-\tau(a)}\} =& E_{Q_{\lambda w}}[e^{-\tau}E^x_{\lambda w}\{e^{-\tau(a)}\}|_{x=X(\tau)}; \tau < \tau^+(a)] \\ &+ E_{Q_{\lambda w}}[e^{-\tau(a)}; \tau \ge \tau^+(a)]. \end{aligned} \tag{4.11}$$

By (3.22), the right-hand side of (4.11) is dominated by

$$(4.12)\quad E_{Q_{\lambda w}}[e^{-\tau}E_R^x\{e^{-\tau^+(a)}\}|_{x=X(\tau)\vee 0};\tau<\tau^+(a)]+E_{Q_{\lambda w}}[e^{-\tau^+(a)};\tau\ge\tau^+(a)]$$
$$=E_{Q_{\lambda w}}[e^{-\tau}E_R^x\{e^{-\tau^+(a)}\}|_{x=X^+(\tau)};\tau<\tau^+(a)]+E_{Q_{\lambda w}}[e^{-\tau^+(a)};\tau\ge\tau^+(a)]$$
$$-E_{Q_{\lambda w}}[e^{-\tau}\{E_R^x\{e^{-\tau^+(a)}\}|_{x=X^+(\tau)}-E_R^x\{e^{-\tau^+(a)}\}|_{x=X(\tau)\vee 0}\};\tau<\tau^+(a)].$$

Using (3.23) (ii) and (4.3), we see that the right-hand side of (4.12) is equal to

$$E_{Q_{\lambda w}}\{e^{-\tau^+(a)}\}-E_{Q_{\lambda w}}[e^{-\tau}\{f(a,X^+(\tau))-f(a,X(\tau)\vee 0)\};\tau<\tau^+(a)].$$

Hence we obtain (4.10).

PROOF OF PROPOSITION 4.5. For any $a>0$ the left-hand side of (4.5) is dominated by

$$(4.13)\qquad Q_{\lambda w}\left\{\max_{0\le t\le T}|X^+(t)-X(t)|>\varepsilon,\tau^+(a)>T\right\}+Q_{\lambda w}\{\tau^+(a)\le T\}.$$

Using τ defined in (4.9), we can estimate the first term in (4.13) as follows:

$$(4.14)\quad Q_{\lambda w}\left\{\max_{0\le t\le T}|X^+(t)-X(t)|>\varepsilon,\tau^+(a)>T\right\}$$
$$\le Q_{\lambda w}\{\tau<T<\tau^+(a)\}$$
$$\le e^T E_{Q_{\lambda w}}\{e^{-\tau};\tau<\tau^+(a),\tau<T\}$$
$$\le e^T[E_{Q_{\lambda w}}\{e^{-\tau};\tau<\tau^+(a),\tau<T,X(\tau)>-\varepsilon/2\}+Q_{\lambda w}\{X(\tau)\le-\varepsilon/2,\tau<T\}].$$

In the case $X(\tau)>-\varepsilon/2$, we have $X^+(\tau)-(X(\tau)\vee 0)\ge\varepsilon/2$ and therefore

$$(4.15)\qquad f(a,X^+(\tau))-f(a,X(\tau)\vee 0)\ge f(a,\varepsilon/2)-f(a,0).$$

By (4.15), the expectation in the right-hand side of (4.14) can be estimated as follows:

$$(4.16)\quad E_{Q_{\lambda w}}\{e^{-\tau};\tau<\tau^+(a),\tau<T,X(\tau)>-\varepsilon/2\}$$
$$\le\frac{1}{f(a,\varepsilon/2)-f(a,0)}E_{Q_{\lambda w}}[e^{-\tau}\{f(a,X^+(\tau))-f(a,X(\tau)\vee 0)\};\tau<\tau^+(a)]$$
$$\le\frac{K_{a,\varepsilon}}{\log\lambda}.$$

Here $K_{a,\varepsilon}$ is a positive constant. The last inequality follows from Lemma 4.7 and Lemma 4.8 since $w\in A_\lambda$. As for the second term of the right-hand side of (4.14) we have

$$(4.17)\qquad Q_{\lambda w}\{X(\tau)\le-\varepsilon/2,\tau<T\}\le P_{\lambda w}^0\left\{\min_{0\le t\le T}X(t)\le-\varepsilon/2\right\}\le\frac{1}{n(\lambda)},$$

by Lemma 4.3, since $T<n(\lambda)$, $n(\lambda)>2/\varepsilon$ and $w\in\Gamma_\lambda$. The second term in (4.13) can be estimated as follows:

$$(4.18)\qquad Q_{\lambda w}\{\tau^+(a)\le T\}\le e^T E_{Q_{\lambda w}}\{e^{-\tau^+(a)}\}=e^T f(a,0)\le 2e^T e^{-\sqrt{2}a}.$$

By (4.13),(4.14) and (4.16)~(4.18), we obtain (4.5).

PROOF OF PROPOSITION 4.6. We have

$$P^0_{\lambda w}\left\{\max_{0\le t\le T} X(t) > \varepsilon\right\} \le P^0_{\lambda w}\{\tau(\varepsilon) < T\} \le e^T E^0_{\lambda w}\{e^{-\tau(\varepsilon)}\} = e^T g(\varepsilon, U^\lambda_1(0)). \tag{4.19}$$

Since $w \in B_\lambda$, the right-hand side of (4.19) is dominated by $e^T g(\varepsilon, \log\lambda)$. Combining this with Lemma 4.3, we obtain (4.6).

5. Proof of Theorem 2.

We begin with the following lemma.

LEMMA 5.1. *Let $w \in \mathbf{W}$ and $a < 0$. Assume $w(a) > w(x)$ for all $x > a$. Then for any $\varepsilon > 0$*

$$\lim_{\lambda\to\infty} P^0_{\lambda w}\{e^{\lambda(J-\varepsilon)} < \tau(a) < e^{\lambda(J+\varepsilon)}\} = 1, \tag{5.1}$$

where $\tau(a)$ is defined by (2.2) *and*

$$J = \max\{J_0, 2w(a)\}, \quad J_0 = w(a) - \min\{w(x) : x \ge a\}.$$

PROOF. Recalling the notation in Section 1, we set

$$\tau(a; 0, \lambda w) = \inf\{t > 0 : X(t; 0, \lambda w) = a\},$$

which is defined on the probability space $(\tilde\Omega, \tilde P)$. Since $\{\tau(a), P^0_{\lambda w}\}$ is identical in law to $\{\tau(a; 0, \lambda w), \tilde P\}$, (5.1) is equivalent to

$$\lim_{\lambda\to\infty} \tilde P\{e^{\lambda(J-\varepsilon)} < \tau(a; 0, \lambda w) < e^{\lambda(J+\varepsilon)}\} = 1. \tag{5.2}$$

We employ the method of [1] to prove (5.2). Let

$$T(z) = \inf\{t > 0 : B(t) = z\}, \quad z \in \mathbf{R}.$$

Then the expression (1.1) yields

$$\begin{aligned}\tau(a; 0, \lambda w) &= A_\lambda(T(S_\lambda(a))) \\ &= \int_{\mathbf{R}} e^{-2\lambda w(S_\lambda^{-1}(x))} L(T(S_\lambda(a)), x)dx \\ &= \int_a^\infty e^{-\lambda w(y)} L(T(S_\lambda(a)), S_\lambda(y))dy.\end{aligned} \tag{5.3}$$

The self-similarity of the Brownian motion $B(t)$ implies that for each fixed $b \in \mathbf{R}$ and $c > 0$ the process $\{L(T(cb), cu)\}$ is identical in law to $\{cL(T(b), u)\}$, where $u \in [b, \infty)$ is considered as a time parameter. Using this scaling relation with $b = -1$ and $c = |S_\lambda(a)|$, we have

$$\{L(T(S_\lambda(a)), |S_\lambda(a)|u), u \ge -1, \tilde P\} \stackrel{\mathrm{d}}{=} \{|S_\lambda(a)|L(T(-1), u), u \ge -1, \tilde P\}.$$

Introducing a new time y via $u = S_\lambda(y)/|S_\lambda(a)|$, we have

$$\{L(T(S_\lambda(a)), S_\lambda(y)), y \geq a, \tilde{P}\} \stackrel{d}{=} \{|S_\lambda(a)|L(T(-1), S_\lambda(y)/|S_\lambda(a)|), y \geq a, \tilde{P}\},$$

and hence (5.3) yields

$$\begin{aligned}(5.4)\quad & \tau(a; 0, \lambda w)\\ &\stackrel{d}{=} |S_\lambda(a)| \int_a^\infty e^{-\lambda w(y)} L\left(T(-1), \frac{S_\lambda(y)}{|S_\lambda(a)|}\right) dy\\ &= |S_\lambda(a)| \int_a^0 e^{-\lambda w(y)} L\left(T(-1), \frac{S_\lambda(y)}{|S_\lambda(a)|}\right) dy + |S_\lambda(a)| \int_0^\infty L\left(T(-1), \frac{y}{|S_\lambda(a)|}\right) dy\\ &= \int_a^0 \int_a^0 e^{\lambda w(x,y)} L\left(T(-1), \frac{S_\lambda(y)}{|S_\lambda(a)|}\right) dxdy + S_\lambda(a)^2 \int_0^\infty L(T(-1), x)dx\\ &= I_\lambda + II_\lambda ,\end{aligned}$$

where $w(x, y) = w(x) - w(y)$. Since $S_\lambda(y)/|S_\lambda(a)|$ tends to 0 as $\lambda \to \infty$ uniformly on any closed interval contained in $(a, 0]$,

$$L(T(-1), S_\lambda(y)/|S_\lambda(a)|) \to L(T(-1), 0) > 0 \quad (\tilde{P}\text{-a.s.})$$

as $\lambda \to \infty$ uniformly on any closed interval contained in $(a, 0]$. Therefore by the classical Laplace method we have

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log I_\lambda = \max_{a\leq x\leq 0,\, a\leq y\leq 0} w(x, y) = J_0 , \quad \tilde{P}\text{-a.s.},$$

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log II_\lambda = 2 \max_{a\leq x\leq 0} w(x) = 2w(a) , \quad \tilde{P}\text{-a.s.}$$

Therefore from (5.4) we have

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log \tau(a; 0, \lambda w) = \max\{J_0, 2w(a)\} = J ,$$

in probability with respect to $\tilde{P}$. This proves the lemma.

We now recall the definition of M in (0.6) and set

$$\tau(M) = \inf\{t > 0 : X(t) = M\} .$$

PROPOSITION 5.2. *For any* $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P^0_{\lambda w}\{e^{\lambda(1-\varepsilon)} < \tau(M) < e^{\lambda(1+\varepsilon)}\} = 1 , \quad P\text{-a.s.} \tag{5.5}$$

PROOF. First we consider the case $\sigma(-1/2) < \sigma(1/2)$. In this case $M = \sigma(1/2)$, $w(M) = 1/2$ and $w(M) > w(x)$ for all $x > M$. Therefore we can apply Lemma 5.1 with $a = M$ and $J = 1$ to obtain (5.5).

Next we consider the case $\sigma(-1/2) > \sigma(1/2)$. In this case $M = \zeta$. Define σ_1 in $(\zeta, 0)$ by $w(\sigma_1) = \min\{w(x) : \zeta \leq x \leq 0\}$ and then σ_2 in $(\sigma_1, 0)$ by $w(\sigma_2) = \max\{w(x) : \sigma_1 \leq x \leq 0\}$. If $w(\zeta) > w(\sigma_2)$, then an application of Lemma 5.1 with $a = \zeta$ and $J = 1$ immediately implies (5.5). Therefore from now on we assume $w(\zeta) < w(\sigma_2)$. We take $\sigma_3 \in (\sigma_1, \sigma_2)$ and define $\tilde{w} \in \mathbf{W}$ by

$$\tilde{w}(x) = \begin{cases} w(x), & \text{for } x \geq \sigma_3, \\ -x + w(\sigma_3) + \sigma_3, & \text{for } x < \sigma_3. \end{cases}$$

Next take $\sigma_4 \in (-\infty, \sigma_3)$ in such a way that

$$\begin{cases} \tilde{w}(\sigma_2) < \tilde{w}(\sigma_4) < 1/2, \\ \tilde{J} \equiv \{\tilde{w}(\sigma_4) - \min_{\sigma_4 \leq x \leq 0} \tilde{w}(x)\} \vee \{2\tilde{w}(\sigma_4)\} < 1. \end{cases} \tag{5.6}$$

When $\sigma_2 < \sigma(-1/2)$, such a σ_4 exists provided that σ_3 is close to σ_2 enough; when $\sigma_2 > \sigma(-1/2)$, such a σ_4 exists provided that $\sigma_3 \in (\sigma(-1/2), \sigma_2)$. With σ_4 taken in this way we now apply Lemma 5.1. Then for any $\varepsilon > 0$

$$\lim_{\lambda \to \infty} P^0_{\lambda\tilde{w}}\{\tau(\sigma_4) < e^{\lambda(\tilde{J}+\varepsilon)}\} = 1. \tag{5.7}$$

Since

$$\begin{aligned} P^0_{\lambda\tilde{w}}\{\tau(\sigma_4) < e^{\lambda(\tilde{J}+\varepsilon)}\} &\leq P^0_{\lambda\tilde{w}}\{\tau(\sigma_3) < e^{\lambda(\tilde{J}+\varepsilon)}\} \\ &= P^0_{\lambda w}\{\tau(\sigma_3) < e^{\lambda(\tilde{J}+\varepsilon)}\}, \end{aligned}$$

(5.7) implies

$$\lim_{\lambda \to \infty} P^0_{\lambda w}\{\tau(\sigma_3) < e^{\lambda J'}\} = 1 \quad \text{for some } J' < 1. \tag{5.8}$$

On the other hand we see that $w_{[\zeta, \sigma_2]}$ is a valley with depth 1. Therefore by Lemma 4.1 we have

$$\lim_{\lambda \to \infty} P^{\sigma_3}_{\lambda w}\{e^{\lambda(1-\varepsilon)} < \tau(\zeta, \sigma_2) < e^{\lambda(1+\varepsilon)}\} = 1. \tag{5.9}$$

But the inequality $w(x) < w(\sigma_2)$ for all $x \in [\zeta, \sigma_2)$ implies

$$P^{\sigma_3}_{\lambda w}\{\tau(\zeta) < \tau(\sigma_2)\} = \frac{\int_{\sigma_3}^{\sigma_2} e^{\lambda w(x)} dx}{\int_{\zeta}^{\sigma_2} e^{\lambda w(x)} dx} \to 1, \quad \text{as } \lambda \to \infty,$$

so (5.9) yields

$$\lim_{\lambda \to \infty} P^{\sigma_3}_{\lambda w}\{e^{\lambda(1-\varepsilon)} < \tau(\zeta) < e^{\lambda(1+\varepsilon)}\} = 1. \tag{5.10}$$

Taking account of (5.8) and (5.10), we can obtain (5.5) by a routine use of the strong Markov property of $\{X(t), P^{\cdot}_{\lambda w}\}$. The proof of the proposition is finished.

We proceed to the proof of Theorem 2. First we notice that Proposition 5.2 can be rephrased as

$$P^0_{\lambda w}\left\{\min_{t \leq e^{\lambda(1+\varepsilon)}} X(t) < M(w) < \min_{t \leq e^{\lambda(1-\varepsilon)}} X(t)\right\} \to 1, \quad P\text{-a.s.},$$

as $\lambda \to \infty$. Therefore, by (1.2),

$$P^0_{\lambda w_\lambda}\left\{\min_{t \leq e^{\lambda(1+\varepsilon)}} X(t) < M(w_\lambda) < \min_{t \leq e^{\lambda(1-\varepsilon)}} X(t)\right\} \to 1,$$

in probability with respect to P as $\lambda \to \infty$. Therefore, by using Lemma 1.1, we obtain for any $\varepsilon > 0$

$$\lim_{\lambda\to\infty} \mathcal{P}^0 \left\{ \min_{t\le e^{\lambda(1+\varepsilon)}} \lambda^{-2} X(\lambda^4 t) < M(w_\lambda) < \min_{t\le e^{\lambda(1-\varepsilon)}} \lambda^{-2} X(\lambda^4 t) \right\} = 1 . \tag{5.11}$$

But it is easy to see that the validness of (5.11) for all $\varepsilon > 0$ is equivalent to the validness of

$$\lim_{\lambda\to\infty} \mathcal{P}^0 \left\{ \min_{t\le e^{\lambda(1+\varepsilon)}} \lambda^{-2} X(t) < M(w_\lambda) < \min_{t\le e^{\lambda(1-\varepsilon)}} \lambda^{-2} X(t) \right\} = 1 \tag{5.12}$$

for all $\varepsilon > 0$. For $\lambda > 0$ and $0 < \varepsilon < 1$ we put

$$U_{\lambda,\varepsilon} = \lambda^{-2} \min\{X(t) : 0 \le t \le e^{\lambda(1+\varepsilon)}\}, \quad V_{\lambda,\varepsilon} = \lambda^{-2} \min\{X(t) : 0 \le t \le e^{\lambda(1-\varepsilon)}\} .$$

Then, by (5.12), we have

$$\lim_{\lambda\to\infty} \mathcal{P}^0\{U_{\lambda,\varepsilon} < M(w_\lambda) < V_{\lambda,\varepsilon}\} = 1 . \tag{5.13}$$

Next we prove that for any $\delta > 0$

$$\lim_{\varepsilon\downarrow 0} \limsup_{\lambda\to\infty} \mathcal{P}^0\{V_{\lambda,\varepsilon} - U_{\lambda,\varepsilon} > \delta\} = 0 . \tag{5.14}$$

Take $c > 1$ and then $\varepsilon > 0$ so small that $c(1-\varepsilon) > 1+\varepsilon$. Then, because $U_{\lambda,\varepsilon} \ge c^2 V_{c\lambda,\varepsilon}$ and $V_{\lambda,\varepsilon} \le c^{-2} U_{\lambda/c,\varepsilon}$, we have

$$\begin{aligned} \mathcal{P}^0\{V_{\lambda,\varepsilon} - U_{\lambda,\varepsilon} > \delta\} &\le \mathcal{P}^0\{c^{-2} U_{\lambda/c,\varepsilon} - c^2 V_{c\lambda,\varepsilon} > \delta\} \\ &\le P\{c^{-2} M(w_{\lambda/c}) - c^2 M(w_{c\lambda}) > \delta\} + \mathcal{P}^0\{U_{\lambda/c,\varepsilon} > M(w_{\lambda/c})\} + \mathcal{P}^0\{M(w_{c\lambda}) > V_{c\lambda,\varepsilon}\} . \end{aligned}$$

Since $(w_{\lambda/c}, w_{c\lambda}) \overset{\mathrm{d}}{=} (w_{1/c}, w_c)$, we have

$$P\{c^{-2} M(w_{\lambda/c}) - c^2 M(w_{c\lambda}) > \delta\} = P\{c^{-2} M(w_{1/c}) - c^2 M(w_c) > \delta\} \to 0 ,$$

as $c \downarrow 1$. Moreover, by (5.13) we have

$$\mathcal{P}^0\{U_{\lambda/c,\varepsilon} > M(w_{\lambda/c})\} \to 0 , \quad \mathcal{P}^0\{M(w_{c\lambda}) > V_{c\lambda,\varepsilon}\} \to 0$$

as $\lambda \to \infty$. Therefore we have

$$\begin{aligned} &\limsup_{\varepsilon\downarrow 0} \limsup_{\lambda\to\infty} \mathcal{P}^0\{V_{\lambda,\varepsilon} - U_{\lambda,\varepsilon} > \delta\} \\ &\quad \le P\{c^{-2} M(w_{1/c}) - c^2 M(w_c) > \delta\} \to 0 , \quad \text{as } c \downarrow 1 , \end{aligned}$$

proving (5.14). Now it follows from (5.13) and (5.14) that

$$\lim_{\lambda\to\infty} \mathcal{P}^0 \left\{ \left| \lambda^{-2} \min_{t\le e^\lambda} X(t) - M(w_\lambda) \right| > \delta \right\} = 0 ,$$

which, by virtue of $M(w_\lambda) \overset{\mathrm{d}}{=} M(w)$, immediately implies the assertion (i) of Theorem 2.

Finally we prove the assertion (ii) of Theorem 2. We define ζ' by

$$\zeta' = \sup\{x < \sigma(-1/2) : w(x) - \min\{w(y) : x \le y \le \sigma(-1/2)\} = 1\}$$

and set $\zeta'' = \zeta' - \sigma(-1/2)$. Then (0.6) yields

$$M = \begin{cases} \sigma(1/2), & \text{if } \sigma(-1/2) < \sigma(1/2), \\ \sigma(-1/2) + \zeta'', & \text{if } \sigma(1/2) < \sigma(-1/2). \end{cases} \tag{5.15}$$

Moreover, ζ'' is independent of $\{w(x), \sigma(-1/2) \leq x \leq 0\}$ and $-\zeta''$ is identical in law to the exit time from $[-1, 1]$ for a Brownian motion starting from 0. Therefore the expression (5.15) shows that $-M$ is identical in law to the exit time from $[0, 2]$ for a Brownian motion starting from $1/2$. As for (0.7) see [2, p. 342], for example. The proof of Theorem 2 is finished.

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