TOKYO J. MATH.
VOL. 24, NO. 1, 2001

A Remark on Torsion Euler Classes of Circle Bundles

Shigeaki MIYOSHI

Komazawa University

(Communicated by K. Kobayasi)

Abstract. We show that any torsion class $e \in H^2(M; \mathbf{Z})$ of any closed manifold M is realized as the Euler class of a smoothly foliated orientable circle bundle over M. In the case where M is a 3-manifold, we construct the homomorphism $\pi_1(M) \to SO(2) \subset \mathrm{Diff}_+^\infty(S^1)$ explicitly whose Euler class is the given torsion class.

1. Introduction and statement of the result.

Let M be a closed orientable manifold and $\xi = \{E \to M\}$ an orientable circle bundle over M. We denote by $e(\xi)$ the Euler class of ξ. As is well known, orientable circle bundles are classified by their Euler classes. On the other hand, *foliated* orientable circle bundles are classified by their *total holonomy* homomorphisms. Namely, there is a natural bijection between the set of all smoothly (C^∞) foliated orientable circle bundles over M modulo leaf preserving bundle isomorphism and the set of all homomorphisms $\pi_1(M) \to \mathrm{Diff}_+^\infty(S^1)$ modulo conjugacy (cf. [HH]). Here, $\mathrm{Diff}_+^\infty(S^1)$ denotes the group of all orientation preserving diffeomorphisms of the circle. We consider a homomorphism $\pi_1(M) \to \mathrm{Diff}_+^\infty(S^1)$ as an equivalent of a smoothly foliated orientable circle bundle over M.

In [My] we studied the problem of the existence of a codimension-one foliation transverse to the fibers of a given circle bundle $E \to M$, that is, the question when is a circle bundle foliated, in the case where the base space M is a 3-manifold. In case the base space is a surface $M = \Sigma$, the necessary and sufficient condition for the existence of a transverse foliation was obtained by J. Milnor and J. W. Wood in [M] and [W]. Assume Σ is connected. Denote by $\chi(\xi)$ the Euler number of the circle bundle ξ and set $\chi_-(\Sigma) = \max\{0, -\chi(\Sigma)\}$, where $\chi(\Sigma)$ denotes the Euler characteristic of Σ. Then, there exists a transverse foliation if and only if $|\chi(\xi)| \le \chi_-(\Sigma)$. Here, Σ is a closed orientable surface and we omit the non-orientable case for simplicity. We call this inequality *Milnor-Wood inequality*. In higher dimensions, Milnor-Wood inequality induces a necessary condition for the existence as follows (cf. [M], [W], [My]): If there exists a transverse foliation on the total space E, then the following condition is satisfied:

(MW) : $|\langle e(\xi), z\rangle| \le x(z)$ for any $z \in H_2(M; \mathbf{Z})$.

Received December 8, 1999
Revised May 24, 2000

Here, $\langle\,,\,\rangle$ denotes Kronecker product and x is Thurston norm, that is, the pseudonorm on $H_2(M;\mathbf{Z})$ defined as follows: for any $z \in H_2(M;\mathbf{Z})$, $x(z)$ is defined to be the minimum $\chi_-(\Sigma)$ of all surfaces Σ in M each of which represents the given homology class z (cf. [Th]). In case Σ is not connected, here, we set $\chi_-(\Sigma) = \sum_i \max\{0, \chi_-(\Sigma_i)\}$ with respect to the decomposition into connected components $\Sigma = \coprod_i \Sigma_i$.

We showed in [My] there exists a family of circle bundles each of which has a transverse foliation of class C^0 but none of class C^3. Also we proved with some exceptions the condition (MW) is sufficient for the existence of a C^∞ transverse foliation if the base space is a closed Seifert fibred manifold.

In this paper, we consider the case where the condition (MW) is trivial. In fact, we show the following:

THEOREM. *Suppose $\xi = \{E \to M\}$ is an orientable circle bundle over a closed manifold M. The dimension of the base space M is arbitrary. If the Euler class $e(\xi)$ is a torsion class in $H^2(M;\mathbf{Z})$, then there exists a codimension-one C^∞ foliation on E which is transverse to the fibres. In fact, we can construct the transverse foliation whose total holonomy group is contained in $SO(2)$.*

In [M], Milnor showed that the Euler class of a flat $SO(m)$-bundle is a torsion element.

We will prove Theorem in §2. In §3, in connection with the results of the paper [My], we explicitly construct the homomorphism $\pi_1(M) \to SO(2) \subset \mathrm{Diff}^\infty_+(S^1)$ whose Euler class is the given torsion class if M is a 3-manifold.

2. Proof of Theorem.

In this section we prove Theorem. We identify the rotation group $SO(2)$ with $S^1 = \mathbf{R}/\mathbf{Z}$. Suppose that M is a closed orientable manifold. Consider the short exact sequence:

$$0 \to \mathbf{Z} \to \mathbf{R} \to SO(2) \to 0\,,$$

then we have the following exact sequence:

$$\cdots \to H^1(M;SO(2)) \xrightarrow{\beta} H^2(M;\mathbf{Z}) \to H^2(M;\mathbf{R}) \to H^2(M;SO(2)) \to \cdots\,,$$

where β is the Bockstein cohomology homomorphism corresponding to the coefficient sequence above (cf. [S]). By the exactness of the sequence above, we have

$$\begin{aligned}\mathrm{Im}(\beta) &= \mathrm{Ker}(H^2(M;\mathbf{Z}) \to H^2(M;\mathbf{R}))\\ &= \mathrm{Tor}(H^2(M;\mathbf{Z}))\,,\end{aligned}$$

where Tor denotes the torsion subgroup. On the other hand, any homomorphism $\varphi : \pi_1(M) \to SO(2)$ can be considered as a cohomology class of $H^1(M;SO(2))$ by the universal coefficient theorem

$$\begin{aligned}\mathrm{Hom}(\pi_1(M), SO(2)) &\cong \mathrm{Hom}(H_1(M), SO(2))\\ &\cong H^1(M;SO(2))\,.\end{aligned}$$

From now on, we identify these groups via these natural isomorphisms. Recall that a homomorphism $\pi_1(M) \to SO(2)$ is considered as an equivalent of a C^∞ foliated orientable circle bundle over M whose total holonomy is contained in $SO(2)$. Now we claim the following:

CLAIM (cf. [M]). The Euler class of a homomorphism $\varphi : \pi_1(M) \to SO(2)$ is equal to $-\beta(\varphi) \in H^2(M; \mathbf{Z})$.

PROOF OF CLAIM. First, note that a cochain $C_1(M) \to SO(2)$ is a cocycle if and only if its restriction to the boundaries $B_1(M)$ is zero. Suppose a homomorphism $\varphi : H_1(M; \mathbf{Z}) \to SO(2)$ is given. We will define a 1-cocycle which represents $\varphi \in \mathrm{Hom}(H_1(M; \mathbf{Z}), SO(2)) \cong H^1(M; SO(2))$. Since the short exact sequence of 1-cycles, 1-chains and 0-boundaries

$$0 \to Z_1(M) \to C_1(M) \xrightarrow{\partial} B_0(M) \to 0$$

is split, we have a direct sum decomposition $C_1(M) = Z_1(M) \oplus B$, where $B \subset C_1(M)$ is a subgroup isomorphic to $B_0(M)$. We define a cochain $c : C_1(M) \to SO(2)$ as $\varphi \cdot \pi$ on $Z_1(M)$ and 0 on B, where $\pi : Z_1(M) \to H_1(M; \mathbf{Z})$ is the natural quotient homomorphism. Then c is a cocycle, that is, $\delta c = 0$ and c represents the class $\varphi \in H^1(M; SO(2))$. For, we have $\delta c = c \cdot \partial = \varphi \cdot \pi \cdot \partial = \varphi \cdot \pi | B_1(M) = 0$ and also $c(\zeta) = \varphi \cdot \pi(\zeta) = \varphi[\zeta]$ for any $\zeta \in Z_1(M)$.

Next, we show that $-\beta(\varphi) = e(\varphi)$, where $\beta : H^1(M; SO(2)) \to H^2(M; \mathbf{Z})$ is the Bockstein homomorphism. Indeed, we will show that a representative cocycle of $-\beta(\varphi)$ also represents the primary obstruction class $e(\varphi)$ of the circle bundle defined by φ. First, the homomorphism β is defined through the snake diagram as follows:

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Hom}(C_1(M), \mathbf{Z}) & \to & \mathrm{Hom}(C_1(M), \mathbf{R}) & \to & \mathrm{Hom}(C_1(M), SO(2)) & \to 0 \\ & \downarrow \partial^* & & \downarrow \partial^* & & \downarrow \partial^* & \\ 0 \to & \mathrm{Hom}(C_2(M), \mathbf{Z}) & \to & \mathrm{Hom}(C_2(M), \mathbf{R}) & \to & \mathrm{Hom}(C_2(M), SO(2)) & \to 0 \end{array}$$

For any 1-cocycle $h \in \mathrm{Hom}(C_1(M), SO(2))$, there is $\tilde{h} \in \mathrm{Hom}(C_1(M), \mathbf{R})$ which maps to h. Since $\partial^* \tilde{h} \in \mathrm{Hom}(C_2(M), \mathbf{R})$ goes to 0 in $\mathrm{Hom}(C_2(M), SO(2))$, $\partial^* \tilde{h}$, in fact, lies in $\mathrm{Hom}(C_2(M), \mathbf{Z})$. Denote it by $g \in \mathrm{Hom}(C_2(M), \mathbf{Z})$. This 2-cochain g is a cocycle and $\beta[h]$ is defined to be the cohomology class of g.

Let $f : C_2(M) \to \mathbf{Z}$ be the 2-cocycle defined by chasing the snake diagram from the 1-cocycle $c \in \mathrm{Hom}(C_1(M), SO(2))$, the representative cocycle of φ. Thus, f represents $\beta[c] = \beta(\varphi)$. Indeed, the 2-cocycle f is defined as follows: Fix a triangulation of M and suppose $\pi_1(M) = \langle G | R \rangle$ is the presentation associated with the triangulation of M. Namely, each element of G corresponds to an oriented edge which is not contained in a fixed maximal tree and each word of R corresponds to an oriented 2-simplex. From now on we consider each generator $g \in G$ as an edge which is not contained in the maximal tree. Then, by the definition of c we have $c(g) = \varphi(g)$ for $g \in G$. Choose a lift $\widetilde{\varphi(g)} \in \mathbf{R}$ of $\varphi(g) \in SO(2)$ for each $g \in G$. We define a lift $\tilde{c} : C_1(M) \to \mathbf{R}$ of $c : C_1(M) \to SO(2)$ by setting $\tilde{c}(g) = \widetilde{\varphi(g)}$ for each $g \in G$. Note that for an edge contained in the maximal tree the value of $\tilde{c}$ is defined to be zero. By definition, $f = \partial^* \tilde{c}$. Now, let Δ be any oriented 2-simplex and suppose its boundary $\partial\Delta$ corresponds to $h_1^{\varepsilon_1}h_2^{\varepsilon_2}h_3^{\varepsilon_3}$ ($h_i \in G\cup\{1\}$, $\varepsilon_i = \pm 1$). Then since $\partial\Delta$ determines a word consists of the letters of G which belongs to (the normal closure of) R, $\varepsilon_1\tilde{c}(h_1)+\varepsilon_2\tilde{c}(h_2)+\varepsilon_3\tilde{c}(h_3) = \varepsilon_1\widetilde{\varphi(h_1)}+\varepsilon_2\widetilde{\varphi(h_2)}+\varepsilon_3\widetilde{\varphi(h_3)}$ is an integer. Thus, we have

$$\begin{aligned} f(\Delta) &= \partial^*\tilde{c}(\Delta) \\ &= \tilde{c}(\partial\Delta) \\ &= \varepsilon_1\tilde{c}(h_1)+\varepsilon_2\tilde{c}(h_2)+\varepsilon_3\tilde{c}(h_3) \\ &= \varepsilon_1\widetilde{\varphi(h_1)}+\varepsilon_2\widetilde{\varphi(h_2)}+\varepsilon_3\widetilde{\varphi(h_3)}\,. \end{aligned}$$

This implies that $-f$ is the Euler cocycle of φ. Q.E.D. of Claim.

Recall that $\mathrm{Im}(\beta)$ is the torsion subgroup of $H^2(M;\mathbf{Z})$. Thus, by this Claim the Euler class of a foliated circle bundle whose total holonomy $\pi_1(M) \to SO(2)$ is a torsion class and conversely any torsion class in $H^2(M;\mathbf{Z})$ can be the Euler class of a total holonomy $\pi_1(M) \to SO(2)$. Now the proof is completed.

3. Explicit construction in dimension three.

In the case of dimension three we explicitly construct the homomorphism $\pi_1(M) \to \mathrm{Diff}_+^\infty(S^1)$ which represents the given torsion class. In this section, every coefficient group of homology group is $\mathbf{Z}$.

First, we see how the Euler class of an orientable circle bundle describes the twist of the bundle. Suppose an orientable circle bundle $\xi = \{E \to M\}$ over a closed 3-manifold M is given. For simplicity we assume M is orientable. We choose an orientation on M and take an oriented embedded loop K in M which represents the Poincaré dual of $e(\xi)$. Denote by $\mathcal{E}_M(K)$ the exterior of K, that is, $\mathcal{E}_M(K) = M - \mathrm{int}N(K)$ where $N(K)$ denotes a small tubular neighbourhood of K in M. Since $e(\xi)|\mathcal{E}_M(K) = 0$, the restriction $\xi|\mathcal{E}_M(K)$ is trivial. Fix trivializations $\mathcal{E}_M(K)\times S^1 \cong E|\mathcal{E}_M(K)$ and $N(K)\times S^1 \cong E|N(K)$. Then the gluing diffeomorphism $g : \partial(\mathcal{E}_M(K))\times S^1 \to \partial N(K)\times S^1$ is defined to be the map which makes the following diagram commutative:

$$\begin{array}{ccc} \partial(E|\mathcal{E}_M(K)) & \cong & \partial(\mathcal{E}_M(K))\times S^1 \\ \| & & \downarrow g \\ \partial(E|N(K)) & \cong & \partial N(K)\times S^1\,. \end{array}$$

Now fix a framing $S^1\times D^2 \cong N(K)$ so that the gluing diffeomorphism g is represented as a diffeomorphism $S^1\times\partial D^2\times S^1 \to S^1\times\partial D^2\times S^1$, which is expressed as follows:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ m & n & 1 \end{pmatrix}.$$

Note that on the boundary tori, the framings are the same: $S^1\times\partial D^2 \cong \partial N(K) = \partial(\mathcal{E}_M(K))$. Here m is an ambiguity of the choices and we can assume $m = 0$ by choosing another trivialization over $N(K)$. If the Euler class $e(\xi)$ is a torsion element, then the integer n in the above expression is determined modulo the order of $e(\xi)$. Namely, suppose $pe(\xi) = 0$ $(p \in \mathbf{Z}, p > 0)$ and $qe(\xi) \neq 0$ if $0 < q < p$ $(q \in \mathbf{Z})$, then n changes into $n + lp$ $(l \in \mathbf{Z})$ by changing trivialization over $\mathcal{E}_M(K)$. Consequently, if the Euler class $e(\xi)$ is a torsion element, then the integer n modulo the order of $e(\xi)$ depends only on the Euler class $e(\xi)$. This representation of the gluing map g implies that the meridian loop of K on the cross section over $\mathcal{E}_M(K)$ winds up n times in the fibre direction.

Now we construct the homomorphism. Denote by $\widetilde{SO(2)}$ the universal covering group of $SO(2)$. Recall that we identify $SO(2)$ with $S^1 = \mathbf{R}/\mathbf{Z}$ and $\widetilde{SO(2)}$ with $\mathbf{R}$. It is sufficient for our task that we define a homomorphism from $\pi_1(\mathcal{E}_M(K))$ into $\widetilde{SO(2)}$ such that $[\mu]$ is forced to be mapped to the translation by n, where μ denotes the meridian loop of K. Then the homomorphism goes down to $\pi_1(M) = \pi_1(\mathcal{E}_M(K))/\langle[\mu]\rangle \to SO(2)$ as desired.

Since $[K] \in H_1(M)$ is a torsion element, there is no $z \in H_2(M)$ such that the intersection number $[K] \cdot z \neq 0$. Therefore, $H_2(M) \to H_2(M, \mathcal{E}_M(K))$ is the zero map so that $\partial : H_2(M, \mathcal{E}_M(K)) \to H_1(\mathcal{E}_M(K))$ is injective. It is obvious that the meridian loop μ of K represents an element of infinite order of $H_1(\mathcal{E}_M(K))$.

We will define a homomorphism $\tilde{\psi} : \pi_1(\mathcal{E}_M(K)) \to \widetilde{SO(2)}$ via $H_1(\mathcal{E}_M(K))$ by choosing a homomorphism $H_1(\mathcal{E}_M(K)) \to \widetilde{SO(2)}$ such that $[\mu]$ is forced to be mapped to the translation by n. First, we choose a direct sum decomposition of $H_1(\mathcal{E}_M(K)) = F \oplus T$, where T is the torsion subgroup and F is a complementary free part. We assume that $[\mu] \in F$ and choose free basis $\alpha_1, \cdots, \alpha_r$ for the free part F. Then, changing signs of α_i's if necessary, we have an expression $[\mu] = \sum_{i=1}^r a_i\alpha_i$ where $a_i \in \mathbf{Z}, a_i \geq 0$ and $\sum_{i=1}^r a_i \geq 1$. Note that $[\mu] \neq 0$. Define a homomorphism $\rho : H_1(\mathcal{E}_M(K)) \to \widetilde{SO(2)}$ by

$$\begin{aligned}\rho(\alpha_i) &= \mathrm{sh}\left(\frac{n}{\sum_{i=1}^r a_i}\right)\\ \rho|T &= \mathrm{id},\end{aligned}$$

where sh(t) denotes the translation by t. Composing ρ with the natural quotient homomorphism $\pi_1(\mathcal{E}_M(K)) \to H_1(\mathcal{E}_M(K))$ we have the desired homomorphism. This completes our construction.

ACKNOWLEDGEMENTS. Thanks are due to the referee whose comments improve and simplify the presentation of the paper.

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Present address:
DEPARTMENT OF MATHEMATICS, CHUO UNIVERSITY,
KASUGA, BUNKYO, TOKYO, 112–8551 JAPAN.
e-mail: miyoshi@math.chuo-u.ac.jp