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Remarks on Cusp Forms for Fricke Groups

Ryuji ABE

Keio University

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1. Introduction.

We consider the Teichmüller spaces of the closed torus and the once punctured torus. This is a part of a series of papers in which we investigate explicit relations between these spaces and we will give some remarks on our obtained results. In [A1] and [A2] we gave correspondences of subsets of these Teichmüller spaces and constructed holomorphic mappings between once punctured tori and closed tori based on these correspondences. In this paper we will show that constructions of these holomorphic mappings are closely related to constructions of cusp forms of weight 1.

First we recall a coordinate system for the Teichmüller space of the closed torus. We describe a closed torus by $R_\tau = \mathbf{C}/\Gamma_\tau$, $\Gamma_\tau = \{m + n\tau \mid m, n \in \mathbf{Z}\}$, then the Teichmüller space $\mathcal{T}_{1,0}$ of the closed torus is the upper half-plane $\mathbf{H}$, *i.e.,* a point in the Teichmüller space $\mathcal{T}_{1,0}$ of the closed torus is denoted by $\tau \in \mathbf{H}$. (See, for example, [IT].) We introduce the three subsets of $\mathcal{T}_{1,0}$: $L_1 = \{\tau \in \mathbf{H} \mid |\tau| \geq 1 \text{ and } \mathrm{Re}(\tau) = 0\}$, $L_2 = \{\tau \in \mathbf{H} \mid |\tau| = 1 \text{ and } -1/2 \leq \mathrm{Re}(\tau) \leq 0\}$ and $L_3 = \{\tau \in \mathbf{H} \mid |\tau| \geq 1 \text{ and } \mathrm{Re}(\tau) = -1/2\}$. These sets are characterized by the fact that in a fundamental domain for the modular group, $\tau \in L_1 \cup L_2 \cup L_3$ if and only if τ is a closed torus associated with a real lattice, that is, $\overline{\mu\Gamma_\tau} = \{\overline{\mu\gamma} \mid \mu\gamma \in \mu\Gamma_\tau\} = \mu\Gamma_\tau$ for some $\mu \in \mathbf{C}$.

Next we recall a coordinate system for the Teichmüller space of the once punctured torus. We use the convention that an element in PSL(2, $\mathbf{R}$) represents the Möbius transformation induced by it. In this paper we consider a Fuchsian group G consisting of Möbius transformations of PSL(2, $\mathbf{R}$) and having the following properties: (i) G is discontinuous in the upper half-plane $\mathbf{H}$, (ii) every real number is a limit point for G, (iii) G is finitely generated. A Fuchsian group $\Gamma = \langle A, B\rangle$ which is a free group generated by $A, B \in$ PSL(2, $\mathbf{R}$) is called a *Fricke group* if $X^2 + Y^2 + Z^2 = XYZ$ and $X, Y, Z > 2$, where $X = \mathrm{tr}\, A$, $Y = \mathrm{tr}\, B$ and $Z = \mathrm{tr}\, AB$. We consider a once punctured torus which is uniformized by a Fricke group Γ and take a normalized form for the representation of Γ (see [Sc]), then the Teichmüller space $\mathcal{T}_{1,1}$ of the once punctured torus can be identified with the set of all Fricke groups (see

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[K]), that is, a point in the Teichmüller space $\mathcal{T}_{1,1}$ of the once punctured torus is denoted by a triple (X, Y, Z). Associated with L_1, L_2 and L_3 we define the three subsets of $\mathcal{T}_{1,1}$: $M_1 = \{(X, Y, Z) \in \mathcal{T}_{1,1} \mid 2 < X \leq Y \leq Z = XY/2\}$, $M_2 = \{(X, Y, Z) \in \mathcal{T}_{1,1} \mid 2 < X = Y \leq Z\}$ and $M_3 = \{(X, Y, Z) \in \mathcal{T}_{1,1} \mid 2 < X \leq Y = Z\}$.

Now we summarize the results obtained in [A1] and [A2]. We represent a point in the upper half-plane $\mathbf{H}$ and a point in the complex plane $\mathbf{C}$ by z and u, respectively. We call $\mathbf{H}$ the *z-plane* and $\mathbf{C}$ the *u-plane*. Then a once punctured torus (X, Y, Z) can be identified with a fundamental domain in the z-plane and a closed torus τ can be identified with a fundamental domain in the u-plane. A holomorphic mapping from $(3, 3, 3) \in \mathcal{T}_{1,1}$ to $\rho_3 = e^{2\pi i/3} \in \mathcal{T}_{1,0}$ is given by the relation

$$1 - J(z) = \wp'(u)^2 = 4\wp(u)^3 + 1\,, \tag{1.1}$$

and a holomorphic mapping from $(2\sqrt{2}, 2\sqrt{2}, 4) \in \mathcal{T}_{1,1}$ to $i \in \mathcal{T}_{1,0}$ is given by the relations

$$J_4(z) = \wp(u)^2 \quad \text{and} \quad \wp'(u)^2 = 4\wp(u)^3 - 4\wp(u)\,, \tag{1.2}$$

where $\wp(u)$ are the Weierstrass $\wp$-functions defined by the above equations, $J(z)$ is the modular function and $J_4(z)$ is a function having similar properties to $J(z)$ (see Proposition 3.1). The relation (1.1) was first used in [C1]. Generalizing these relations, we obtained the following theorems. A basic idea of our proof is abelianization of Fricke group.

THEOREM 1.1. *For any $(X, Y, Z) \in M_1$ there uniquely exists an element $\tau \in L_1$ satisfying the following conditions: if $\tau \in L_1$ then $p(x) = 4x^3 - g_2(\tau)x - g_3(\tau)$ has three distinct real roots and a holomorphic mapping between (X, Y, Z) and τ is given by the relation*

$$\wp(u) = (x_2 - x_1)J_{(X,Y,Z)}(z) + x_2\,, \tag{1.3}$$

where $x_1 < x_2 < x_3$ are the three real roots of $p(x)$, $\wp(u)$ is the Weierstrass $\wp$-function defined by $\wp'(u)^2 = 4\wp(u)^3 - g_2(\tau)\wp(u) - g_3(\tau)$ and $J_{(X,Y,Z)}(z)$ is a function having similar properties to the modular function $J(z)$.

The precise definition of $J_{(X,Y,Z)}(z)$ will be recalled in §3.2. A proof of this theorem was shown in §5 of [A1].

THEOREM 1.2. *For any $(X, Y, Z) \in M_k, k = 2, 3$ there uniquely exist an element $\tau \in L_k$ and a number P satisfying $7 - 4\sqrt{3} \leq P \leq 1$ if $\tau \in L_2$ and $P \geq 7 + 4\sqrt{3}$ if $\tau \in L_3$ such that a holomorphic mapping between (X, Y, Z) and τ is given by the relation*

$$\wp(u) = J_{(X,Y,Z)}(z) - \frac{P}{J_{(X,Y,Z)}(z)} + \frac{1-P}{3}\,, \tag{1.4}$$

where $\wp(u)$ is the Weierstrass $\wp$-function defined by

$$\wp'(u)^2 = 4\left(\wp(u) - \frac{2}{3}(P-1)\right)\left(\wp(u) - \left(\frac{1-P}{3} + 2\sqrt{P}i\right)\right)\left(\wp(u) - \left(\frac{1-P}{3} - 2\sqrt{P}i\right)\right)$$

and $J_{(X,Y,Z)}(z)$ is a function having similar properties to the modular function $J(z)$.

For the precise definition of $J_{(X,Y,Z)}(z)$ and a proof of this theorem we refer the reader to [A2].

The results described above were proved geometrically, that is, by investigating fundamental domains identified with once punctured and closed tori. The first aim in this paper is to point out an analytic approach to our problem, that is, to show that the problem of constructing a holomorphic mapping between a closed torus and a once punctured torus uniformized by Γ is equivalent to the problem of constructing a cusp form of weight 1 for Γ. Then we obtain the following assertion:

THEOREM 1.3. *Let Γ be a Fricke group associated with a once punctured torus (X, Y, Z). A cusp form of weight* 1 *for Γ is given by* (1.1) *if* $(X, Y, Z) = (3, 3, 3)$, *by* (1.2) *if* $(X, Y, Z) = (2\sqrt{2}, 2\sqrt{2}, 4)$, *by* (1.3) *if* $(X, Y, Z) \in M_1$ *and by* (1.4) *if* $(X, Y, Z) \in M_2 \cup M_3$.

It is important that such cusp forms give holomorphic quadratic differentials on once punctured tori, because the Teichmüller geodesic is defined by using them. Remarks on these points will be given in §2.

The second aim is to show explicit representations of cusp forms mentioned above. The Fricke groups associated with (3, 3, 3) and $(2\sqrt{2}, 2\sqrt{2}, 4)$ are subgroups of the moular group SL(2, **Z**) and a Hecke group, respectively. For these special cases some constructions of automorphic forms have been studied. By using them we will show the following result:

THEOREM 1.4. *For the Fricke group Γ_{ρ_3} associated with* (3, 3, 3) *we can obtain an explicit representation of a cusp form of weight* 1 *for Γ_{ρ_3} determined by* (1.1) *which is a sixth root of a cusp form of weight* 6 *for the modular group* SL(2, **Z**).

THEOREM 1.5. *For the Fricke group Γ_i associated with* $(2\sqrt{2}, 2\sqrt{2}, 4)$ *the cusp form of weight* 1 *for Γ_i determined by* (1.2) *is a fourth root of a cusp form of weight* 4 *for the Hecke group generated by* $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix}$.

In the case of $(X, Y, Z) \in M_1$ we can obtain a similar result by using the extended Fricke group [Sc] associated with (X, Y, Z).

THEOREM 1.6. *For the Fricke group Γ_α associated with $(X, Y, Z) \in M_1$ and generated by*

$$\begin{pmatrix} 0 & -1 \\ 1 & \frac{2\sqrt{1+\alpha^2}}{\alpha} \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} \sqrt{1+\alpha^2} & -\alpha \\ -\alpha & \sqrt{1+\alpha^2} \end{pmatrix} \quad \textit{for some } \alpha \geq 1\,,$$

the cusp form of weight 1 *for Γ_α determined by* (1.3) *is a square root of a cusp form of weight* 2 *for the group generated by*

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \sqrt{1+\alpha^2} & \frac{2+\alpha^2}{\alpha} \\ -\alpha & -\sqrt{1+\alpha^2} \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 1 & \frac{2\sqrt{1+\alpha^2}}{\alpha} \\ 0 & 1 \end{pmatrix}.$$

Generally, an explicit construction of a cusp form of weight 1 for a given Fuchsian group is not easy to obtain. Our results give a construction of such a cusp form for a Fricke group Γ

associated with $(3, 3, 3)$, $(2\sqrt{2}, 2\sqrt{2}, 4)$ or $(X, Y, Z) \in M_1$. The basic ideas are summarized as follows: find a Fuchsian group G in which Γ is a subgroup of index $l = 6, 4$ or 2, construct a cusp form of weight l for G, for example by using the Poincaré series, and take its l-th root.

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2. Cusp forms and quadratic differentials.

In this section by using cusp forms and quadratic differentials we will give interpretations of our problem of constructing holomorphic mappings between closed tori and once punctured tori.

We begin by recalling definitions of an automorphic form and a cusp form. We can refer the reader to [Mi] for further details. Let G be a Fuchsian group. We take $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and define

$$C_{T,k}(z) = \left(\frac{dT}{dz}\right)^{-k} = (cz+d)^{2k}, \quad \text{where } k \text{ is some integer}$$

for the Möbius transformation $T(z)$, $z \in \mathbf{H}$. Recall that a point z is a *cusp* of G if and only if G has a parabolic element whose fixed point is z.

DEFINITION 2.1. Let f be a function of $\mathbf{H}$ satisfying $f(Tz) = C_{T,k}(z)f(z)$ for all $z \in \mathbf{H}$ and $T \in G$. If f is holomorphic on $\mathbf{H}$ and is finite at each cusp of G, f is called an *automorphic form* of weight k for G. And if the automorphic form f vanishes at each cusp, we call f a *cusp form* of weight k for G. An automorphic form for the modular group is called a *modular form*.

By using the Riemann-Roch theorem we obtain the following theorem which guarantees the existence of an automorphic form and a cusp form.

THEOREM 2.1. *The dimension $\delta_k^a(G)$ of the space of automorphic forms of weight k for G is*

$$\delta_k^a(G) = (2k-1)(g-1) + tk + \sum_j \left[k\left(1 - \frac{1}{e_j}\right)\right] \quad \text{if } k > 1,$$

$$\delta_1^a(G) = \begin{cases} g & \text{if } t = 0, \\ g + t - 1 & \text{if } t > 0, \end{cases}$$

and the dimension $\delta_k^c(G)$ of the space of cusp forms of weight k for G is

$$\delta_k^c(G) = (2k-1)(g-1) + t(k-1) + \sum_j \left[k\left(1 - \frac{1}{e_j}\right)\right] \quad \text{if } k > 1 \quad \text{and} \quad \delta_1^c(G) = g,$$

where g is the genus of the Riemann surface R of G, i.e., the compactification of $\mathbf{H}/G$ obtained by adding parabolic points with the appropriate local coordinates, t is the number of parabolic fixed points which are not equivalent to each other and the sum with respect to j runs through the elliptic fixed points of G in R whose periods are e_j.

COROLLARY 2.1. *Let $\Gamma = \langle A, B\rangle$ be a Fricke group. Then $\delta_1^a(\Gamma) = \delta_1^c(\Gamma) = 1$.*

PROOF. Let R be the once punctured torus associated with Γ. Since the genus of R is 1 and four parabolic fixed points are equivalent to each other (see [Sc]), we obtain the assertion. □

Now we state a relation between an automorphic form for a Fricke group $\Gamma = \langle A, B\rangle$ and a holomorphic mapping between elements of $T_{1,0}$ and $T_{1,1}$. Our aimed mapping is a holomorphic function $\Psi : \mathbf{H} \to \mathbf{C}$ satisfying the following conditions:

(i) $\Psi(Az) = \Psi(z) + \omega_1$ and $\psi(Bz) = \Psi(z) + \omega_2$ for all $z \in \mathbf{H}$ and some $\omega_1, \omega_2 \in \mathbf{C}$.

(ii) Each cusp of Γ is mapped to a lattice point of $\Omega(\omega_1, \omega_2) = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbf{Z}\}$.

PROPOSITION 2.1. *Ψ is the function as above if and only if the derivative of Ψ with respect to z is a cusp form of weight 1 for Γ.*

PROOF. First we show that if Ψ is the function satisfying (i) and (ii) then Ψ' is a cusp form of weight 1 for Γ. Since an automorphic form of weight 1 for Γ must be a cusp form by using Corollary 2.1, we only show that Ψ' is an automorphic form of weight 1 for Γ. Taking the derivative of the first equation of (i) with respect to z, we have

$$A'(z)\Psi'(Az) = \Psi'(z). \tag{2.1}$$

If we set $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A'(z) = 1/(cz+d)^2$, so the relation (2.1) is transformed into $\Psi'(Az) = (cz+d)^2\Psi'(z)$. We can apply the same argument as above to another equation of (i). It is easily checked by using (ii) that Ψ' is finite at each cusp of Γ.

The other direction is easily proved by using the fact that the constants ω_1 and ω_2 satisfying

$$\int_z^{Az} f(w)dw = \omega_1 \quad \text{and} \quad \int_z^{Bz} f(w)dw = \omega_2$$

do not depend on z, which comes from Cauchy's theorem. □

Therefore one approach to our problem is to construct a cusp form $f = \Psi'$ of weight 1 for Γ. A construction of such a cusp form has been tried for a long time; however, unfortunately an explicit result has not been found yet. From Proposition 2.1 we readily get that the relations (1.1), (1.2), (1.3) and (1.4) give a cusp form of weight 1 for an associated Fricke group.

Next we recall a definition of quadratic differential.

DEFINITION 2.2. Let R be a Riemann surface with a given complex structure $\{(U_i, z_i)\}_{i\in I}$. A *quadratic differential* $\varphi = \{\varphi_i\}$ on R is a set of meromorphic functions φ_i on $z_i(U_i)$ which satisfy $\varphi_i(z_i) = \varphi_j(z_j)(dz_j/dz_i)^2$ whenever $U_i \cap U_j \neq \emptyset$. We write $\varphi_i(z_i)dz_i^2 = \varphi_j(z_j)dz_j^2$ and $\varphi = \varphi(z)dz^2$ for simplicity. If all the φ_i are holomorphic, the quadratic differential φ is called *holomorphic*. The function φ_i is called the *representation* of the quadratic differential φ in terms of the local parameter z_i. The *norm* of a quadratic differential $\varphi = \varphi(z)dz^2$, $z = x + iy$ is defined as $\|\varphi\| = \iint_R |\varphi(z)|dxdy$.

Quadratic differentials play an important role in a construction of the Teichmüller geodesic. For the Teichmüller space of the closed torus we can summarize a construction of the Teichmüller geodesic as follows. A representation of a holomorphic quadratic differential in terms of a local parameter on a closed torus R_τ for any $\tau \in \mathbf{H}$ is a complex constant, which comes from the fact that a doubly periodic holomorphic function in $\mathbf{C}$ must be a complex constant. If we suppose that the value of the norm of a holomorphic quadratic differential defined on R_τ is equal to one, a representation of a holomorphic quadratic differential on R_τ in terms of a local parameter z is $\varphi(\theta) = (\operatorname{Im}\tau)^{-1}(e^{i\theta})^2dz^2$. An orientation preserving affine diffeomorphism between two tori R_τ and $R_{\tau'}$ for $\tau, \tau' \in \mathbf{H}$ can be considered as a Teichmüller mapping for a holomorphic quadratic differential $\varphi(\theta)$ on R_τ and a number corresponding to the Teichmüller distance between R_τ and $R_{\tau'}$. The Teichmüller geodesic determined by $\varphi(\theta)$ on R_τ is defined by using such Teichmüller mappings. Therefore the task of finding holomorphic quadratic differentials on once punctured tori is also important in our problem of investigating explicit relations between the Teichmüller spaces of the closed torus and the once punctured torus.

PROPOSITION 2.2. *Let f be an automorphic form of weight 1 for a Fuchsian group Γ. Then f^2dz^2 is a holomorphic quadratic differential on a Riemann surface $\mathbf{H}/\Gamma$.*

PROOF. If we set $w = Tz$ for $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ then it is easily obtained that $f^2(w)dw^2 = f^2(z)dz^2$ since $dz = (cz+d)^2dw$ and $f^2(w) = f^2(Tz) = (cz+d)^4f^2(z)$. □

Therefore Theorem 1.3 can be changed into the following form:

THEOREM 2.2. *Let Γ be a Fricke group associated with a once punctured torus (X, Y, Z). A holomorphic quadratic differential on (X, Y, Z) is given by (1.1) if $(X, Y, Z) = (3, 3, 3)$, by (1.2) if $(X, Y, Z) = (2\sqrt{2}, 2\sqrt{2}, 4)$, by (1.3) if $(X, Y, Z) \in M_1$ and by (1.4) if $(X, Y, Z) \in M_2 \cup M_3$.*

Moreover, such a holomorphic quadratic differential spans the space of holomorphic quadratic differentials on a once punctured torus, because it has dimension 1.

3. Explicit representations of cusp forms.

3.1. Two special cases. In this subsection we will show explicit representations of cusp forms coming from the relation (1.1) and (1.2) by using some facts on Hecke groups and will show another representation for the cusp form coming from (1.1) by using Eisenstein series. We begin by recalling Hecke groups. The following arguments are due to Chapter III in [H].

DEFINITION 3.1. The group Γ_q generated by the following matrices S and U is called a *Hecke group*

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \quad \text{for } \lambda = 2\cos\frac{\pi}{q} \quad \text{with } q = 3, 4, \cdots .$$

Note that if $q = 3$ the Hecke group Γ_3 is equal to the modular group SL(2, $\mathbf{Z}$). We take the following domain:

$$F_q^* = \left\{ z \in \mathbf{H} \,\middle|\, |z| \geq 1 \text{ and } -\frac{\lambda}{2} \leq |\mathrm{Re}(z)| \leq 0 \right\}.$$

By the Riemann mapping theorem there exists a mapping from F_q^* to $\mathbf{H}$ and by applying Schwarz' reflection principle we can construct the mapping J_q satisfying the following proposition:

PROPOSITION 3.1. (i) *J_q is invariant under the action of the Hecke group Γ_q, i.e., $J_q(T(z)) = J_q(z)$ for all $z \in \mathbf{H}$ and $T \in \Gamma_q$, where $T(z)$ is a Möbius transformation.*

(ii) *J_q maps L_q onto $\mathbf{R}$ where $L_q = L_{q1} \cup L_{q2} \cup L_{q3}$,*

$$L_{q1} = \{z \in \mathbf{H} \mid |z| \geq 1 \textit{ and } \mathrm{Re}(z) = 0\},$$

$$L_{q2} = \left\{ z \in \mathbf{H} \,\middle|\, |z| = 1 \textit{ and } -\frac{\lambda}{2} \leq \mathrm{Re}(z) \leq 0 \right\},$$

$$L_{q3} = \left\{ z \in \mathbf{H} \,\middle|\, |z| \geq 1 \textit{ and } \mathrm{Re}(z) = -\frac{\lambda}{2} \right\}.$$

Especially, $J_q(i\infty) = \infty$, $J_q(i) = 1$ and $J_q(\rho_q) = 0$ where $\rho_q = e^{(1-1/q)\pi i}$.

(iii) *J_q maps F_q onto $\mathbf{C}$ where*

$$F_q = \left\{ z \in \mathbf{H} \,\middle|\, |z| \geq 1 \textit{ and } |\mathrm{Re}(z)| \leq \frac{\lambda}{2} \right\}$$

is a fundamental domain for the Hecke group Γ_q.

(iv) *The mapping $J_q : \mathbf{H} \to \mathbf{C}$ is holomorphic on $\mathbf{H}$.*

Let f be a holomorphic function on the upper half-plane $\mathbf{H}$ and satisfying the following conditions:

$$f(z+\lambda) = f(z) \quad \text{and} \quad f\left(-\frac{1}{z}\right) = (-iz)^k \gamma f(z) \quad \text{for some } k > 0 \text{ and } \gamma = 1 \text{ or } -1.$$

Then we say that the function f has *signature* (k, γ). Such functions were used by Hecke in connection with finding correspondences between automorphic forms for Hecke groups and Dirichlet series satisfying a functional equation. Then we can construct explicitly the following functions by using J_q:

$$f_{\rho_q,q}(z) = \left(\frac{J_q'^2}{J_q(J_q - 1)} \right)^{\frac{1}{q-2}} \quad \text{with } (k, \gamma) = \left(\frac{4}{q-2}, 1 \right),$$

$$f_{i,q}(z) = \left(\frac{J_q'^q}{J_q{}^{q-1}(J_q-1)}\right)^{\frac{1}{q-2}} \quad \text{with } (k,\gamma) = \left(\frac{2q}{q-2}, -1\right),$$

$$f_{i\infty,q}(z) = \left(\frac{J_q'^{2q}}{J_q^{2q-2}(J_q-1)^q}\right)^{\frac{1}{q-2}} \quad \text{with } (k,\gamma) = \left(\frac{4q}{q-2}, 1\right).$$

Moreover, $f_{\rho_q,q}$, $f_{i,q}$ and $f_{i\infty,q}$ have only one zero at points ρ_q, i and $i\infty$, respectively, and their orders are equal to 1. We easily get the relation connecting them: $f_{i\infty,q} = f_{\rho_q,q}^q - f_{i,q}^2$.

If $q = 3$ then we have

$$f_{\rho_3,3}(z) = \frac{J_3'^2}{J_3(J_3-1)}, \quad f_{i,3}(z) = \frac{J_3'^3}{J_3{}^2(J_3-1)}, \quad f_{i\infty,3}(z) = \frac{J_3'^6}{J_3^4(J_3-1)^3}$$

with signatures (4, 1), (6, −1) and (12, 1), respectively. It follows from the definition of the signature that $f_{\rho_3,3}$ and $f_{i,3}$ are automorphic forms of weight 2 and 3 for $\Gamma_3 = \mathrm{SL}(2, \mathbf{Z})$, respectively, and that $f_{i\infty,3}$ is a cusp form of weight 6 for Γ_3. Note that J_3 is equal to J used in the relation (1.1) and $J_3 = f_{\rho_3,3}^3/f_{i\infty,3}$.

We give the same argument as above for the case $q = 4$. The following functions are obtained:

$$f_{\rho_4,4}(z) = \left(\frac{J_4'^2}{J_4(J_4-1)}\right)^{\frac{1}{2}}, \quad f_{i,4}(z) = \left(\frac{J_4'^4}{J_4{}^3(J_4-1)}\right)^{\frac{1}{2}}, \quad f_{i\infty,4}(z) = \left(\frac{J_4'^8}{J_4^6(J_4-1)^4}\right)^{\frac{1}{2}}$$

with signatures (2, 1), (4, −1) and (8, 1), respectively. Then $f_{i\infty,4}$ is a cusp form of weight 4 for Γ_4 and we get $J_4 = f_{\rho_4,4}^4/f_{i\infty,4}$.

Now we study the relations (1.1) and (1.2). We take a representation of (3, 3, 3) as follows (see [C1],[A1]):

$$\Gamma_{\rho_3} = \langle A_{\rho_3}, B_{\rho_3}\rangle \quad \text{with} \quad A_{\rho_3} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad B_{\rho_3} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Then Γ_{ρ_3} is a subgroup of index 6 in $\Gamma_3 = \mathrm{SL}(2, \mathbf{Z})$. From Corollary 2.1 there must be a cusp form of weight 1 for Γ_{ρ_3}. Note that J in the relation (1.1) is equal to J_3 and we use J in the following discussion for simplicity. Taking the derivative of $1 - J(z) = 4\wp(u)^3 + 1$ in (1.1) with respect to u, we have

$$-\frac{dJ}{du} = 12\wp^2\frac{d\wp}{du}. \tag{3.1}$$

From (1.1) we get $-J(z) = 4\wp^3(u)$ and $1 - J(z) = (\wp'(u))^2$. On the one hand ρ_3 and i are elliptic fixed points of Γ_3 in F_3 and have periods 3 and 2, respectively. On the other hand the group Γ_{ρ_3} does not have an elliptic fixed point, that is, a fundamental domain for Γ_{ρ_3} is a 3-sheeted covering of the fundamental domain F_3 for Γ_3 around ρ_3 and is a 2-sheeted covering of F_3 around i. (More precisely, see §4 in [A1].) We can make the same assertion for equivalent points of ρ_3 and i under the action of Γ_3. Since $J(\rho_3) = 0$ and $J(i) = 1$, we can take a cube root of J and a square root of $1 - J$. Then $\wp(u) = (const) \cdot J(z)^{1/3}$ and $\wp'(u) = \pm(1 - J(z))^{1/2}$. Therefore (3.1) is transformed into the following:

$$du = (const) \cdot \frac{1}{J^{\frac{2}{3}}(1-J)^{\frac{1}{2}}} \frac{dJ}{dz} dz \stackrel{\text{def}}{=} (const) \cdot \varphi_3(z) dz \,.$$

THEOREM 3.1. *φ_3 is a cusp form of weight 1 for Γ_{ρ_3} and is a sixth root of $f_{i\infty,3}$.*

PROOF. Let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of Γ_{ρ_3}. As J is T-invariant and satisfies $dJ(Tz)/d(Tz) = (cz+d)^2 dJ(z)/dz$, it is easily obtained that φ_3 satisfies $\varphi_3(Tz) = (cz+d)^2\varphi_3(z)$. We easily have that $\varphi_3(z)$ is a sixth root of $f_{i\infty,3}$. Since $f_{i\infty,3}$ vanishes at $i\infty$, $\varphi_3(z)$ is a cusp form. □

Next we will give a similar discussion on the relation (1.2). We take a representation of $(2\sqrt{2}, 2\sqrt{2}, 4)$ as follows (see [C2],[A1]):

$$\Gamma_i = \langle A_i, B_i \rangle \quad \text{with} \quad A_i = \begin{pmatrix} 0 & -1 \\ 1 & 2\sqrt{2} \end{pmatrix} \quad \text{and} \quad B_i = \begin{pmatrix} \sqrt{2} & -1 \\ -1 & \sqrt{2} \end{pmatrix} .$$

Then Γ_i is a subgroup of index 4 in the Hecke group Γ_4. From Corollary 2.1 there must be a cusp form of weight 1 for Γ_i. Taking the derivative of the first equation $J_4(z) = \wp(u)^2$ in (1.2) with respect to u, we have

$$\frac{dJ_4}{du} = 2\wp \frac{d\wp}{du} . \tag{3.2}$$

On the one hand it follows from the construction of J_4 in Proposition 3.1 that ρ_4 and i are elliptic fixed points of Γ_4 in F_4 and have periods 4 and 2, respectively. On the other hand the group Γ_i does not have an elliptic fixed point. By using the same argument as in the case for J_3 we can take a fourth root of J_4 and a square root of $J_4 - 1$. Then $\wp(u) = \pm J_4(z)^{1/2}$ and $\wp'(u) = (const) \cdot J_4(z)^{1/4}(J_4(z) - 1)^{1/2}$ from (1.2). Therefore (3.2) is transformed into the following:

$$du = (const) \cdot \frac{1}{J_4^{\frac{3}{4}}(J_4 - 1)^{\frac{1}{2}}} \frac{dJ_4}{dz} \stackrel{\text{def}}{=} (const) \cdot \varphi_4(z) dz \,.$$

THEOREM 3.2. *φ_4 is a cusp form of weight 1 for Γ_i and is a fourth root of $f_{i\infty,4}$.*

The proof of this theorem is almost the same as the proof of Theorem 3.1.

In the rest of this subsection we will introduce a more explicit representation of a cusp form of weight 1 for the Fricke group Γ_{ρ_3} associated with $(3,3,3)$. Let G be a Fuchsian group. The function f in Definition 2.1 is represented by using the Fourier series

$$f(z) = \sum_{n \in \mathbf{Z}} a_n q^n \quad \text{with} \quad q = e^{2\pi i z} ,$$

which is called its *q-expansion*. If $a_n = 0$ for all $n < 0$, f is an automorphic form of weight k for G. And if we further have $a_0 = 0$, then f is a cusp form of weight k for G.

We recall the normalized Eisenstein series as examples of modular forms. For $k \geq 1$ and $z \in \mathbf{H}$ we define

$$E_k(z) = \frac{1}{2} \sum_{\substack{m,n \in \mathbf{Z} \\ (m,n)=1}} \frac{1}{(mz+n)^{2k}},$$

where (m, n) denotes the greatest common divisor of m and n. Then $E_k(z)$ is a modular form of weight k. For $k = 2$ and 3 we obtain their q-expansions as follows:

$$E_2(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \quad \text{and} \quad E_3(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n,$$

where $\sigma_k(n) = \sum_{d|n} d^k$. Moreover, we define

$$J(z) = \frac{E_2(z)^3}{E_2(z)^3 - E_3(z)^2}.$$

Then $\Delta(z) = E_2(z)^3 - E_3(z)^2$ is a cusp form of weight 6 for SL(2, $\mathbf{Z}$) and $J(z)$ is a modular function of weight 0, which we have used in the arguments above. From Theorem 2.1 we have $\delta_2^a(\Gamma_3) = \delta_3^a(\Gamma_3) = 1$ and $\delta_6^c(\Gamma_3) = 1$. Therefore E_2, E_3 and Δ must be equal to $f_{\rho_3,3}$, $f_{i,3}$ and $f_{i\infty,3}$, respectively, modulo a complex constant multiple. Now we have another explicit representation of a cusp form of weight 1 for Γ_{ρ_3}.

COROLLARY 3.1. *φ_3 is a sixth root of Δ.*

We will study the q-expansion of $\varphi_3(z)$ in order to check directly that φ_3 vanishes at $i\infty$. Representing the normalized Eisenstein series for $k = 2$ and 3 by

$$E_2(z) = 1 + a_1 q + a_2 q^2 + a_3 q^3 + \cdots \quad \text{and} \quad E_3(z) = 1 + b_1 q + b_2 q^2 + b_3 q^3 + \cdots,$$

we have

$$\begin{aligned} J(z) &= \frac{1 + 3a_1 q + 3(a_1^2 + a_2) q^2 + \cdots}{(3a_1 - 2b_1) q + (3(a_1^2 + a_2) - (b_1^2 + 2b_2)) q^2 + \cdots} \\ &= c_{-1} \frac{1}{q} + c_0 + c_1 q + c_2 q^2 + \cdots, \end{aligned}$$

where $c_{-1} = 1/1728$ and $c_i, i = 0, 1, 2, \cdots$ are some real constants determined by the above equation. Moreover, we get

$$\begin{aligned} \frac{1}{J^{\frac{2}{3}}(1-J)^{\frac{1}{2}}} &= \frac{1}{(c_{-1}q^{-1} + c_0 + c_1 q + \cdots)^{\frac{2}{3}} (1 - c_{-1}q^{-1} - c_0 - c_1 q - \cdots)^{\frac{1}{2}}} \\ &= -ic_{-1}^{-\frac{7}{6}} q^{\frac{7}{6}} + \cdots, \end{aligned}$$

and

$$\begin{aligned} \frac{dJ}{dz} &= \frac{dJ}{dq}\frac{dq}{dz} = \left(-c_{-1}\frac{1}{q^2} + c_1 + 2c_2 q + \cdots\right)(2\pi i q) \\ &= 2\pi i \left(-c_{-1}\frac{1}{q} + c_1 q + 2c_2 q^2 + \cdots\right). \end{aligned}$$

Therefore

$$\varphi_3(z) = (-ic_{-1}^{-\frac{7}{6}} q^{\frac{7}{6}} + \cdots) \cdot 2\pi i \left(-c_{-1}\frac{1}{q} + c_1 q + 2c_2 q^2 + \cdots \right) = 2\pi i (ic_{-1}^{-\frac{1}{6}} q^{\frac{1}{6}} + \cdots).$$

This means that φ_3 vanishes at infinity. Therefore we can prove that φ_3 is a cusp form of weight 1 for Γ_{ρ_3} without using results of Hecke groups.

3.2. The case (X, Y, Z) in M_1. We will show explicit representations of cusp forms coming from the relation (1.3). We begin by recalling some facts (see §5 in [A1]). A representation of $(X, Y, Z) \in M_1$ is given by

$$\Gamma_\alpha = \langle A_\alpha, B_\alpha \rangle \quad \text{with} \quad A_\alpha = \begin{pmatrix} 0 & -1 \\ 1 & \frac{2\sqrt{1+\alpha^2}}{\alpha} \end{pmatrix} \quad \text{and} \quad B_\alpha = \begin{pmatrix} \sqrt{1+\alpha^2} & -\alpha \\ -\alpha & \sqrt{1+\alpha^2} \end{pmatrix},$$

where $\alpha \geq 1$ is a parameter. A fundamental domain identified with the once punctured torus (X, Y, Z) can be represented as follows:

$$D(\Gamma_\alpha) = \left\{ z \in \mathbf{H} \;\middle|\; \left| z + \frac{3\sqrt{1+\alpha^2}}{\alpha} \right| \geq \frac{1}{\alpha},\ \left| z + \frac{2\sqrt{1+\alpha^2}}{\alpha} \right| \geq 1, \right.$$
$$\left. \left| z + \frac{\sqrt{1+\alpha^2}}{\alpha} \right| \geq \frac{1}{\alpha},\ |z| \geq 1,\ -\frac{4+3\alpha^2}{\alpha\sqrt{1+\alpha^2}} \leq \mathrm{Re}(z) \leq \frac{\alpha}{\sqrt{1+\alpha^2}} \right\}$$

(the part shaded by lines downward to the right in Fig. 3.1). Moreover, we used the following notations:

$$F_\alpha^* = \left\{ z \in \mathbf{H} \;\middle|\; \left| z + \frac{\sqrt{1+\alpha^2}}{\alpha} \right| \geq \frac{1}{\alpha},\ |z| \geq 1,\ -\frac{\sqrt{1+\alpha^2}}{\alpha} \leq \mathrm{Re}(z) \leq 0 \right\},$$

$$\eta = \left(-\frac{\sqrt{1+\alpha^2}}{\alpha}, \frac{1}{\alpha} \right) \quad \text{and} \quad \zeta = \left(-\frac{\alpha}{\sqrt{1+\alpha^2}}, \frac{1}{\sqrt{1+\alpha^2}} \right).$$

The region F_α^* (shown in broken line in Fig. 3.1) is a quadrangle with angles $0, \pi/2, \pi/2, \pi/2$.

We introduce the following transformations:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S_\alpha = \begin{pmatrix} \sqrt{1+\alpha^2} & \frac{2+\alpha^2}{\alpha} \\ -\alpha & -\sqrt{1+\alpha^2} \end{pmatrix}, \quad U_\alpha = \begin{pmatrix} 1 & \frac{2\sqrt{1+\alpha^2}}{\alpha} \\ 0 & 1 \end{pmatrix}.$$

Relations among A_α, B_α, S, S_α and U_α are summarized in:

$$A_\alpha = SU_\alpha, \quad B_\alpha = S_\alpha U_\alpha^{-1}, \tag{3.3}$$

$$U_\alpha^2 = B_\alpha^{-1} A_\alpha^{-1} B_\alpha A_\alpha. \tag{3.4}$$

PROPOSITION 3.2. *Γ_α is a subgroup of index 2 in $\hat{\Gamma}_\alpha = \langle S, S_\alpha, U_\alpha \rangle$.*

PROOF. Since $U_\alpha \notin \Gamma_\alpha$ and $\Gamma_\alpha \cap U_\alpha \Gamma_\alpha = \emptyset$, we will prove $\hat{\Gamma}_\alpha = \Gamma_\alpha \cup U_\alpha \Gamma_\alpha$. It immediately follows from (3.3) that $\hat{\Gamma}_\alpha \supset \Gamma_\alpha \cup U_\alpha \Gamma_\alpha$. Then we only show that $\hat{\Gamma}_\alpha \subset \Gamma_\alpha \cup U_\alpha \Gamma_\alpha$. Let g be an element of $\hat{\Gamma}_\alpha$. We can represent g by using A_α, B_α and U_α, for we

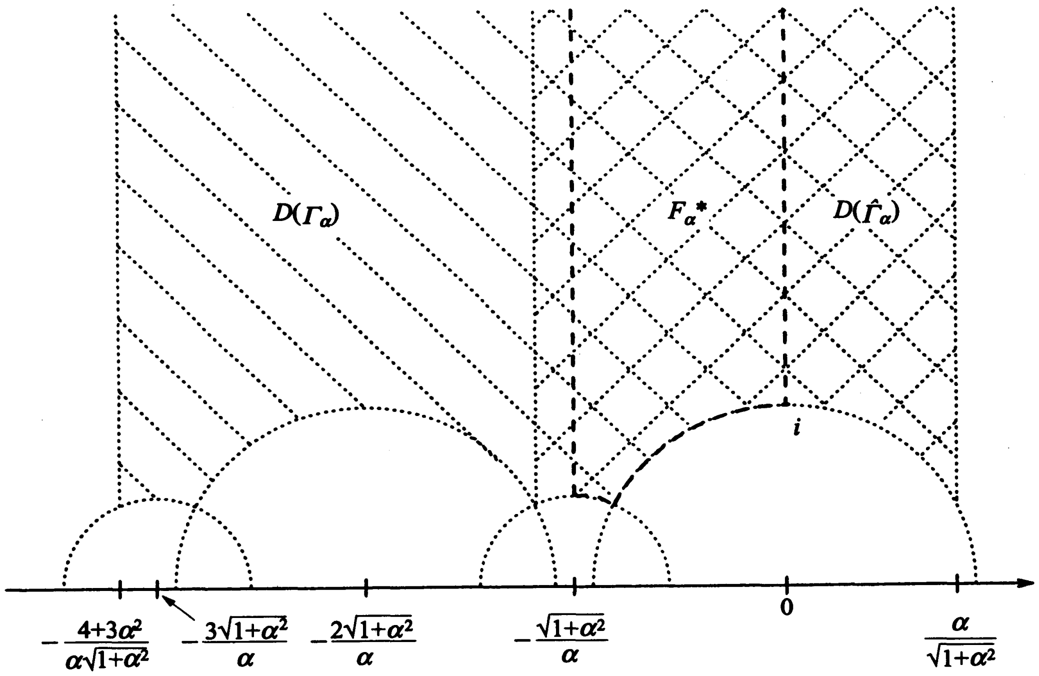

Fig. 3.1

get $S = A_\alpha U_\alpha^{-1}$ and $S_\alpha = B_\alpha U_\alpha$ from (3.3). Generally, A_α, B_α and U_α satisfy the following relations:

$$A_\alpha U_\alpha = U_\alpha A_\alpha^{-1} B_\alpha^{-1} A_\alpha^{-1} B_\alpha A_\alpha\,, \quad A_\alpha U_\alpha^{-1} = U_\alpha A_\alpha^{-1}\,,$$
$$B_\alpha U_\alpha = U_\alpha^{-1} B_\alpha^{-1}\,, \quad B_\alpha U_\alpha^{-1} = U_\alpha^{-1} B_\alpha^{-1} A_\alpha^{-1} B_\alpha^{-1} A_\alpha B_\alpha\,.$$

By using these relations and (3.4), the element g can be changed into either the form not including U_α and U_α^{-1} or the form having U_α or U_α^{-1} at the left end. The former case means $g \in \Gamma_\alpha$ and the latter case means $g \in U_\alpha \Gamma_\alpha$. □

Note that $\hat{\Gamma}_\alpha$ is an extended Fricke group of Γ_α defined in §3.2 of [Sc].

Using this we can introduce a fundamental domain for $\hat{\Gamma}_\alpha$:

$$D(\hat{\Gamma}_\alpha) = \left\{ z \in \mathbf{H} \;\middle|\; \left| z + \frac{\sqrt{1+\alpha^2}}{\alpha} \right| \geq \frac{1}{\alpha},\ |z| \geq 1,\ -\frac{2+\alpha^2}{\alpha\sqrt{1+\alpha^2}} \leq \mathrm{Re}(z) \leq \frac{\alpha}{\sqrt{1+\alpha^2}} \right\}$$

(the part shaded by lines upward to the right in Fig. 3.1). Note that $D(\Gamma_\alpha)$ is a 2-sheeted covering of $D(\hat{\Gamma}_\alpha)$ and that η, ζ and i are elliptic fixed points of $\hat{\Gamma}_\alpha$ in $D(\hat{\Gamma}_\alpha)$ and their periods are equal to 2. Moreover, we obtain that the genus of the Riemann surface of $\hat{\Gamma}_\alpha$ is equal to 0 and the number of cusps of $\hat{\Gamma}_\alpha$ is equal to1. Then it follows from Theorem 2.1 that $\delta_1^a(\hat{\Gamma}_\alpha) = \delta_1^c(\hat{\Gamma}_\alpha) = 0$ and $\delta_2^c(\hat{\Gamma}_\alpha) = 1$.

Next we recall the function $J_{(X,Y,Z)}$ used in the relation (1.3). In the following discussion $J_{(X,Y,Z)}$ is written in the form J_α, since the representation of (X, Y, Z) is described by using the parameter α. By the Riemann mapping theorem we get a holomorphic mapping from F_α^*

to **H** and by applying Schwartz' reflection principle with respect to the reflections in the four circles:

$$\Sigma_1 : \mathrm{Re}(z) = 0\,, \quad \Sigma_2 : |z| = 1\,, \quad \Sigma_3 : \left| z + \frac{\sqrt{1+\alpha^2}}{\alpha} \right| = \frac{1}{\alpha}\,, \quad \Sigma_4 : \mathrm{Re}(z) = -\frac{\sqrt{1+\alpha^2}}{\alpha}\,,$$

we can construct the mapping J_α satisfying the following proposition:

PROPOSITION 3.3. (i) *J_α is invariant under the action of $\hat{\Gamma}_\alpha$, i.e., $J_\alpha(T(z)) = J_\alpha(z)$ for all $z \in \mathbf{H}$ and $T \in \hat{\Gamma}_\alpha$, where $T(z)$ is a Möbius transformation.*

(ii) *J_α maps L_α onto $\mathbf{R}$ where $L_\alpha = L_{\alpha 1} \cup L_{\alpha 2} \cup L_{\alpha 3} \cup L_{\alpha 4}$ and*

$$L_{\alpha 1} = L_1\,, \quad L_{\alpha 2} = \left\{ z \in \mathbf{H} \;\middle|\; |z| = 1 \text{ and } -\frac{\alpha}{\sqrt{1+\alpha^2}} \le \mathrm{Re}(z) \le 0 \right\},$$

$$L_{\alpha 3} = \left\{ z \in \mathbf{H} \;\middle|\; \left| z + \frac{\sqrt{1+\alpha^2}}{\alpha} \right| = \frac{1}{\alpha} \text{ and } -\frac{\sqrt{1+\alpha^2}}{\alpha} \le \mathrm{Re}(z) \le -\frac{\alpha}{\sqrt{1+\alpha^2}} \right\},$$

$$L_{\alpha 4} = \left\{ z \in \mathbf{H} \;\middle|\; \left| z + \frac{\sqrt{1+\alpha^2}}{\alpha} \right| \ge \frac{1}{\alpha} \text{ and } \mathrm{Re}(z) = -\frac{\sqrt{1+\alpha^2}}{\alpha} \right\}.$$

Especially, $J_\alpha(i\infty) = \infty$, $J_\alpha(i) = P$ for some $P \ge 1$, $J_\alpha(\zeta) = 0$ and $J_\alpha(\eta) = -1$.

(iii) *J_α maps F_α onto $\mathbf{C}$ where*

$$F_\alpha = \left\{ z \in \mathbf{H} \;\middle|\; \left| z + \frac{\sqrt{1+\alpha^2}}{\alpha} \right| \ge \frac{1}{\alpha},\ |z| \ge 1,\ \left| z - \frac{\sqrt{1+\alpha^2}}{\alpha} \right| \ge \frac{1}{\alpha}, \right.$$
$$\left. -\frac{\sqrt{1+\alpha^2}}{\alpha} \le \mathrm{Re}(z) \le \frac{\sqrt{1+\alpha^2}}{\alpha} \right\}.$$

(iv) *The mapping $J_\alpha : \mathbf{H} \to \mathbf{C}$ is holomorphic on $\mathbf{H}$.*

For the proof of this proposition we refer the reader to the proof of Proposition 5.1 in [A1]. We note that the assertion *(i)* comes from the following relations:

$$S = \Sigma_1\Sigma_2 = \Sigma_2\Sigma_1\,, \quad S_\alpha = \Sigma_3\Sigma_4 = \Sigma_4\Sigma_3\,, \quad U_\alpha = \Sigma_1\Sigma_4\,.$$

Now we recall relations used in Theorem 1.1. Let τ be an element in L_1 corresponding to (X, Y, Z) in M_1. If $\tau \in L_1$ the polynomial $p(x) = 4x^3 - g_2(\tau)x - g_3(\tau)$ has three distinct real roots. Let $x_1 < x_2 < x_3$ be these roots. Then we have

$$\wp'(u)^2 = 4(\wp(u) - x_1)(\wp(u) - x_2)(\wp(u) - x_3)\,. \tag{3.5}$$

The relation giving a holomorphic mapping between (X, Y, Z) and τ is

$$\wp(u) = (x_2 - x_1)J_\alpha(z) + x_2\,. \tag{3.6}$$

We study the relations (3.5) and (3.6). Taking the derivative of (3.6) with respect to u, we have

$$\frac{dJ_\alpha}{du} = \frac{1}{x_2 - x_1}\frac{d\wp}{du}\,. \tag{3.7}$$

By using (3.6) and the relation $(x_2 - x_3)/(x_2 - x_1) = -P$ (see Lemma 5.3 [A1]) we get the following equation:

$$(\wp(u) - x_1)(\wp(u) - x_2)(\wp(u) - x_3) = (x_2 - x_3)^3 (J_\alpha + 1) J_\alpha (J_\alpha - P).$$

On the one hand η, ζ and i are elliptic fixed points of $\hat{\Gamma}_\alpha$ in $D(\hat{\Gamma}_\alpha)$ and their periods are equal to 2, but on the other hand the group Γ_α does not have an elliptic fixed point, that is, the fundamental domain $D(\Gamma_\alpha)$ is a 2-sheeted covering of the fundamental domain $D(\hat{\Gamma}_\alpha)$ around η, ζ and i. We can make the same assertion for equivalent points of η, ζ and i under the action of $\hat{\Gamma}_\alpha$. Since $J_\alpha(\eta) = -1$, $J_\alpha(\zeta) = 0$ and $J_\alpha(i) = P$, we can take square roots of $J_\alpha + 1$, J_α and $J_\alpha - P$. Then the equation (3.5) is changed into the following:

$$\frac{d\wp}{du} = \pm 2(x_2 - x_1)^{\frac{3}{2}} (J_\alpha + 1)^{\frac{1}{2}} J_\alpha{}^{\frac{1}{2}} (J_\alpha - P)^{\frac{1}{2}}. \tag{3.8}$$

Comparing (3.7) and (3.8), we obtain

$$du = (const) \cdot \frac{1}{(J_\alpha + 1)^{\frac{1}{2}} J_\alpha{}^{\frac{1}{2}} (J_\alpha - P)^{\frac{1}{2}}} \frac{dJ_\alpha}{dz} dz \overset{\text{def}}{=} (const) \cdot \varphi_\alpha(z) dz.$$

THEOREM 3.3. *φ_α is a cusp form of weight 1 for Γ_α and φ_α^2 is a cusp form of weight 2 for $\hat{\Gamma}_\alpha$.*

PROOF. We have already known the existence of these cusp forms. We show the latter assertion. Let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\hat{\Gamma}_\alpha$. As $J_\alpha(Tz) = J_\alpha(z)$ from Proposition 3.3 (i), we get $J'_\alpha(Tz) = (cz+d)^2 J'_\alpha(z)$. Then it is easily checked that $\varphi_\alpha^2(Tz) = (cz+d)^4 \varphi_\alpha^2(z)$.

Next we will show that φ_α^2 vanishes at infinity. We introduce local coordinates around $z = i\infty$ defined by $w = e^{2\pi i z/\lambda}$ with $\lambda = 2\sqrt{1+\alpha^2}/\alpha$. Then an expansion of J_α around $i\infty$ is represented by $J_\alpha(z) = a_{-1}/w + a_0 + a_1 w + \cdots$ and its derivative with respect to z is $J'_\alpha(z) = b_{-1}/w + b_0 + b_1 w + \cdots$, where a_i and b_i for $i = -1, 0, 1, \cdots$ are some constants. We substitute these expansions in $\varphi_\alpha^2(z)$:

$$\varphi_\alpha^2(z) = \frac{J'^2_\alpha}{(J_\alpha + 1) J_\alpha (J_\alpha - P)} = \frac{(const)(1/w)^2 + \cdots}{(const)(1/w)^3 + \cdots} = (const) w + \cdots.$$

This means $\varphi_\alpha^2(i\infty) = 0$. Therefore φ_α^2 is a cusp form of weight 2 for $\hat{\Gamma}_\alpha$.

By using the same argument as above the first assertion is also proved. □

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Present Address:
DEPARTMENT OF MATHEMATICS, KEIO UNIVERSITY,
YOKOHAMA, 223–8522 JAPAN.
e-mail: ryu2@math.keio.ac.jp