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# Distinguished Bases of Non-simple Singularities

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Dedicated to Professor Takuo Fukuda on his sixtieth birthday

## 0. Introduction.

Let $f : (\mathbf{C}^n, \mathbf{0}) \to (\mathbf{C}, \mathbf{0})$ be an arbitrary function germ, with an isolated critical point at zero. Let $\Delta_1, \Delta_2, \cdots, \Delta_\mu$ be a distinguished basis of vanishing cycles in the homology group $H_{n-1}(V_\varepsilon; \mathbf{Z}) \cong \mathbf{Z}^\mu$ of the non-singular level manifold. With respect to such a basis the variation operator $Var$ (resp. $Var^{-1}$) of the singularity $f$ is represented by an upper triangular matrix. In [5], Gusein-Zade gave the following converse result for simple singularities.

GUSEIN-ZADE THEOREM 1. *Let $f : (\mathbf{C}^n, \mathbf{0}) \to (\mathbf{C}, \mathbf{0})$ be one of the simple singularities $A_k$, $D_k$, $E_6$, $E_7$ and $E_8$ and $\Delta_1, \Delta_2, \cdots, \Delta_\mu$ be an integral basis in the homology group $H_{n-1}(V_\varepsilon; \mathbf{Z}) \cong \mathbf{Z}^\mu$, in which the matrix of the operator $Var$ (resp. $Var^{-1}$) is upper triangular. Then $\Delta_1, \Delta_2, \cdots, \Delta_\mu$ is a distinguished basis of vanishing cycles.*

For the proof of this, the following result for simple singularities is used which is of interest in its own right.

GUSEIN-ZADE THEOREM 2. *Let $f : (\mathbf{C}^n, \mathbf{0}) \to (\mathbf{C}, \mathbf{0})$ be one of the simple singularities $A_k$, $D_k$, $E_6$, $E_7$ and $E_8$ in an odd number of variables $n$. For any vanishing cycle $\Delta$ and any distinguished basis $\Delta_1, \Delta_2, \cdots, \Delta_\mu$ for $f$, there exists a sequence of elementary substitutions, turning it into a distinguished basis $\Delta'_1, \Delta'_2, \cdots, \Delta'_\mu$ with the first element $\Delta'_1 = \pm\Delta$.*

In [3] page 103, V. I. Arnol'd et als propose as an open problem to study whether an analogous theorem to Gusein-Zade Theorem 2 is true for non-simple singularities. The purpose of the present paper is to give a negative answer to this problem. Two distinguished bases of vanishing cycles in the homology group $H_{n-1}(V_\varepsilon; \mathbf{Z})$ are said to be *elementary equivalent* if one of the two bases can be transfered into the other by a (finite) sequence of elementary substitutions and changing of the orientation of some of the elements of the basis. Then the main theorem in this paper can be stated as follows.

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MAIN THEOREM. *For any function $f$ in an odd number of variables with non-simple singularity, there exists a distinguished basis $\{\Delta_1, \Delta_2, \cdots, \Delta_\mu\}$ and a vanishing cycle $\Delta$ such that there does not exist any distinguished basis $\{\Delta'_1, \Delta'_2, \cdots, \Delta'_\mu\}$ which is elementary equivalent to $\{\Delta_1, \Delta_2, \cdots, \Delta_\mu\}$ with $\Delta'_1 = \pm\Delta$.*

In §1, we explain *elementary substitution* described in the Main Theorem. In §2 we prove our theorem for the singularity $\tilde{E}_6$. In §3 and §4, we show the same content for the singularities $\tilde{E}_7$ and $\tilde{E}_8$. Finally, in §5 we prove the theorem for arbitrary non-simple singularity.

## 1. Elementary substitutions.

We describe two elementary operations of distinguished bases [3]. Let $f : (\mathbf{C}^n, \mathbf{0}) \to (\mathbf{C}, \mathbf{0})$ be an arbitrary function germ, with an isolated critical point at zero. By Milnor's theorem there exists a number $\rho > 0$ such that the sphere $S_r \subset \mathbf{C}^n$ of radius $r \leq \rho$ with center at zero intersects the level set $f^{-1}(0)$ transversally. For sufficiently small $\varepsilon_0 > 0$ the level manifold $f^{-1}(\varepsilon)$ is also transversal to the sphere $S_\rho$ for $|\varepsilon| \leq \varepsilon_0$. We set $B_\rho = \{x \in \mathbf{C}^n \mid \|x\| \leq \rho\}$ and $D_{\varepsilon_0} = \{x \in \mathbf{C} \mid \|x\| \leq \varepsilon_0\}$. For a sufficiently small perturbation $\tilde{f}$ of $f$, we see that $\tilde{f}^{-1}(\varepsilon) \cap B_\rho$ is diffeomorphic to $V_\varepsilon = f^{-1}(\varepsilon) \cap B_\rho$ for $0 < |\varepsilon| \leq \varepsilon_0$. We may suppose that $\tilde{f}$ has in $B_\rho$ only non-degenerate critical points with distinct critical values $\{z_i\}$ in $D_{\varepsilon_0}$. Let $\{\Delta_1, \Delta_2, \cdots, \Delta_\mu\}$ be a distinguished basis of vanishing cycles in $H_{n-1}(V_\varepsilon; \mathbf{Z}) \cong \mathbf{Z}^\mu$. Let $\{u_i\}$ be a system of paths defining the distinguished basis. This means that the following conditions hold:

(i) the $u_i(t)$ is non-self-intersecting path in $D_{\varepsilon_0}$, joining the critical values $z_i$ of the perturbation $\tilde{f}$ of the function $f$ to the non-critical value $z_0 (u(0) = z_i, u(1) = z_0)$ and not passing through any critical values of $\tilde{f}$ for $t \neq 0$;

(ii) the path $u_i$ and $u_j$ intersect each other only at the point $u_i(1) = u_j(1) = z_0 (|z_0| = \varepsilon_0)$;

(iii) the paths $u_1, u_2, \cdots, u_\mu$ are numbered in the order such that they enter the point $z_0$, counting clockwise, begining at the boundary $\partial D_{\varepsilon_0}$ of $D_{\varepsilon_0}$ (see Figure 1).

Let $\tau_i$ be a *simple loop* corresponding to the path $u_i$. The *simple loop* corresponding to $u_i$ is a loop going from $z_0$ to $z_i$ by the path $u_i$, going round $z_i$ in positive direction (unticlockwise) and returning to $z_0$ by the path $u_i$.

DEFINITION of the operation $\alpha_m$ $(1 \leq m < \mu)$. We define a new system of paths $\{\tilde{u}_i\}$ in the following manner:

$$\tilde{u}_i = u_i \quad \text{for} \quad i \neq m, m+1;$$
$$\tilde{u}_{m+1} = u_m;$$
$$\tilde{u}_m = u_{m+1}\tau_m ,$$

where $u_{m+1}\tau_m$ means first go along $u_{m+1}$ then along $\tau_m$.

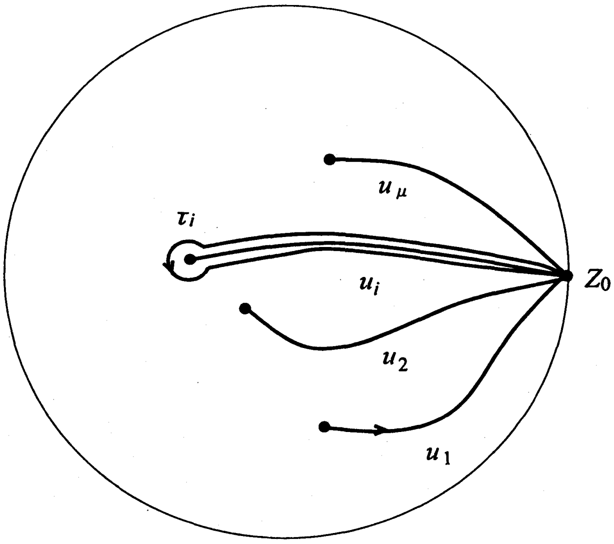


FIGURE 1. The paths $\{u_i\}$ and the loop $\tau_i$.

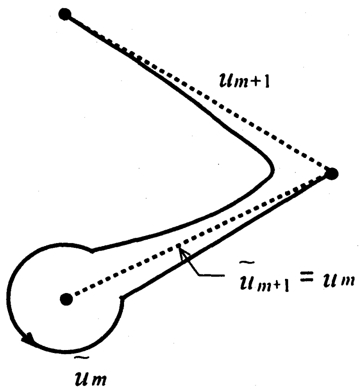


FIGURE 2. $\tilde{u}_{m+1}$ and $\tilde{u}_m$.

This system of paths defines a distinguished basis $\{\tilde{\Delta}_i\}$, related to the basis $\{\Delta_i\}$ by the formulae:

$$\begin{aligned}
&\tilde{\Delta}_i = \Delta_i \quad \text{for} \quad i \neq m, m+1; \\
&\tilde{\Delta}_{m+1} = \Delta_m; \\
&\tilde{\Delta}_m = h_m(\Delta_{m+1}) = \Delta_{m+1} + (-1)^{n(n+1)/2}(\Delta_{m+1}, \Delta_m)\Delta_m ,
\end{aligned}$$

where $(\Delta_{m+1}, \Delta_m)$ is the intersection index of the vanishing cycles $\Delta_{m+1}$ and $\Delta_m$. The operation of transferring the distinguished basis $\{\Delta_i\}$ to the distinguished basis $\{\tilde{\Delta}_i\}$, described by these formulae, is denoted by $\alpha_m$.

DEFINITION of the operation $\beta_{m+1}$ ($1 \leq m < \mu$). We define a new system of paths $\{\tilde{u}'_i\}$ in the following manner:

$$\tilde{u}'_i = u_i \quad \text{for} \quad i \neq m, m+1;$$
$$\tilde{u}'_m = u_{m+1};$$
$$\tilde{u}'_{m+1} = u_m \tau_{m+1}^{-1},$$

where $u_m \tau_{m+1}^{-1}$ means first go along $u_m$ then along $\tau_{m+1}^{-1}$.

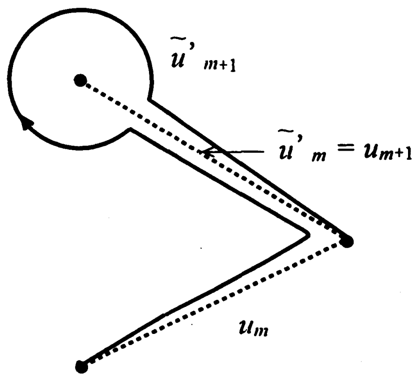


FIGURE 3. $\tilde{u}'_m$ and $\tilde{u}'_{m+1}$.

This system of paths defines a distinguished basis $\{\tilde{\Delta}'_i\}$, related to the basis $\{\Delta_i\}$ by the formulae:

$$\tilde{\Delta}'_i = \Delta_i \quad \text{for} \quad i \neq m, m+1;$$
$$\tilde{\Delta}'_m = \Delta_{m+1};$$
$$\tilde{\Delta}'_{m+1} = h_{m+1}^{-1}(\Delta_m) = \Delta_m + (-1)^{n(n+1)/2}(\Delta_{m+1}, \Delta_m)\Delta_{m+1},$$

The operation of transferring the distinguished basis $\{\Delta_i\}$ to the distinguished basis $\{\tilde{\Delta}'_i\}$, described by these formulae, is denoted by $\beta_{m+1}$.

In [6] Gusein-Zade has proved the following important assersion.

THEOREM ([6], page 44). *Any two distinguished bases can be obtained from each other by iteration of the operations $\alpha_m$ and $\beta_m$, with subsequent change of orientation of some of the elements of the basis.*

## 2. A distinguished basis of the singularity $\tilde{E}_6$.

In this section we prove our theorem for the singularity $\tilde{E}_6 : x^3 + y^3 + z^3$: We construct a vanishing cycle $\Delta$ and a distinguished basis $\{\Delta_1, \Delta_2, \cdots, \Delta_8\}$ which can never be turned into a distinguished basis $\{\Delta'_1, \Delta'_2, \cdots, \Delta'_8\}$ with $\Delta'_1 = \pm\Delta$ by a sequence of elementary substitutions.

**2.1. Construction of a distinguished basis $\{\Delta_i\}$.** There exists a distinguished basis $\delta_1, \cdots, \delta_8$ of the singularity $\tilde{E}_6 : x^3 + y^3 + z^3$ with the following Dynkin diagram (see [4]).

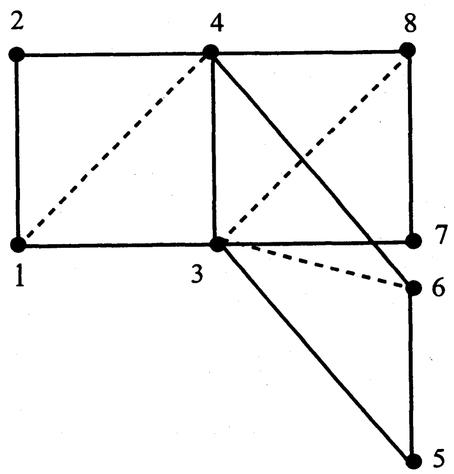


FIGURE 4. The Dynkin diagram of the basis $\{\delta_i\}$.

Consider the sequence of elementary substitutions

$$\alpha_2, \alpha_1, \alpha_3, \alpha_4, \alpha_3, \alpha_6, \alpha_5, \alpha_7, \alpha_6, \beta_6, \alpha_4, (\delta_2 \mapsto -\delta_2), (\delta_7 \mapsto -\delta_7),$$

where first operate $\alpha_2$, then $\alpha_1$ and $\alpha_3$ and so on, and where $(\delta_2 \mapsto -\delta_2)$ and $(\delta_7 \mapsto -\delta_7)$ are operations which change the orientation of $\delta_2$ and $\delta_7$ respectively. Then it sends the basis $\{\delta_i\}$ to a distinguished basis $\{\Delta_i\}$ with the Dynkin diagram:

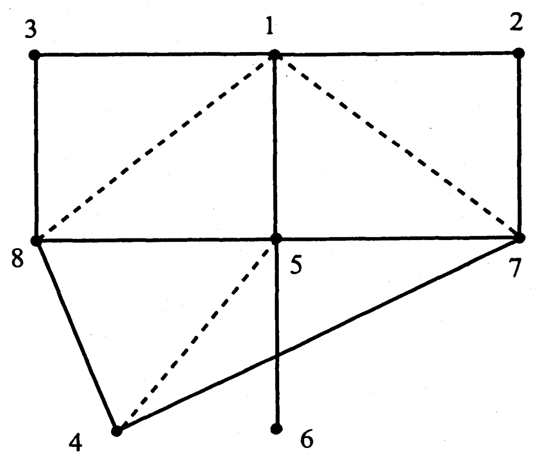


FIGURE 5. The Dynkin diagram of the basis $\{\Delta_i\}$.

Therefore, the matrix $L$ of the inverse variation operator

$$Var^{-1} : H_2(V_\varepsilon; \mathbf{Z}) \to H_2(V_\varepsilon, \partial V_\varepsilon; \mathbf{Z})$$

in the basis $\{\Delta_i\}$ and its dual basis $\{\nabla_i\}$, which is determined by the relation $(\nabla_i, \Delta_j) = \delta_{ij}$, is

$$L = \begin{bmatrix} 1 & -1 & -1 & 0 & -1 & 0 & 1 & 1 \\ & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ & & 1 & 0 & 0 & 0 & 0 & -1 \\ & & & 1 & 1 & 0 & -1 & -1 \\ & \text{\huge 0} & & & 1 & -1 & -1 & -1 \\ & & & & & 1 & 0 & 0 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{bmatrix}.$$

**2.2. Construction of a vanishing cycle $\Delta$.** There exists a system of paths $u_1, \cdots, u_8$ by which the distinguished basis $\{\Delta_i\}$ of vanishing cycles is defined. This means that the cycle $\Delta_i$ vanishes along the path $u_i$ joining some critical value $z_i$ with the non-critical value $z_0$. Let $\{\tau_i\}$ be the simple loops corresponding to the paths $\{u_i\}$. Consider the path $\tilde{u}$ represented in terms of the path $u_5$ and the loops $\{\tau_i\}$ in the following manner:

$$\tilde{u} = u_5 \cdot \tau_6^{-1} \cdot \tau_7^{-1} \cdot \tau_8^{-1} \cdot \tau_1^{-1} \cdot \tau_4^{-1}\tau_5^{-1},$$

where $u_5 \cdot \tau_6^{-1} \dots$ means first go along $u_5$ then along $\tau_6^{-1}$ and so on.

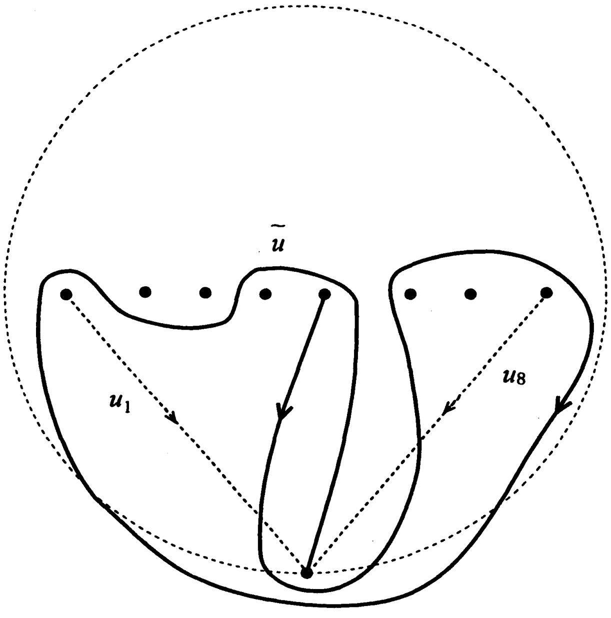


FIGURE 6. The path $\tilde{u}$ defining $\Delta$.

Let $\Delta$ be the vanishing cycle defined by the path $\tilde{u}$. Then the vanishing cycle $\Delta$ is represented in terms of the basis $\{\Delta_i\}$ and the transformations

$$h_i = (\tau_i)_* : H_2(V_\varepsilon; \mathbf{Z}) \to H_2(V_\varepsilon; \mathbf{Z})$$

as follows:

$$\Delta = h_5^{-1} \circ h_4^{-1} \circ h_1^{-1} \circ h_8^{-1} \circ h_7^{-1} \circ h_6^{-1}(\Delta_5) = -\Delta_1 + \Delta_4 + \Delta_6 + \Delta_7 + \Delta_8 .$$

By the matrix $L$ of the inverse variation operator $Var^{-1}$ defined in 2.1, we have

$$(2.2)\qquad \begin{aligned} Var^{-1}(\Delta) &= Var^{-1}(-\Delta_1 + \Delta_4 + \Delta_6 + \Delta_7 + \Delta_8) \\ &= \nabla_1 - \nabla_2 - \nabla_3 - \nabla_4 - 3\nabla_5 + \nabla_6 + \nabla_7 + \nabla_8 , \end{aligned}$$

where $\{\nabla_i\}$ is the dual basis of $\{\Delta_i\}$.

**2.3. Elements satisfing $(Var^{-1}\delta, \delta) = 1$ and $(Var^{-1}\Delta, \delta) = 0$.** For the vanishing cycle $\Delta$ and the distinguished basis $\{\Delta_1, \Delta_2, \cdots, \Delta_8\}$, if there exists a sequence of elementary substitutions which turns the basis $\{\Delta_1, \Delta_2, \cdots, \Delta_8\}$ into a distinguished basis $\{\Delta'_1, \Delta'_2, \cdots, \Delta'_8\}$ with $\Delta'_1 = \pm\Delta$, then the matrix $L$ of the inverse variation operater $Var^{-1}$ with respect to the distinguished basis $\{\Delta'_i\}$ is an upper triangular matrix with diagonal entries equal $(-1)^{n(n+1)/2} = 1$, since $n = 3$. Therefore we have the following two equalities;

$$(Var^{-1}\Delta'_i, \Delta'_i) = 1 , \quad (Var^{-1}\Delta, \Delta'_i) = 0 \quad (i = 2, \cdots, 8) .$$

We shall determine conditions in terms of $\Delta_1, \Delta_2, \cdots, \Delta_8$ for a cycle $\delta$ to be an element of such a distinguished basis $\{\Delta'_i\}$. If $\delta \neq \Delta$ is any element of the distinguished basis $\{\Delta'_i\}$, similarly the following equalities hold;

$$(2.3.1)\qquad (Var^{-1}\delta, \delta) = 1 ,$$

$$(2.3.2)\qquad (Var^{-1}\Delta, \delta) = 0 .$$

Let us represent such a $\delta$ as a linear combination of the distinguished basis $\{\Delta_i\}$:

$$\delta = \sum_{i=1}^{8} a_i \Delta_i (a_i \in \mathbf{Z}) .$$

Then, by the matrix $L$, we have

$$\begin{aligned} Var^{-1}(\delta) =&(a_1 - a_2 - a_3 - a_5 + a_7 + a_8)\nabla_1 + (a_2 - a_7)\nabla_2 \\ &+ (a_3 - a_8)\nabla_3 + (a_4 + a_5 - a_7 - a_8)\nabla_4 \\ &+ (a_5 - a_6 - a_7 - a_8)\nabla_5 + a_6\nabla_6 + a_7\nabla_7 + a_8\nabla_8 , \end{aligned}$$

where $\{\nabla_i\}$ is the dual basis of $\{\Delta_i\}$. Therefore we have

$$(2.3.3)\qquad \begin{aligned} (Var^{-1}\delta, \delta) =& a_1(a_1 - a_2 - a_3 - a_5 + a_7 + a_8) + a_2(a_2 - a_7) + a_3(a_3 - a_8) \\ &+ a_4(a_4 + a_5 - a_7 - a_8) + a_5(a_5 - a_6 - a_7 - a_8) + a_6^2 + a_7^2 + a_8^2 \\ =& \left\{a_1 - \frac{1}{2}(a_2 + a_3 + a_5 - a_7 - a_8)\right\}^2 \\ &+ \frac{3}{4}\left\{a_2 - \frac{1}{3}(a_3 + a_5 + a_7 - a_8)\right\}^2 + \frac{2}{3}\left\{a_3 - \frac{1}{4}(2a_5 - a_7 + a_8)\right\}^2 \\ &+ \left\{a_4 + \frac{1}{2}(a_5 - a_7 - a_8)\right\}^2 + \frac{1}{4}(a_5 - 2a_6)^2 + \frac{3}{8}(a_7 - a_8)^2 . \end{aligned}$$

From (2.3.1) and (2.3.3), we have

$$\begin{aligned}(2.3.4)\qquad & 6\{2a_1 - (a_2 + a_3 + a_5 - a_7 - a_8)\}^2 \\ & + 2\{3a_2 - (a_3 + a_5 + a_7 - a_8)\}^2 + \{4a_3 - (2a_5 - a_7 + a_8)\}^2 \\ & + 6\{2a_4 + (a_5 - a_7 - a_8)\}^2 + 6(a_5 - 2a_6)^2 \\ & + 9(a_7 - a_8)^2 = 24\,.\end{aligned}$$

On the other hand, from (2.2) and (2.3.2) we have

$$\begin{aligned}(2.3.5)\qquad (Var^{-1}\Delta, \delta) &= \left(\nabla_1 - \nabla_2 - \nabla_3 - \nabla_4 - 3\nabla_5 + \nabla_6 + \nabla_7 + \nabla_8, \sum_{i=1}^{8} a_i \Delta_i\right) \\ &= a_1 - a_2 - a_3 - a_4 - 3a_5 + a_6 + a_7 + a_8 = 0\,.\end{aligned}$$

LEMMA 2.3. *The common integral solutions of* (2.3.4) *and* (2.3.5) *are exhausted by the following seven types*;

(1) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = \pm(1 - t_1, 0, 0, t_1 - 1, 1, 1, t_1, t_1)$ $(t_1 \in \mathbf{Z})$
(2) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = \pm(1 - t_2, 0, 1, t_2, 0, 0, t_2, t_2)$ $(t_2 \in \mathbf{Z})$
(3) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = \pm(1 - t_3, 1, 0, t_3, 0, 0, t_3, t_3)$ $(t_3 \in \mathbf{Z})$
(4) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = \pm(1 - t_4, 0, 0, t_4, 0, 0, t_4, t_4 - 1)$ $(t_4 \in \mathbf{Z})$
(5) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = \pm(-t_5, 0, 0, t_5 + 1, 0, 0, t_5, t_5 + 1)$ $(t_5 \in \mathbf{Z})$
(6) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = \pm(-t_6, 0, 1, t_6, 0, 0, t_6, t_6 + 1)$ $(t_6 \in \mathbf{Z})$
(7) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = \pm(1 - t_7, 1, 0, t_7 - 1, 0, 0, t_7, t_7 - 1)$ $(t_7 \in \mathbf{Z})$

OUTLINES OF PROOF. Since $a_i \in \mathbf{Z}(i = 1, \cdots, 8)$, we have $(a_5 - 2a_6)^2 \le 4$ from (2.3.4). There are three cases to be considered: Case 1, $a_5 - 2a_6 = \pm 2$; Case 2, $a_5 - 2a_6 = \pm 1$; Case 3, $a_5 - 2a_6 = 0$.

Case 1. $a_5 - 2a_6 = \pm 2$. In this case, there are no common solutions to (2.3.4) and (2.3.5).

Case 2. $a_5 - 2a_6 = \pm 1$. From (2.3.4), it follows that

$$\begin{aligned}(\mathrm{II})\qquad & 6\{2a_1 - (a_2 + a_3 + a_5 - a_7 - a_8)\}^2 + 2\{3a_2 - (a_3 + a_5 + a_7 - a_8)\}^2 \\ & + \{4a_3 - (2a_5 - a_7 + a_8)\}^2 + 6\{2a_4 - (-a_5 + a_7 + a_8)\}^2 \\ & + 9(a_7 - a_8)^2 = 18\,.\end{aligned}$$

Since $a_i \in \mathbf{Z}(i = 1, \cdots, 8)$, we have $(a_7 - a_8)^2 \le 1$. Therefore there are two cases to be considered: Case 2.1, $a_7 - a_8 = 0$; Case 2.2, $a_7 - a_8 = \pm 1$.

Case 2.1. $a_7 - a_8 = 0$. From (II), it follows that

$$\begin{aligned} & 3\{2a_1 - (a_2 + a_3 + a_5 - 2a_7)\}^2 + \{3a_2 - (a_3 + a_5)\}^2 \\ & + 2(2a_3 - a_5)^2 + 3\{2a_4 - (-a_5 + 2a_7)\}^2 = 9\,.\end{aligned}$$

In case $(a_2 + a_3) \equiv 0 \pmod 2$, we obtain the integral solutions of the type (1) in Lemma 2.3. In the other cases, there are no common solutions to (2.3.4) and (2.3.5).

Case 2.2. $a_7 - a_8 = \pm 1$. In this case, there are no common solutions to (2.3.4) and (2.3.5).

Case 3. $a_5 - 2a_6 = 0$. From (2.3.4), it follows that

$$\text{(III)} \qquad \begin{aligned} &6\{2a_1 - (a_2 + a_3 + a_5 - a_7 - a_8)\}^2 + 2\{3a_2 - (a_3 + a_5 + a_7 - a_8)\}^2 \\ &+ \{4a_3 - (2a_5 - a_7 + a_8)\}^2 + 6\{2a_4 + (a_5 - a_7 - a_8)\}^2 \\ &+ 9(a_7 - a_8)^2 = 24 . \end{aligned}$$

Since $a_i \in \mathbf{Z}(i = 1, \cdots, 8)$, we have $(a_7 - a_8)^2 \leq 1$. Therefore there are two cases to be considered: Case 3.1, $a_7 - a_8 = 0$; Case 3.2, $a_7 - a_8 = \pm 1$.

Case 3.1. $a_7 - a_8 = 0$. From (III), it follows that

$$\begin{aligned} &3\{2a_1 - (a_2 + a_3 + a_5 - 2a_7)\}^2 + \{3a_2 - (a_3 + a_5)\}^2 \\ &+ 2(2a_3 - a_5)^2 + 3\{2a_4 + (a_5 - 2a_7)\}^2 = 12 . \end{aligned}$$

In case $(a_2 + a_3) \equiv 1 \pmod 2$, we obtain the integral solutions of the type (2), (3) in Lemma 2.3. In the other cases, there are no common solutions to (2.3.4) and (2.3.5).

Case 3.2. $a_7 - a_8 = \pm 1$, that is $a_8 = a_7 + \varepsilon_1$ $(\varepsilon_1 = \pm 1)$. From (III), it follows that

$$\begin{aligned} &6\{2a_1 - (a_2 + a_3 + a_5 - 2a_7 - \varepsilon_1)\}^2 + 2\{3a_2 - (a_3 + a_5 - \varepsilon_1)\}^2 \\ &+ \{4a_3 - (2a_5 + \varepsilon_1)\}^2 + 6\{2a_4 - (2a_7 - a_5 + \varepsilon_1)\}^2 = 15 . \end{aligned}$$

In case $(a_2 + a_3) \equiv 0 \pmod 2$, we obtain the integral solutions of the type (4), (5) in Lemma 2.3.
In case $(a_2 + a_3) \equiv 1 \pmod 2$, we obtain the integral solutions of the type (6), (7) in Lemma 2.3. This completes the proof of Lemma 2.3.

The above seven solutions are not $\mathbf{Z}$-linear independent, for

$$(t_3 - t_5 - t_7)(-(2) + (4) + (6)) - (t_2 - t_4 - t_6)(-(3) + (5) + (7)) = 0 .$$

So there are only six $\mathbf{Z}$-linear independent solutions. Therefore together with $\Delta$, one has at most seven $\mathbf{Z}$-linear independent elements satisfying the requirements (2.3.1) and (2.3.2). So they can not form a basis for $H_2(V_\varepsilon; \mathbf{Z})$. Thus we have

THEOREM 2.3. *Let $\{\Delta_1, \Delta_2, \cdots, \Delta_8\}$ be the distinguished basis constructed in* §2.1 *and let $\Delta$ be the vanishing cycle constructed in* §2.2. *Then the distinguished basis $\{\Delta_1, \Delta_2, \cdots, \Delta_8\}$ can never be turned into a distinguished basis $\{\Delta_i'\}$; $(i = 1, \cdots, 8)$ with $\Delta_1' = \pm\Delta$ by a sequence of elementary substitutions.*

## 3. A distinguished basis of the singularity $\tilde{E}_7$.

In this section we prove our theorem for the singularity $\tilde{E}_7 : x^4 + y^4 + z^2$: We construct a vanishing cycle $\Delta$ and a distinguished basis $\{\Delta_1, \Delta_2, \cdots, \Delta_9\}$ which can never be turned into a distinguished basis $\{\Delta'_1, \Delta'_2, \cdots, \Delta'_9\}$ with $\Delta'_1 = \pm\Delta$ by a sequence of elementary substitutions.

**3.1. Selection of distinguished basis $\{\Delta_i\}$.** There exists a distinguished basis $\Delta_1, \cdots, \Delta_9$ of the singularity $\tilde{E}_7 : x^4 + y^4 + z^2$ with the following Dynkin diagram (see [2], [8]).

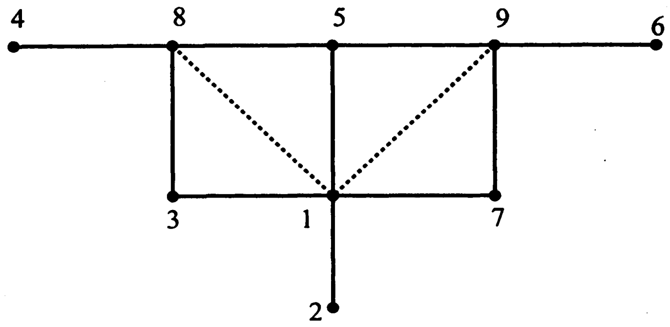


FIGURE 7. The Dynkin diagram of the basis $\{\Delta_i\}$.

Therefore, the matrix $L$ of the inverse variation operator

$$Var^{-1} : H_2(V_\varepsilon; \mathbf{Z}) \to H_2(V_\varepsilon, \partial V_\varepsilon; \mathbf{Z})$$

in the basis $\{\Delta_i\}$ and its dual basis $\{\nabla_i\}$ is

$$L = \begin{bmatrix} 1 & -1 & -1 & 0 & -1 & 0 & -1 & 1 & 1 \\ & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ & & & 1 & 0 & 0 & 0 & -1 & 0 \\ & 0 & & & 1 & 0 & 0 & -1 & -1 \\ & & & & & 1 & 0 & 0 & -1 \\ & & & & & & 1 & 0 & -1 \\ & & & & & & & 1 & 0 \\ & & & & & & & & 1 \end{bmatrix}.$$

**3.2. Construction of a vanishing cycle $\Delta$.** There exists a system of paths $u_1, u_2, \cdots, u_9$ by which the distinguished basis $\{\Delta_i\}$ of vanishing cycles is defined. This means that the cycle $\Delta_i$ vanishes along the path $u_i$ joining some critical value $z_i$ with the non-critical value $z_0$. Let $\{\tau_i\}$ be the simple loops corresponding to the paths $\{u_i\}$. Consider the path $\tilde{u}$ represented in terms of the path $u_1$ and the loops $\{\tau_i\}$ in the following manner:

$$\begin{aligned} \tilde{u} =& u_1 \cdot \tau_2^{-1} \cdot \tau_3^{-1} \cdot \tau_4^{-1} \cdot \tau_5^{-1} \cdot \tau_6^{-1} \cdot \tau_7^{-1} \cdot \tau_8^{-1} \\ & \cdot \tau_9^{-1} \cdot \tau_4^{-1} \cdot \tau_5^{-1} \cdot \tau_6^{-1} \cdot \tau_8^{-1} \cdot \tau_9^{-1} \cdot \tau_5 \cdot \tau_9 \cdot \tau_8 \cdot \tau_6 \cdot \tau_5 \cdot \tau_4 \cdot \tau_1^{-1}, \end{aligned}$$

where $u_1 \cdot \tau_2^{-1} \ldots$ means first go along $u_1$ then along $\tau_2^{-1}$ and so on.

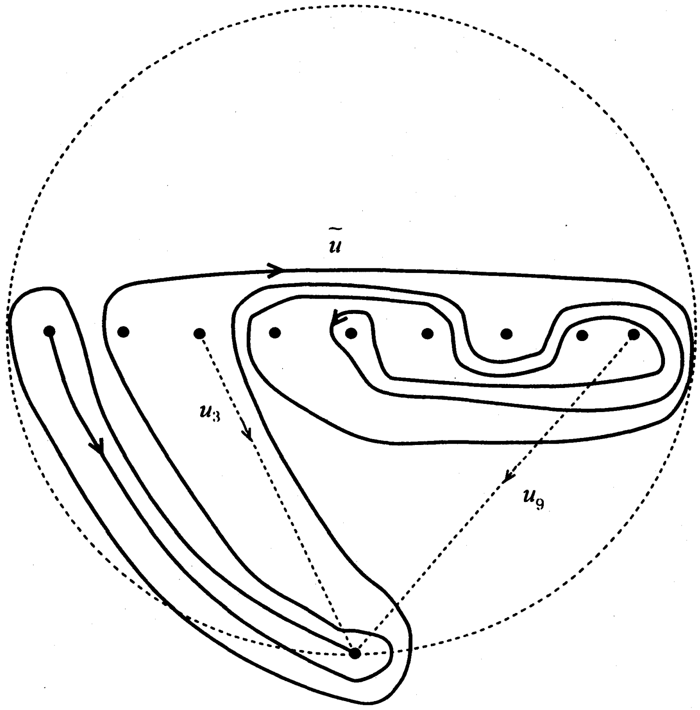


FIGURE 8. The path $\tilde{u}$ defining $\Delta$.

Let $\Delta$ be the vanishing cycle defined by the path $\tilde{u}$. Then the vanishing cycle $\Delta$ is represented by the basis $\{\Delta_i\}$ and the transformations

$$h_i = (\tau_i)_* : H_2(V_\varepsilon; \mathbf{Z}) \to H_2(V_\varepsilon; \mathbf{Z})$$

as follows:

$$\begin{aligned}\Delta =& h_1^{-1} \circ h_4 \circ h_5 \circ h_6 \circ h_8 \circ h_9 \circ h_5 \circ h_9^{-1} \circ h_8^{-1} \circ h_6^{-1} \circ h_5^{-1} \circ h_4^{-1} \\ &\circ h_9^{-1} \circ h_8^{-1} \circ h_7^{-1} \circ h_6^{-1} \circ h_5^{-1} \circ h_4^{-1} \circ h_3^{-1} \circ h_2^{-1}(\Delta_1) \\ =& \Delta_2 + \Delta_3 + \Delta_4 + 2\Delta_5 + \Delta_6 + \Delta_7 + 2\Delta_8 + 2\Delta_9 .\end{aligned}$$

By the matrix $L$ of the inverse variation operater $Var^{-1}$ defined in 3.1, we have

$$\begin{aligned} Var^{-1}(\Delta) &= Var^{-1}(\Delta_2 + \Delta_3 + \Delta_4 + 2\Delta_5 + \Delta_6 + \Delta_7 + 2\Delta_8 + 2\Delta_9) \\ &= -\nabla_1 + \nabla_2 - \nabla_3 - \nabla_4 - 2\nabla_5 - \nabla_6 - \nabla_7 + 2\nabla_8 + 2\nabla_9 , \end{aligned} \tag{3.2}$$

where $\{\nabla_i\}$ is the dual basis of $\{\Delta_i\}$.

**3.3. Elements satisfing $(Var^{-1}\delta, \delta) = 1$ and $(Var^{-1}\Delta, \delta) = 0$.** For the vanishing cycle $\Delta$ and the distinguished basis $\{\Delta_1, \Delta_2, \cdots, \Delta_9\}$, if there exists a sequence of elementary substitutions which turns $\{\Delta_1, \Delta_2, \cdots, \Delta_9\}$ into a distinguished basis $\{\Delta_1', \Delta_2', \cdots, \Delta_9'\}$ with $\Delta_1' = \pm\Delta$, then the matrix $L$ of the inverse variation operater $Var^{-1}$ with respect to the distinguished basis $\{\Delta_i'\}$ is an upper triangular matrix with diagonal entries equal

$(-1)^{n(n+1)/2} = 1$, since $n = 3$. Therefore we have the following two equalities;

$$(Var^{-1}\Delta'_i, \Delta'_i) = 1\,, \quad (Var^{-1}\Delta, \Delta'_i) = 0 \quad (i = 2, \cdots, 9)\,.$$

We shall determine conditions in terms of $\Delta_1, \Delta_2, \cdots, \Delta_9$ for a cycle $\delta$ to be an element of such a distinguished basis $\{\Delta'_i\}$. If $\delta \neq \Delta$ is any element of the distinguished basis $\{\Delta'_i\}$, similarly the following equalities hold;

$$(Var^{-1}\delta, \delta) = 1\,, \tag{3.3.1}$$

$$(Var^{-1}\Delta, \delta) = 0\,. \tag{3.3.2}$$

Let us represent such a $\delta$ as a linear combination of the distinguished basis $\{\Delta_i\}$:

$$\delta = \sum_{i=1}^{9} a_i \Delta_i (a_i \in \mathbf{Z})\,.$$

Then, by the matrix $L$, we have

$$\begin{aligned} Var^{-1}(\delta) =& (a_1 - a_2 - a_3 - a_5 - a_7 + a_8 + a_9)\nabla_1 + a_2\nabla_2 \\ &+ (a_3 - a_8)\nabla_3 + (a_4 - a_8)\nabla_4 \\ &+ (a_5 - a_8 - a_9)\nabla_5 + (a_6 - a_9)\nabla_6 + (a_7 - a_9)\nabla_7 + a_8\nabla_8 + a_9\nabla_9\,, \end{aligned}$$

where $\{\nabla_i\}$ is the dual basis of $\{\Delta_i\}$. Therefore we have

$$\begin{aligned} (Var^{-1}\delta, \delta) =& a_1(a_1 - a_2 - a_3 - a_5 - a_7 + a_8 + a_9) + a_2^2 + a_3(a_3 - a_8) \\ &+ a_4(a_4 - a_8) + a_5(a_5 - a_8 - a_9) + a_6(a_6 - a_9) \\ &+ a_7(a_7 - a_9) + a_8^2 + a_9^2 \\ =& \left(a_2 - \frac{1}{2}a_1\right)^2 + \left\{a_3 - \frac{1}{2}(a_1 + a_8)\right\}^2 \\ &+ \left(a_4 - \frac{1}{2}a_8\right)^2 + \left\{a_5 - \frac{1}{2}(a_1 + a_8 + a_9)\right\}^2 \\ &+ \left(a_6 - \frac{1}{2}a_9\right)^2 + \left\{a_7 - \frac{1}{2}(a_1 + a_9)\right\}^2 + \frac{1}{4}(a_8 - a_9)^2 \\ =& \frac{1}{4}(2a_2 - a_1)^2 + \frac{1}{4}\{2a_3 - (a_1 + a_8)\}^2 + \frac{1}{4}(2a_4 - a_8)^2 \\ &+ \frac{1}{4}\{2a_5 - (a_1 + a_8 + a_9)\}^2 + \frac{1}{4}(2a_6 - a_9)^2 \\ &+ \frac{1}{4}\{2a_7 - (a_1 + a_9)\}^2 + \frac{1}{4}(a_8 - a_9)^2\,. \end{aligned} \tag{3.3.3}$$

From (3.3.1) and (3.3.3), we have

$$\begin{aligned} &(2a_2 - a_1)^2 + \{2a_3 - (a_1 + a_8)\}^2 \\ &+ (2a_4 - a_8)^2 + \{2a_5 - (a_1 + a_8 + a_9)\}^2 \\ &+ (2a_6 - a_9)^2 + \{2a_7 - (a_1 + a_9)\}^2 + (a_8 - a_9)^2 = 4\,. \end{aligned} \tag{3.3.4}$$

On the other hand, from (3.2) and (3.3.2) we have

$$(Var^{-1}\Delta, \delta) = \left(-\nabla_1 + \nabla_2 - \nabla_3 - \nabla_4 - 2\nabla_5 - \nabla_6 - \nabla_7 + 2\nabla_8 + 2\nabla_9, \sum_{i=1}^{9} a_i \Delta_i\right)$$
$$= -a_1 + a_2 - a_3 - a_4 - 2a_5 - a_6 - a_7 + 2a_8 + 2a_9 = 0\,. \tag{3.3.5}$$

LEMMA 3.3. *The integral common solutions of* (3.3.4) *and* (3.3.5) *are exhausted by the following ten types*;

(1) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$
$= \pm(1, 1, t_1, t_1, 2t_1, t_1, t_1, 2t_1, 2t_1)$ $\quad (t_1 \in \mathbf{Z})$

(2) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$
$= \pm(1, 1, t_2, t_2 - 1, 2t_2 - 1, t_2 - 1, t_2, 2t_2 - 1, 2t_2 - 1)$ $\quad (t_2 \in \mathbf{Z})$

(3) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$
$= \pm(0, 0, t_3, t_3, 2t_3, t_3 - 1, t_3 - 1, 2t_3, 2t_3 - 1)$ $\quad (t_3 \in \mathbf{Z})$

(4) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$
$= \pm(0, 0, t_4, t_4, 2t_4 + 1, t_4, t_4, 2t_4, 2t_4 + 1)$ $\quad (t_4 \in \mathbf{Z})$

(5) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$
$= \pm(0, 0, t_5, t_5, 2t_5 - 2, t_5 - 1, t_5 - 1, 2t_5 - 1, 2t_5 - 2)$ $\quad (t_5 \in \mathbf{Z})$

(6) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$
$= \pm(0, 0, t_6, t_6, 2t_6 - 1, t_6, t_6, 2t_6 - 1, 2t_6)$ $\quad (t_6 \in \mathbf{Z})$

(7) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$
$= \pm(0, 0, t_7, t_7, 2t_7 - 1, t_7 - 1, t_7 - 1, 2t_7 - 1, 2t_7 - 1)$ $\quad (t_7 \in \mathbf{Z})$

(8) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$
$= \pm(0, 0, t_8, t_8 - 1, 2t_8 - 1, t_8, t_8 - 1, 2t_8 - 1, 2t_8 - 1)$ $\quad (t_8 \in \mathbf{Z})$

(9) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$
$= \pm(0, 0, t_9, t_9 - 1, 2t_9 - 1, t_9 - 1, t_9, 2t_9 - 1, 2t_9 - 1)$ $\quad (t_9 \in \mathbf{Z})$

(10) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$
$= \pm(0, 0, t_{10}, t_{10}, 2t_{10} - 1, t_{10} - 1, t_{10} - 1, 2t_{10}, 2t_{10} - 2)$ $\quad (t_{10} \in \mathbf{Z})$

The proof of Lemma 3.3 is similar to that of Lemma 2.3. The above ten solutions are not **Z**-linear independent, for

$$(t_2 - t_1 - t_9)(6) = (t_2 - t_1 - t_9)((5) + (4) - (3)) + (t_6 - t_5 - t_4 + t_3)((2) - (1) - (9))\,,$$
$$(t_2 - t_1 - t_9)(7) = (t_2 - t_1 - t_9)((5) + (4)) + (t_7 - t_5 - t_4)((2) - (1) - (9))\,,$$
$$(t_2 - t_1 - t_9)(10) = (t_2 - t_1 - t_9)((3) - (4)) + (t_{10} + t_4 - t_3)((2) - (1) - (9))\,.$$

So there are only seven **Z**-linear independent solutions. Therefore together with $\Delta$, one has at most eight **Z**-linear independent elements satisfying the requirements (3.3.1) and (3.3.2). So they can not form a basis for $H_2(V_\varepsilon; \mathbf{Z})$. Thus we have

THEOREM 3.3. *Let* $\{\Delta_1, \Delta_2, \cdots, \Delta_9\}$ *be the distinguished basis constructed in* §3.1 *and let* $\Delta$ *be the vanishing cycle constructed in* §3.2. *Then the distinguished basis* $\{\Delta_1, \Delta_2, \cdots, \Delta_9\}$ *can never be turned into a distinguished basis* $\{\Delta'_i\}$; $(i = 1, \cdots, 9)$ *with* $\Delta'_1 = \pm\Delta$ *by a sequence of elementary substitutions.*

## 4. A distinguished basis of the singularity $\tilde{E}_8$.

In this section we prove our theorem for the singularity $\tilde{E}_8 : x^3 + y^6 + z^2$: We construct a vanishing cycle $\Delta$ and a distinguished basis $\{\Delta_1, \Delta_2, \cdots, \Delta_{10}\}$ which can never be turned into a distinguished basis $\{\Delta'_1, \Delta'_2, \cdots, \Delta'_{10}\}$ with $\Delta'_1 = \pm\Delta$ by a sequence of elementary substitutions.

**4.1. Construction of a distinguished basis $\{\Delta_i\}$.** There exists a distinguished basis $\{\Delta_1, \cdots, \Delta_{10}\}$ of the singularity $\tilde{E}_8 : x^3 + y^6 + z^2$ with the following Dynkin diagram (see [1], [8]).

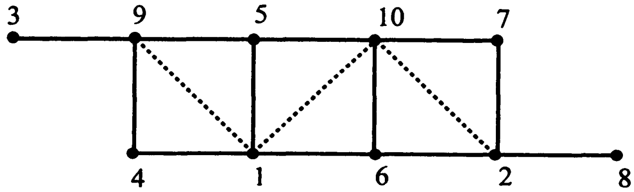


FIGURE 9. The Dynkin diagram of the basis $\{\Delta_i\}$.

Therefore, the matrix $L$ of the inverse variation operator

$$Var^{-1} : H_2(V_\varepsilon; \mathbf{Z}) \to H_2(V_\varepsilon, \partial V_\varepsilon; \mathbf{Z})$$

in the basis $\{\Delta_i\}$ and its dual basis $\{\nabla_i\}$ which is determined by the relation $(\nabla_i, \Delta_j) = \delta_{ij}$, is

$$L = \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 1 & 1 \\ & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 1 \\ & & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ & & & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ & \text{\Large 0} & & & 1 & 0 & 0 & 0 & -1 & -1 \\ & & & & & 1 & 0 & 0 & 0 & -1 \\ & & & & & & 1 & 0 & 0 & -1 \\ & & & & & & & 1 & 0 & 0 \\ & & & & & & & & 1 & 0 \\ & & & & & & & & & 1 \end{bmatrix}.$$

**4.2. Construction of a vanishing cycle $\Delta$.** There exists a system of paths $u_1, u_2, \cdots, u_{10}$ by which the distinguished basis $\{\Delta_i\}$ of vanishing cycles is defined. This means that the cycle $\Delta_i$ vanishes along the path $u_i$ joining the critical value $z_i$ with the non-critical value $z_0$. Let $\{\tau_i\}$ be the simple loops corresponding to the paths $\{u_i\}$. Consider the path $\tilde{u}$ represented by the path $u_2$ and the loops $\{\tau_i\}$ in the following manner:

$$\begin{aligned}\tilde{u} =& u_2 \cdot \tau_3^{-1} \cdot \tau_4^{-1} \cdot \tau_5^{-1} \cdot \tau_6^{-1} \cdot \tau_7^{-1} \cdot \tau_8^{-1} \cdot \tau_9^{-1} \cdot \tau_{10}^{-1} \cdot \tau_5^{-1} \cdot \tau_9 \cdot \tau_5 \cdot \tau_3^{-1} \cdot \tau_4^{-1} \cdot \tau_5^{-1} \\ & \cdot \tau_9^{-1} \cdot \tau_5^{-1} \cdot \tau_{10} \cdot \tau_9 \cdot \tau_5 \cdot \tau_4 \cdot \tau_3 \cdot \tau_2^{-1},\end{aligned}$$

where $u_2 \cdot \tau_3^{-1} \ldots$ means first go along $u_2$ then along $\tau_3^{-1}$ and so on.

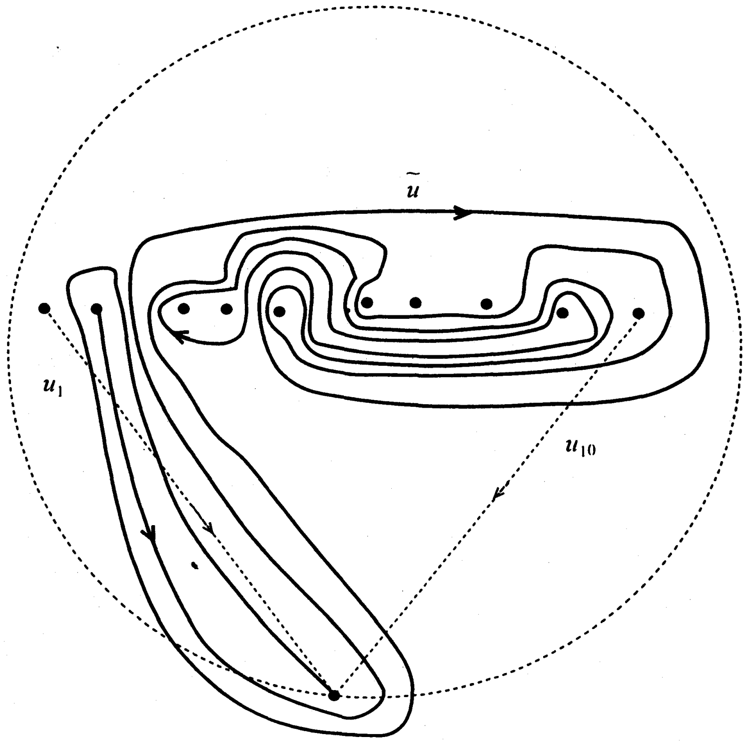


FIGURE 10. The path $\tilde{u}$ defining $\Delta$.

Let $\Delta$ be the vanishing cycle defined by the path $\tilde{u}$. Then the vanishing cycle $\Delta$ is represented in terms of the basis $\{\Delta_i\}$ and the transformations

$$h_i = (\tau_i)_* : H_2(V_\varepsilon; \mathbf{Z}) \to H_2(V_\varepsilon; \mathbf{Z})$$

as follow:

$$\begin{aligned}\Delta =& h_2^{-1} \circ h_3 \circ h_4 \circ h_5 \circ h_9 \circ h_{10} \circ h_5^{-1} \circ h_9^{-1} \circ h_5^{-1} \circ h_4^{-1} \circ h_3^{-1} \circ h_5 \circ h_9 \circ h_5^{-1} \\ & \circ h_{10}^{-1} \circ h_9^{-1} \circ h_8^{-1} \circ h_7^{-1} \circ h_6^{-1} \circ h_5^{-1} \circ h_4^{-1} \circ h_3^{-1}(\Delta_2) \\ =& \Delta_3 + \Delta_4 + 2\Delta_5 + \Delta_6 + \Delta_7 + \Delta_8 + 2\Delta_9 + 2\Delta_{10}.\end{aligned}$$

By the matrix $L$ of the inverse variation operator $Var^{-1}$ defined in 4.1, we have

$$\begin{aligned}Var^{-1}(\Delta) =& Var^{-1}(\Delta_3 + \Delta_4 + 2\Delta_5 + \Delta_6 + \Delta_7 + \Delta_8 + 2\Delta_9 + 2\Delta_{10}) \\ =& -\nabla_2 - \nabla_3 - \nabla_4 - 2\nabla_5 - \nabla_6 - \nabla_7 + \nabla_8 + 2\nabla_9 + 2\nabla_{10},\end{aligned} \tag{4.2}$$

where $\{\nabla_i\}$ is the dual basis of $\{\Delta_i\}$.

**4.3. Elements satisfying $(Var^{-1}\delta, \delta) = 1$ and $(Var^{-1}\Delta, \delta) = 0$.** For the vanishing cycle $\Delta$ and the distinguished basis $\{\Delta_1, \Delta_2, \cdots, \Delta_{10}\}$, if there exists a sequence of elementary substitutions which turn the basis $\{\Delta_1, \Delta_2, \cdots, \Delta_{10}\}$ into a distinguished basis $\{\Delta'_1, \Delta'_2, \cdots, \Delta'_{10}\}$ with $\Delta'_1 = \pm\Delta$, then the matrix $L$ of the inverse variation operator $Var^{-1}$

with respect to the distinguished basis $\{\Delta_i'\}$ must be an upper triangular matrix with diagonal entries equal $(-1)^{n(n+1)/2} = 1$, since $n = 3$. Therefore we have the following two equalities;

$$(Var^{-1}\Delta_i', \Delta_i') = 1\,, \quad (Var^{-1}\Delta, \Delta_i') = 0 \quad (i = 2, \cdots, 10)\,.$$

We shall now determine conditions in terms of $\Delta_1, \Delta_2, \cdots, \Delta_{10}$ for a cycle $\delta$ to be an element of such a distinguished basis $\{\Delta_i'\}$. If $\delta \neq \Delta$ is any element of the distinguished basis $\{\Delta_i'\}$, similarly the following equalities must hold;

$$(4.3.1) \qquad (Var^{-1}\delta, \delta) = 1\,,$$

$$(4.3.2) \qquad (Var^{-1}\Delta, \delta) = 0\,.$$

Let us represent such a $\delta$ as a linear combination of the distinguished basis $\{\Delta_i\}$:

$$\delta = \sum_{i=1}^{10} a_i \Delta_i (a_i \in \mathbf{Z})\,.$$

Then by the matrix $L$, we have

$$\begin{aligned} Var^{-1}(\delta) =&(a_1 - a_4 - a_5 - a_6 + a_9 + a_{10})\nabla_1 + (a_2 - a_6 - a_7 - a_8 + a_{10})\nabla_2 \\ &+ (a_3 - a_9)\nabla_3 + (a_4 - a_9)\nabla_4 + (a_5 - a_9 - a_{10})\nabla_5 \\ &+ (a_6 - a_{10})\nabla_6 + (a_7 - a_{10})\nabla_7 + a_8\nabla_8 + a_9\nabla_9 + a_{10}\nabla_{10}\,, \end{aligned}$$

where $\{\nabla_i\}$ is the dual basis of $\{\Delta_i\}$. Therefore we have

$$\begin{aligned} (Var^{-1}\delta, \delta) =&a_1(a_1 - a_4 - a_5 - a_6 + a_9 + a_{10}) + a_2(a_2 - a_6 - a_7 - a_8 + a_{10}) \\ &+ a_3(a_3 - a_9) + a_4(a_4 - a_9) + a_5(a_5 - a_9 - a_{10}) + a_6(a_6 - a_{10}) \\ &+ a_7(a_7 - a_{10}) + a_8^2 + a_9^2 + a_{10}^2 \\ =&\frac{1}{4}(a_2 - a_1)^2 + \left(a_3 - \frac{1}{2}a_9\right)^2 \\ &+ \left\{a_4 - \frac{1}{2}(a_1 + a_9)\right\}^2 + \left\{a_5 - \frac{1}{2}(a_1 + a_9 + a_{10})\right\}^2 \\ &+ \left\{a_6 - \frac{1}{2}(a_1 + a_2 + a_{10})\right\}^2 + \left\{a_7 - \frac{1}{2}(a_2 + a_{10})\right\}^2 \qquad (4.3.3) \\ &+ \left(a_8 - \frac{1}{2}a_2\right)^2 + \frac{1}{4}(a_9 - a_{10})^2 \\ =&\frac{1}{4}(a_2 - a_1)^2 + \frac{1}{4}(2a_3 - a_9)^2 + \frac{1}{4}\{2a_4 - (a_1 + a_9)\}^2 \\ &+ \frac{1}{4}\{2a_5 - (a_1 + a_9 + a_{10})\}^2 + \frac{1}{4}\{2a_6 - (a_1 + a_2 + a_{10})\}^2 \\ &+ \frac{1}{4}\{2a_7 - (a_2 + a_{10})\}^2 + \frac{1}{4}(2a_8 - a_2)^2 + \frac{1}{4}(a_9 - a_{10})^2\,. \end{aligned}$$

From (4.3.1) and (4.3.3), we have

$$\begin{aligned}(a_2 - a_1)^2 + (2a_3 - a_9)^2 + \{2a_4 - (a_1 + a_9)\}^2 \\ + \{2a_5 - (a_1 + a_9 + a_{10})\}^2 + \{2a_6 - (a_1 + a_2 + a_{10})\}^2 \\ + \{2a_7 - (a_2 + a_{10})\}^2 + (2a_8 - a_2)^2 + (a_9 - a_{10})^2 = 4 .\end{aligned} \tag{4.3.4}$$

On the other hand, from (4.2) and (4.3.2) we have

$$\begin{aligned}(Var^{-1}\Delta, \delta) &= \left(-\nabla_2 - \nabla_3 - \nabla_4 - 2\nabla_5 - \nabla_6 - \nabla_7 + \nabla_8 + 2\nabla_9 + 2\nabla_{10}, \sum_{i=1}^{10} a_i \delta_i\right) \\ &= -a_2 - a_3 - a_4 - 2a_5 - a_6 - a_7 + a_8 + 2a_9 + 2a_{10} = 0 .\end{aligned} \tag{4.3.5}$$

LEMMA 4.3. *The common integral solutions of* (4.3.4) *and* (4.3.5) *are exhausted by the following fourteen types*;

(1) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$
$= \pm(0, 1, t_1, t_1, 2t_1, t_1, t_1, 1, 2t_1, 2t_1)$ $(t_1 \in \mathbf{Z})$,

(2) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$
$= \pm(0, 1, t_2, t_2, 2t_2 - 1, t_2, t_2, 1, 2t_2, 2t_2 - 1)$ $(t_2 \in \mathbf{Z})$,

(3) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$
$= \pm(0, 1, t_3, t_3, 2t_3, t_3 + 1, t_3 + 1, 1, 2t_3, 2t_3 + 1)$ $(t_3 \in \mathbf{Z})$,

(4) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$
$= \pm(0, 1, t_4, t_4, 2t_4 + 1, t_4 + 1, t_4 + 1, 1, 2t_4 + 1, 2t_4 + 1)$ $(t_4 \in \mathbf{Z})$,

(5) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$
$= \pm(0, 0, t_5, t_5, 2t_5 - 1, t_5 - 1, t_5 - 1, 0, 2t_5, 2t_5 - 2)$ $(t_5 \in \mathbf{Z})$,

(6) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$
$= \pm(0, 0, t_6, t_6, 2t_6, t_6 - 1, t_6 - 1, 0, 2t_6, 2t_6 - 1)$ $(t_6 \in \mathbf{Z})$,

(7) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$
$= \pm(0, 0, t_7, t_7, 2t_7 + 1, t_7, t_7, 0, 2t_7, 2t_7 + 1)$ $(t_7 \in \mathbf{Z})$,

(8) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$
$= \pm(0, 0, t_8, t_8, 2t_8 - 2, t_8 - 1, t_8 - 1, 0, 2t_8 - 1, 2t_8 - 2)$ $(t_8 \in \mathbf{Z})$,

(9) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$
$= \pm(0, 0, t_9, t_9, 2t_9 - 1, t_9, t_9, 0, 2t_9 - 1, 2t_9)$ $(t_9 \in \mathbf{Z})$,

(10) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$
$= \pm(0, 0, t_{10}, t_{10}, 2t_{10} - 1, t_{10} - 1, t_{10} - 1, 0, 2t_{10} - 1, 2t_{10} - 1)$ $(t_{10} \in \mathbf{Z})$,

(11) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$
$= \pm(0, 0, t_{11} + 1, t_{11}, 2t_{11} + 1, t_{11}, t_{11} + 1, 0, 2t_{11} + 1, 2t_{11} + 1)$ $(t_{11} \in \mathbf{Z})$,

(12) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$
$= \pm(0, 0, t_{12} + 1, t_{12}, 2t_{12} + 1, t_{12} + 1, t_{12}, 0, 2t_{12} + 1, 2t_{12} + 1)$ $(t_{12} \in \mathbf{Z})$,

(13) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$
$= \pm(-1, 0, t_{13}, t_{13}, 2t_{13}, t_{13}, t_{13}, 0, 2t_{13}, 2t_{13})$ $(t_{13} \in \mathbf{Z})$,

(14) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$
$= \pm(1, 0, t_{14} - 1, t_{14}, 2t_{14} - 1, t_{14}, t_{14} - 1, 0, 2t_{14} - 1, 2t_{14} - 1)$ $(t_{14} \in \mathbf{Z})$.

The proof of Lemma 4.3 is similar to that of Lemma 2.3. The above fourteen solutions are not **Z**-linear independent, for

$$\begin{aligned}
(t_2+t_7-t_1)(3) &= (t_3+t_5+t_7-t_1)((2)+(7)-(1))+(t_2+t_7-t_1)((1)-(5)-(7)),\\
(t_2+t_7-t_1)(4) &= (t_4+t_5+t_7+t_9-t_1)((2)+(7)-(1))\\
&\quad +(t_2+t_7-t_1)((1)-(5)-(7)-(9)),\\
(t_2+t_7-t_1)(6) &= (t_6-t_5-t_7)((2)+(7)-(1))+(t_2+t_7-t_1)((5)+(7)),\\
(t_2+t_7-t_1)(8) &= (t_8-t_5-t_9)((2)+(7)-(1))+(t_2+t_7-t_1)((5)+(9)),\\
(t_2+t_7-t_1)(10) &= (t_{10}-t_5-t_7-t_9)((2)+(7)-(1))+(t_2+t_7-t_1)((5)+(7)+(9)),\\
(t_2+t_7-t_1)(14) &= (t_{14}+t_{13}+t_{11})((2)+(7)-(1))-(t_2+t_7-t_1)((13)+(11)).
\end{aligned}$$

So there are only eight **Z**-linear independent solutions. Therefore together with $\Delta$, one has at most nine **Z**-linear independent elements satisfying the requirements (4.3.1) and (4.3.2). So they can not form a basis for $H_2(V_\varepsilon;\mathbf{Z})$. Thus we have

THEOREM 4.4. *Let $\{\Delta_1,\Delta_2,\cdots,\Delta_{10}\}$ be the distinguished basis constructed in* §4.1 *and let $\Delta$ be the vanishing cycle constructed in* §4.2. *Then the distinguished basis $\{\Delta_1,\Delta_2,\cdots,\Delta_{10}\}$ can never be turned into a distinguished basis $\{\Delta'_i\}$; $(i=1,\cdots,10)$ with $\Delta'_1=\pm\Delta$ by a sequence of elementary substitutions.*

## 5. A distinguished basis of non-simple singularities.

In this section we prove our Main Theorem for arbitrary non-simple singularities.

DEFINITION 5.1 [7]. A germ $\hat{g}$ is *adjacent* to $\hat{f}$ if in any neighborhood of $\hat{f}$ there are germs of the orbit of $\hat{g}$. We denote this situation by $\hat{g}\le\hat{f}$.

PROPOSITION 5.2 [7]. *If $\hat{g}\le\hat{f}$, then there exists an injection $i_*: H_{n-1}(V_g)\to H_{n-1}(V_f)$ which preserves the intersection form and maps a distinguished basis of $H_{n-1}(V_g)$ into a distinguished basis of vanishing cycles of $H_{n-1}(V_f)$ such that the intersection matrix of $\hat{g}$ can be identified with a diagonal submatrix of the intersection matrix of $\hat{f}$.*

PROOF OF THE MAIN THEOREM. If $f$ is not simple, then at least one of the following three families is adjacent of $\hat{f}$. (see [7])

$$\begin{aligned}
\tilde{E}_6 &: \quad z_1^3+z_2^3+z_3^3+tz_1z_2z_3+z_4^2+\cdots+z_n^2\\
\tilde{E}_7 &: \quad z_1^4+z_2^4+z_3^2+tz_1^2z_2^2+z_4^2+\cdots+z_n^2\\
\tilde{E}_8 &: \quad z_1^6+z_2^3+z_3^2+tz_1^4z_2+z_4^2+\cdots+z_n^2
\end{aligned}$$

Since these families of germs have constant Milnor numbers, their intersection forms are also constant and can be computed from those of

$$x^3+y^3+z^3,\quad x^4+y^4+z^2,\quad x^6+y^3+z^2.$$

There are three cases to be considered: $\tilde{E}_6 \leq f$, $\tilde{E}_7 \leq f$, $\tilde{E}_8 \leq f$. Let us first consider the case $\tilde{E}_6 \leq f$. For stably equivalent singularities, their desitnguished bases are in one-to-one correspondence and the matrices of the operators $Var^{-1}$ differ by the facter $\pm 1$. Therefore, without loss of generality, we will assume that the number of variables $n \equiv 3 \bmod 4$. The singularity $\tilde{E}_6$ $(x^3 + y^3 + z^3)$ has the distinguished basis $\{\Delta_1, \Delta_2, \Delta_3, \cdots, \Delta_8\}$ with Dynkin diagram shown in Figure 5 and the vanishing cycle $\Delta$ constructed in §2.2. By Proposition 5.2, it can be shown that the image of this distinguished basis in $H_{n-1}(V_f)$ can be extended to the distinguished basis $\{\widetilde{\Delta_1} = i_*(\Delta_1), \widetilde{\Delta_2} = i_*(\Delta_2), \cdots, \widetilde{\Delta_8} = i_*(\Delta_8), \widetilde{\Delta_9}, \cdots, \widetilde{\Delta_\mu}\}$ with the following relation: $(\Delta_i, \Delta_j) = (\widetilde{\Delta_i}, \widetilde{\Delta_j})$ for $1 \leq i \leq j \leq 8$. Therefore the matrix $\tilde{L}$ of the inverse variation operator in the basis $\{\tilde{\Delta}_i\}$ and its dual basis $\{\tilde{\nabla}_i\}$ is:

$$\tilde{L} = \left[\begin{array}{c|ccc} & * & & \\ L & & \ddots & \\ & & & * \\ \hline & 1 & & * \\ & & \ddots & \\ 0 & 0 & & 1 \end{array}\right],$$

where $L$ is the matrix defined in §2.1.

Consider the vanishing cycle $\tilde{\Delta} = i_*(\Delta)$:

$$\begin{aligned} \tilde{\Delta} &= -i_*(\Delta_1) + i_*(\Delta_4) + i_*(\Delta_6) + i_*(\Delta_7) + i_*(\Delta_8) \\ &= -\widetilde{\Delta_1} + \widetilde{\Delta_4} + \widetilde{\Delta_6} + \widetilde{\Delta_7} + \widetilde{\Delta_8}\,. \end{aligned}$$

We are going to prove that the vanishing cycle $\tilde{\Delta}$ and the distinguished basis $\{\widetilde{\Delta_1}, \widetilde{\Delta_2}, \cdots, \widetilde{\Delta_8}, \widetilde{\Delta_9}, \cdots, \widetilde{\Delta_\mu}\}$ thus constructed are the ones whose existence is claimed in Main theorem: $\{\widetilde{\Delta_1}, \cdots, \widetilde{\Delta_\mu}\}$ can never be elementary equivalent to any distinguished basis $\{\widetilde{\Delta'_i}\}$ with $\widetilde{\Delta'_1} = \pm\tilde{\Delta}$.

Suppose that there exists a sequence of elementary substitutions which turn $\{\widetilde{\Delta_1}, \widetilde{\Delta_2}, \cdots, \widetilde{\Delta_8}, \widetilde{\Delta_9}, \cdots, \widetilde{\Delta_\mu}\}$ into a distinguished basis $\{\widetilde{\Delta'_i}\}$ with the first element $\widetilde{\Delta'_1} = \pm\tilde{\Delta}$. Then the distinguished basis $\{\widetilde{\Delta'_i}\}$ must satisfy the conditions that $(Var^{-1}\tilde{\Delta}, \widetilde{\Delta'_i}) = 0$ $(i = 2, \cdots, \mu)$, and by $\tilde{L}$ we have $(Var^{-1}\tilde{\Delta}, \widetilde{\Delta_i}) = 0$ $(i = 9, \cdots, \mu)$. Therefore the cycles $\widetilde{\Delta_9}, \widetilde{\Delta_{10}}, \cdots, \widetilde{\Delta_\mu}$ belong to the lattice $\{x : x \in H_{n-1}(V_f), (Var^{-1}\tilde{\Delta}, x) = 0\}$ that is spanned by the vectors $\widetilde{\Delta'_2}, \cdots, \widetilde{\Delta'_\mu}$. Thus, let $\{\tilde{u}_i\}$ $(1 \leq i \leq \mu - 7)$ be a system of paths defining an ordered set of vanishing cycles $\{\tilde{\Delta}, \widetilde{\Delta_9}, \cdots, \widetilde{\Delta_\mu}\}$. The system of paths $\{\tilde{u}_i\}$ $(1 \leq i \leq \mu - 7)$ satisfy the conditions (i), (ii) defined in §1 and are numbered as (iii) in §1. Therefore the system of paths $\{\tilde{u}_i\}$ $(1 \leq i \leq \mu - 7\}$ can be extended to a system $\{\tilde{u}_i\}$ $(1 \leq i \leq \mu)$ defining a distinguished basis $\{\tilde{\Delta}, \widetilde{\Delta_9}, \cdots, \widetilde{\Delta_\mu}, \overline{\Delta_{\mu-6}}, \cdots, \overline{\Delta_\mu}\}$. We change the distinguished basis $\{\tilde{\Delta}, \widetilde{\Delta_9}, \cdots, \widetilde{\Delta_\mu}, \overline{\Delta_{\mu-6}}, \cdots, \overline{\Delta_\mu}\}$ by the sequence of elementaly substitutions $\alpha_{\mu-7}, \alpha_{\mu-8}, \cdots, \alpha_2, \alpha_{\mu-6}, \cdots, \alpha_3, \cdots, \alpha_{\mu-1}, \alpha_{\mu-2}, \cdots, \alpha_8$. So we obtain a distinguished basis $\{\widehat{\Delta_i}\}$ of the form $\{\widehat{\Delta_1} = \tilde{\Delta}, \widehat{\Delta_2}, \cdots, \widehat{\Delta_8}, \widehat{\Delta_9} = \widetilde{\Delta_9}, \cdots, \widehat{\Delta_\mu} = \widetilde{\Delta_\mu}\}$. Since $\{\widehat{\Delta_i}\}$ is a distinguished basis, it follows that $(Var^{-1}\widehat{\Delta_i}, \widehat{\Delta_j}) = 0$ $(1 \leq i < j \leq \mu)$. Therefore

$\{\widehat{\Delta_i}\}$, $(i = 2, \cdots, 8)$ can be described as a linear combination of the distinguished basis $\{\widetilde{\Delta_i}\}$ as follows: $\widehat{\Delta_i} = \sum_{j=1}^{8} a_{ij}\widetilde{\Delta_j}$ $(i = 2, \cdots, 8)$. Set $\Delta_i' = \sum_{j=1}^{8} a_{ij}\Delta_j (a_{ij} \in \mathbf{Z})$. Then $\{\Delta_1', \cdots, \Delta_8'\}$ is a distinguished basis for $\tilde{E}_6$ with $\Delta_1' = \Delta$. This contradicts to Theorem 2.3. In the other cases we obtain the same result. This completes the proof of Main Theorem.

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