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An Estimate for the Bochner-Riesz Operator on Functions of Product Type in $\mathbf{R}^2$

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Abstract. In this paper we shall give $L^p(\mathbf{R}^2)$-boundedness of the Bochner-Riesz operator S_δ for $2 < p < \infty$ and $\delta > 0$, restricting it to functions of product type. In this range, $2 < p < \infty$ and $\delta > 0$, the strong L^p-estimate is valid for functions of product type but not for general functions.

1. Introduction and main theorem.

In this paper we shall prove a certain estimate for the Bochner-Riesz summing operator S_δ, $\delta > 0$, for functions of product type. We first recall definitions and state the main theorem.

For a function f on the d-dimensional Euclidean space $\mathbf{R}^d$, $d \geq 2$, the Bochner-Riesz operator $S_\delta f$ is defined by

$$(S_\delta f)(x) = \int_{\mathbf{R}^d} e^{2\pi i\langle x,\xi\rangle}(1 - |\xi|^2)_+^\delta \hat{f}(\xi)d\xi\ .$$

Here, $t_+^\delta = t^\delta$ for $t > 0$ and zero otherwise, and $\hat{f}$ is the Fourier transform of f:

$$\hat{f}(\xi) = \int_{\mathbf{R}^d} e^{-2\pi i\langle \xi,x\rangle} f(x)dx\ .$$

$\mathcal{S}(\mathbf{R}^d)$, $d \geq 1$, will denote the set of all Schwartz-class functions on $\mathbf{R}^d$.

THEOREM 1. *Let $d = 2$. If $\delta > 0$ and $2 < p < \infty$, then the inequality*

$$\|S_\delta f\|_{L^p(\mathbf{R}^2)} \leq C_{p,\delta}\|f\|_{L^p(\mathbf{R}^2)} \tag{1}$$

holds for all f in $\mathcal{S}(\mathbf{R}^2)$ of the form $f(x) = f_1(x_1)f_2(x_2)$ with a constant $C_{p,\delta}$.

REMARK 2. By a standard approximation argument based on (1) we see that $S_\delta f$ can be defined for $f(x) = f_1(x_1)f_2(x_2)$ with $f_j \in L^p(\mathbf{R})$ and (1) holds for such f.

Before proceeding we shall make some remarks on the relation between known results and the above theorem. Early history of the boundedness problem for the Bochner-Riesz operator is summarized in [Fe]. In [Fe] C. Fefferman suggested a possible connection with the Kakeya maximal operator. For the dimension two A. Córdoba [Co1], [Co2] gave a proof

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of $L^4(\mathbf{R}^2)$-boundedness of S_δ using a boundedness estimate for the Kakeya maximal operator. The method of the proof in the present paper is based on the idea of [Ta2] combined with an idea of Córdoba developed in [Co1], [Co2]. However, the Kakeya maximal operator itself does not appear explicitly in this work.

We shall now review some more recent results which are relevant to our work. (See [So] and [St].)

The critical index $\delta(p)$ for $L^p(\mathbf{R}^d)$ is defined by

$$\delta(p) = \max\left\{d\left|\frac{1}{2}-\frac{1}{p}\right|-\frac{1}{2},\ 0\right\}, \quad 1 \le p \le \infty .$$

Note that $\delta(p) > 0 \Leftrightarrow p \notin [2d/(d+1), 2d/(d-1)]$. It is known that a necessary condition in order that $f \to S_\delta f$ is bounded in $L^p(\mathbf{R}^d)$, $p \neq 2$, is that $\delta > \delta(p)$. When $\delta(p) = 0$ this is a theorem of C. Fefferman. In other cases it follows from the fact that the kernel of S_δ is in $L^p(\mathbf{R}^d)$ only when $\delta > \delta(p)$ for $1 \le p \le 2d/(d+1)$. Indeed, the kernel of S_δ has the asymptotic form

$$K_\delta(x) = |x|^{-(d+1)/2-\delta} a(x) + O(|x|^{-(d+3)/2-\delta})$$

with

$$a(x) = C_\delta \cos(2\pi|x| - (\pi/2)(d/2+\delta) - \pi/4) .$$

Choose f in $C_0^\infty(\mathbf{R}^d)$ as an approximation of Dirac function. Then for $1 \le p \le 2d/(d+1)$

$$S_\delta f \in L^p(\mathbf{R}^d) \ \Rightarrow\ d < p\left(\frac{d+1}{2}+\delta\right) \iff \delta > d\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2} = \delta(p) . \qquad (2)$$

REMARK 3. We note that f in (2) can be choosen in the form $f(x) = \prod_{l=1}^d f_l(x_l)$. This shows that Theorem 1 cannot be extended to the range $p \in [1, 4/3)$.

As for sufficient conditions we quote the following theorem which is due to Carleson and Sjölin [CS] in the two-dimensional case and Tomas [To] in the higher-dimensional case.

THEOREM 4 ([So, Theorem 2.3.1]). *If*

(i) $d \ge 3$ *and* $p \in [1, (2d+1)/(d+3)] \cup [2(d+1)/(d-1), \infty]$ *or*

(ii) $d = 2$ *and* $1 \le p \le \infty$,

it follows that

$$\|S_\delta f\|_{L^p(\mathbf{R}^d)} \le C_{p,\delta}\|f\|_{L^p(\mathbf{R}^d)}$$

when $\delta > \delta(p)$.

J. Bourgain and T. Wolff improved the range of (i) (see [Bo] and [Wo]).

For functions of product type the following theorem is known.

THEOREM 5 ([Ig, Theorem 6]). *If* $\delta > 0$ *and* $p \in [2d/(d+1), 2]$, *then the inequality*

$$\|S_\delta f\|_{L^p(\mathbf{R}^d)} \le C_{p,\delta}\|f\|_{L^p(\mathbf{R}^d)}$$

holds for all f *in* $L^p(\mathbf{R}^d)$ *of the form* $f(x) = \prod_{l=1}^d f_l(x_l)$.

Thus, the really new part of Theorem 1 is the case of $d = 2$, $4 < p < \infty$, and $0 < \delta \le \delta(p)$ with f being of product type. We might emphasize, however, that this range of p,

δ is outside the range of necessary condition for the L^p-boundedness of S_δ. In this range the strong L^p-estimate is valid for functions of product type but not for general functions.

This paper is a part of the thesis of the doctor of science [Ta1] Chapter 6 submitted to Gakushuin university.

2. Reduction of the proof of Theorem 1.

In this section we shall reduce the proof of Theorem 1 to Theorem 6 below. This type of argument is essentially known and we basically follow [Mi].

In the following $\check{f}$ will denote the inverse Fourier transform of f.

Consider $\zeta(\xi)$ in $C^\infty(\mathbf{R}^2)$ such that $\zeta(\xi)$ equals 0 in some neighborhood of 0 and equals 1 in some neighborhood of $|\xi| = 1$. If we can prove that for one such ζ the inequality

$$\|(\zeta(\xi)(1-|\xi|)_+^\delta \hat{f}(\xi))^\vee\|_p \le C_{p,\delta}\|f\|_p \tag{3}$$

holds for all f in $\mathcal{S}(\mathbf{R}^2)$ of product type with a constant $C_{p,\delta}$ which is independent of f, then we will obtain the boundedness of S_δ in $L^p(\mathbf{R}^2)$ for such f by decomposing the multiplier as

$$(1-|\xi|^2)_+^\delta = (1-\zeta(\xi))(1-|\xi|^2)_+^\delta + (1+|\xi|)^\delta \cdot \zeta(\xi)(1-|\xi|)_+^\delta .$$

Let $\alpha(t)$ in $C^\infty(\mathbf{R})$ be

$$\alpha(t) = \begin{cases} 1, & t \le 1, \\ 0, & t > 2. \end{cases}$$

Put $\beta(t) = \alpha(t) - \alpha(2t)$. Note that $\operatorname{supp}\beta \subset [1/2, 2]$ and

$$\sum_{k=k_0}^\infty \beta(2^k t) = \alpha(2^{k_0} t) = \begin{cases} 1, & 0 < t \le 2^{-k_0}, \\ 0, & t > 2^{-k_0+1}. \end{cases}$$

It follows from this equality that

$$\begin{aligned} \alpha(2^{k_0}(1-|\xi|))(1-|\xi|)_+^\delta &= \sum_{k=k_0}^\infty \beta(2^k(1-|\xi|))(1-|\xi|)^\delta \\ &= \sum_{k=k_0}^\infty 2^{-k\delta}\beta(2^k(1-|\xi|))(2^k(1-|\xi|))^\delta . \end{aligned}$$

Put $\varphi(t) = \beta(t)t^\delta$. If we can prove that for every $\varepsilon > 0$ there exist $k_0 \ge 2$ and a constant $C = C_{\varepsilon,\varphi,p}$ such that

$$\|(\varphi(2^k(1-|\xi|))\hat{f}(\xi))^\vee\|_p \le C 2^{k\varepsilon}\|f\|_p , \quad \forall k \ge k_0 , \tag{4}$$

holds for all f in $\mathcal{S}(\mathbf{R}^2)$ of product type, then we obtain

$$\|(\alpha(2^{k_0}(1-|\xi|))(1-|\xi|)_+^\delta \hat{f}(\xi))^\vee\|_p \le C\sum_{k=k_0}^\infty 2^{(\varepsilon-\delta)k}\|f\|_p \le C_{p,\delta}\|f\|_p . \tag{5}$$

by choosing $\varepsilon < \delta$. Thus, by (3) and (5) the proof of Theorem 1 is reduced to proving (4) (choose $\zeta(\xi) = \alpha(2^{k_0}(1 - |\xi|))$).

We introduce the operator T_a, $0 < a < 1/4$, as follows. Let φ be a function in $C_0^\infty(\mathbf{R})$ with support in $[1/2, 2]$. For a function f in $\mathcal{S}(\mathbf{R}^2)$ define $T_a f$ by

$$(T_a f)(x) = \int_{\mathbf{R}^2} e^{2\pi i \langle x, \xi \rangle} \varphi\left(\frac{1 - |\xi|}{a}\right) \hat{f}(\xi) d\xi.$$

Then (4) follows from the next theorem. In fact, take $a = 2^{-k}$ in (6) and choose k_0 so that $2^{k_0 \varepsilon} > \sqrt{k_0}$.

THEOREM 6. *Let $d = 2$. For every $2 < p < \infty$ there exists a constant C_p independent of f and a such that*

$$\|T_a f\|_{L^p(\mathbf{R}^2)} \le C_p \left(\log\left(\frac{1}{a}\right)\right)^{1/2} \|f\|_{L^p(\mathbf{R}^2)} \tag{6}$$

holds for all f in $\mathcal{S}(\mathbf{R}^2)$ of the form $f(x) = f_1(x_1) f_2(x_2)$.

In the following C's will denote constants independent of f and a. It will be different in each occasion.

3. Proof of Theorem 6.

3.1. Decomposition of T_a by an angular partition of unity. Hereafter, we denote by $[x]$ the largest integer not greater than x.

Fix a, $0 < a < 1/4$. We shall consider a decomposition of T_a.

For the integers

$$k \in \left[1, \left[\frac{\pi}{2\sqrt{a}}\right] - 1\right],$$

and $m = 0, 1, 2, 3$ let the sequence $\{p_{k,m}\}$ on the unit circle S^1 be

$$p_{k,m} = \left(\cos\left(\frac{\pi m}{2} + \sqrt{a}k\right), \sin\left(\frac{\pi m}{2} + \sqrt{a}k\right)\right).$$

Choose $\psi \ge 0$ in $C_0^\infty(\mathbf{R})$ which equals 1 for $0 \le t \le 4$. Define the function $\psi_{k,m}(\omega)$ on S^1 as

$$\psi_{k,m}(\omega) = \psi\left(\frac{|\omega - p_{k,m}|^2}{a}\right).$$

If $\omega \in S^1$, then $\psi_{k,m}(\omega) \ne 0$ for some k, m and the number of such k, m is uniformly bounded. If we put $\Psi_{k,m}(\omega) = \psi_{k,m}(\omega)/(\sum_{k',m'} \psi_{k',m'}(\omega))$ where denominator does not vanish, then $\{\Psi_{k,m}\}$ is a partition of unity on S^1.

Let $\varphi_{k,m}(\xi)$ be

$$\varphi_{k,m}(\xi) = \varphi\left(\frac{1 - |\xi|}{a}\right) \Psi_{k,m}\left(\frac{\xi}{|\xi|}\right).$$

Let $\tau_{k,m}$ be

$$(\tau_{k,m} f)(x) = (\varphi_{k,m}(\xi)\hat{f}(\xi))^{\vee}(x).$$

Thus, we have reduced the problem to the estimate

$$\Big\| \sum_{k,m} \tau_{k,m} f \Big\|_{L^p(\mathbf{R}^2)} \le C_p \left(\log\left(\frac{1}{a}\right)\right)^{1/2} \|f\|_{L^p(\mathbf{R}^2)} . \tag{7}$$

Let N_0 be $N_0 = [\pi/4\sqrt{a}]$. Then without loss of generality we may restrict k and m to $1 \le k \le N_0$ and $m = 0$. For simplicity we will write $\tau_{k,0}$, $\varphi_{k,0}$ and $p_{k,0}$ as τ_k, φ_k and p_k, respectively.

3.2. What product type implies, 1. Let $I = \chi_{(-11,11)}$. For every $\varepsilon > 0$ and an integer $j \in \mathbf{Z}$ the partial sum operator $P_{\varepsilon,j}$ is defined by

$$(P_{\varepsilon,j} f)(x) = \int_{\mathbf{R}} e^{2\pi i x\xi} I\left(\frac{\xi}{\varepsilon} - j\right) \hat{f}(\xi) d\xi , \quad f \in \mathcal{S}(\mathbf{R}) . \tag{8}$$

Then we have the following lemma, where our assumption $2 < p < \infty$ is essential.

LEMMA 7. *Suppose that $2 < p < \infty$. There exists a constant C_p depending only on p such that*

$$\Big\| \Big(\sum_{j\in\mathbf{Z}} |P_{\varepsilon,j} f|^2 \Big)^{1/2} \Big\|_{L^p(\mathbf{R})} \le C_p \|f\|_{L^p(\mathbf{R})} .$$

PROOF. By a dilation argument it suffices to consider only the case $\varepsilon = 1$. Then this lemma is a special case of Theorem 2.16 in Chapter V of [GR] (p489). □

Let $N_1 = [\log N_0/\log 2]$. For every k with $2^l \le k < 2^{l+1}$, $l = 0, 1, \cdots, N_1$, and $k \le N_0$ let the integer γ_1^k be

$$\gamma_1^k = \left[\frac{\cos\sqrt{a}k}{2^{l+1}a} \right] .$$

For $1 \le k \le N_0$ let the integer γ_2^k be

$$\gamma_2^k = \left[\frac{\sin\sqrt{a}k}{\sqrt{a}} \right] .$$

Then the following proposition holds.

PROPOSITION 8. *Fix $0 \le l \le N_1$. For every k with $2^l \le k < 2^{l+1}$ and $k \le N_0$ the operator P_1^k is defined by*

$$(P_1^k f)(x) = (P_{2^{l+1}a,\gamma_1^k} f)(x) , \quad f \in \mathcal{S}(\mathbf{R}) ,$$

where $P_{\varepsilon,j}$ is defined in (8). For every k with $1 \le k \le N_0$ the operator P_2^k is defined by

$$(P_2^k f)(x) = (P_{\sqrt{a},\gamma_2^k} f)(x) , \quad f \in \mathcal{S}(\mathbf{R}) .$$

Then, if f in $\mathcal{S}(\mathbf{R}^2)$ is of the form $f_1(x_1)f_2(x_2)$, we have

$$(\tau_k f)(x) = (\tau_k(P_1^k f_1 P_2^k f_2))(x) .$$

PROOF. It suffices to show that

$$I\left(\frac{\xi_1}{2^{l+1}a} - \gamma_1^k\right) I\left(\frac{\xi_2}{\sqrt{a}} - \gamma_2^k\right) = 1\,, \quad \forall \xi \in \operatorname{supp}\varphi_k\,.$$

Fix $\xi \in \operatorname{supp}\varphi_k$. It suffices to show that

$$\left|\frac{\xi_1}{2^{l+1}a} - \gamma_1^k\right| \le 11\,, \tag{9}$$

$$\left|\frac{\xi_2}{\sqrt{a}} - \gamma_2^k\right| \le 11\,. \tag{10}$$

We prove only (9). (10) can be proved similarly.

PROOF OF (9). It follows that

$$|\xi_1 - 2^{l+1}a\gamma_1^k| \le \left|\xi_1 - \frac{\xi_1}{|\xi|}\right| + \left|\frac{\xi_1}{|\xi|} - \cos\sqrt{a}k\right| + |\cos\sqrt{a}k - 2^{l+1}a\gamma_1^k|\,. \tag{11}$$

We have

$$\left|\xi_1 - \frac{\xi_1}{|\xi|}\right| = \frac{|\xi_1|}{|\xi|}||\xi| - 1| \le 2a\,, \tag{12}$$

because $a/2 \le 1 - |\xi| \le 2a$ for $\xi \in \operatorname{supp}\varphi_k$, and

$$|\cos\sqrt{a}k - 2^{l+1}a\gamma_1^k| = 2^{l+1}a\left|\frac{\cos\sqrt{a}k}{2^{l+1}a} - \gamma_1^k\right| \le 2^{l+1}a \tag{13}$$

by the definition of γ_1^k. Define θ as $\xi_1/|\xi| = \cos\theta$. Then we have $|\theta - \sqrt{a}k| < 3\sqrt{a}$ for $\xi \in \operatorname{supp}\varphi_k$. It follows from this inequality that

$$\left|\frac{\xi_1}{|\xi|} - \cos\sqrt{a}k\right| \le \cos\sqrt{a}k - \cos\sqrt{a}(k+3) \tag{14}$$

$$= \int_{\sqrt{a}k}^{\sqrt{a}(k+3)} \sin t\,dt \le 3\sqrt{a}\sin\sqrt{a}(k+3) \le 3(k+3)a \le 9\cdot 2^{l+1}a\,.$$

Here, the last inequality follows from $k < 2^{l+1}$. From (11)–(14) we have proved (9). □

3.3. Analysis in the x-space. Let U_k be the orthogonal transformation in $\mathbf{R}^2$ defined by

$$U_k = \begin{pmatrix} \cos\sqrt{a}k & -\sin\sqrt{a}k \\ \sin\sqrt{a}k & \cos\sqrt{a}k \end{pmatrix}.$$

Then $U_k^{-1}p_k = (1, 0)$. Let the rectangle R_a be

$$R_a = \left\{(x_1, x_2) \,\middle|\, |x_1| \le \frac{1}{a}, \quad |x_2| \le \frac{1}{\sqrt{a}}\right\}.$$

Let $R_{a,k}$ be $R_{a,k} = U_k R_a$. Then we have the following basically known lemma (cf. [Co2]).

LEMMA 9. *In the situation above we have*

$$|\check{\varphi}_k(x)| \le C\sum_{m=1}^{\infty} 2^{-m}\frac{1}{|2^m R_{a,k}|}\chi_{2^m R_{a,k}}(x) \equiv K_k(x). \tag{15}$$

The proof of this lemma can be found in [Mi] and is reproduced in Section 4.
Let $F_k(x)$ and $G_k(x)$ be

$$F_k(x) = (P_1^k f_1)(x), \quad G_k(x) = (P_2^k f_2)(x).$$

Then it follows from Proposition 8, Lemma 9 and $K_k \in L^1$ that

$$|(\tau_k f)(x)| = |(\tau_k(F_k G_k))(x)| = |(\check{\varphi}_k *(F_k G_k))(x)| \le (K_k * (|F_k||G_k|))(x). \tag{16}$$

3.4. What product type implies, 2. Using the same idea as in [Ta2], we shall prove the following proposition.

PROPOSITION 10. *Put* $R = 2^m R_{a,k}$, $N = 1/\sqrt{a}$, $\alpha = 2^m/\sqrt{a}$ *and* $(\omega_1, \omega_2) = (\cos\sqrt{a}k, \sin\sqrt{a}k)$. *If* $h(x) \ge 0$ *is a locally integrable function of the form* $h(x) = h_1(x_1)h_2(x_2)$, *then we have*

$$\frac{1}{|R|}\int_R h(y)dy$$
$$\le C\left\{\frac{1}{6\omega_1 N\alpha}\int_{-3\omega_1 N\alpha}^{3\omega_1 N\alpha} h_1(y_1)^2 dy_1\right\}^{1/2}\left\{\frac{1}{6\omega_2 N\alpha}\int_{-3\omega_2 N\alpha}^{3\omega_2 N\alpha} h_2(y_2)^2 dy_2\right\}^{1/2}.$$

PROOF. By Fubini's theorem we can select s, $0 \le |s| \le \alpha$, such that

$$\int_R h(y)dy \le 2\alpha\int_{-N\alpha}^{N\alpha} h(s(\omega_2, -\omega_1) + t(\omega_1, \omega_2))dt.$$

By the Schwarz inequality we have

$$\begin{aligned}
\text{RHS} &= 2\alpha\int_{-N\alpha}^{N\alpha} h_1(s\omega_2 + t\omega_1)h_2(-s\omega_1 + t\omega_2)dt \\
&\le 2\alpha\left(\int_{-N\alpha}^{N\alpha} h_1(s\omega_2 + t\omega_1)^2 dt\right)^{1/2}\left(\int_{-N\alpha}^{N\alpha} h_2(-s\omega_1 + t\omega_2)^2 dt\right)^{1/2} \\
&= 2\alpha\left(\frac{1}{\omega_1}\int_{-\omega_1 N\alpha}^{\omega_1 N\alpha} h_1(s\omega_2 + t)^2 dt\right)^{1/2}\left(\frac{1}{\omega_2}\int_{-\omega_2 N\alpha}^{\omega_2 N\alpha} h_2(-s\omega_1 + t)^2 dt\right)^{1/2}.
\end{aligned}$$

Note that for $1 \le k \le N_0$ we have $1/2N \le \omega_2 \le \omega_1$. Hence we have

$$|s\omega_2| + |\omega_1 N\alpha| \le 2\omega_1 N\alpha \quad \text{and} \quad |s\omega_1| + |\omega_2 N\alpha| \le 3\omega_2 N\alpha.$$

Thus, we obtain

$$\frac{1}{|R|}\int_R h(y)dy \le C\left\{\frac{1}{6\omega_1 N\alpha}\int_{-3\omega_1 N\alpha}^{3\omega_1 N\alpha} h_1(y_1)^2 dy_1\right\}^{1/2}\left\{\frac{1}{6\omega_2 N\alpha}\int_{-3\omega_2 N\alpha}^{3\omega_2 N\alpha} h_2(y_2)^2 dy_2\right\}^{1/2}. \quad \square$$

It follows that

$$\frac{1}{|2^m R_{a,k}|}(\chi_{2^m R_{a,k}} * (|F_k||G_k|))(x) = \frac{1}{|2^m R_{a,k}|}\int_{2^m R_{a,k}} |F_k(x_1 - y_1)||G_k(x_1 - y_1)|dy\,.$$

By putting $h(y) = |F_k(x_1 - y_1)||G_k(x_1 - y_1)|$ in Proposition 10, we obtain

$$\begin{aligned}
&\frac{1}{|2^m R_{a,k}|}(\chi_{2^m R_{a,k}} * (|F_k||G_k|))(x) \\
&\le C\left\{\frac{1}{6\frac{2^m}{a}\cos\sqrt{a}k}(\chi_{[-3\frac{2^m}{a}\cos\sqrt{a}k,3\frac{2^m}{a}\cos\sqrt{a}k]} * |F_k|^2)(x_1)\right\}^{1/2} \\
&\quad \left\{\frac{1}{6\frac{2^m}{a}\sin\sqrt{a}k}(\chi_{[-3\frac{2^m}{a}\sin\sqrt{a}k,3\frac{2^m}{a}\sin\sqrt{a}k]} * |G_k|^2)(x_2)\right\}^{1/2} \\
&\equiv C X_{k,m}(x_1)^{1/2} Y_{k,m}(x_2)^{1/2}\,.
\end{aligned} \tag{17}$$

Using Hölder's inequality and the Schwarz inequality, we have from (16), (15) and (17) that

$$\begin{aligned}
\left|\sum_{k=1}^{N_0}\tau_k f(x)\right|^p &\le \left(\sum_k |(\tau_k(F_k G_k))(x)|\right)^p \\
&\le C\left(\sum_k\sum_{m=1}^{\infty} 2^{-m} X_{k,m}(x_1)^{1/2} Y_{k,m}(x_2)^{1/2}\right)^p \\
&\le C'\sum_m 2^{-m}\left(\sum_k X_{k,m}(x_1)^{1/2} Y_{k,m}(x_2)^{1/2}\right)^p \\
&\le C'\sum_m 2^{-m}\left\{\left(\sum_k X_{k,m}(x_1)\right)\cdot\left(\sum_k Y_{k,m}(x_2)\right)\right\}^{p/2}.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\int_{\mathbf{R}^2}\left|\sum_{k=1}^{N_0}\tau_k f(x)\right|^p dx \\
&\le C'\sum_{m=1}^{\infty} 2^{-m}\int_{\mathbf{R}}\left(\sum_k X_{k,m}(x_1)\right)^{p/2} dx_1\cdot\int_{\mathbf{R}}\left(\sum_k Y_{k,m}(x_2)\right)^{p/2} dx_2.
\end{aligned} \tag{18}$$

Fix $w \geq 0$ in $L^{p/(p-2)}(\mathbf{R})$ (conjugate exponent of $p/2$). Let M be the Hardy-Littlewood maximal operator. Then we have

$$\begin{aligned}
&\int_{\mathbf{R}} \left(\sum_{k=1}^{N_0} X_{k,m}(x) \right) w(x)dx \\
&= \int_{\mathbf{R}} \sum_k |F_k(y)|^2 \left\{ \frac{1}{6\frac{2^m}{a}\cos\sqrt{a}k} (\chi_{[-3\frac{2^m}{a}\cos\sqrt{a}k, 3\frac{2^m}{a}\cos\sqrt{a}k]} * w)(y) \right\} dy \\
&\leq \left\{ \int_{\mathbf{R}} \left(\sum_k |F_k(y)|^2 \right)^{p/2} dy \right\}^{2/p} \cdot \left\{ \int_{\mathbf{R}} ((Mw)(y))^{p/(p-2)} dy \right\}^{(p-2)/p} \\
&\leq C \left\{ \int_{\mathbf{R}} (\sum_k |F_k(y)|^2 \Big)^{p/2} dy \right\}^{2/p} \|w\|_{L^{p/(p-2)}(\mathbf{R})} .
\end{aligned}$$

Here, the last inequality follows from $L^{p/(p-2)}$ boundedness of M. Allowing $w \geq 0$ to vary in $L^{p/(p-2)}(\mathbf{R})$ freely, we obtain

$$\int_{\mathbf{R}} \left(\sum_{k=1}^{N_0} X_{k,m}(x) \right)^{p/2} dx \leq C \int_{\mathbf{R}} \left(\sum_k |F_k(x)|^2 \right)^{p/2} dx . \tag{19}$$

Obviously, the same inequality holds for $Y_{k,m}$.

In the process of estimating the RHS of (19) and similar one for G_k we need a property of γ_j^k.

3.5. A property of γ_j^k.

PROPOSITION 11. (i) *Fix $0 \leq l \leq N_1$. For every m, $2^l \leq m < 2^{l+1}$, the number of n such that*

$$\gamma_1^m = \gamma_1^n , \quad 2^l \leq n < 2^{l+1}$$

is at most 7.

(ii) *For every m, $1 \leq m \leq N_0$ the number of n such that*

$$\gamma_2^m = \gamma_2^n , \quad 1 \leq n \leq N_0$$

is at most 3.

PROOF OF (i). Note that γ_1^k is a non-increasing sequence. We first assume that $m \leq n$. Then we have

$$\begin{aligned}
\gamma_1^m 2^{l+1} a \leq \cos\sqrt{a}m < (\gamma_1^m + 1)2^{l+1} a , \\
\gamma_1^m 2^{l+1} a \leq \cos\sqrt{a}n < (\gamma_1^m + 1)2^{l+1} a
\end{aligned}$$

and hence

$$0 \leq \cos\sqrt{a}m - \cos\sqrt{a}n < 2^{l+1} a .$$

We see that

$$\cos\sqrt{a}m - \cos\sqrt{a}n = \int_{\sqrt{a}m}^{\sqrt{a}n} \sin t dt \geq (n-m)\sqrt{a}\sin(\sqrt{a}2^l) .$$

Note that $\sin(\sqrt{a}2^l) \geq \sqrt{a}2^{l-1}$ because $\sqrt{a}2^l < \pi/2$. Therefore, we have

$$(n-m)2^{l-1}a < 2^{l+1}a$$

and hence n must satisfy $m \leq n \leq m+3$. Exchanging the role of m, n, we have $m-3 \leq n \leq m$ if $n \leq m$.

PROOF OF (ii). Note that γ_2^k is a non-decreasing sequence. We first assume that $m \leq n$. Proceeding as above we have

$$0 \leq \sin\sqrt{a}n - \sin\sqrt{a}m < \sqrt{a}\,.$$

We see that

$$\sin\sqrt{a}n - \sin\sqrt{a}m = \int_{\sqrt{a}m}^{\sqrt{a}n} \cos t\,dt \geq (n-m)\frac{\sqrt{a}}{\sqrt{2}}\,.$$

Therefore, we have

$$(n-m)\frac{\sqrt{a}}{\sqrt{2}} < \sqrt{a}$$

and hence n must satisfy $m \leq n \leq m+1$. Exchanging the role of m, n, we have $m-1 \leq n \leq m$ if $n \leq m$. □

3.6. Completion of the proof. Now, using Propositions 11 and Lemma 7, the RHS of (19) is estimated as

$$\begin{aligned}
&\int_{\mathbf{R}}\Big(\sum_{k=1}^{N_0}|F_k(x)|^2\Big)^{p/2}dx \qquad (20)\\
&\leq \int_{\mathbf{R}}\Big(\sum_{l=0}^{N_1}\sum_{k=2^l}^{2^{l+1}-1}|F_k(x)|^2\Big)^{p/2}dx\\
&\leq C(N_1+1)^{p/2-1}\sum_{l=0}^{N_1}\int_{\mathbf{R}}\Big(\sum_{j\in\mathbf{Z}}|P_{2^{l+1}a,j}f_1(x)|^2\Big)^{p/2}dx\\
&\leq C(N_1+1)^{p/2}\|f_1\|_p^p \leq C\Big(\log\Big(\frac{1}{a}\Big)\Big)^{p/2}\|f_1\|_p^p\,.
\end{aligned}$$

The same inequality, but not including the logarithm factor, holds for G_k.

Thus, combining estimates (18), (19) and (20) we have finally proved (7) and proved Theorem 6.

4. Proof of Lemma 9.

The argument basically follows [Mi, p. 109–110].

Put $\kappa(\xi) = \psi\left(\frac{(\xi_1 - |\xi|)^2 + \xi_2^2}{|\xi|^2 a}\right)\varphi\left(\frac{1-|\xi|}{a}\right)$ for $\xi \in \{1 - 2a \leq |\xi| \leq 1 - \frac{a}{2},\ |\xi_2| \leq \sqrt{5a}\}$. If we can prove

$$|\overset{\vee}{\kappa}(x)| \leq C\sum_{m=0}^{\infty} 2^{-m}\frac{1}{|2^m R_a|}\chi_{2^m R_a}(x)\,, \tag{21}$$

then by the rotation argument everything reduces to this inequality.

Now, for every $N \in \mathbf{N}$ we shall prove

$$|\overset{\vee}{\kappa}(x)| \leq C_N a^{3/2}(1 + a|x_1| + \sqrt{a}|x_2|)^{-N}\,. \tag{22}$$

If this can be done, (21) follows from the following observation.

$$\begin{aligned} a^{3/2}(1 + a|x_1| + \sqrt{a}|x_2|)^{-N} &\leq a^{3/2}(1 + \max(a|x_1|,\ \sqrt{a}|x_2|))^{-N} \\ &\leq a^{3/2}\sum_{m=0}^{\infty}\chi_{\{\max(a|y_1|,\ \sqrt{a}|y_2|)\leq 2^m\}}(x)2^{-mN} = a^{3/2}\sum_{m=0}^{\infty}2^{-mN}\chi_{2^m R_a}(x) \\ &= \sum_{m=0}^{\infty}2^{-m(N-2)}\frac{1}{|2^m R_a|}\chi_{2^m R_a}(x)\,. \end{aligned}$$

Putting $N = 3$, we have (21).

PROOF OF (22). By the elementary computations for every multi-indices $\alpha = (\alpha_1,\ \alpha_2)$ we see that

$$\left|\left(\frac{\partial}{\partial\xi_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial\xi_2}\right)^{\alpha_2}\kappa(\xi)\right| \leq C_\alpha a^{-\alpha_1-(1/2)\alpha_2}\,.$$

It follows from this inequality and $|\mathrm{supp}\kappa| \leq Ca^{3/2}$ that

$$|(ax_1)^{\alpha_1}(\sqrt{a}x_2)^{\alpha_2}\overset{\vee}{\kappa}(x)| \leq C_\alpha a^{3/2}\,.$$

Therefore, we obtain

$$|\overset{\vee}{\kappa}(x)| \leq C_N a^{3/2}((1 + a|x_1|)(1 + \sqrt{a}|x_2|))^{-N} \leq C_N a^{3/2}(1 + a|x_1| + \sqrt{a}|x_2|)^{-N}\,.$$

Thus, we have proved (22).

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