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The Connectivities of Leaf Graphs of Sets of Points in the Plane

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Abstract. Let U be a finite set of points in general position in the plane. We consider the following graph $\mathcal{G}$ determined by U. A vertex of $\mathcal{G}$ is a spanning tree of U whose edges are straight line segments and do not cross. Two such trees $\mathbf{t}$ and $\mathbf{t}'$ are adjacent if for some vertex $u \in U$, $\mathbf{t} - u$ is connected and coincides with $\mathbf{t}' - u$. We show that $\mathcal{G}$ is 2-connected, which is the best possible result.

1. Introduction.

Let G be a connected graph and $\mathcal{V}_G$ the set of all the spanning trees of G. We define an adjacency relation on $\mathcal{V}_G$ so that two spanning trees $\mathbf{t}_1$ and $\mathbf{t}_2 \in \mathcal{V}_G$ are adjacent if and only if there exist edges $e_i \in E(\mathbf{t}_i)$ such that

$$\mathbf{t}_1 - e_1 = \mathbf{t}_2 - e_2 . \tag{1}$$

The graph thus obtained is called *a tree graph*. The lower bound of the connectivities of a tree graph was shown by Liu.

THEOREM 1 (Liu [8]). *The tree graph of a connected graph $G = (V, E)$ is $2(|E| - |V| + 1)$-connected.*

We can consider two subgraphs of a tree graph as follows. If an edge is incident to endvertices in a spanning tree $\mathbf{t}$, then we call it an *outer edge*. An edge is not outer is called *inner*. In the equation (1), the edge e_1 is an outer edge in $\mathbf{t}_1$ if and only if e_2 is also outer in $\mathbf{t}_2$. A *leaf graph* is defined on $\mathcal{V}_G$ as follows; $\mathbf{t}_1$ and $\mathbf{t}_2 \in \mathcal{V}_G$ are said to be adjacent if there exist outer edges $e_i \in E(\mathbf{t}_i)$ which satisfy the equation (1). The authors showed the following theorem.

THEOREM 2 (Kaneko and Yoshimoto [7]). *Let G be a 2-connected graph of minimum degree δ. Then the leaf graph of G is $(2\delta - 2)$-connected.*

We can define adjacency relation of a leaf graph as follows; $\mathbf{t}_1$ and $\mathbf{t}_2$ are adjacent if there exists a vertex $u \in V(G)$ such that $\mathbf{t}_1 - u$ is connected and coincides with $\mathbf{t}_2 - u$. On the other hand, *a trunk graph* is defined on the set $\mathcal{V}_G^*$ of all the spanning trees except stars as follows; $\mathbf{t}_1$ and $\mathbf{t}_2 \in \mathcal{V}_G^*$ are said to be adjacent if there exist inner edges $e_i \in E(\mathbf{t}_i)$ which satisfy the

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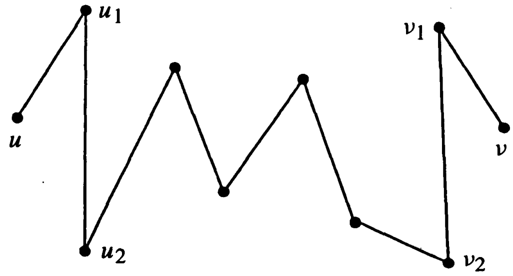

FIGURE 1.

equation (1). Yoshimoto [9] showed that if G is a 2-connected graph with at least five vertices and if G is k-edge connected, then the trunk graph of G is $(k-1)$-connected.

In this paper, we consider a geometric version of a leaf graph. Let U be a set of n points in the plane which is in general position, i.e., no three points in U are collinear. A graph on U whose edges are straight line segments joining two vertices in U and do not cross is called *a non-crossing graph* on U. Let $\mathcal{V}_U$ be the set of all the non-crossing spanning trees on U. Ikebe et al. [6] showed that any rooted tree with n vertices can be embedded as a non-crossing spanning tree on a given set U, the root being mapped to an arbitrary specified point of U.

A geometric tree graph on U is defined on the set $\mathcal{V}_U$ as follows; $\mathbf{t}_1$ and $\mathbf{t}_2 \in \mathcal{V}_U$ are said to be adjacent if there exist edges $e_i \in E(\mathbf{t}_i)$ which satisfy the equation (1). Avis and Fukuda [1] showed that the geometric tree graph on U is connected. In [4], Hernando et al. showed hamiltonicity and connectivity of a geometric tree graph on U whose points are in convex position. *A geometric leaf graph* on U is defined by $\mathcal{V}_U$ as follows; $\mathbf{t}_1$ and $\mathbf{t}_2 \in \mathcal{V}_U$ are said to be adjacent if there exists $u \in U$ such that $\mathbf{t}_1 - u$ is connected and coincides with $\mathbf{t}_2 - u$. We shall prove the following theorem in this paper.

THEOREM 3. *Let U be the set of points in the plane in general position. Then the geometric leaf graph on U is 2-connected.*

Let $\mathbf{t}$ be the non-crossing spanning tree in Figure 1. Let $\mathbf{t}' = (\mathbf{t} - uu_1) \cup uu_2$ and $\mathbf{t}'' = (\mathbf{t} - vv_1) \cup vv_2$. Then since $\mathbf{t}' - u = \mathbf{t} - u$ and this graph is connected, the non-crossing spanning tree $\mathbf{t}'$ is adjacent to $\mathbf{t}$ in the geometric leaf graph on U. Similarly, $\mathbf{t}''$ is adjacent to $\mathbf{t}$. Because any other non-crossing spanning tree on U is not adjacent to $\mathbf{t}$, the degree of $\mathbf{t}$ in the geometric leaf graph is two. Thus the lower bound of the theorem is the best possible.

Finally, we introduce concepts and notations used in the subsequent arguments. Let G be a non-crossing graph on U and $u \in U$. Let $\tilde{G}$ be a maximal non-crossing graph (i.e. any edge except edges in $E(\tilde{G})$ intersects this graph) on $U \setminus u$ which includes $G - u$ as a subgraph. The vertex u is included in some triangulate region or the infinite region of $\tilde{G}$. In either case, u can be adjacent to at least two vertices in $\tilde{G}$. Since $\tilde{G}$ includes $G - u$, it holds for $G - u$. We denote by $S_G(u)$ the set of all the vertices which can be adjacent to u in $G - u$. It is a plain fact that if $S_G(u)$ includes only two vertices, then these are adjacent in G. We call the edge the *shield* of the vertex u. Since a non-crossing spanning treet $\mathbf{t}$ does not include a cycle, there

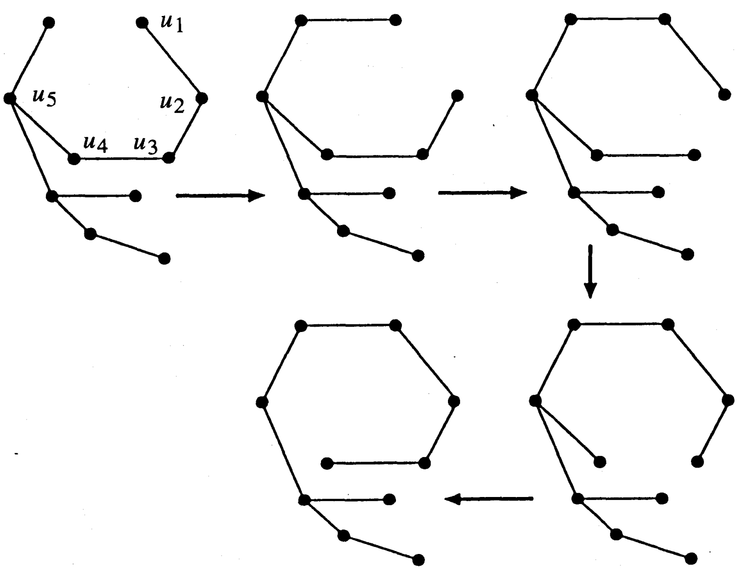

FIGURE 2.

exists exactly one path between any vertices u and $v \in U$, denoted by $P_{\mathbf{t}}(u, v)$. A *simple path* $P = (u_1, u_2, \cdots, u_l)$ is a path in a non-crossing spanning tree $\mathbf{t}$ such that u_1 is an endpoint of $\mathbf{t}$ and the degree of any vertex u_i is two in $\mathbf{t}$ for $2 \leq i < l$. Let $x \in S_{\mathbf{t}}(u_1) \setminus \{u_2, u_3, \cdots, u_{l-1}\}$. Then there exists a natural path from $\mathbf{t}$ to $\mathbf{t}' = (\mathbf{t} - u_{l-1}u_l) \cup u_1x$. See Figure 2.

In fact, let $\mathbf{r}_1 = (\mathbf{t} - u_1u_2) \cup u_1x$ and

$$\mathbf{r}_i = (\mathbf{r}_{i-1} - u_iu_{i+1}) \cup u_iu_{i-1}$$

for any $i \leq l$. Then $\mathbf{r}_i$ is a non-crossing spanning tree and $\mathbf{r}_i$ is adjacent to $\mathbf{r}_{i-1}$ in the geometric leaf graph for any $i \leq l$. If $i \neq j$, then $\mathbf{r}_i \neq \mathbf{r}_j$. Thus

$$(\mathbf{t}, \mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_{l-1} = \mathbf{t}')$$

is a path between $\mathbf{t}$ and $\mathbf{t}'$ in the leaf graph. We call the path a *short-cut passage* determined by the edge u_1x and the simple path P.

2. The proof of Theorem 3.

In the following, we call a geometric leaf graph simply a leaf graph. At first, we shall show that the leaf graph $\mathcal{G}$ of U is connected. Let $\mathbf{t}_1$ and $\mathbf{t}_n$ be any non-crossing spanning trees on U. We find out a path between the graphs by an induction on the number of vertices in U.

Suppose that there exists $u \in U$ such that u is an endpoint of $\mathbf{t}_1$ and $\mathbf{t}_n$. Then there is a path

$$(\mathbf{t}_1 - u = \mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_n = \mathbf{t}_n - u)$$

in the leaf graph of $U \setminus u$ by the hypothesis. Let us assume that $\mathbf{s}_{i+1} = (\mathbf{s}_i - v_1 v_i') \cup v_i v_i''$ for any i.

Since the interior of the edge $v_i v_i''$ does not intersect $\mathbf{s}_i$, we have that $\mathbf{s}_i \cup v_i v_i''$ is non-crossing. Thus there exists a vertex $u_i \in S_{\mathbf{s}_i \cup v_i v_i''}(u)$ which is not v_i. Then the graph $\mathbf{s}_i \cup v_i v_i'' \cup uu_i$ is non-crossing. Let $\mathbf{r}_i = \mathbf{s}_i \cup uu_i$ and $\mathbf{t}_{i+1} = \mathbf{s}_{i+1} \cup uu_i$. These are non-crossing because $\mathbf{r}_i$ and $\mathbf{t}_{i+1}$ are subgraphs of $\mathbf{s}_i \cup v_i v_i'' \cup uu_i$. Furthermore $\mathbf{r}_i$ is adjacent to $\mathbf{t}_i$ and $\mathbf{t}_{i+1}$ since $\mathbf{t}_i - u = \mathbf{s}_i = \mathbf{r}_i - u$ and $\mathbf{r}_i - v_i = \mathbf{t}_{i+1} - v_i$. Especially we denote the non-crossing spanning tree $\mathbf{s}_n \cup uu_{n-1}$ by $\mathbf{t}_n'$. Then we have found out the path

$$(\mathbf{t}_1, \mathbf{r}_1, \mathbf{t}_2, \mathbf{r}_2, \cdots, \mathbf{t}_{n-1}, \mathbf{r}_{n-1}, \mathbf{t}_n', \mathbf{t}_n).$$

Assume that $\mathbf{t}_1$ and $\mathbf{t}_n$ does not have a common endpoint. Let u and v be endpoints of $\mathbf{t}_1$ and $\mathbf{t}_n$ respectively. Let $\mathbf{s}$ be an non-crossing spanning tree on $U \setminus \{u, v\}$. Let $u' \in S_{\mathbf{s}}(u)$ and $\mathbf{s}' = \mathbf{s} \cup uu'$. Since $\mathbf{s}'$ is non-crossing, there exists a vertex $v' \in S_{\mathbf{s}'}(v)$ which is not u. The non-crossing spanning tree $\mathbf{s}'' = \mathbf{s}' \cup vv'$ on U has u and v as endpoints. Since $\mathbf{t}_1$ and $\mathbf{s}''$ include the common endpoint u, there exists a path between the non-crossing spanning trees by the previous argument. Similarly there is a path from $\mathbf{s}''$ to $\mathbf{t}_n$, showing the connectivity of the leaf graph $\mathcal{G}$.

Next, we shall show the 2-connectivity of the leaf graph by a contradiction. Suppose that $\mathbf{t}$ is a cut vertex of $\mathcal{G}$, with $\mathcal{C}_1$ and $\mathcal{C}_2$ the connected components of $\mathcal{G} - \mathbf{t}$. Let $\mathbf{t}_i \in \mathcal{C}_i$ be adjacent to $\mathbf{t}$ in such a way that $\mathbf{t}_1 = (\mathbf{t} - uu_1) \cup uu_2$ and $\mathbf{t}_2 = (\mathbf{t} - vv_1) \cup vv_2$. If $u = v$, then $\mathbf{t}_1$ is adjacent to $\mathbf{t}_2$. Therefore we have $u \neq v$. Let us find out a path joining $\mathbf{t}_1$ and $\mathbf{t}_2$ which is internally disjoint from $\mathcal{P} = (\mathbf{t}_1, \mathbf{t}, \mathbf{t}_2)$. (i.e., does not pass through $\mathbf{t}$.) Notice that the interiors of the edges uu_i and vv_i do not intersect $\mathbf{t} - \{u, v\}$. Furthermore the interior of the edge uu_1 does not intersect vv_1 and vv_2 and the interior of the edge uu_2 does not intersect vv_1.

We divide the arguments into three cases.

Case 1. $u_2 \neq v$ and $v_2 \neq u$

Suppose that the interior of the edge uu_2 does not intersect vv_2. Then, since the interior of uu_2 does not intersect uu_1, $\mathbf{t}_1 \cup uu_1 \cup vv_2$ is non-crossing. Thus $\mathbf{s} = (\mathbf{t}_1 - vv_1) \cup vv_2 \subset \mathbf{t}_1 \cup uu_1 \cup vv_2$ is non-crossing and is adjacent to $\mathbf{t}_1$. Since v_2 is not u, the vertex u is an endpoint in $\mathbf{s}$. Therefore $\mathbf{s}$ is adjacent to $\mathbf{t}_2$ in the leaf graph. Because $\mathbf{s} \neq \mathbf{t}$, the path $\mathcal{Q} = (\mathbf{t}_1, \mathbf{s}, \mathbf{t}_2)$ is internally disjoint from $\mathcal{P}$.

Assume that uu_2 intersects vv_2. Let $\mathbf{r} = (\mathbf{t}_1 - uu_2) \cup vv_2$. If there exists a vertex $x \in S_r(u)$ which is not v and u_1, then there is a path

$$\mathcal{Q} = (\mathbf{t}_1, \mathbf{s}, \mathbf{s}', \mathbf{t}_2),$$

where $\mathbf{s} = (\mathbf{t}_1 - uu_2) \cup ux$ and $\mathbf{s}' = (\mathbf{s} - vv_1) \cup vv_2$. Because $\mathbf{s}$ and $\mathbf{s}'$ include the edge ux, the path does not pass through $\mathbf{t}$. If $S_{\mathbf{r}}(u) = \{v, u_1\}$, then vu_1 is a shield of u in $\mathbf{r}$. Because v is adjacent to exactly two vertices in $\mathbf{r}$, the vertex u_1 is v_1 or v_2. If $u_1 = v_1$, then the interior of the shield $vv_1 = vu_1$ intersect uu_2. A contradiction. Thus the vertex u_1 is v_2.

Let $\mathbf{r}' = \mathbf{t}_1 \cup uu_1$. If $S_{\mathbf{r}'}(v) \setminus \{u, v_1\} \neq \emptyset$, then there exists a path between $\mathbf{t}_1$ and $\mathbf{t}_2$ which is internally disjoint from $\mathcal{P}$ as before. If such a vertex does not exist, then uv_1 is a shield of

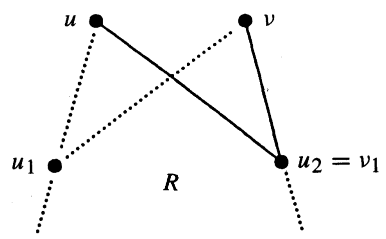

FIGURE 3.

v in $\mathbf{r}'$. Then vertex u is adjacent to exactly u_1 and u_2 in $\mathbf{r}'$. If $v_1 = u_1$, then the interior of the shield $uv_1 = uu_1$ intersects vv_2. Thus we have $v_1 = u_2$. Because the edge vv_2 is a shield of u in $\mathbf{r}$ and the edge uu_2 is a shield of v in $\mathbf{r}'$, any points in $U \setminus \{u, v, u_1 = v_2, u_2 = v_1\}$ are contains in the region R in Figure 3.

If there is not an endpoint except u and v and $u_1 = v_2$ in $\mathbf{t}_1$, then $\mathbf{s}_1 = (\mathbf{t}_1 - uu_2) \cup uv$ is a Hamiltonian path. Therefore there exists a short-cut passage determined by uu_1 and $P_{\mathbf{s}_1}(v, v_2 = u_1)$, denoted by $(\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_l)$. The non-crossing spanning tree $\mathbf{s}_l$ is $(\mathbf{s}_1 - vv_1) \cup uu_1 = (\mathbf{t}_2 - vv_2) \cup vu$. Because v is an endpoint of $\mathbf{s}_l$, the non-crossing spanning tree is adjacent to $\mathbf{t}_2$. Thus we obtained a path

$$\mathcal{Q} = (\mathbf{t}_1, \mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_l, \mathbf{t}_2)$$

which does not pass through $\mathbf{t}$.

Suppose that there is an endpoint w other than u and v and $u_1 = v_2$ in $\mathbf{t}_1$. Since $u_2 = v_1$ is not an endpoint in $\mathbf{t}_1$, U includes at least five points. Two different vertices do not have a common shield if the number of vertices in a graph is greater than four. Thus $uu_2 = uv_1$ is not a shield of w. Furthermore because $w \in R$, $S_{\mathbf{t}_1}(w)$ contains at least two vertices which are not u and v. Assume that $ww_1 \in E(\mathbf{t}_1)$ and let $w_2 \in S_{\mathbf{t}_1}(w) \setminus \{u, v, w_1\}$. Then the interior of the edge $ww_2 \subset R$ does not intersect uu_i and vv_i. Therefore, after transferring the edge ww_1 to ww_2, we move the edges uu_2 and vv_1 to the desired place. It is clear that the transformations induces a path from $\mathbf{t}_1$ to $\mathbf{t}_2$ which does not pass through $\mathbf{t}$.

Case 2. $u_2 = v$ and $v_2 \neq u$

The interior of the edge uu_2 does not intersect vv_2 in the present case. Thus $\mathbf{r} = \mathbf{t}_1 \cup uu_1 \cup vv_2$ is non-crossing. If $S_{\mathbf{r}}(u) \setminus \{v = u_2, u_1\} \neq \emptyset$, then there exists a path from $\mathbf{t}_1$ to $\mathbf{t}_2$ which does not pass through $\mathbf{t}$ as before. If such a vertex does not exist, then the edge vu_1 is a shield of u in $\mathbf{r}$. Thus we have that the vertex u_1 is v_1 or v_2.

If there is not an endpoint in $\mathbf{r}$, then $\mathbf{t}_1$ is a Hamiltonian path. See Figure 4. Thus it is easy to find out a path between $\mathbf{t}_1$ and $\mathbf{t}_2$ which is internally disjoint from $\mathcal{P}$.

Therefore we suppose that there exists an endpoint w in $\mathbf{r}$. If $S_{\mathbf{r}}(w) \setminus \{v, v_i \neq u_1\} \neq \emptyset$, then we can find out a path between $\mathbf{t}_1$ and $\mathbf{t}_2$ as follows. Assume that $ww_1 \in E(\mathbf{t}_1)$ and let $w_2 \in S_{\mathbf{r}}(w)$ be neither v nor w_1 and let $\mathbf{s} = (\mathbf{t}_1 - ww_1) \cup ww_2$. Since $w \notin S_{\mathbf{r}}(u)$, the vertex w_2 is not u. Therefore u is also an endpoint of $\mathbf{s}$. Thus the non-crossing spanning tree $\mathbf{s}' = (\mathbf{s} - uu_2) \cup uu_1$ is adjacent to $\mathbf{s}$. Furthermore since u_1 is not $v = u_2$, the non-crossing

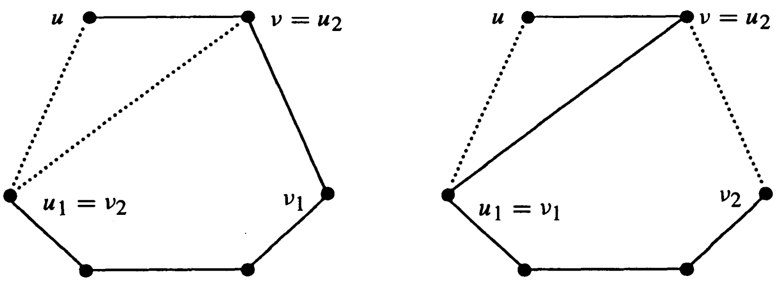

FIGURE 4.

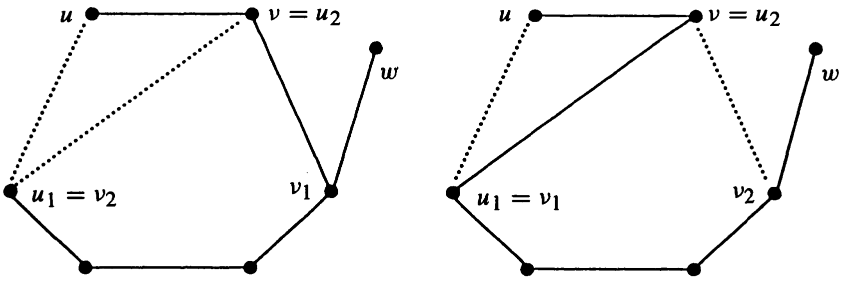

FIGURE 5.

spanning tree $\mathbf{s}'$ is adjacent to $\mathbf{s}'' = (\mathbf{s}' - vv_1) \cup vv_2$. Then $\mathbf{s}'' = (\mathbf{t}_2 - ww_1) \cup ww_2$. Thus there is a path

$$\mathcal{Q} = (\mathbf{t}_1, \mathbf{s}, \mathbf{s}', \mathbf{s}'', \mathbf{t}_2)$$

which does not pass through $\mathbf{t}$.

Let $v_i \in \{v_1, v_2\}$ be not u_1. If $S_{\mathbf{r}}(w) = \{v, v_i\}$, then vv_i is a shield of w. It is clear that $\mathbf{r}$ contains at least five vertices. Thus the only endpoint in $\mathbf{r}$ is w because no two vertices admit a common shield. Since v is not adjacent to w, we have $wv_i \in E(\mathbf{t}_1)$. Let $\mathbf{s} = (\mathbf{t}_1 - wv_i) \cup wv$. Then $\mathcal{P}_{\mathbf{s}}(v, v_2)$ is a simple path. See Figure 5. Thus there exists a short-cut passage determined by the edge vv_2 and the simple path. The short-cut passage is a path from $\mathbf{s}$ to $\mathbf{s}' = (\mathbf{s} - vv_1) \cup vv_2$. Since u is also an endpoint of $\mathbf{s}'$, it is adjacent to $\mathbf{t}_2$. Now we get a path between $\mathbf{t}_1$ and $\mathbf{t}_2$ which does not pass through $\mathbf{t}$.

Case 3. $u_2 = v$ and $v_2 = u$

If there is not an endpoint in $\mathbf{r} = \mathbf{t}_1 \cup uu_1$, then the non-crossing spanning tree $\mathbf{t}_1$ is a Hamiltonian path. Therefore there exists a short-cut passage determined by the edge uu_1 and the path $P_{\mathbf{t}_1}(v, u_1)$. The short-cut passage is a path from $\mathbf{t}_1$ to $\mathbf{t}_2 = (\mathbf{t}_1 - vv_1) \cup uu_1$ which is internally disjoint from $\mathcal{P}$.

Suppose that there exists an endpoint w in $\mathbf{r}$ such that $S_{\mathbf{r}}(w)$ contains at least three vertices. Assume that $ww_1 \in E(\mathbf{r})$ and let $w_2 \in S_{\mathbf{r}}(w)$ be neither w_1 nor u. Then the non-crossing spanning tree $\mathbf{s}_1 = (\mathbf{t}_1 - ww_1) \cup ww_2$ is adjacent to $\mathbf{t}_1$. Since u is also an endpoint, we transfer the edge uu_2 to uu_1 to obtain $\mathbf{s}_2 = (\mathbf{s}_1 - uu_2) \cup uu_1$. Let $w_3 \in S_{\mathbf{r}}(w)$ be neither

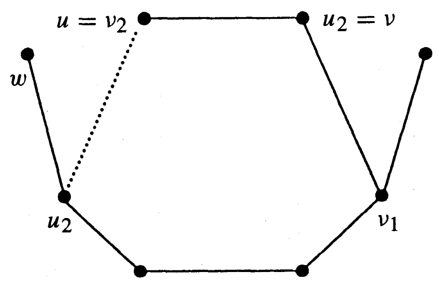

FIGURE 6.

w_1 nor v and let $\mathbf{s}_3 = (\mathbf{s}_2 - ww_2) \cup ww_3$. Because it is adjacent to $\mathbf{s}_4 = (\mathbf{s}_3 - vv_1) \cup vv_2 = (\mathbf{t}_2 - ww_1) \cup ww_3$, we have found out a path

$$\mathcal{Q} = (\mathbf{t}_1, \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4, \mathbf{t}_2)$$

which does not pass through $\mathbf{t}$.

Assume that any endpoint of $\mathbf{r}$ can be adjacent to exactly two vertices. If there exists an endpoint in $\mathbf{r}$ whose shield is not incident to u and v, then a desired path between $\mathbf{t}_1$ and $\mathbf{t}_2$ is easy to find out.

Thus we suppose that such an endpoint does not exist in $\mathbf{r}$. Notice that there is not an endpoint with shield uv because u and v are not adjacent to an endpoint in $\mathbf{r}$. Therefore the endpoints in $\mathbf{r}$ whose shield is incident to u or v are at most two. See Figure 6. We transfer the edge $ww_1 \in E(\mathbf{t}_1)$ to wu or wv for any endpoint w in $\mathbf{r}$. Then the path between u_1 and v in the non-crossing spanning tree is simple. Thus there exists a short-cut passage determined by the edge uu_1 and this simple path. At the endpoint of the short-cut passage, we transfer the edge wu or wv back to the original place. Then we get the non-crossing spanning tree $\mathbf{t}_2$. Therefore we have found out the desired path.

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