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# Chaotic Maps on a Measure Space and the Behavior of the Orbit of a State

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## Introduction.

This paper is a continuation of [5]. As well known, a continuous map $\varphi$ on a metric space is considered as a chaotic map if $\varphi$ has the following properties.

(1) The set of all periodic points for $\varphi$ is dense.

(2) $\varphi$ is one-sided topologically transitive.

(3) $\varphi$ depends sensitively on initial conditions.

These properties are concerned with the orbit of a given initial point and the interesting relation among these three conditions was shown in [1]. Moreover, the chaotic property of the orbit of a point for the logistic map $\lambda(x) = 4x(1-x)$ has been pointed out since old times (cf. [8]). In [5], we considered how a probability density function changed by the iteration of a unimodal chaotic map and more generally we studied the behavior of a state changed by the iteration of the transposed map $\alpha^*_{V(\varphi)}$ of the $*$-endomorphism $\alpha_{V(\varphi)}$ of a von Neumann algebra associated with a chaotic map $\varphi$. In the present paper, we develop this study into the study in the case of a family of maps which includes not only unimodal maps but also those measurable maps which have $n$ laps for $n \geq 1$. We call each of those maps on a measure space $X$ a map with $n$ laps (MW$n$L for short, Definition 2.1) and show some results concerning the limit of the orbit of a state. The following is one of the statements concerning our main subject, which is considered as an important property of chaotic maps in addition to the above three properties.

(4) For any state, its orbit determined by an MW$n$L $\varphi$ satisfying some conditions converges to a unique state *with respect to a norm topology*.

We express Property (4) more precisely. The following is the most familiar form of (4) for some chaotic maps $\varphi$'s, which are MW$n$L's defined on a measure space $(X, m)$;

$$\lim_{k\to\infty} \int_X f(\varphi^k(x))\eta(x)dm = \int_X f(x)|e(x)|^2 dm \tag{4-1}$$

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for any $f$ in $L^\infty(X)$ and $\eta \geq 0$ with $\int_X \eta(x)dm = 1$, where $e$ is a function in $L^2(X)$ with $\int_X |e(x)|^2 dm = 1$. In the context of the duality between $L^1(X)$ and $L^\infty(X)$, the equality in (4.1) is written as follows:

$$\text{(4-2)} \quad \lim_{k\to\infty}\int_X ((A_\varphi^*)^k f)(x)\eta(x)dm = \lim_{k\to\infty}\int_X f(x)(A_\varphi^k \eta)(x)dm = \int_X f(x)|e(x)|^2 dm\,,$$

where $A_\varphi$ is Perron Frobenius operator on $L^1(X)$. This means that the sequence $\{A_\varphi^k \eta\}$ converges to $|e|^2$ in $L^1(X)$ with respect to the $\sigma(L^1(X), L^\infty(X))$-topology, and it is our important purpose to show that this convergence holds with respect to the norm topology in $L^1(X)$. By considering $\eta$ as a vector state $\omega_{\sqrt{\eta}}$, the equality is formally written as follows:

$$\text{(4-3)} \qquad \lim_{k\to\infty}(A_\varphi^k(\omega_{\sqrt{\eta}}))(f) = \omega_{\sqrt{e}}(f)\,.$$

The operator $A_\varphi$ is extended to the transposed map $\alpha_{V(\varphi)}^*$, on the predual $M_*$ of a von Neumann algebra $M$ on $L^2(X)$, if $M$ is invariant for $\alpha_{V(\varphi)}$. Moreover (4-3) is extended to the following form: for any $\xi$ in some Hilbert subspace $L^2(X)_e$ of $L^2(X)$, we have

$$\text{(4-4)} \qquad \lim_{k\to\infty}(\alpha_{V(\varphi)}^*)^k(\omega_\xi) = \omega_e$$

*with respect to the norm topology* in the predual $M_*$ of a sufficiently large von Neumann algebra $M$ on $L^2(X)$.

In Section 1, we discuss the theorems which yield Property (4). Though those theorems were studied in [5], we show in the present paper some more precise and generalized results and give them complete proofs. Furthermore, in the first part of Section 2 until Example 2.10, we show Property (4) concerning MW$n$L's by applying the theorems in Section 1 and giving some concrete examples. In this discussion, the families of isometries associated with MW$n$L's play an important role. In the remainder of Section 2, we discuss the relation between the functional analytical property of the isometries associated with a given MW$n$L and the measure theoretical property of the map. As mentioned above, Property (4) is closely related to the property of strong-mixing about measurable maps. Of course, in the case of (4.4), if the considering Hilbert space $L^2(X)_e$ coincides with the whole space $L^2(X)$, the former is stronger than the latter and in general the latter does not imply the former, that is, the property of strong-mixing do not imply norm convergence. In Section 3, we discuss the property of norm convergence of the orbit of a state for the case of MW1L (MW$n$L for $n = 1$), and analyze the difference between property of strong-mixing and norm convergence for the case of Baker's transformation.

Finally, we note that the notation **N** and **C** denote the set of all positive integers and the set of all complex numbers respectively.

**1. The $*$-endomorphism of a von Neumann algebra associated with a family of isomorphisms.**

Let $\mathcal{H}$ be a separable complex Hilbert space with inner product $\langle \xi, \eta \rangle$ and norm $\|\xi\| = \sqrt{\langle \xi, \xi \rangle}$. Moreover let $\{V_i\}_{i=1}^n$ be a family of isometries on $\mathcal{H}$ satisfying the following property.

(A-1) $\{V_i V_i^*\}_{i=1}^n$ is a set of mutually orthogonal projections and $\sum_{i=1}^n V_i V_i^* = I$.

Of course, this family $\{V_i\}_{i=1}^n$ on $\mathcal{H}$ is the generators of the image of a representation of Cuntz-algebra $\mathcal{O}_n$ (cf. [3]) and thus we call it a family of isometries satisfying Cuntz property (f.i.c. for short). Moreover we can define a $*$-endomorphism $\alpha_V$ of the full operator algebra $B(\mathcal{H})$ as follows:

(A-2)
$$\alpha_V(T) = \sum_{i=1}^n V_i T V_i^*, \quad (T \in B(\mathcal{H})).$$

For positive integers $n$ and $k$, we denote by $I(n)$ the set $\{1, 2, \cdots, n\}$ and $I(n)^k$ the set of all $k$-tuples $\mu = (i_1, \cdots, i_k)$ with $i_j$ in $I(n)$. For $\mu$ in $I(n)^k$ we denote by $V(\mu)$ the isometry $V_{i_1} V_{i_2} \cdots V_{i_k}$ on $\mathcal{H}$. Then $\{V(\mu) \mid \mu \in I(n)^k\}$ is a family of isometries whose final projections are mutually orthogonal and $\alpha_V^k$ is written as follows:

(A-3)
$$\alpha_V^k(T) = \sum_{\mu \in I(n)^k} V(\mu) T V(\mu)^*, \quad (T \in B(\mathcal{H})).$$

Here we show an important property concerning unit vectors which are fixed by $V_1$.

LEMMA 1.1. *Let $\{V_i\}_{i=1}^n$ be an f.i.c. on $\mathcal{H}$ and $e$ a unit vector in $\mathcal{H}$ such that $V_1 e = e$. We put*
$$ONS(e, V) = \bigcup_{k=1}^{\infty} \{V(\mu)e \mid \mu \in I(n)^k\}.$$
*Then $ONS(e, V)$ is an orthonormal system.*

PROOF. First, let $k$ and $\ell$ are positive integers such that $k < \ell$. For $\mu = (i_1, \cdots, i_k)$ in $I(n)^k$, we put $\mu' = (i_1, \cdots, i_k, 1, \cdots, 1)$ in $I(n)^\ell$. Then, since $V_1 e = e$, we have $V(\mu')e = V(\mu)e$. Next, suppose that $V(\mu_1)e \neq V(\mu_2)e$ for $\mu_1$ in $I(n)^{k_1}$, $\mu_2$ in $I(n)^{k_2}$ and $k_1 \leq k_2$. By the first mention we can assume that $k_1 = k_2$, because, if necessary, we can take $\mu_1'$ in $I(n)^{k_2}$ such that $V(\mu_1')e = V(\mu_1)e$. Therefore we have $\mu_1 \neq \mu_2$ in $I(n)^{k_2}$. Hence the final projections of $V(\mu_1)$ and $V(\mu_2)$ are orthogonal, and thus we have the conclusion. q.e.d.

In the case of $n \geq 2$, an orthonormal system $ONS(e, V)$ in the above lemma is regarded as a sequence $\{e_\ell\}_{\ell=1}^\infty$ which is inductively defined as follows:

(B)
$$e_1 = e \quad \text{and} \quad e_{(\ell-1)n+i} = V_i e_\ell, \quad (i \in I(n), \ell \in \mathbf{N}).$$

REMARK. (1) In the case where $V_i e = e$ for $i \neq 1$, the set $ONS(V, e)$ is also an orthonormal system and canonically ordered as $e_1 = V_i e$ and so on in the same fashion as in the case of $i = 1$.

(2) $ONS(V, e)$ is a function system of order $n$ in the sense of [2: Definition 2.1].

(3) Let $e$ and $f$ be two orthogonal unit vectors in $\mathcal{H}$ such that $V_i e = e$ and $V_j f = f$. Then if $i \neq j$, $ONS(e, V)$ and $ONS(f, V)$ are orthogonal. However if $i = j$, $ONS(e, V)$ and $ONS(f, V)$ are not necessarily orthogonal (cf. Examples 2.19, 2.20).

(4) In the case of $n = 1$, $ONS(e, V)$ consists of only one vector $\{e\}$.

The following is a key lemma in the present paper, so we give a complete proof.

LEMMA 1.2. *Let $ONS(e, V) = \{e_\ell\}_{\ell=1}^{\infty}$ be as in Lemma 1.1 and suppose $n \geq 2$. Then, for a fixed positive integer $k$ and an arbitrary positive integer $\ell \leq k$, there exists a unique $k$-tuple $(i_1, \cdots, i_k)$ in $I(n)^k$ such that*

$$V(\mu)^* e_\ell = \begin{cases} e_1 & \text{if } \mu = (i_1, \cdots, i_k), \\ 0 & \text{otherwise}, \end{cases}$$

*where $\mu$ is in $I(n)^k$.*

PROOF. First we write $\ell = (\ell_1 - 1)n + i_1 \leq k$ and obtain inductively a finite decreasing sequence $\{\ell_j\}_{j=1}^{k'}$ $(k' \leq k)$ such that

$$\ell_j = (\ell_{(j+1)} - 1)n + i_{(j+1)} \quad \text{and} \quad \ell_{k'} = 1.$$

By Definition (B) of $e_\ell$'s we have $V_{i_{(j+1)}} e_{\ell_{(j+1)}} = e_{\ell_j}$ and thus $V^*_{i_{(j+1)}} e_{\ell_j} = e_{\ell_{(j+1)}}$. Put $\mu_0 = (i_1, \cdots, i_{k'}, 1, \cdots, 1) \in I(n)^k$. Then we have

$$V(\mu_0)^* e_\ell = V_1^* \cdots V_1^* V_{i_{k'}}^* \cdots V_{i_1}^* e_\ell = e_1 = e.$$

Furthermore, for $\mu$ in $I(n)^k$ with $\mu \neq \mu_0$, we have $V(\mu)^* e_\ell = V(\mu)^*(\mu_0) e_1 = 0$. q.e.d.

Let $M$ be a von Neumann algebra on $\mathcal{H}$ which is invariant for $\alpha_V$. Then $\alpha_V^k$ is also a $*$-endomorphism of $M$ for any positive integer $k$. We denote by $M_*$ the predual of $M$ with norm $\|\cdot\|_1$ and by $\alpha_V^*$ the transposed map of $\alpha_V$ with respect to the duality $M$ and $M_*$, that is, $\alpha_V^*(\omega)(T) = \omega(\alpha_V(T))$, $(T \in M, \omega \in M_*)$. Then we have that, for any *positive* element $\omega$ in $M_*$,

$$\|\alpha_V^*(\omega)\|_1 = \|\alpha_V^*(\omega)(I)\| = |\omega(\alpha_V(I))| = |\omega(I)| = \|\omega\|_1,$$

where $I$ is the identity operator on $\mathcal{H}$. In particular, if $\omega$ is a state, then $\alpha_V^*(\omega)$ is also a state in $M_*$. For a vector $\xi$ in $\mathcal{H}$, we denote by $\omega_\xi$ a positive linear functional of $M$ defined by $\omega_\xi(T) = \langle T\xi, \xi\rangle$, $(T \in M)$. Especially, in the case of $\|\xi\| = 1$, $\omega_\xi$ is a state and called a vector state in $M_*$ associated with the unit vector $\xi$, and it follows that

$$\begin{aligned}\alpha_V^*(\omega_\xi)(T) &= \langle \alpha_V(T)\xi, \xi\rangle = \omega_\xi(\alpha_V(T)) = \omega_\xi\left(\sum_{i=1}^n V_i T V_i^*\right) \\ &= \sum_{i=1}^n \langle T V_i^* \xi, V_i^* \xi\rangle = \sum_{i=1}^n \omega_{V_i^*\xi}(T).\end{aligned}$$

Hence we have

$$\alpha_V^*(\omega_\xi) = \sum_{i=1}^n \omega_{V_i^*\xi}, \tag{C-1}$$

$$(\alpha_V^*)^k(\omega_\xi) = \sum_{\mu\in I(n)^k} \omega_{V(\mu)^*\xi}. \tag{C-2}$$

Hence we are interested in the behavior of the sequence $\{\sum_{\mu\in I(n)^k} \omega_{V(\mu)^*\xi}\}_{k=1}^\infty$. Now, in the case where $e$ is a unit vector such that $V_1 e = e$, that is, it is an eigenvector for the eigenvalue 1 of $V_1$, we denote by $\mathcal{H}_e$ the subspace of $\mathcal{H}$ spanned by $ONS(e, V)$. Then $\mathcal{H}$ is decomposed to the subspaces $\mathcal{H}_e$ and $\mathcal{H}_e^\perp$, which are invariant for $\{V_i V_i^*\}_{i=1}^n$. Needless to say, when $\mathcal{H} = L^2(X)$ for a measure space $(X, m)$, the notation $L^2(X)_e$ means $\mathcal{H}_e$.

PROPOSITION 1.3. *Let $\{V_i\}_{i=1}^n$ be an f.i.c. on $\mathcal{H}$. If there exists a unit vector $e$ such that $V_1 e = e$, then for any unit vector $\xi$ in the subspace $\mathcal{H}_e$ it follows that*

$$\lim_{k\to\infty} (\alpha_V^*)^k(\omega_\xi) = \omega_e \quad (\textit{w.r.t. the norm topology in } B(\mathcal{H})_*).$$

PROOF. In the case of $n = 1$, we have $\mathcal{H}_e = \{ce \mid c \text{ is a complex number}\}$, and thus $\alpha_V^*(\omega_\xi) = \alpha_V^*(\omega_e) = \omega_e$ for any unit vector $\xi$ in $\mathcal{H}_e$. Now suppose that $n \geq 2$. Let $\xi$ be a unit vector in $\mathcal{H}_e$ and $\varepsilon$ an arbitrary small positive number ($\varepsilon < 3$). Then $\xi$ has the Fourier expansion $\xi = \sum_{l=1}^\infty c_\ell e_\ell$ with respect to $ONS(e, V) = \{e_\ell\}_{\ell=1}^\infty$ and there exists a positive integer $K$ such that

$$\left\|\xi - \sum_{\ell=1}^k c_\ell e_\ell\right\| = \left(1 - \sum_{\ell=1}^k |c_\ell|^2\right)^{\frac{1}{2}} < \frac{\varepsilon}{3}$$

for all $k \geq K$. Let $k \geq K$ and put $\xi_k = \sum_{\ell=1}^k c_\ell e_\ell$. For an operator $T$ in $B(\mathcal{H})$ with $\|T\| \leq 1$, by Property (A.3) and Lemma 1.2, we have

$$\langle \alpha_V^k(T)\xi_k, \xi_k\rangle = \sum_{\mu\in I(n)^k} \sum_{\ell,j=1}^k \langle T V(\mu)^* c_\ell e_\ell, V(\mu)^* c_j e_j\rangle = \left(\sum_{\ell=1}^k |c_\ell|^2\right)\langle T e_1, e_1\rangle.$$

Thus it follows that

$$
\begin{aligned}
&|(\alpha_V^*)^k(\omega_\xi)(T) - \omega_e(T)| \\
&\quad = |\langle \alpha_V^k(T)\xi, \xi\rangle - \langle Te_1, e_1\rangle| \\
&\quad \le |\langle \alpha_V^k(T)\xi, \xi\rangle - \langle \alpha_V^k(T)\xi_k, \xi\rangle| + |\langle \alpha_V^k(T)\xi_k, \xi\rangle - \langle \alpha_V^k(T)\xi_k, \xi_k\rangle| \\
&\qquad + |\langle \alpha_V^k(T)\xi_k, \xi_k\rangle - \langle Te_1, e_1\rangle| \\
&\quad \le \|\alpha_V^k(T)\| \cdot \|\xi - \xi_k\| \cdot \|\xi\| + \|\alpha_V^k(T)\| \cdot \|\xi_k\| \cdot \|\xi - \xi_k\| \\
&\qquad + |\textstyle\sum_{\ell=1}^k |c_\ell|^2 - 1| \cdot \|T\| \cdot \|e_1\|^2 \\
&\quad < (\varepsilon/3) + (\varepsilon/3) + (\varepsilon/3)^2 < \varepsilon .
\end{aligned}
$$

Hence we have

$$\|(\alpha_V^*)^k(\omega_\xi) - \omega_e\|_1 < \varepsilon . \qquad \text{q.e.d.}$$

Consequently we obtain the following theorem, which has more precise form than that of [5, Theorem 2.2.3]

THEOREM 1.4. *Let $\{V_i\}_{i=1}^n$ be an f.i.c. on $\mathcal{H}$. If there exists a unit vector $e$ such that $V_1 e = e$, then for any state $\omega$ of the form $\omega = \sum_{\ell=1}^\infty \omega_{\xi_\ell}$ where $\xi_\ell$'s are in $\mathcal{H}_e$, it follows that*

$$\lim_{k\to\infty} (\alpha_V^*)^k(\omega) = \omega_e \quad (\textit{w.r.t. the norm topology in } B(\mathcal{H})_*) .$$

PROOF. Let $\varepsilon > 0$ be given. Since $\|\omega\|_1 = \sum_{\ell=1}^\infty \|\xi_\ell\|^2 = 1$, there exists a positive integer $L$ such that

$$\left|1 - \sum_{\ell=1}^L \|\xi_\ell\|^2\right| < \varepsilon/3 .$$

Moreover Proposition 1.3 implies that there exists an positive integer $K$ such that for each $\omega_{\xi_\ell}$ ($\ell = 1, \cdots, L$) and any $k > K$ it follows that

$$\|(\alpha_V^*)^k(\omega_{\xi_\ell}) - \|\xi_\ell\|^2\omega_e\|_1 < \|\xi_\ell\|^2\varepsilon/3 .$$

Hence for any $k > K$ we have

$$
\begin{aligned}
&\|(\alpha_V^*)^k(\omega) - \omega_e\|_1 \\
&\quad \le \|(\alpha_V^*)^k(\omega) - (\alpha_V^*)^k(\textstyle\sum_{\ell=1}^L \omega_{\xi_\ell})\|_1 + \|(\alpha_V^*)^k(\sum_{\ell=1}^L \omega_{\xi_\ell}) - \sum_{\ell=1}^L \|\xi_\ell\|^2\omega_e\|_1 \\
&\qquad + \|\textstyle\sum_{\ell=1}^L \|\xi_\ell\|^2\omega_e - \omega_e\|_1 \\
&\quad \le \|\omega - \textstyle\sum_{\ell=1}^L \omega_{\xi_\ell}\|_1 + \sum_{\ell=1}^L \|(\alpha_V^*)^k(\omega_{\xi_\ell}) - \|\xi_\ell\|^2\omega_e\|_1 + |\sum_{\ell=1}^L \|\xi_\ell\|^2 - 1| \cdot \|\omega_e\|_1 \\
&\quad < (\varepsilon/3) + (\textstyle\sum_{\ell=1}^L \|\xi_\ell\|^2\varepsilon/3) + (\varepsilon/3)| \le \varepsilon . \qquad \text{q.e.d.}
\end{aligned}
$$

Since every state in the predual of $B(\mathcal{H})$ is of the form in Theorem 1.4 ([7, Chapter II, Proposition 3.20]), we have the following.

COROLLARY 1.5. *Let $\{V_i\}_{i=1}^n$ be an f.i.c. on $\mathcal{H}$ and $e$ a unit vector such that $V_1 e = e$. If $ONS(e, V)$ is complete, then for any state $\omega$ in the predual $B(\mathcal{H})_*$ of $B(\mathcal{H})$ it follows that*

$$\lim_{k\to\infty} (\alpha_V^*)^k(\omega) = \omega_e \quad (\text{w.r.t. the norm topology in } B(\mathcal{H})_*)\,.$$

Before stating the following proposition, we confirm a notation about tensor product. Let $M$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $M_n$ the full matrix algebra on $n$-dimensional complex Hilbert space $\mathbf{C}^n$. The notation $M_n \otimes M$ means the von Neumann algebra which is the tensor product of $M_n$ and $M$ on the Hilbert space $\mathbf{C}^n \otimes \mathcal{H}$. Moreover $M'$ means the commutant of $M$ on $\mathcal{H}$.

PROPOSITION 1.6. *Let $M$ be a von Neumann algebra on $\mathcal{H}$ and $\{V_i\}_{i=1}^n$ and $\{W_i\}_{i=1}^n$ a couple of f.i.c.'s on $\mathcal{H}$. Suppose that $M$ is invariant for $\alpha_V$ and $\alpha_W$. Then the following conditions are equivalent.*

(1) $\alpha_W = \alpha_V$ *on* $M$.

(2) $(W_1, \cdots, W_n) = (V_1, \cdots, V_n)\begin{pmatrix} H_{11} & \cdots & H_{1n} \\ \vdots & \ddots & \vdots \\ H_{n1} & \cdots & H_{nn} \end{pmatrix}$, *where the matrix $[H_{i,j}]$ is a unitary element in $M_n \otimes M'$.*

The proof is omitted since it can be given similarly to that of [5, Proposition 2.2.4].

COROLLARY 1.7. *Let $M$ and $\{V_i\}_{i=1}^n$, $\{W_i\}_{i=1}^n$ be the same as in Proposition 1.6. If $W_1$ has an eigenvalue 1 with a unit eigenvector $e$ such that $ONS(e, W)$ is complete, then for any state $\omega$ in $M_*$ we have*

$$\lim_{k\to\infty} (\alpha_V^*)^k(\omega) = \omega_e \quad (\text{w.r.t. the norm topology in } M_*)\,.$$

REMARK. Each $W_i$ in Proposition 1.7 is of the form $W_i = \sum_{j=1}^n V_j H_{ji}$, $(i \in I(n))$. The equation (2) concerning matrix in the above proposition is sometimes written for short such as $W = VH$.

## 2. Chaotic maps and the behavior of the orbit of a state.

In this paper $(X, m)$ means a $\sigma$-finite measure space. For a measurable subset $Y$ of $X$, we denote by $(Y, m)$ the canonical measure subspace. Let $(Y, m)$ and $(Z, m)$ be two measure subspaces of $(X, m)$. A map $\varphi$ of $Y$ into $Z$ is called a *non-singular* map of $Y$ onto $Z$ if $\varphi$ satisfies the following condition.

(D-1) There exist two null sets $N_Y$ and $N_Z$ such that $\varphi$ is a bijective map of $Y \setminus N_Y$ onto $Z \setminus N_Z$.

(D-2) $\varphi$ is bimeasurable.

(D-3) Two measures $m$ and $m \circ \varphi$ on $X$ are mutually absolutely continuous in the sense that, for any measurable set $E$ in $Y$, $m(E) = 0$ if and only if $m \circ \varphi(E) = 0$, where $m \circ \varphi(E) = m(\varphi(E))$.

Here we note notation concerning non-singular maps $\varphi$'s of $(Y, m)$ onto $(Z, m)$.

(E-1) The Radon-Nikodym derivative for $m \circ \varphi$ with respect to $m$ is denoted by $\frac{dm\circ\varphi}{dm}$.

(E-2) $\varphi^0(x) = x$ and $\varphi^k(x) = \varphi(\varphi^{k-1}(x))$ for a positive integer $k$.

(E-3) When $Y = X$, $\alpha_\varphi$ denotes the $*$-endomorphism of $L^\infty(X)$ ($= L^\infty(X, m)$) defined by $\alpha_\varphi(f)(x) = f(\varphi(x))$, (a.a. $x \in X$) for $f$ in $L^\infty(X)$.

(E-4) For a measurable function $\xi$ on $Z$, $T_\varphi\xi$ denotes the measurable function on $Y$ defined by $(T_\varphi\xi)(x) = \xi(\varphi(x))$, (a.a. $x \in Y$).

Moreover we note notation concerning multiplication operators on $L^2(X)$ ($= L^2(X, m)$).

(F-1) For a measurable function $f$ on $X$, $M_f$ denotes the multiplication operator by $f$ on $L^2(X)$, that is, $M_f\xi = f\xi$ for $\xi$ in $L^2(X)$.

(F-2) For a function $f$ in $L^\infty(X)$, $\pi(f)$ denotes the bounded multiplication operator $M_f$ on $L^2(X)$, that is, $\pi(f)\xi = M_f\xi = f\xi$ for $\xi$ in $L^2(X)$.

REMARK. (1) For a function $\zeta$ on $X$, the notation $|\zeta|$ and $|\zeta|^2$ means $|\zeta|(x) = |\zeta(x)|$, ($x \in X$) and $|\zeta|^2(x) = |\zeta(x)|^2$, ($x \in X$).

(2) The $L^2$-norm of $\xi$ in $L^2(X)$ and the $L^1$-norm of $\eta$ in $L^1(X)$ are denoted by $\|\xi\|$ and $\|\eta\|_1$ respectively.

(3) In the present paper, we use both $M_f$ and $\pi(f)$ for $f$ in $L^\infty(X)$ and $\pi$ is considered as a canonical representation of $L^\infty(X)$ into $B(L^2(X))$.

(4) $\pi(L^\infty(X))$ ($= \{\pi(f) \mid f \in L^\infty(X)\}$) is a von Neumann algebra on $L^2(X)$.

Finally we note notation concerning the duality between $\pi(L^\infty(X))$ and its predual.

$$\text{(G)} \qquad \rho_\eta(\pi(f)) = \int_X f(x)\eta(x)dm\,, \quad (f \in L^\infty(X), \eta \in L^1(X))\,.$$

The following is our definition, with which we begin the discussion on the chaotic property of measurable functions.

DEFINITION 2.1. Let $(X, m)$ be a measure space. A measurable map $\varphi$ of $X$ into $X$ is called a map with $n$ laps, MW$n$L for short, if there exist $n$ measurable subsets $\{X_i\}_{i=1}^n$ of $X$ such that

(i) $\bigcup_{i=1}^n X_i = X$, $X_i \cap X_j = \emptyset$ for $i \neq j$ and $m(X_i) > 0$ for all $i$,

(ii) each restriction $\varphi_i$ of $\varphi$ to $X_i$ is a non-singular map of $X_i$ onto $X$.

REMARK. (1) If $\varphi$ is an MW$n$L on $X$, then for each $i$ in $I(n)$ it follows that

(i) $\frac{dm\circ\varphi_i}{dm}(x) \neq 0$ for a.a. $x$ in $X_i$ and $\frac{dm\circ\varphi_i^{-1}}{dm}(x) \neq 0$ for a.a. $x$ in $X$,

(ii) $\frac{dm\circ\varphi_i}{dm}(\varphi_i^{-1}(x))\,\frac{dm\circ\varphi_i^{-1}}{dm}(x) = 1$ for a.a. $x$ in $X$ and $\frac{dm\circ\varphi_i^{-1}}{dm}(\varphi_i(x))\,\frac{dm\circ\varphi_i}{dm}(x) = 1$ for a.a. $x$ in $X_i$.

(2) For a measure space $(X, m)$ and a measurable map $\varphi$ of $X$ into itself, $M_f$ and $T_\varphi$ are not necessarily defined on the full space $L^2(X)$. Then, if necessary, each operator $V_i$ in

the following definition is considered as a uniquely extended bounded linear operator on the full Hilbert space $L^2(X)$.

DEFINITION 2.2. Let $\varphi$ be an MW$n$L on a measure space $(X, m)$. A family of isometries $V(\varphi) = \{V(\varphi)_i\}_{i=1}^n$ on $L^2(X)$ associated with $\varphi$ is defined as follows:

$$V(\varphi)_i = M_{\sqrt{dm\circ\varphi_1/dm}} M_{\chi_{X_i}} T_\varphi\,, \quad (i \in I(n))\,,$$

where each $\chi_{X_i}$ is the characteristic function of $X_i$.

By the definition we can see that

(1) $V(\varphi)_i^* = M_{\sqrt{dm\circ\varphi_i^{-1}/dm}} T_{\varphi_i^{-1}}\,, \quad (i \in I(n))\,,$

(2) $V(\varphi)_i^* V(\varphi)_i = I\,, \quad (i \in I(n))\,,$

(3) $V(\varphi)_i V(\varphi)_i^* = M_{\chi_{X_i}}\,, \quad (i \in I(n))\,.$

Hence we have the following. The proof is omitted because it is given by routine calculation.

PROPOSITION 2.3. *Let $\varphi$ be an MWnL on a measure space $(X, m)$ and $V(\varphi) = \{V(\varphi)_i\}_{i=1}^n$ the family of isometries associated with $\varphi$ defined in the above definition. Then it follows that*

(1) *$V(\varphi)$ is an f.i.c. (cf. (A-1)) on $L^2(X)$.*

(2) *$\alpha_{V(\varphi)}$ is a $*$-endomorphism of $B(L^2(X))$.*

(3) *$\alpha_{V(\varphi)}$ is a $*$-endomorphism of $\pi(L^\infty(X))$ and $\pi(\alpha_\varphi(f)) = \alpha_{V(\varphi)}(\pi(f))$ for all $f$ in $L^\infty(X)$.*

From Proposition 2.3 (1) and Theorem 1.4 we have the following, which contains the case of a lot of unimodal maps on the unit interval [0,1].

THEOREM 2.4. *Let $\varphi$ be an MWnL on a measure space $(X, m)$ and $M$ a von Neumann algebra which is invariant for $\alpha_{V(\varphi)}$. Suppose that there exist an f.i.c. $W = \{W_i\}_{i=1}^n$ on $L^2(X)$ and a unit vector $e$ in $L^2(X)$ such that*

(1) *$\alpha_W = \alpha_{V(\varphi)}$ on $M$,*

(2) *$W_1 e = e$.*

*Then, for any state $\omega$ of the form $\omega = \sum_{k=1}^\infty \omega_{\xi_k}$, where $\xi_k$'s are in $L^2(X)_e$, it follows that*

$$\lim_{n\to\infty} (\alpha_V^*)^n(\omega) = \omega_e \quad (\text{w.r.t. the norm topology in } M_*)\,.$$

REMARK. In the above theorem, if $ONS(e, W)$ is complete, the conclusion holds for any state $\omega$ in $M_*$. However we have the case only if $n \geq 2$ except the trivial case where $X$ consists of only single point. In fact, when $n = 1$, the isometry $W_1$ in the theorem becomes a unitary operators and thus $ONS(e, W) = \{e\}$ consists of only one vector.

As mentioned above, $\alpha_{V(\varphi)}$ is a $*$-endomorphism of the von Neumann algebra $\pi(L^\infty(X))$ and thus we can define the transposed map of the restriction of $\alpha_{V(\varphi)}$ to $\pi(L^\infty(X))$, which is denoted by $((\alpha_{V(\varphi)})_{|\pi_1(L^\infty(X))})^*$. For a while we discuss this transposed map. First we note that the transposed map is equal to the restriction of $\alpha_{V(\varphi)}^*$ to the predual of $\pi(L^\infty(X))$.

Namely we have

$$((\alpha_{V(\varphi)})_{|\pi_1(L^\infty(X))})^* = (\alpha^*_{V(\varphi)})_{|\pi(L^\infty(X))_*},$$

where the vertical bar means the restriction of a map. We use the notation $\alpha^*_{V(\varphi)}$ instead of $(\alpha^*_{V(\varphi)})_{|\pi(L^\infty(X))_*}$ unless we have confusion. In Section 1, we have already shown that $\alpha^*_{V(\varphi)}(\omega_\xi) = \sum_{i=1}^n \omega_{V(\varphi)_i^*\xi}$ for $\xi$ in $L^2(X)$ (cf. (C) after Lemma 1.2). Hence we have, for any non-negative function $\eta$ in $L^1(X)$ and $f$ in $L^\infty(X)$,

$$\begin{aligned}\alpha^*_{V(\varphi)}(\rho_\eta)(\pi(f)) = \alpha^*_{V(\varphi)}(\omega_{\sqrt{\eta}})(\pi(f)) &= \sum_{i=1}^n \omega_{V(\varphi)_i^*\sqrt{\eta}}(\pi(f))\\ &= \sum_{i=1}^n \langle \pi(f)V(\varphi)_i^*\sqrt{\eta},\ V(\varphi)_i^*\sqrt{\eta}\rangle\\ &= \sum_{i=1}^n \int_X f(x)\frac{dm\circ\varphi_i^{-1}}{dm}(x)\eta(\varphi_i^{-1}(x))dm\,.\end{aligned}$$

Since the predual $\{\rho(\eta) \mid \eta \in L^1(X)\}$ of $\pi(L^\infty(X))$ is spanned by the set $\{\omega_\xi | \xi \in L^2(X)\}$, the above equation holds for all functions $\eta \in L^1(x)$. Since $\alpha^*_{V(\varphi)}(\rho_\eta)(\pi(f)) = \rho_\eta(\alpha_{V(\varphi)}(\pi(f))) = \rho_\eta(\pi(\alpha_\varphi(f)))$ for any function $\eta$ in $L^1(X)$ and $f$ in $L^\infty(X)$, we have

$$\alpha^*_{V(\varphi)}(\rho_\eta)(\pi(f)) = \int_X f(\varphi(x))\eta(x)dm = \sum_{i=1}^n \int_X f(x)\frac{dm\circ\varphi_i^{-1}}{dm}(x)\eta(\varphi_i^{-1}(x))dm\,.$$

Now we put

$$\text{(H)}\qquad (A_\varphi\eta)(x) = \sum_{i=1}^n \frac{dm\circ\varphi_i^{-1}}{dm}(x)\eta(\varphi_i^{-1}(x)), \quad (\eta \in L^1(X))\,.$$

Then we have $\alpha^*_{V(\varphi)}(\rho_\eta)(\pi(f)) = \rho_{A_\varphi\eta}(\pi(f))$, that is,

$$\text{(I)}\qquad \alpha^*_{V(\varphi)}(\rho_\eta) = \rho_{A_\varphi\eta}\,.$$

Here we note that if $\alpha_W = \alpha_{V(\varphi)}$ on $\pi(L^\infty(X))$ as in Theorem 2.4, we have $\alpha^*_W(\eta) = \alpha^*_{V(\varphi)}(\eta) = \rho_{A_\varphi\eta}$ for all $\eta$ in $L^1(X)$. Moreover we note that the map $A_\varphi$ is a bounded linear operator of $L^1(X)$ into itself, which is known as Perron-Frobenius operator, and isometric on the set of all *non-negative* functions in $L^1(X)$. Indeed, for any non-negative function $\eta$ in $L^1(X)$, it follows that

$$\begin{aligned}\|A_\varphi\eta\|_1 = \|\rho_{A_\varphi\eta}\|_1 = \|\alpha^*_{V(\varphi)}(\rho_\eta)\|_1 &= (\alpha^*_{V(\varphi)}(\rho_\eta))(I)\\ &= \rho_\eta(\alpha_{V(\varphi)}(I)) = \rho_\eta(I) = \|\rho_\eta\|_1 = \|\eta\|_1\,.\end{aligned}$$

The study of the sequence $\{A_\varphi^k(\eta)\}_{k=1}^\infty$ is an important viewpoint for the study of the sequence $\{(\alpha^*_{V(\varphi)})^k(\omega)\}_{k=1}^\infty$. Hence we are interested in the von Neumann algebra $M$ which contains $\pi(L^\infty(X))$ and is invariant for $\alpha_{V(\varphi)}$.

COROLLARY 2.5. *Let $\varphi$ be an MWnL on a measure space $(X, m)$. Suppose that there exist a f.i.c. $W = \{W_i\}_{i=1}^n$ and a unit vector $e$ in $L^2(X)$ such that*

(1) $\alpha_W = \alpha_{V(\varphi)}$ *on* $\pi(L^\infty(X))$,
(2) $W_1 e = e$.

*Then, for any function* $\xi$ *in* $L^2(X)_e$ *with* $\|\xi\| = 1$, *it follows that*

$$\lim_{k\to\infty} \|A_\varphi^k |\xi|^2 - |e|^2\|_1 = 0\,.$$

*In addition, if* $ONS(e, W)$ *is complete, then for any non-negative function* $\eta$ *in* $L^1(X)$ *with* $\|\eta\|_1 = 1$, *it follows that*

$$\lim_{k\to\infty} \|A_\varphi^k \eta - |e|^2\|_1 = 0\,.$$

PROOF. For $\xi$ satisfying the condition in the statement, by Theorem 2.4, it follows that

$$\lim_{k\to\infty} \|A_\varphi^k |\xi|^2 - |e|^2\|_1 = \lim_{n\to\infty} \|(\alpha_{V(\varphi)}^*)^k(\omega_\xi) - \omega_e\|_1 = 0\,.$$

The second conclusion obviously follows from the first one. q.e.d.

COROLLARY 2.6. *For an* $MWnL$ $\varphi$ *on a measure space* $(X, m)$, *let* $W = \{W_i\}_{i=1}^n$, $W' = \{W_i'\}_{i=1}^n$ *be f.i.c.'s on* $L^2(X)$ *and* $e$, $f$ *unit vectors in* $L^2(X)$ *such that*

(1) $\alpha_W = \alpha_{W'} = \alpha_{V(\varphi)}$ *on* $\pi(L^\infty(X))$,
(2) $W_1 e = e$ *and* $W_1' f = f$,
(3) $L^2(x)_e \cap L^2(X)_f \neq \{0\}$.

*Then we have* $|e| = |f|$.

PROOF. Let $\xi$ be a function in $L^2(x)_e \cap L^2(X)_f$ with $\|\xi\| = 1$. Then, by Corollary 2.5, we have

$$|e|^2 = \lim_{k\to\infty} A_\varphi^k |\xi|^2 = |f|^2\,.$$

q.e.d.

Now let $\varphi$ be an MW$n$L on a probability measure space $(X, m)$. As in the case of measure preserving bijective transformation on $X$, a map $\varphi$ is said to be *strong-mixing* if

$$\lim_{k\to\infty} m(\varphi^{-k}(E) \cap F) = m(E)m(F)$$

for each pair of measurable sets $E$ and $F$. Moreover, in the same manner as in [9, Lemma 6.11], we can see that this is equivalent to that, for any $\eta$ in $L^1(X)$ and any $f$ in $L^\infty(X)$, it follows that

$$\lim_{k\to\infty} \int_X f(\varphi^k(x))\eta(x)dm = \int_X f(x)dm \int_X \eta(x)dm\,.$$

This equation can be derived by the conclusion of Corollary 2.5, in which $e$ is the case where $e(x) = 1$, $(x \in X)$ and $ONS(e, W)$ is complete. Thus we have the following corollary.

COROLLARY 2.7. *Let $\varphi$ be an MWnL as in Corollary 2.5. If the eigenvector $e$ is the constant function $e(x) = 1$ and $ONS(e, W)$ is complete, then $\varphi$ is strong-mixing.*

The following is typical examples of MW$n$L, which is well known in the theory of chaotic maps. Here we show that the example yields operators which have strange and interesting properties in the viewpoint of the probability theory or the theory of functional analysis.

EXAMPLE 2.8. Let $X = [0, 1]$ with $X_i = [(i-1)/n, i/n)$ for $i = 1, 2, \cdots, n-1$, $X_n = [(n-1)/n, 1]$ and $m$ the Lebesgue measure on $[0, 1]$, $\varphi_n$ the piecewise continuous map on [0,1] defined by

$$\varphi_n(x) = nx - (i-1), \quad (x \in X_i, i \in I(n)).$$

Then $\varphi_n$ is an MW$n$L on $[0, 1)$ with the partition $\{X_i\}_{i=1}^n$ and it follows that

$$V(\varphi_n)_i = M_{\sqrt{n}} M_{\chi_{I_i}} T_{\varphi_n}, \quad (i \in I(n)).$$

Let $W = \{W_i\}_{i=1}^n$ be the family of isometries defined by

$$(W_1, \cdots, W_n) = (V(\varphi_n)_1, \cdots, V(\varphi_n)_n) \begin{pmatrix} \sqrt{1/n} & a_{12} & \cdots & a_{1n} \\ \sqrt{1/n} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{1/n} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

where $\{a_{ij} \mid i, j = 2, 3, \cdots, n\}$ are complex numbers such that the matrix in the right hand side is an element in the unitary group $U(n)$. Then $W = \{W_i\}_{i=1}^n$ is an f.i.c. on $L^2[0, 1]$ with $W_1 = T_{\varphi_n}$, and, by virtue of Proposition 1.6, we have

$$\alpha_W(T) = \sum_{i=1}^n W_i T W_i^* = \sum_{i=1}^n V(\varphi_n)_i T V(\varphi_n)_i^* = \alpha_{V(\varphi_n)}(T)$$

for all $T$ in $B(L^2[0, 1])$. Moreover for $e = 1$ we have $W_1 e = e$ and $ONS(e, W)$ is complete. Indeed, for each $j = 1, 2, \cdots, n$, the vector $W_j e$ is of the form $W_j e = \sqrt{n} \sum_{i=1}^n a_{ij} \chi_{X_i}$. Since the unit vectors $\{(a_{1j}, \cdots, a_{nj})\}_{j=1}^n$ in $\mathbf{C}^n$ are linearly independent, $L^2[0, 1]_e$ contains the vectors $\{\chi_{X_i}\}_{i=1}^n$. Similarly, by considering $\{W(\mu)e\}_{\mu \in I(n)^k}$, we can see that $L^2[0, 1]_e$ contains the vectors $\{\chi_{[(i-1)/n^k, i/n^k)}\}_{i=1}^{n^k}$ for all $k$ in $\mathbf{N}$. Namely we have $L^2[0, 1]_e = L^2[0, 1]$. Now, by virtue of Corollary 1.7, for any state $\omega$ in $B(L^2[0, 1])_*$, it follows that $\{(\alpha_{V(\varphi_n)}^*)^k(\omega)\}_{k=1}^\infty$ converges to the vector state $\omega_e$ with respect to the norm topology in $B(L^2[0, 1])_*$. Especially, it follows that

$$\lim_{k \to \infty} \|A_{\varphi_n}^k \eta - e\|_1 = 0$$

for each non-negative function $\eta$ in $L^1[0, 1]$. Moreover we note that

$$(A_\varphi \eta)(x) = \frac{1}{n}\sum_{i=1}^{n} \eta\left(\frac{x+i-1}{n}\right), \quad (\eta \in L^1(X)).$$

EXAMPLE 2.9 (Generalized tent maps). Let $X = [0, 1]$, $m$ the Lebesgue measure on $[0, 1]$ and $\tau_c$ $(0 < c < 1)$ the continuous map on $[0, 1]$ defined by

$$\tau_c(x) = \begin{cases} \dfrac{1}{c}x & \text{for } 0 \le x < c, \\ \dfrac{1}{1-c}(1-x) & \text{for } c \le x \le 1. \end{cases}$$

Then $\tau_c$ is an MW2L on $[0, 1]$ with the partition $X_1 = [0, c)$, $X_2 = [c, 1]$ and it follows that

$$V(\tau_c)_1 = M_{\sqrt{1/c}} M_{\chi_{[0,c)}} T_{\tau_c}, \quad V(\tau_c)_2 = M_{\sqrt{1/(1-c)}} M_{\chi_{[c,1]}} T_{\tau_c}.$$

Let $W_1$ and $W_2$ be the isometries defined by

$$(W_1, W_2) = (V(\tau_c)_1, V(\tau_c)_2) \begin{pmatrix} \sqrt{c} & \sqrt{1-c} \\ \sqrt{1-c} & -\sqrt{c} \end{pmatrix}.$$

That is,

$$\begin{aligned} W_1 &= \sqrt{c}V(\tau_c)_1 + \sqrt{1-c}V(\tau_c)_2 = T_{\tau_c}, \\ W_2 &= \sqrt{1-c}V(\tau_c)_1 - \sqrt{c}V(\tau_c)_2 = \sqrt{(1-c)/c}M_{\chi_{[0,c)}}T_{\tau_c} - \sqrt{c/(1-c)}M_{\chi_{[c,1]}}T_{\tau_c}. \end{aligned}$$

Hence, by virtue of Proposition 1.6, we have $\alpha_{V(\tau_c)}(T) = \alpha_W(T)$ for all $T$ in $B(L^2[0, 1])$. Moreover for $e = 1$ we have $W_1 e = e$ and $ONS(e, W)$ is complete. By virtue of Corollary 1.7, for any state $\omega$ in $B(L^2[0, 1])_*$, it follows that $\{(\alpha_V^*)^k(\omega)\}_{k=1}^\infty$ converges to the vector state $\omega_e$ with respect to the norm topology in $B(L^2[0, 1])_*$. Moreover we have

$$(A_\varphi \eta)(x) = c\eta(cx) + (1-c)\eta(1-(1-c)x), \quad (\eta \in L^1[0, 1]).$$

REMARK. In Example 2.9, when $c = 1/2$, the map $\tau_c$ is so-called the tent map on [0,1] and each $e_\ell$ in $ONS(e, W) = \{e_\ell\}_{\ell=1}^\infty$, which is defined by (B), is of the form

$$e_1 = e, \quad e_2 = \chi_{[0,1/2)} - \chi_{[1/2,1]} \text{ and } e_{(2^\ell-1)+i} = \sum_{j=0}^{2^\ell-1} c_j \chi_{[j/2^\ell,(j+1)/2^\ell)}, \quad (c_j \in \{1, -1\}).$$

Namely, $\{e_\ell\}_{\ell=1}^\infty$ is Walsh series in $L^2[0, 1]$. Like this, the study of $ONS(e, W)$ is closely related to wavelet theory (cf. [4]).

In the definition of MW$n$L, we gave the condition that, each $i$ in $I(n)$, $m \circ \varphi_i^{-1}$ is absolutely continuous with respect to $m$ in addition to $m \circ \varphi_i$. We cannot drop this condition even if $\varphi_i$ is absolutely continuous on the real line with the Lebesgue measure. This is shown by the following example. Let $c(x)$ be the Cantor function on $[0, 1]$. We put $k(x) = \frac{x+c(x)}{2}$. Then $k$ is a homeomorphism of $[0, 1]$ but not absolutely continuous on $[0, 1]$. Let $h(x) = k^{-1}(x)$,

($x \in [0, 1]$). Then $h$ is an absolutely continuous homeomorphism of $[0, 1]$. We define a map $\varphi$ on $[0, 1]$ as follows:

$$\varphi(x) = \begin{cases} h(2x) & \text{for } 0 \le x < 1/2\,, \\ h(-2x+2) & \text{for } 1/2 \le x \le 1\,. \end{cases}$$

Then $\varphi$ is an absolutely continuous function on $[0, 1]$, which satisfy the condition of Definition 2.1 except the absolutely continuity of $m \circ \varphi_i^{-1}$, ($i = 1, 2$) in (2).

The following is an example of MW4L on two-dimensional space.

EXAMPLE 2.10 (Maps on two-dimensional space). Let $X = \mathbf{I} \times \mathbf{I} = [0, 1] \times [0, 1]$, $m$ the Lebesgue measure on $\mathbf{I} \times \mathbf{I}$ and $\tau_{c,d}$ the map defined by $\tau_{c,d}(x, y) = (\tau_c(x), \tau_d(y))$, where $\tau_c$ and $\tau_d$ are generalized tent maps defined in Example 2.9. Then $\tau_{c,d}$ is a MW4L with the partition $X_1 = [0, c) \times [0, d)$, $X_2 = [c, 1] \times [0, d)$, $X_3 = [c, 1] \times [d, 1]$, $X_4 = [0, c) \times [d, 1]$ and it follows that

$$\begin{aligned} V(\tau_{c,d})_1 &= M_{\sqrt{1/cd}} M_{\chi[0,c)\times[0,d)} T_{\tau_{c,d}}\,, \\ V(\tau_{c,d})_2 &= M_{\sqrt{1/(1-c)d}} M_{\chi[c,1]\times[0,d)} T_{\tau_{c,d}}\,, \\ V(\tau_{c,d})_3 &= M_{\sqrt{1/c(1-d)}} M_{\chi[0,c)\times[d,1]} T_{\tau_{c,d}}\,, \\ V(\tau_{c,d})_4 &= M_{\sqrt{1/(1-c)(1-d)}} M_{\chi[c,1]\times[d,1]} T_{\tau_{c,d}}\,. \end{aligned}$$

Moreover we define a family of isometries $\{W_i\}_{i=1}^4$ as follows:

$$\begin{aligned} &(W_1, W_2, W_3, W_4) \\ &= (V(\tau_{c,d})_1, V(\tau_{c,d})_2, V(\tau_{c,d})_3, V(\tau_{c,d})_4) \begin{pmatrix} \sqrt{cd} & a_{12} & a_{13} & a_{14} \\ \sqrt{(1-c)d} & a_{22} & a_{23} & a_{24} \\ \sqrt{c(1-d)} & a_{32} & a_{33} & a_{34} \\ \sqrt{(1-c)(1-d)} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \end{aligned}$$

where $\{a_{ij} \mid i, j = 2, 3, 4\}$ are complex numbers such that the matrix in the right hand side is an element in $U(4)$. Then we have $W_1 = T_{\tau_{c,d}}$. Hence for $e(x, y) = 1$, ($(x, y) \in \mathbf{I} \times \mathbf{I}$) we have $W_1 e = e$ and $ONS(e, W)$ is complete. Therefore, for any state $\omega$ in $B(L^2(\mathbf{I} \times \mathbf{I}))_*$, it follows that $\{(\alpha^*_{V(\tau_{c,d})})^k(\omega)\}_{k=1}^\infty$ converges to the vector state $\omega_e$ with respect to the norm topology in the predual of $B(L^2(I \times I))$. Moreover we have

$$\begin{aligned} (A_\varphi\eta)(x, y) = &cd\eta(cx, dy) + (1-c)d\eta(1-(1-c)x, dy) + c(1-d)\eta(cx, 1-(1-d)y) \\ &+ (1-c)(1-d)\eta(1-(1-c)x, 1-(1-d)y)\,, \quad (\eta \in L^1(\mathbf{I} \times \mathbf{I}))\,. \end{aligned}$$

In the following, we show some relationship between the measure theoretical property of a given map $\varphi$ and the functional analytical property of the operators related to $\varphi$.

THEOREM 2.11. *Let $\varphi$ be an MWnL on a measure space $(X, m)$ and $e$ a function in $L^2(X)$ such that $e(x) \neq 0$ for a.a. $x$ in $X$. Then the following conditions are equivalent.*

(1) *There exists an f.i.c. $W = \{W_i\}_{i=1}^n$ on $L^2(X)$ such that $\alpha_W = \alpha_{V(\varphi)}$ on $\pi(L^\infty(X))$ and $W_1 e = e$.*

(2) $\sum_{i=1}^{n} |e_\varphi(\varphi_i^{-1}(x))|^2 \frac{dm\circ\varphi_i^{-1}}{dm}(x) = 1$ *on* $X$ *(a.e.), where* $e_\varphi(x) = \frac{e(x)}{e(\varphi(x))}$.
(3) $M_{e_\varphi} T_\varphi$ *is an isometry.*

PROOF. (1) $\Rightarrow$ (2), (3). By Proposition 1.6, we can see that $W = V(\varphi)H$ where each entry $H_{ij}$ of $H$ is of the form $H_{ij} = \pi(h_{ij})$ for some $h_{ij}$ in $L^\infty(X)$, because $\pi(L^\infty(X))' = \pi(L^\infty(X))$. Since

$$W_1 = \sum_{i=1}^{n} V(\varphi)_i H_{i1} \quad \text{and} \quad W_1 e = e\,,$$

we have

$$(W_1 e)(x) = \sum_{i=1}^{n} \sqrt{\frac{dm\circ\varphi_i}{dm}(x)}\chi_{X_i}(x) h_{i1}(\varphi(x)) e(\varphi(x)) = e(x)\,,$$

for a.a. $x$ in $X$. Thus, for each $i \in I(n)$, it follows that

$$\sqrt{\frac{dm\circ\varphi_i}{dm}(x)} h_{i1}(\varphi(x)) e(\varphi(x)) = e(x)\,, \quad (\text{a.a. } x \in X_i)\,.$$

Hence we have

$$h_{i1}(x) = e_\varphi(\varphi_i^{-1}(x)) \Big/ \sqrt{\frac{dm\circ\varphi_i}{dm}(\varphi^{-1}(x))} = e_\varphi(\varphi_i^{-1}(x)) \sqrt{\frac{dm\circ\varphi_i^{-1}}{dm}(x)}$$

for a.a. $x$ in $X$. Since $H = [\pi(h_{ij})]$ is a unitary element in $M_n \otimes \pi(L^\infty(X))$, it follows that

$$\sum_{i=1}^{n} |h_{i1}(x)|^2 = \sum_{i=1}^{n} |e_\varphi(\varphi_i^{-1}(x))|^2 \frac{dm\circ\varphi_i^{-1}}{dm}(x) = 1$$

for a.a. $x$ in $X$. Moreover we have

$$W_1 = \sum_{i=1}^{n} M_{\sqrt{dm\circ\varphi_i/dm}} M_{\chi_{X_i}} T_\varphi \pi(h_{i1}) = \sum_{i=1}^{n} M_{e_\varphi} M_{\chi_{X_i}} T_\varphi = M_{e_\varphi} T_\varphi\,.$$

(2) $\Rightarrow$ (1), (3). Let $h_{i1}(x) = e_\varphi(\varphi_i^{-1}(x))\sqrt{\frac{dm\circ\varphi_i^{-1}}{dm}(x)}$, $(x \in X)$ for $i = 1, 2, \cdots, n$. Using Condition (2), we can get a unitary element $H = [H_{ij}]$ in $M_n \otimes \pi(L^\infty(X))$ such that $H_{i1} = \pi(h_{i1})$ $(i = 1, 2, \cdots, n)$ as follows. Let $\{f_j\}_{j=1}^{n}$ be a c.o.n.s. of $\mathbf{C}^n$. We define $n+1$ $\mathbf{C}^n$-valued measurable functions $\{s_j\}_{j=1}^{n+1}$ on $X$ by

$$s_1(x) = (h_{11}(x), h_{21}(x), \cdots, h_{n1}(x))\,,$$
$$s_j(x) = f_{j-1}\,, \quad (j = 2, 3, \cdots, n+1)\,.$$

Then, for $x$ in $X$ except a measurable set $N$ with $m(N) = 0$, $\{s_j(x)\}_{j=1}^{n+1}$ is a family of unit vectors whose linear span is the whole space $\mathbf{C}^n$. Hence, as in the way of Schmidt's orthogonalization, for $x$ in $X \setminus N$, we can get an orthogonal system $\{t'_j(x)\}_{j=1}^{n+1}$ such that $t'_1(x) = s_1(x)$ and $t'_j(x)$'s are unit vectors except one zero vector $t'_{k(x)}(x)$, where $k(x)$ is determined by $x$. For each $x$ in $X \setminus N$, we put $t_j(x) = t'_j(x)$ for $j = 1, 2, \cdots, k(x) - 1$ and

$t_j(x) = t'_{j+1}(x)$ for $j = k(x), \cdots, n$. Moreover we put $t_j(x) = 0$ for $j = 1, 2, \cdots, n$ and $x$ in $N$. Then each $t_j$ is a $\mathbf{C}^n$-valued $L^\infty$-function on $X$ and thus it can be expressed as follows:

$$t_j(x) = (h_{1j}(x), h_{2j}(x), \cdots, h_{nj}(x)) \in \mathbf{C}^n, \quad (x \in X),$$

where $\{h_{ij}\}_{i=1}^n$ are $\mathbf{C}^n$-valued $L^\infty$-functions on $X$. Since the matrix $[h_{ij}(x)]$ is a unitary element in $M_n$ for all $x$ in $X \setminus N$, the family of operators $\{\pi(h_{ij})\}_{i,j=1}^n$ is a desired set of $\{H_{ij}\}_{i,j=1}^n$. Now, let $W = (W_1, \cdots, W_n)$ be the f.i.c. defined by $W = V(\varphi)H$. Then , by Proposition 1.6, it follows that $\alpha_W = \alpha_{V(\varphi)}$ on $\pi(L^\infty(X))$. Moreover we have, for any $\xi$ in $L^2(X)$,

$$(W_1\xi)(x) = \sum_{i=1}^n (M_{\sqrt{dm\circ\varphi^{-1}/dm}} M_{\chi_{X_i}} T_\varphi H_{1i}\xi)(x) = e_\varphi(x)\xi(\varphi(x)) = (M_{e_\varphi} T_\varphi \xi)(x)$$

for a.a. $x$ in $X$. This implies that $W_1 = M_{e_\varphi} T_\varphi$ and we have $M_{e_\varphi} T_\varphi e = e$.

(3) $\Rightarrow$ (2). Let $E$ be a measurable set in $X$ with finite measure. By virtue of (3), it follows that $\|M_{e_\varphi} T_\varphi \chi_E\|_2 = \|\chi_E\|_2$ and we have that

$$\begin{aligned}\|M_{e_\varphi} T_\varphi \chi_E\|_2^2 &= \int_X |e_\varphi(x)\chi_E(\varphi(x))|^2 dm = \sum_{i=1}^n \int_{X_i} |e_\varphi(x)\chi_E(\varphi_i(x))|^2 dm \\ &= \sum_{i=1}^n \int_{X_i} |e_\varphi(\varphi_i^{-1}(x))\chi_E(x)|^2 \frac{dm\circ\varphi_i^{-1}}{dm}(x) dm \\ &= \int_E \sum_{i=1}^n |e_\varphi(\varphi_i^{-1}(x))|^2 \frac{dm\circ\varphi_i^{-1}}{dm}(x) dm, \\ \|\chi_E\|_2^2 &= \int_X |\chi_E(x)|^2 dm = \int_E 1 dm.\end{aligned}$$

Hence we have

$$\int_E \left(\sum_{i=1}^n |e_\varphi(\varphi_i^{-1}(x))|^2 \frac{dm\circ\varphi_i^{-1}}{dm}(x) - 1\right) dm = 0.$$

This implies (2), because $E$ is an arbitrary measurable subset of $X$ with finite measure and $(X, m)$ is a $\sigma$-finite measure space. q.e.d.

EXAMPLE 2.12. Let $\lambda$ be the logistic map on $X = [0, 1]$ defined by $\lambda(x) = 4x(1-x)$. Then $\lambda$ is an MW2L with the partition $X_1 = [0, 1/2)$, $X_2 = [1/2, 1]$ and there exists an f.i.c. $W = \{W_1, W_2\}$ such that $\alpha_V = \alpha_{V(\lambda)}$ on $B(L^2[0, 1])$ and $W_1 e = e$ for $e(x) = 1/(\pi(x(1-x))^{1/2})^{1/2}$ in $L^2[0, 1]$ (cf. [6], [5, Example 3.2.5]). Moreover we have that $\lambda_1^{-1}(x) = (1-\sqrt{1-x})/2$, $\lambda_2^{-1}(x) = (1+\sqrt{1-x})/2$, $\frac{dm\circ\lambda_1^{-1}}{dm}(x) = \frac{dm\circ\lambda_2^{-1}}{dm}(x) = 1/(4\sqrt{1-x})$ and $e_\lambda(x) = \sqrt{2|2x-1|}$. Hence we can check that $\sum_{i=1}^2 |e_\lambda(\lambda_i^{-1}(x))|^2 \frac{dm\circ\lambda_1^{-1}}{dm}(x) = 1$ and $W_1 = M_{e_\lambda} T_\lambda = M_{\sqrt{2|2x-1|}} T_\lambda$.

COROLLARY 2.13. *Let $\varphi$ be an MWnL on a measure space $(X, m)$ with $m(X) = 1$ and $e$ a non-zero constant function on $X$. Then the following conditions are equivalent.*

(1) *There exists an f.i.c.* $W = \{W_i\}_{i=1}^n$ *on* $L^2(X)$ *such that* $\alpha_W = \alpha_{V(\varphi)}$ *on* $\pi(L^\infty(X))$ *and* $W_1 e = e$.

(2) $\sum_{i=1}^n \frac{dm\circ\varphi_i^{-1}}{dm}(x) = 1$ *on* $X$ *(a.e.)*.

(3) $T_\varphi$ *is an isometry.*

The following is an example such that $\{T \in B(\mathcal{H}) \mid \alpha_W(T) = \alpha_{V(\varphi)}(T)\} = \pi(L^\infty(X))$.

EXAMPLE 2.14 (Square root maps). Let $X = [0, 1]$, $m$ the Lebesgue measure on $[0, 1]$ and $\rho_c$, $(0 < c \leq 1/2)$ a continuous map on $[0, 1]$ defined by

$$\rho_c(x) = \begin{cases} \sqrt{x/c} & \text{for } 0 \leq x < c\,, \\ (1 - \sqrt{4cx + 1 - 4c})/2c & \text{for } c \leq x \leq 1\,. \end{cases}$$

Then $\rho_c$ is a MW2L on $[0, 1]$ with the partition $X_1 = [0, c)$, $X_2 = [c, 1]$ and it follows that

$$(\rho_c)_1^{-1}(x) = cx^2 \quad \text{and} \quad (\rho_c)_2^{-1}(x) = cx^2 - x + 1\,.$$

Hence Condition (2) in Corollary 2.13 is satisfied. We put

$$(W_1, W_2) = (V_1, V_2)\begin{pmatrix} \pi(f_1) & \pi(f_2) \\ \pi(f_2) & -\pi(f_1) \end{pmatrix},$$

where $f_1(x) = \sqrt{2cx}$ and $f_2(x) = \sqrt{1 - 2cx}$. Then, by Proposition 1.6 and the fact the von Neumann algebra generated by the entries in the matrix is equal to $\pi(L^\infty[0, 1])$, we can see that the von Neumann algebra $\{T \in B(L^2[0, 1]) | \alpha_W(T) = \alpha_{V(\varphi)}(T)\}$ is just equal to $\pi(L^\infty[0, 1])$. Thus we have that $\alpha_W(T) = \alpha_{V(\varphi)}(T)$ for $T$ in $\pi(L^\infty[0, 1])$ and $W_1 e = e$ where $e(x) = 1$.

REMARK. If an MW$n$L $\varphi$ satisfies one of the conditions in Theorem 2.11, $W_1$ is always equal to $M_{e_\varphi} T_\varphi$, and thus $W_1 = V(\varphi)_i$ for some $i$ in $I(n)$ only if $n = 1$. In particular, in the case where $e$ is a constant function, $W_1$ is always equal to $T_\varphi$.

Now, in Examples 2.9, 2.10, 2.12 and 2.14, we had a family of those f.i.c.'s $W = \{W_i\}_{i=1}^n$ which satisfy the conditions in Theorem 2.11. Thus we can see that each $V(\varphi)_i$ in the examples cannot have eigenvalue 1. In the following, we study the condition under which $V(\varphi)_i$ has eigenvalue 1. For $\varphi$ which is an MW$n$L on $(X, m)$ and $k$ in $\mathbf{N}$, we put

$$X_1(k) = \left\{ x \in X_1 \mid \varphi^k(x) = x,\, \varphi^j(x) \in X_1 \text{ for } j = 1, \cdots, k-1 \text{ and } \frac{dm \circ \varphi^k}{dm}(x) = 1 \right\}.$$

Then we have the following theorem.

THEOREM 2.15. *Let $\varphi$ be an MWnL on a measure space $(X, m)$. If there exists a measurable subset $E$ in $X_1(k)$ for same $k$ such that $m(E) > 0$ and $\{\varphi^j(E)\}_{j=0}^{k-1}$ are mutually disjoint, then there exists a non-zero vector $e$ in $L^2(X)$ such that $V(\varphi)_1 e = e$.*

PROOF. Using $E$ in the statement of the theorem, we define a function $e$ in $L^2(X)$ as follows:

$$e(x) = \begin{cases} 1 & \text{if } x = y \text{ for } y \in E\,, \\ 1\Big/\sqrt{\frac{dm\circ\varphi}{dm}(y)\frac{dm\circ\varphi}{dm}(\varphi(y))\cdots\frac{dm\circ\varphi}{dm}(\varphi^{j-1}(y))} & \text{if } x = \varphi^j(y) \text{ for} \\ & j \in \{1, \cdots, k-1\} \text{ and } y \in E\,, \\ 0 & \text{if } x \notin \bigcup_{j=0}^{k-1}\varphi^j(E)\,. \end{cases}$$

By virtue of the condition $\frac{dm\circ\varphi^k}{dm}(x) = 1$ on $E$ and chain rule of derivative $\frac{dm\circ\varphi^k}{dm}$, it follows that

$$\frac{dm\circ\varphi^k}{dm}(x) = \frac{dm\circ\varphi}{dm}(\varphi^{k-1}(x))\frac{dm\circ\varphi}{dm}(\varphi^{k-2}(x))\cdots\frac{dm\circ\varphi}{dm}(x) = 1\,, \quad (x \in E)\,.$$

Thus we obtain that

$$V(\varphi)_1 e = M_{\sqrt{\frac{dm\circ\varphi}{dm}}} M_{\chi_{X_1}} T_\varphi e = e\,.$$

q.e.d.

COROLLARY 2.16. *Suppose that $X$ is a Hausdorff topological space and $m$ a regular measure on the $\sigma$-field of Borel sets in $X$. Let $\varphi$ be an MWnL on a measure space $(X, m)$ which is continuous on $X$. If $m(X_1(k)) > 0$ for some $k$ in $\mathbf{N}$, then there exists a non-zero vector $e$ in $L^2(X)$ such that $V(\varphi)_1 e = e$.*

PROOF. It is sufficient to confirm that there exists a subset $E$ in $X_1(k)$ satisfying the condition in Theorem 2.15. By the assumption and the regularity of $m$ there exists a compact set $K$ in $X_1(k)$ with $m(K) > 0$. For each $x$ in $K$, there exists an open neighborhood $U(x)$ of $x$ such that $\{\varphi^j(U(x))\}_{j=0}^{k-1}$ are mutually disjoint. Since the family $\{U(x)\}_{x\in K}$ is a covering of $K$, the set $K$ is covered by finitely many $U(x)$'s. Thus we have $m(U(x_0) \cap K) > 0$ for some $x_0$ in $X_1$, so that $E = U(x_0) \cap K$ is a desired subset of $X_1(k)$. q.e.d.

The following is a typical example satisfying the condition in the above theorem and $ONS(e, V(\varphi))$ is complete.

EXAMPLE 2.17 (MW$n$L on the set $\mathbf{N}$ with the discrete topology). Let $X = \mathbf{N}$, $m$ the counting measure on $X$ and $\delta$ the map on $X$ defined by

$$\delta(n(k-1)+i) = k\,, \quad (k \in \mathbf{N}, i \in I(n))\,.$$

Then $\delta$ is an MW$n$L on $X$ with $X_i = n(\mathbf{N}-1)+i$, $(i \in I(n))$ and

$$\frac{dm\circ\delta_i}{dm} = \chi_{X_i}\,, \quad \frac{dm\circ\delta_i^{-1}}{dm} = \chi_X = 1 \quad \text{and} \quad V(\varphi)_i = M_{\chi_{X_i}} T_\delta\,, \quad (i \in I(n))\,.$$

The point 1 in $X_1$ is the only periodic point for $\delta$ and $e = \chi_{\{1\}}$ is a unit eigenvector for eigenvalue 1 of $V(\delta)_1$. Obviously $ONS(e, V(\delta)) = \{\chi_{\{k\}} \mid k \in \mathbf{N}\}$ is complete in $\ell^2(\mathbf{N})$. Thus, by Corollary 1.5, $\{(\alpha^*_{V(\delta)})^k(\omega)\}_{k=1}^{\infty}$ converges to $\omega_e$ with respect to the norm topology in $B(\ell^2(\mathbf{N}))_*$. Moreover we have

$$(A_\delta \eta)(k) = \sum_{i=1}^{n} \eta(n(k-1)+i), \quad (\eta \in \ell^1(\mathbf{N})).$$

REMARK. If an MW$n$L $\varphi$ satisfies the assumption of Theorem 2.4 and $ONS(e, W)$ is complete, then the f.i.c. $W = \{W_i\}_{i=1}^n$ on $L^2(X) = L^2(X)_e$ is regarded as the f.i.c. $V(\delta) = \{W(\delta)_i\}_{i=1}^n$ on the Hilbert space $\ell^2(\mathbf{N})$, where $\delta$ is the MW$n$L defined in the above example and $L^2(X)$ and $\ell^2(\mathbf{N})$ are identified.

Here, we show the behavior of the orbit $\{(\alpha^*_{V(\varphi)})^k(\omega)\}_{k=1}^{\infty}$ in the case where $\varphi$ is an MW$n$L on a measure space $(X, m)$ and there exists an f.i.c. $W = \{W_i\}_{i=1}^n$ such that

(1) $\alpha_W(T) = \alpha_{V(\varphi)}(T)$ for all $T$ in $B(L^2(X))$,

(2) $W_1 e = e$ for some unit vector $e$ in $L^2(X)$,

(3) $ONS(e, W)$ is complete, that is, $L^2(X)_e = L^2(X)$.

From these conditions it follows that for any state $\omega$ in $B(L^2(X))_*$ we have

$$\lim_{k\to\infty} (\alpha^*_V)^k(\omega) = \lim_{k\to\infty} (\alpha^*_W)^k(\omega) = \omega_e \quad (\text{w.r.t. the norm topology in } B(L^2(X))_*).$$

Moreover we find that $W = \{W_i\}_{i=1}^n$ can be considered as the f.i.c. $V(\delta) = \{V(\delta)_i\}_{i=1}^n$ where $\delta$ is the map defined in Example 2.17 and that $ONS(e, V(\delta)) = ONS(e, W) = \{e_\ell\}_{\ell=1}^{\infty}$, $(e_1 = e)$. For $f$ in $L^\infty(X)$, $a$ in $\ell^\infty(\mathbf{N})$ and $\eta$ in $L^1(X)$, $\zeta$ in $\ell^1(\mathbf{N})$, we denote by $\pi_1(f)$, $\pi_2(a)$ and $\rho^1_\eta$, $\rho^2_\zeta$ the corresponding operators and elements in the preduals according to (F-2) and (G). In this case, $\pi_1(L^\infty(X))$ and $\pi_2(\ell^\infty(\mathbf{N}))$ are subalgebras of the same von Neumann algebra $B(L^2(X))$. Namely, for $\omega$ in $B(L^2(X))_*$, there exist $\eta$ in $L^1(X)$ and $a$ in $\ell^1(N)$ such that

$$\omega(\pi_1(f)) = \rho^1_\eta(\pi_1(f)) = \int_X f(x)\eta(x)dm, \quad (f \in L^\infty(X)),$$

$$\omega(\pi_2(a)) = \rho^2_\zeta(\pi_2(a)) = \sum_{\ell=1}^{\infty} a(\ell)\zeta(\ell), \quad (a \in \ell^\infty(\mathbf{N})).$$

Then the above convergency implies the following:

$$\lim_{k\to\infty} (\alpha^*_{V(\varphi)})^k(\omega)_{|\pi_1(L^\infty(x))} = \omega_{e|\pi_1(L^\infty(X))} = \rho^1_{|e|^2}$$

$$(\text{w.r.t. the norm topology in } L^1(X)),$$

$$\lim_{k\to\infty} (\alpha^*_{V(\varphi)})^k(\omega)_{|\pi_2(\ell^\infty(\mathbf{N}))} = \omega_{e|\pi_2(\ell^\infty(\mathbf{N}))} = \rho^2_{|\hat{e}|^2} = \rho^2_{\hat{e}}$$

$$(\text{w.r.t. the norm topology in } \ell^1(\mathbf{N})),$$

where $\hat{e}$ is the characteristic function $\chi_{\{1\}}$ on $\mathbf{N}$. Namely we observed that the limit of the sequence $\{(\alpha^*_{V(\varphi)})^k(\omega)\}_{k=1}^{\infty}$ converges to a unique state $\rho_{\hat{e}}$ on $\pi_2(\ell^\infty(\mathbf{N})) \subset B(L^2(X))$ for any $\omega$ if $\varphi$ satisfies the condition mentioned above. This is a property of the behavior of the orbit of a state concerning MW$n$L's.

Now, in the following proposition and examples, we show that the converse statement of Theorem 2.15 holds in the case $X = [0, 1]$ and $X_1$ is an interval in $[0, 1]$, though it does not hold in general.

PROPOSITION 2.18. *Let $X = [0, 1]$, $m$ the Lebesgue measure on $X$ and $\varphi$ an MWnL on $(X, m)$ which is continuous on $X$ with $X_1 = [0, c)$. Then we have the following*:

(1) $\bigcup_{k=1}^{\infty} X_1(k) = X_1(1) \cup X_1(2)$,

(2) $m(X_1(1) \cup X_1(2)) > 0$ *if and only if there exists a non-zero vector $e$ such that* $V(\varphi)_1 e = e$.

PROOF. (1) By the property of MW$n$L and the continuity of $\varphi$, the map $\varphi$ is a homeomorphism of $[0, c]$ onto $[0, 1]$. Hence $\varphi_1$ is monotonically increasing or monotonically decreasing on $[0, c]$. In the first case, all periodic points are fixed points. On the other hand, in the second case, there exists one fixed point and the period of the other periodic points are nothing but 2.

(2) It is sufficient to prove if part bacause of Corollary 2.16. We assume that $m(X_1(1) \cup X_1(2)) = 0$ and there exists a vector $e$ in $L^2(X)$ such that $V(\varphi)_1 e = e$. In the following we show that $e = 1$. By the assumption, we have

$$(V(\varphi)_1 e)(x) = \sqrt{\frac{dm \circ \varphi}{dm}}(x)\chi_{[0,c)}(x)e(\varphi(x)) = e(x)\,, \qquad (\text{a.a. } x \in X)\,.$$

Hence we have $e = \chi_{[0,c)}e = \chi_{X_1}e$ and inductively it follows that

$$e(x) = \sqrt{\frac{dm \circ \varphi^k}{dm}}(x)e(\varphi^k(x))\chi_{[0,c)}(x)\chi_{[0,c)}(\varphi(x)) \cdots \chi_{[0,c)}(\varphi^{k-1}(x))\,, \qquad (\text{a.a. } x \in X, k \geq 1)\,.$$

Now we put

$$Y_1(1) = \left\{x \in X_1 \mid \varphi(x) = x, \text{ and } \frac{dm \circ \varphi}{dm}(x) \neq 1\right\},$$

$$Y_1(2) = \left\{x \in X_1 \mid \varphi^2(x) = x, \varphi(x) \in X_1 \text{ and } \frac{dm \circ \varphi^2}{dm}(x) \neq 1\right\},$$

$$X_1(\infty) = \{x \in X_1 \mid \varphi^k(x) \in X_1 \text{ and } \varphi^k(x) \neq x \text{ for all } k \geq 1\}\,,$$

$$Z = \{x \in X_1 \mid \varphi^k(x) \notin X_1 \text{ for some } k\}\,.$$

Then we have

$$X_1 = X_1(1) \cup X_1(2) \cup Y_1(1) \cup Y_1(2) \cup X_1(\infty) \cup Z\,.$$

By the above equality, it follows that $e(x) = 0$ for a.a. $x$ in $Y_1(1) \cup Y_1(2) \cup Z$. Let $x_0$ be a point in $X_1(\infty)$. Then there exists an open interval $J$ containing $x_0$ such that $\{\varphi^k(J)\}_{k=1}^{\infty}$ are mutually disjoint. Hence we have

$$\int_J |e(x)|^2 dm = \int_J \left| \sqrt{\frac{dm \circ \varphi_1^k}{dm}}(x) e(\varphi_1^k(x)) \right|^2 dm = \int_{\varphi^k(J)} |e(x)|^2 dm$$

for all $k \geq 1$ and thus

$$\|e\|^2 \geq \int_{\bigcup_{K=1}^{\infty} \varphi^k(J)} |e(x)|^2 dm = \sum_{k=1}^{\infty} \int_{\varphi^k(J)} |e(x)|^2 dm = \sum_{k=1}^{\infty} \int_J |e(x)|^2 dm \,.$$

Hence $e(x) = 0$ on $J$ (a.e.) and so on $X_1(\infty)$ (a.e.). Combining the assumption $m(X_1(1) \cup X_1(2)) = 0$ with this, we have $e = 0$. q.e.d.

EXAMPLE 2.19 (MW2L on $[0, 1]$ such that $m(X_1(1)) > 0$). Let $X = [0, 1]$, $m$ the Lebesgue measure on $[0, 1]$ and $\varphi$ the map defined by

$$\varphi(x) = \begin{cases} x & \text{for } 0 \leq x < 1/4\,, \\ (6x - 1)/2 & \text{for } 1/4 \leq x < 1/2\,, \\ -2x + 2 & \text{for } 1/2 \leq x \leq 1\,. \end{cases}$$

Then $\varphi$ is an MW2L on $[0, 1]$ and $V(\varphi)_1 = M_{\chi_{[0,1/4)}} T_\varphi + \sqrt{3} M_{\chi_{[1/4,1/2)}} T_\varphi$, $V(\varphi)_2 = \sqrt{2} M_{\chi_{[1/2,1]}} T_\varphi$. Put $e = 2\chi_{[0,1/4)}$. Then $V(\varphi)_1 e = e$ and $ONS(e, V(\varphi))$ is not complete. Moreover we have

$$(A_\varphi \eta)(x) = \chi_{[0,1/4)}(x)\eta(x) + \frac{1}{3}\chi_{[1/4,1]}(x)\eta\left(\frac{2x+1}{6}\right) + \frac{1}{2}\eta\left(\frac{-x+2}{2}\right).$$

EXAMPLE 2.20 (MW2L on $[0, 1]$ such that $m((X_1(2)) > 0$). Let $X = [0, 1]$, $m$ the Lebesgue measure on $[0, 1]$ and $\varphi$ the map defined by

$$\varphi(x) = \begin{cases} -5x + 1 & \text{for } 0 \leq x < 1/8\,, \\ -x + (1/2) & \text{for } 1/8 \leq x < 1/2\,, \\ 2x - 1 & \text{for } 1/2 \leq x \leq 1\,. \end{cases}$$

Then $\varphi$ is an MW2L on $[0, 1]$ and $V(\varphi)_1 = \sqrt{5} M_{\chi_{[0,1/8)}} T_\varphi + M_{\chi_{[1/8,1/2)}} T_\varphi$, $V(\varphi)_2 = \sqrt{2} M_{\chi_{[1/2,1]}} T_\varphi$. Put $e = 2\chi_{[1/8,3/8]}$. Then $V(\varphi)_1 e = e$ and $ONS(e, V(\varphi))$ is not complete. Moreover we have

$$(A_\varphi \eta)(x) = \chi_{[0,1/8)}(x)\eta\left(\frac{-2x+1}{2}\right) + \frac{1}{5}\chi_{[1/8,1]}(x)\eta\left(\frac{-x+1}{5}\right) + \frac{1}{2}\eta\left(\frac{x+1}{2}\right).$$

REMARK. Though we showed one invariant unit vector $e$ for $V(\varphi)_1$ in Examples 2.19 and 2.20, it is easy to see that there exist infitely many invariant unit vectors $f$'s for $V(\varphi)_1$ and, for every $f$, $ONS(f, V(\varphi))$ is not complete in these examples. This holds for any continuous MW$n$L on $[0, 1]$ with $m(X_1(1)) > 0$ or $m(X_1(2)) > 0$.

## 3. MW1L and the behavior of the orbit of a state.

Let $\varphi$ be an MW1L on a measure space $(X, m)$. Then $V(\varphi)_1 = M_{\sqrt{dm\circ\varphi/dm}}T_\varphi$ is a unitary operator on $L^2(X)$ and we have

$$(A_\varphi\eta)(x) = \frac{dm \circ \varphi^{-1}}{dm}(x)\eta(\varphi^{-1}(x)), \quad (x \in X)$$

for $\eta$ in $L^1(X)$. Suppose that there exists a non-negative function $\eta_0$ in $L^1(X)$ such that $\|\eta_0\|_1 = 1$ and $\lim_{k\to\infty} A^k_\varphi\eta = \eta_0$ for all non-negative function $\eta$ in $L^1(X)$ with $\|\eta\|_1 = 1$. Then $A_\varphi\eta_0 = \eta_0$, that is,

$$\frac{dm \circ \varphi^{-1}}{dm}(x)\eta_0(\varphi^{-1}(x)) = \eta_0(x), \quad (\text{a.a. } x \in X).$$

We put $e = \sqrt{\eta_0}$. Then the above equation implies $V(\varphi)_1 e = e$. However, since $ONS(e, V(\varphi)) = \{e\}$, we have no information for finding whether the sequence $\{A^k_\varphi\eta\}_{k=1}^\infty$ converges to $\eta_0$ with respect to the norm topology in $L^1(X)$ or not. Indeed, in many cases, these sequences do not converge in the sense of the norm topology. In case of MW1L's, the behavior of $\{A^k_\varphi\eta\}_{k=1}^\infty$ seems to be very complicated. So it is one of the our purpose in the next step to study the behavior of these sequences. In the present paper, we show the behavior of the sequences $\{A^k_\beta\eta\}_{k=1}^\infty$ for Baker's transformation $\beta$, which is a typical example of strong-mixing MW1L.

EXAMPLE 3.1 (Baker's transformation). Let $X = \mathbf{I} \times \mathbf{I} = [0, 1] \times [0, 1]$ and $m$ the Lebesgue measure on $\mathbf{I} \times \mathbf{I}$. Let $\beta$ is Baker's transformation on $\mathbf{I} \times \mathbf{I}$, which is defined by

$$\beta(x, y) = \begin{cases} (2x, y/2) & \text{for } 0 \le x < 1/2, \\ (2x - 1, (y + 1)/2) & \text{for } 1/2 \le x \le 1. \end{cases}$$

Then $\beta$ is an MW1L with $\beta^{-1}$, where

$$\beta^{-1}(x, y) = \begin{cases} (x/2, 2y) & \text{for } 0 \le y < 1/2, \\ ((x + 1)/2, 2y - 1) & \text{for } 1/2 \le y \le 1. \end{cases}$$

Namely we have $\beta^{-1}(\beta(x, y)) = \beta(\beta^{-1}(x, y)) = (x, y)$ on $X$ (a.e.). Moreover we have $V(\beta)_1 = T_\beta$ and the constant function $e(x, y) = 1$ is a $V(\beta)_1$-invariant unit vector in $L^2(\mathbf{I} \times \mathbf{I})$. Obviously $ONS(e, V(\beta))(= \{e\})$ is not complete. Moreover it follows that

$$(A_\beta\eta)(x, y) = \eta(\beta^{-1}(x, y)), \quad (\eta \in L^1(\mathbf{I} \times \mathbf{I})).$$

Although Baker's transformation $\beta$ is strong-mixing, it does not satisfy the norm convergence property of the orbit of a state. Indeed, for the non-negative function $\eta$ in $L^1(\mathbf{I} \times \mathbf{I})$ with $\|\eta\|_1 = 1$ defined by

$$\eta(x, y) = \sin 2\pi y + 1,$$

we have $(A^k_\beta\eta)(x, y) = \sin 2^{k+1}\pi y + 1$ and, of course, $\{A^k_\beta\eta\}_{k=1}^\infty$ converges to the constant function $e$ with respect to $\sigma(L^1(\mathbf{I} \times \mathbf{I}), L^\infty(\mathbf{I} \times \mathbf{I}))$-topology, where $e(x, y) = 1$ for all $(x, y)$

in $\mathbf{I}\times\mathbf{I}$, but

$$\|A_\beta^k\eta - |e|^2\|_1 = \frac{2}{\pi}$$

for all $k$. In the following we show precisely how the sequence $\{A_\beta^k\eta\}_{k=1}^\infty$ behaves for functions $\eta$'s in $L^1(\mathbf{I}\times\mathbf{I})$.

PROPOSITION 3.2. *Let $\beta$ be Baker's transformation on $X = \mathbf{I}\times\mathbf{I}$. Then, for $\eta$ in $L^1(\mathbf{I}\times\mathbf{I})$ and $\varepsilon > 0$, there exist positive integers $K$, $s$ and a finite set $\{c_\ell\}_{\ell=0}^{2^s-1}$ of complex numbers such that*

$$\left\|A_\beta^K\eta - \sum_{\ell=0}^{2^s-1} c_\ell \chi_{\mathbf{I}\times[\ell/2^s,(\ell+1)/2^s)}\right\|_1 < \varepsilon .$$

PROOF. First we put

$$\mathbf{I}_{(p,q)}^{(i,j)} = [i/2^p, (i+1)/2^p)\times[j/2^q,(j+1)/2^q)$$

where $p, q$ are non-negative integers and $i = 0, 1, \cdots, 2^p-1$, $j = 0, 1, \cdots, 2^q-1$. Then we have

$$A_\beta\chi_{\mathbf{I}_{(p,q)}^{(i,j)}} = \begin{cases} \chi_{\mathbf{I}_{(p-1,q+1)}^{(i,j)}} & \text{for } i = 0, 1, \cdots, 2^{p-1}-1 , \\ \chi_{\mathbf{I}_{(p-1,q+1)}^{(i-2^{p-1},j+2^q)}} & \text{for } i = 2^{p-1}, 2^{p-1}+1, \cdots, 2^p-1 . \end{cases}$$

Thus for each $\chi_{\mathbf{I}_{(p,q)}^{(i,j)}}$ there exist positive large integers $K$ and $r$ such that

$$A_\beta^K\chi_{\mathbf{I}_{(p,q)}^{(i,j)}} = \sum_{\ell=0}^{2^r-1} a_\ell\chi_{\mathbf{I}\times[\ell/2^r,(\ell+1)/2^r)}, \quad (a_\ell\in\{0,1\}) .$$

Moreover we have

$$\begin{aligned} A_\beta\chi_{\mathbf{I}\times[j/2^q,(j+1)/2^q)} &= A_\beta\chi_{\mathbf{I}_{(0,q)}^{(0,j)}} = \chi_{\mathbf{I}_{(0,q+1)}^{(0,j)}} + \chi_{\mathbf{I}_{(0,q+1)}^{(0,j+2^q)}} \\ &= \chi_{\mathbf{I}\times[j/2^{q+1},(j+1)/2^{q+1})} + \chi_{\mathbf{I}\times[(j+2^q)/2^{q+1},(j+2^q+1)/2^{q+1})} . \end{aligned}$$

Now we put

$$\mathcal{S} = \bigcup_{p,q=0}^{\infty}\left\{\sum_{i=0}^{2^p-1}\sum_{j=0}^{2^q-1} b_{i,j}\chi_{\mathbf{I}_{(p,q)}^{(i,j)}} \mid b_{i,j}\in\mathbf{C}\right\}.$$

Then, for a given function $\eta$ in $L^1(\mathbf{I}\times\mathbf{I})$, since $\mathcal{S}$ is dense in $L^1(\mathbf{I}\times\mathbf{I})$ with respect to the norm topology in $L^1(\mathbf{I}\times\mathbf{I})$, there exist non-negative integers $p, q$ and complex numbers $\{b_{i,j}\}$ such that

$$\left\|\eta - \sum_{i=1}^{2^p-1}\sum_{j=1}^{2^q-1} b_{i,j}\chi_{\mathbf{I}_{(p,q)}^{(i,j)}}\right\|_1 < \varepsilon .$$

By virtue of the above discussion, there exist positive integers $K$ and $s$ such that

$$A_\beta^K\left(\sum_{i=1}^{2^p-1}\sum_{j=1}^{2^q-1} b_{i,j}\chi_{\mathbf{I}_{(p,q)}^{(i,j)}}\right) = \sum_{\ell=0}^{2^s-1} c_\ell\chi_{\mathbf{I}\times[\ell/2^s,(\ell+1)/2^s)} ,$$

where each $c_\ell$ is a complex number. Since $A_\beta$ is isometric on $L^1(\mathbf{I}\times\mathbf{I})$, it follows that

$$\left\| A_\beta^K \eta - \sum_{\ell=0}^{2^s-1} c_\ell \chi_{\mathbf{I}\times[\ell/2^s,(\ell+1)/2^s)} \right\|_1 = \left\| A_\beta^K \left( \eta - \sum_{i=1}^{2^p-1} \sum_{j=1}^{2^q-1} b_{i,j} \chi_{\mathbf{I}_{(p,q)}^{(i,j)}} \right) \right\|_1 < \varepsilon .$$

Now, using the above proposition, the property of strong-mixing for Baker's transformation is derived as follows. For a function $\eta$ in $L^1(\mathbf{I}\times\mathbf{I})$, a function $f$ in $L^\infty(\mathbf{I}\times\mathbf{I}) \subset L^1(\mathbf{I}\times\mathbf{I})$ and a positive real number $\varepsilon > 0$, there exist a large number $k$ in $\mathbf{N}$ and step functions in $L^1(\mathbf{I}\times\mathbf{I})$ such that

$$\left\| A_\beta^K \eta - \sum_{\ell=0}^{2^s-1} c_\ell \chi_{\mathbf{I}\times[\ell/2^s,(\ell+1)/2^s)} \right\|_1 < \varepsilon ,$$

$$\left\| f - \sum_{i=0}^{2^u-1} \sum_{j=0}^{2^v-1} d_{i,j} \chi_{\mathbf{I}_{(u,v)}^{(i,j)}} \right\|_1 < \varepsilon ,$$

where $\{c_\ell\}$ and $\{d_{i,j}\}$ are complex numbers. For the finite set of simple functions generating the step functions in the first approximation formula, we can see that there exists a large number $K'$ in $\mathbf{N}$ such that

$$\int_{\mathbf{I}\times\mathbf{I}} (A_\beta^k \chi_{\mathbf{I}\times[\ell/2^s,(\ell+1)/2^s)}) \cdot \chi_{\mathbf{I}\times[j/2^v,(j+1)/2^v)} dm = \frac{1}{2^{s+v}}$$

for all $k > K'$. Moreover, for this $k > K'$, we have

$$\int_{\mathbf{I}\times\mathbf{I}} (A_\beta^k \chi_{\mathbf{I}\times[\ell/2^s,(\ell+1)/2^s)}) \cdot \chi_{\mathbf{I}_{(u,v)}^{(i,j)}} dm = \frac{1}{2^{s+u+v}} = \int_{\mathbf{I}\times\mathbf{I}} \chi_{\mathbf{I}\times[\ell/2^s,(\ell+1)/2^s)} dm \cdot \int_{\mathbf{I}\times\mathbf{I}} \chi_{\mathbf{I}_{(u,v)}^{(i,j)}} dm .$$

Using these facts, we can show that

$$\lim_{k\to\infty} \int_{\mathbf{I}\times\mathbf{I}} (A_\beta^{K+k}\eta) \cdot f dm = \int_{\mathbf{I}\times\mathbf{I}} A_\beta^K \eta dm \cdot \int_{\mathbf{I}\times\mathbf{I}} f dm = \int_{\mathbf{I}\times\mathbf{I}} \eta dm \cdot \int_{\mathbf{I}\times\mathbf{I}} f dm .$$

Especially, if $\eta$ is a non-negative function with $\|\eta\|_1 = 1$, we have

$$\lim_{k\to\infty} \int_{\mathbf{I}\times\mathbf{I}} A_\beta^k \eta \cdot f dm = \int_{\mathbf{I}\times\mathbf{I}} f dm .$$

Namely we have the following and from this we can see that $\beta$ is strong-mixing.

COROLLARY 3.3. *Let $\beta$ be Baker's transformation on $X = \mathbf{I}\times\mathbf{I}$ and $e$ the unit vector defined by $e(x,y) = 1$ in $L^1(\mathbf{I}\times\mathbf{I})$. Then, for each non-negative function $\eta$ in $L^1(\mathbf{I}\times\mathbf{I})$ with $\|\eta\|_1 = 1$ it follows that*

$$\lim_{k\to\infty} A_\beta^k \eta = e \quad (w.r.t.\ \sigma(L^1(\mathbf{I}\times\mathbf{I}), L^\infty(\mathbf{I}\times\mathbf{I}))\text{-topology in } L^1(\mathbf{I}\times\mathbf{I})) .$$

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## References


[1] J. BANKS, J. BROOKS, G. CAIRNS, G. DAVIS and P. STACEY, On Devaney's definition of chaos, Amer. Math. Monthly **99** (1992), 332–334.

[2] O. BRATTELI and P. E. T. JORGERSEN, *Iterated function systems and permutation representations of the Cuntz algebra*, Mem. Amer. Math. Soc. **663** (1999).

[3] J. CUNTZ, Simple $C^*$-algebras generated by isometries, Comm. Math. Phys. **57** (1977), 173–185.

[4] X. DAI and D. R. LARSON, *Wandering vectors for unitary systems and orthogonal wavelets*, Mem. Amer. Math. Soc. **640** (1998).

[5] S. KAWAMURA, Covariant representations associated with chaotic dynamical systems, Tokyo J. Math. **20** (1997), 205–217.

[6] D. RUELLE, Applications conservant une mesure absolument continue par rapport à $dx$ sur [0, 1], Comm. Math. Phys. **55** (1977), 47–52.

[7] M. TAKESAKI, *Theory of operator algebras I*, Springer (1979).

[8] S. M. ULAM and J. VON NEUMANN, On combination of stochastic and deterministic processes, Preliminary report. Bull. Amer. Math. Soc. **53** (1947), 1120.

[9] P. WALTER, *An introduction to ergodic theory*, GTM **79** (1982), Springer.



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