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A Sharp Existence and Uniqueness Theorem for Linear Fuchsian Partial Differential Equations

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Abstract. In this paper, we will consider the equation $\mathcal{P}u = f$, where $\mathcal{P}$ is the linear Fuchsian partial differential operator

$$\mathcal{P} = (tD_t)^m + \sum_{j=0}^{m-1} \sum_{|\alpha| \le m-j} a_{j,\alpha}(t,z)(\mu(t)D_z)^\alpha (tD_t)^j .$$

We will give a sharp form of unique solvability in the following sense: we can find a domain Ω such that if f is defined on Ω, then we can find a unique solution u also defined on Ω.

1. Introduction and result.

Denote by $\mathbf{N}$ the set of nonnegative integers, and let $(t, z) = (t, z_1, \cdots, z_n) \in \mathbf{R} \times \mathbf{C}^n$. Let $R > 0$ be sufficiently small, and for $\rho \in (0, R]$, let B_ρ be the polydisk $\{z \in \mathbf{C}^n;\ |z_i| < \rho \ \text{ for } i = 1, 2, \cdots, n\}$.

Given any bounded, open subset D of $\mathbf{C}^n$, the space $\mathcal{A}(D)$ of all functions $g(z)$ holomorphic in D and continuous up to $\bar{D}$ forms a Banach space with norm $\|g\|_D = \max_{z\in\bar{D}} |g(z)|$. Let $T > 0$. Then we denote by $C^0([0, T], \mathcal{A}(D))$ the set of functions continuous on the interval $[0, T]$ and valued in the space $\mathcal{A}(D)$.

We say that a continuous, positive-valued function $\mu(t)$ on the interval $(0, T)$ is a *weight function* if $\mu(t)$ is increasing and the function

$$\varphi(t) = \int_0^t \frac{\mu(s)}{s}\, ds \tag{1.1}$$

is well-defined on $(0, T)$, i.e., the integral on the right is finite. (See Tahara [7].)

Consider now the linear partial differential operator

$$\mathcal{P} = (tD_t)^m + \sum_{j=0}^{m-1} \sum_{|\alpha| \le m-j} a_{j,\alpha}(t,z)(\mu(t)D_z)^\alpha (tD_t)^j . \tag{1.2}$$

Here, $D_t = \partial/\partial t$ and $D_z = (\partial/\partial z_1, \cdots, \partial/\partial z_n)$; $\mu(t)$ is a weight function; and the coefficients $a_{j,\alpha}(t, z)$ belong in the space $C^0([0, T], \mathcal{A}(B_R))$, i.e., for any $s \in [0, T]$, each of the

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functions $a_{j,\alpha}(s,z)$, when viewed as a function of z, is holomorphic in B_R and continuous up to $\bar{B}_R$. We associate a polynomial with this operator, called the *characteristic polynomial* of $\mathcal{P}$, and we define it by

$$C(\lambda, z) = \lambda^m + a_{m-1,0}(0,z)\lambda^{m-1} + \cdots + a_{0,0}(0,z)\,. \tag{1.3}$$

Its roots $\lambda_1(z), \cdots, \lambda_m(z)$ will be referred to as *characteristic exponents*. In what follows, we will assume that there exists a positive number L such that

$$\Re\lambda_j(z) \le -L < 0 \qquad \text{for all} \quad z \in B_R \quad \text{and} \quad 1 \le j \le m\,. \tag{1.4}$$

Baouendi and Goulaouic [1] studied the above operator in the case when $\mu(t) = t^a$ $(a > 0)$. They called such operator a Fuchsian partial differential operator, which for them is the "natural" generalization of a Fuchsian ordinary differential operator. In their paper, they gave some generalizations of the classical Cauchy-Kowalewski and Holmgren theorems for this type of operators. Their method has been applied and extended to various cases as can be seen, for example, in Tahara [6], Mandai [5] and Yamane [8].

In a previous paper [4], the author proved existence and uniqueness theorems similar to those given in [1], but for general $\mu(t)$. Essentially, he proved the following unique solvability result.

THEOREM 1. *Let $\mathcal{P}$ be as in* (1.2). *Then given any $\rho \in (0,R)$, there exists an $\varepsilon \in (0,T]$ such that for any $f(t,z) \in C^0([0,T],\mathcal{A}(B_R))$, the equation $\mathcal{P}u = f$ has a unique solution $u(t,z) \in C^0([0,\varepsilon],\mathcal{A}(B_\rho))$ satisfying for $1 \le p \le m$ the relation $(tD_t)^p u \in C^0([0,\varepsilon],\mathcal{A}(B_\rho))$.*

We remark that although $f(t,z)$, viewed as a function of z, is defined on B_R, the existence of the solution $u(t,z)$ is only guaranteed up to B_ρ, with $\rho < R$. Moreover, any two solutions of $\mathcal{P}u = f$ can only be shown to coincide in a neighborhood of the origin which is smaller than the neighborhood on which the two are defined.

In this paper, we shall present a formulation leading to an existence and uniqueness result sharper than the one given above. The result is sharper in the sense that the solution $u(t,z)$ of the equation $\mathcal{P}u = f$ will now have the same domain of definition as the inhomogeneous part $f(t,z)$.

To proceed, we will need the following definitions.

DEFINITION 1. Let $\tau \in (0,T)$, $\gamma > 0$ and $\varphi(t)$ be the one in (1.1). We define

(i) $\omega_\tau[\gamma] = \{z \in \mathbf{C}^n;\ |z_i| < R - \gamma\varphi(\tau) \quad \text{for } i = 1, 2, \cdots, n\}$, and

(ii) $\Omega_T[\gamma] = \{(\tau, z) \in \mathbf{R} \times \mathbf{C}^n;\ 0 \le \tau \le T \text{ and } z \in \omega_\tau[\gamma]\}$.

DEFINITION 2. Let $p \in \mathbf{N}$ and $\gamma > 0$.

(i) We say that $f(t,z)$ belongs in $\mathcal{K}_0(\Omega_T[\gamma])$ if for each $\tau \in [0,T]$, we have $f(t) \in C^0([0,\tau],\mathcal{A}(\omega_\tau[\gamma]))$.

(ii) We say that $w(t,z)$ belongs in $C^0_p([0,\tau],\mathcal{A}(\omega_\tau[\gamma]))$ if for all $0 \le j \le p$, we have $(tD_t)^j w(t) \in C^0([0,\tau],\mathcal{A}(\omega_\tau[\gamma]))$.

(iii) We say that $u(t,z)$ belongs in $\mathcal{K}_p(\Omega_T[\gamma])$ if for each $\tau \in [0,T]$, we have $u(t) \in C_p^0([0,\tau], \mathcal{A}(\omega_\tau[\gamma]))$.

Under the above assumptions, we now state the following main result.

THEOREM 2. *Let $\mathcal{P}$ be the operator given in (1.2). Then there exist constants $T_0 > 0$ and $\gamma_0 > 0$ depending on $\mathcal{P}$ such that for any $f(t,z) \in \mathcal{K}_0(\Omega_{T_0}[\gamma_0])$, the equation*

$$\mathcal{P}u = f \quad \text{in} \quad \Omega_{T_0}[\gamma_0] \tag{1.5}$$

has a unique solution $u(t,z)$ in $\mathcal{K}_m(\Omega_{T_0}[\gamma_0])$.

Moreover, the solution satisfies the a priori estimate

$$\sum_{p=0}^{m} \max_{\Delta} |(tD_t)^p u| \le C \max_{\Delta} |f|, \tag{1.6}$$

where Δ is the closure of $\Omega_{T_0}[\gamma_0]$ and $C > 0$ is some constant dependent on the above equation and on the domain $\Omega_{T_0}[\gamma_0]$.

Note that $f(t,z)$ and $u(t,z)$ both have $\Omega_{T_0}[\gamma_0]$ as their domain of definition. This fact allows us to restate our theorem in the following manner: for any $T,\ \gamma > 0$, let $X_{T,\gamma}$ and $Y_{T,\gamma}$ be the spaces $\mathcal{K}_m(\Omega_T[\gamma])$ and $\mathcal{K}_0(\Omega_T[\gamma])$, respectively. Let $W_{T,\gamma}$ be the subspace of $X_{T,\gamma}$ consisting of functions $u \in X_{T,\gamma}$ such that $\mathcal{P}u$ belongs in $Y_{T,\gamma}$. Define a linear operator Ψ from $X_{T,\gamma}$ to $Y_{T,\gamma}$ with domain $W_{T,\gamma}$ by $\Psi u = \mathcal{P}u$. Let $|||\cdot|||_{T,\gamma}$ denote the maximum norm in the closure of $\Omega_T[\gamma]$. Then $X_{T,\gamma}$ and $Y_{T,\gamma}$ are Banach spaces; given $u \in X_{T,\gamma}$ and $f \in Y_{T,\gamma}$, we define their norms by $\sum_{p=0}^{m} |||(tD_t)^p u|||_{T,\gamma}$ and $|||f|||_{T,\gamma}$, respectively. Note further that the operator Ψ is a closed linear operator from $X_{T,\gamma}$ to $Y_{T,\gamma}$. The above theorem can now be stated as

THEOREM 2$'$. *There exist $T_0,\ \gamma_0 > 0$ depending on $\mathcal{P}$ such that the operator Ψ is a one-one, closed linear operator from X_{T_0,γ_0} onto Y_{T_0,γ_0}.*

Since Ψ is an injection, Ψ^{-1} exists and is also closed. The Closed Graph Theorem further implies that Ψ^{-1} is continuous. The estimate given in (1.6) is just a consequence of the continuity of Ψ^{-1}.

2. Preliminary discussion.

We can rewrite the operator $\mathcal{P}$ as

$$\mathcal{P} = \mathcal{Q} + \sum_{j=0}^{m-1} \sum_{|\alpha| \le m-j} c_{j,\alpha}(t,z)(\mu(t)D_z)^\alpha (tD_t)^j,$$

where the operator $\mathcal{Q}$ is defined by

$$\mathcal{Q} = (tD_t)^m + a_{m-1,0}(0,z)(tD_t)^{m-1} + \cdots + a_{0,0}(0,z) \tag{2.1}$$

and

$$c_{j,\alpha}(t,z)=\begin{cases}a_{j,\alpha}(t,z) & \text{if } |\alpha|\neq 0,\\ a_{j,\alpha}(t,z)-a_{j,\alpha}(0,z) & \text{if } |\alpha|=0.\end{cases}$$

Note that the coefficients of the ordinary differential operator $\mathcal{Q}$ are holomorphic functions of z in B_R. Note further that the characteristic exponents of $\mathcal{Q}$ are the same as that of $\mathcal{P}$, and hence satisfy (1.4).

LEMMA 1. *Fix* $\tau > 0$ *and let* $g(t) \in C^0([0,\tau], \mathcal{A}(\omega_\tau[\gamma]))$. *Then the equation* $\mathcal{Q}u = g$ *has a unique solution* $u(t) \in C^0_m([0,\tau], \mathcal{A}(\omega_\tau[\gamma]))$. *This unique solution is given by*

$$u(t)=\frac{1}{m!}\sum_{\sigma\in S_m}\int_0^t\int_0^{s_m}\cdots\int_0^{s_2}\left(\frac{s_m}{t}\right)^{-\lambda_{\sigma(m)}}\left(\frac{s_{m-1}}{s_m}\right)^{-\lambda_{\sigma(m-1)}}\cdots \times\left(\frac{s_1}{s_2}\right)^{-\lambda_{\sigma(1)}} g(s_1)\,\frac{ds_1}{s_1}\frac{ds_2}{s_2}\cdots\frac{ds_m}{s_m}. \tag{2.2}$$

Here, S_m *is the group of permutations of* $\{1,2,\cdots,m\}$.

A result in symmetric entire functions asserts that the solution $u(t,z)$ is holomorphic with respect to z. The fact that it belongs in $C^0_m([0,\gamma], \mathcal{A}(\omega_\tau[\gamma]))$ is seen in the integral expression, but may actually be obtained *a priori*. (See Baouendi-Goulaouic [1].)

To facilitate computation, we define for $\lambda = (\lambda_1, \cdots, \lambda_m)$ the function

$$G^t_\theta(\lambda)\stackrel{\text{def}}{=}\frac{1}{m!}\sum_{\sigma\in S_m}\left(\frac{s_m}{t}\right)^{-\lambda_{\sigma(m)}}\left(\frac{s_{m-1}}{s_m}\right)^{-\lambda_{\sigma(m-1)}}\cdots\left(\frac{\theta}{s_2}\right)^{-\lambda_{\sigma(1)}}, \tag{2.3}$$

for some dummy variables $s_2, \cdots, s_m$. Define, too, the integral operator

$$\int^{(m)}_{[t;\theta]} g\stackrel{\text{def}}{=}\int_0^t\int_0^{s_m}\cdots\int_0^{s_2} g(\theta)\,\frac{d\theta}{\theta}\frac{ds_2}{s_2}\cdots\frac{ds_m}{s_m}. \tag{2.4}$$

Using the above, we can now write the solution $u(t)$ of the equation $\mathcal{Q}u = g$ as

$$u(t) = \int^{(m)}_{[t;s]} G^t_s(\lambda)\, g.$$

In our proof of the main theorem, it will be necessary to consider the action of the differential operator $(tD_t)^p$ on integral expressions similar to the one in (2.2). One can easily verify the following

LEMMA 2. *Let* $u(t)$ *be the solution of* $\mathcal{Q}u = g$. *Then for a natural number* p *less than* m, *we have*

$$(tD_t)^p u=\sum_{i=m-p}^{m}\int^{(i)}_{[t;s_1]} g\times\left\{\frac{1}{m!}\sum_{\sigma\in S_m} h_i(\sigma,\lambda)\left(\frac{s_i}{t}\right)^{-\lambda_{\sigma(i)}} \times\left(\frac{s_{i-1}}{s_i}\right)^{-\lambda_{\sigma(i-1)}}\cdots\left(\frac{s_1}{s_2}\right)^{-\lambda_{\sigma(1)}}\right\}, \tag{2.5}$$

where the functions $h_i(\sigma, \lambda)$ are suitable polynomial functions of the characteristic exponents $\lambda_1(z), \cdots, \lambda_m(z)$.

For brevity, let us set, for a natural number k,

$$H_\theta^t(k, \lambda) = \frac{1}{m!} \sum_{\sigma \in S_m} h_k(\sigma, \lambda) \Big(\frac{s_k}{t}\Big)^{-\lambda_{\sigma(k)}} \Big(\frac{s_{k-1}}{s_k}\Big)^{-\lambda_{\sigma(k-1)}} \cdots \Big(\frac{\theta}{s_2}\Big)^{-\lambda_{\sigma(1)}}. \tag{2.6}$$

By symmetry, the functions $H_s^t(k, \lambda)$ are holomorphic with respect to z and thus belong in $\mathcal{A}(B_R)$.

The following lemma is useful in evaluating some integral expressions in the proof.

LEMMA 3. *Let k be natural number. Then the following equalities hold:*

(a)
$$\int_0^{s_k} \int_0^{s_{k-1}} \cdots \int_0^{s_1} \Big(\frac{s_0}{s_k}\Big)^L \frac{ds_0}{s_0} \cdots \frac{ds_{k-1}}{s_{k-1}} = \frac{1}{L^k}.$$

(b)
$$\int_0^t \int_0^{s_k} \cdots \int_0^{s_1} \frac{\mu(s_k)}{s_k} \frac{\mu(s_{k-1})}{s_{k-1}} \cdots \frac{\mu(s_1)}{s_1} \times \Big(\frac{s_0}{t}\Big)^L \frac{s_0^{-1}}{[\varphi(t) - \varphi(s_0)]^k} \, ds_0 \cdots ds_k = \frac{1}{L\, k!}.$$

The first equality is obvious. The second can be proved by reversing the order of integration and recalling that $t\varphi'(t) = \mu(t)$.

To estimate the derivatives with respect to z, we have the following lemma. (For a proof, see Hörmander [3, Lemma 5.1.3].)

LEMMA 4. *Let the function $v(z)$ be holomorphic in B_R, and suppose there are positive constants K and c such that*

$$\|v\|_\rho \le \frac{K}{(R-\rho)^c} \quad \textit{for every} \quad \rho \in (0, R). \tag{2.7}$$

Then we have

$$\|D_z^\alpha v\|_\rho \le \frac{K e^{|\alpha|} (c+1)_{|\alpha|}}{(R-\rho)^{c+|\alpha|}} \quad \textit{for every} \quad \rho \in (0, R). \tag{2.8}$$

In the above, we define $(c)_p = (c)(c+1) \cdots (c+p-1)$.

3. Proof of Main Theorem.

Let f be any element of $\mathcal{K}_0(\Omega_{T_0}[\gamma_0])$. Here, the constants $T_0 > 0$ and $\gamma_0 > 0$ satisfy some conditions which will later be specified. For convenience, we will drop the subscript in both and instead use T and γ; we will again use the subscript upon stating the conditions that these constants need to satisfy.

We will use the method of successive approximations to solve the equation $\mathcal{P}u = f$. Define the approximate solutions as follows:

$$u_0(t) = \int_{[t;s]}^{(m)} G_s^t(\lambda)\, f \tag{3.1}$$

and for $k \geq 1$,

$$u_k(t) = \int_{[t;s]}^{(m)} G_s^t(\lambda)[f - \mathcal{S}(s)u_{k-1}]\,. \tag{3.2}$$

Here, $t \in [0, T]$, and for brevity, we have set $\mathcal{S}(t) = \sum_{j=0}^{m-1} \sum_{|\alpha| \leq m-j} c_{j,\alpha}(t, z) \cdot (\mu(t)D_z)^\alpha (tD_t)^j$. Note that for all k, the approximate solutions $u_k(t, z)$ are defined on $\Omega_{T_0}[\gamma_0]$. Furthermore, they are continuous with respect to t and holomorphic with respect to z on this region.

For each natural number k, we also define the sequence of functions $v_k(t) = u_k(t) - u_{k-1}(t)$, where we have set $u_{-1} \equiv 0$. Then the functions $v_k(t, z)$ are also defined on the same region as $u_k(t, z)$, and are also continuous with respect to t and holomorphic with respect to z. Using the expression for $u_k(t)$, we have

$$v_0(t) = \int_{[t;s]}^{(m)} G_s^t(\lambda)\, f \tag{3.3}$$

and for $k \geq 1$,

$$v_k(t) = -\int_{[t;s]}^{(m)} G_s^t(\lambda)\mathcal{S}(s)v_{k-1}\,. \tag{3.4}$$

To prove that the approximate solutions converge to the real solution, we will henceforth fix one $t \in [0, T]$, and estimate the functions $v_k(t)$.

Let C be the bound on $[0, T] \times \overline{B_R}$ of all $c_{j,\alpha}(t, z)$, and K be the bound in $\overline{\Omega_T[\gamma]}$ of $f(t, z)$. As for the functions $G_s^t(\lambda)$ and $H_s^t(k, \lambda)$, we have the following estimates:

$$\sup_{z \in \overline{B_R}} |G_s^t(\lambda)| \leq \left(\frac{s}{t}\right)^L \tag{3.5}$$

and there exists a constant D such that for $1 \leq k \leq m$,

$$\sup_{z \in \overline{B_R}} |H_s^t(k, \lambda)| \leq D\left(\frac{s}{t}\right)^L . \tag{3.6}$$

We can easily see that $\|v_0(t)\|_{\omega_t}$ is bounded by KL^{-m} for any $0 \leq t \leq T$. Here, we have written for convenience $\|\cdot\|_{\omega_t}$ in place of $\|\cdot\|_{\omega_t[\gamma]}$. For general k, we note that $v_k(t)$ is given by the following iterated integral:

$$v_k(t) = (-1)^k \int_{[t;s_k]}^{(m)} G^t_{s_k}(\lambda)\mathcal{S}(s_k) \int_{[s_k;s_{k-1}]}^{(m)} G^{s_k}_{s_{k-1}}(\lambda)\mathcal{S}(s_{k-1}) \cdots \int_{[s_2;s_1]}^{(m)} G^{s_2}_{s_1}(\lambda)\mathcal{S}(s_1) \int_{[s_1;s_0]}^{(m)} G^{s_1}_{s_0}(\lambda) f(s_0)\,. \tag{3.7}$$

The expression above can be expanded using Lemma 2, and thus obtain a finite sum whose number of terms is less than $(mJ)^k$, where J is the cardinality of the set $\{(j,\alpha);\ 0 \le j \le m-1 \text{ and } |\alpha| \le m-j\}$. Each term of the finite sum has the form

$$I = (-1)^k \int_{[t;s_k]}^{(m)} G^t_{s_k}(\lambda)\, c_{j_k,\alpha_k}(\mu D_z)^{\alpha_k} \int_{[s_k;s_{k-1}]}^{(i_k)} H^{s_k}_{s_{k-1}}(i_k,\lambda)\, c_{j_{k-1},\alpha_{k-1}}(\mu D_z)^{\alpha_{k-1}} \cdots \int_{[s_2;s_1]}^{(i_2)} H^{s_2}_{s_1}(i_2,\lambda)\, c_{j_1,\alpha_1}(\mu D_z)^{\alpha_1} \int_{[s_1;s_0]}^{(i_1)} H^{s_1}_{s_0}(i_1,\lambda)\, f(s_0)\,, \tag{3.8}$$

where for each p, the relations $m - j_p \le i_p \le m$ and $|\alpha_p| \le m - j_p$ hold. (Here, α_p is a multi-index and should not be confused with the pth component of α.) The above is further equal to

$$I = (-1)^k \int_{[t;s_k]}^{(m)} \int_{[s_k;s_{k-1}]}^{(i_k)} \cdots \int_{[s_1;s_0]}^{(i_1)} G^t_{s_k}\, c_{j_k,\alpha_k}(s_k)(\mu(s_k)D_z)^{\alpha_k} \times H^{s_k}_{s_{k-1}} c_{j_{k-1},\alpha_{k-1}}(s_{k-1})(\mu(s_{k-1})D_z)^{\alpha_{k-1}} \cdots \times H^{s_2}_{s_1} c_{j_1,\alpha_1}(s_1)(\mu(s_1)D_z)^{\alpha_1} H^{s_1}_{s_0}\, f(s_0)\,. \tag{3.9}$$

Let $F_k(s)$ denote the integrand of the above integral. Let $R_{s_0} = R - \gamma\varphi(s_0)$. Then all the functions above, when viewed as a function of z, belong in $\mathcal{A}(\omega_{s_0}[\gamma])$. (This explains the necessity of the assumption that the coefficients be defined up to B_R, for all t in the interval $[0, T]$.)

We can therefore apply Lemma 4 repeatedly, starting from the rightmost expression, to obtain the following estimate for the integrand: for any $\rho \in (0, R_{s_0})$,

$$\|F_k(s)\|_{B_\rho} \le K(CD)^k \mu(s_1)^{|\alpha_1|} \cdots \mu(s_k)^{|\alpha_k|} \left(\frac{s_0}{t}\right)^L \times \left(\frac{e}{R_{s_0}-\rho}\right)^{|\alpha_1+\cdots+\alpha_k|} |\alpha_1+\cdots+\alpha_k|!\,. \tag{3.10}$$

If $|\alpha_1+\cdots+\alpha_k| = 0$, then for sufficiently small $T = T_0$, the bound for any $c_{j,0}(t,z) = a_{j,0}(t,z) - a_{j,0}(0,z)$ is actually small, since $a_{j,0}(t,z)$ is continuous with respect to t. In other words, by choosing a small $T = T_0$, we could find a small constant δ such that for any $t \in [0, T_0]$ and $0 \le s \le t$, the following holds:

$$\|F_k(s)\|_{\omega_t} \le K\delta^k \left(\frac{s_0}{t}\right)^L. \tag{3.11}$$

Going back to the integral, we have

$$\|I\|_{\omega_t} \leq \int_{[t;s_k]}^{(m)} \int_{[s_k;s_{k-1}]}^{(i_k)} \cdots \int_{[s_1;s_0]}^{(i_1)} K\delta^k \Big(\frac{s_0}{t}\Big)^L \tag{3.12}$$
$$= K\,\frac{\delta^k}{L^{m+i_1+\cdots+i_k}} \qquad \text{(by (a) of Lemma 3)}$$
$$\leq K\Big(\frac{\delta}{L_0}\Big)^k,$$

for some constant L_0 dependent on L. This is possible since $i_p \leq m$ for all p.

If $|\alpha_1 + \cdots + \alpha_k| \neq 0$, then set the ρ in (3.10) to be equal to $R - \gamma\varphi(t)$. This gives

$$\|F_k(s)\|_{\omega_t} \leq K(CD)^k\, \mu(s_1)^{|\alpha_1|} \cdots \mu(s_k)^{|\alpha_k|} \Big(\frac{s_0}{t}\Big)^L \tag{3.13}$$
$$\times\ |\alpha_1 + \cdots + \alpha_k|!\,\Big(\frac{e}{\gamma[\varphi(t) - \varphi(s_0)]}\Big)^{|\alpha_1+\cdots+\alpha_k|}.$$

By renaming if necessary, assume that for $p = 1, \cdots, q$, $|\alpha_p| \neq 0$. Note that $q \geq 1$. We will again use the continuity of $a_{j,0}(t, z)$ to estimate those expressions which are not acted upon by D_z, i.e., the $k - q$ cases when $|\alpha_p| = 0$. Just like before, we can show that for small δ,

$$\|F_k(s)\|_{\omega_t} \leq K(CD)^q \delta^{k-q} \mu(s_1)^{|\alpha_1|} \cdots \mu(s_q)^{|\alpha_q|} \Big(\frac{s_0}{t}\Big)^L \tag{3.14}$$
$$\times\ |\alpha_1 + \cdots + \alpha_q|!\Big(\frac{e}{\gamma[\varphi(t) - \varphi(s_0)]}\Big)^{|\alpha_1+\cdots+\alpha_q|}.$$

Thus, the integral I can now be estimated as follows:

$$\|I\|_{\omega_t} \leq K(CD)^q \delta^{k-q} \Big(\frac{e}{\gamma}\Big)^{|\alpha_1+\cdots+\alpha_q|} |\alpha_1 + \cdots + \alpha_q|! \tag{3.15}$$
$$\times \int_{[t;s_k]}^{(m)} \int_{[s_k;s_{k-1}]}^{(i_k)} \cdots \int_{[s_1;s_0]}^{(i_1)} \Big(\frac{s_0}{t}\Big)^L \frac{\mu(s_1)^{|\alpha_1|} \cdots \mu(s_q)^{|\alpha_q|}}{[\varphi(t) - \varphi(s_0)]^{|\alpha_1+\cdots+\alpha_q|}}.$$

Let $d = m + i_1 + \cdots + i_k$ and $b = |\alpha_1 + \cdots + \alpha_q|$. Note that $b \geq q$. Since for each p, we have $|\alpha_p| \leq m - j_p \leq i_p$, and using the fact that both $\varphi(t)$ and $\mu(t)$ are increasing on $(0, T_0)$, we have

$$\|I\|_{\omega_t} \leq K(CD)^q\, \delta^{k-q} \Big(\frac{e}{\gamma}\Big)^b b! \tag{3.16}$$
$$\times \int_0^t \int_0^{\xi_b} \cdots \int_0^{\xi_1} \frac{\mu(\xi_b)}{\xi_b} \cdots \frac{\mu(\xi_1)}{\xi_1} \Big(\frac{\xi_0}{t}\Big)^L \frac{1}{[\varphi(t) - \varphi(\xi_0)]^b} \frac{d\xi_0}{\xi_0}\, d\xi_1 \cdots d\xi_b$$
$$\times \int_0^{\xi_0} \int_0^{\eta_1} \cdots \int_0^{\eta_{d-b-2}} \Big(\frac{s_0}{\xi_0}\Big)^L \frac{ds_0}{s_0} \cdots \frac{d\eta_1}{\eta_1}$$

By (a) of Lemma 3, the second integral is equal to L^{-d+b+1}. Thus, the above simplifies into

$$\|I\|_{\omega_t} \le K(CD)^q\delta^{k-q}\Big(\frac{e}{\gamma}\Big)^b L^{-d+b+1}b! \times \int_0^t\int_0^{\xi_b}\cdots\int_0^{\xi_1}\frac{\mu(\xi_b)}{\xi_b}\cdots\frac{\mu(\xi_1)}{\xi_1}\Big(\frac{\xi_0}{t}\Big)^L\frac{\xi_0^{-1}}{[\varphi(t)-\varphi(\xi_0)]^b}\,d\xi_0\cdots d\xi_b\,. \tag{3.17}$$

The last integral is equal to $(Lb!)^{-1}$, by (b) of Lemma 3. Meanwhile, since $d \le m(k+1)$, we can find a constant L_1, depending on L, such that $L^{-d} \le L_1^k$. Substituting these results into the above equation, we get

$$\begin{aligned}\|I\|_{\omega_t} &\le K(CD)^q\delta^{k-q}\Big(\frac{eL}{\gamma}\Big)^b L_1^k\\ &= K\Big(\frac{CD}{\delta}\Big)^q(\delta L_1)^k\Big(\frac{eL}{\gamma}\Big)^b\end{aligned} \tag{3.18}$$

By taking a sufficiently small T_0, we can find a constant δ small enough such that δL_1 above and δL_0^{-1} in (3.12) are both less than $(mJ)^{-1}$. Now, since $q \le b$, we can choose and fix a sufficiently large $\gamma = \gamma_0$ to make the remaining expression less than 1.

To summarize, we have shown that if T_0 is sufficiently small and γ_0 is sufficiently large, some constants $K > 0$ and $\delta_0 < 1$ exist such that for all k, we have

$$\|v_k(t)\|_{\omega_t[\gamma_0]} \le K\delta_0^k \qquad \text{for any} \quad t \in [0, T_0]\,. \tag{3.19}$$

It follows that the series $\sum_{k=0}^{\infty} v_k(t, z)$ is majorized by a convergent geometric series, and hence is itself convergent in $C^0([0, \tau], \mathcal{A}(\omega_\tau[\gamma_0]))$ for all $\tau \in [0, T_0]$. This means that $u_k(t)$ converges uniformly to $u(t)$ on $\Omega_{T_0}[\gamma_0]$.

By following the steps above, we can also show that for $1 \le p \le m-1$, the sequence $(tD_t)^p u_k(t)$ converges uniformly to $(tD_t)^p u(t)$ on $\Omega_{T_0}[\gamma_0]$. Thus, it follows that on a compact subset of $\Omega_{T_0}[\gamma_0]$, the sequence $D_z^\alpha(tD_t)^p u_k(t)$ converges to $D_z^\alpha(tD_t)^p u(t)$. This implies the convergence of the approximate solutions to the true solution $u(t)$.

Uniqueness may be proved in a similar manner.

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