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On Some Modules Attached to the Lubin-Tate Formal Groups

Kazuhito KOZUKA and Hirofumi TSUMURA

Miyakonojo National College of Technology and Tokyo Metropolitan College

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Introduction.

Let p be a prime number and denote by $\mathbf{Q}_p$, $\mathbf{Z}_p$ and $\mathbf{C}_p$, the p-adic rational number field, the ring of integers of $\mathbf{Q}_p$ and the completion of the algebraic closure of $\mathbf{Q}_p$, respectively. Let $\mathcal{O}$ denote the ring of integers of $\mathbf{C}_p$.

Let $F(X, Y)$ be a Lubin-Tate formal group over $\mathbf{Z}_p$ and $h(X)$ a meromorphic series in $\mathcal{O}((X))^\times$. In [12], Shiratani and Imada constructed a p-adic zeta function $\zeta_p(s, F, h)$, which explains many well-known p-adic interpolating functions in a unified manner. For example, if $F(X, Y)$ is the formal multiplicative group $\mathbf{G}_m(X, Y) = (X+1)(Y+1)-1$ and $h(X) = X$, then $\zeta_p(s, \mathbf{G}_m, X)$ is the ordinary p-adic zeta function. If F is the formal group associated with an elliptic curve over $\mathbf{Z}$ having complex multiplication with ordinary reduction, then $\zeta_p(s, F, X)$ coincides with the p-adic zeta function for the elliptic curve ([10]).

Let χ be a primitive Dirichlet character with conductor a power of p. In [7], under the slightly generalized situation that $F(X, Y)$ is a relative Lubin-Tate formal group defined over the ring of integers of an unramified extension of $\mathbf{Q}_p$, we constructed a meromorphic function $L_p(s, \chi, F, h)$, which is an extension of $\zeta_p(s, F, h)$. Especially, $L_p(s, \chi, \mathbf{G}_m, X)$ coincides with the Kubota-Leopoldt p-adic L-function $L_p(s, \chi)$.

As is well known, Iwasawa gave the fascinating result that $L_p(s, \chi)$ is closely related to the Galois structure of the local units modulo the closure of the cyclotomic units ([4], [9, Chapter 7], [15, Section 13.8]). This result was extended to abelian fields by Gillard, Tsuji and so on ([3], [13]). It is also well known that Coates and Wiles discovered the analogue of this result for the elliptic units ([1]).

The main purpose of this paper is to generalize the above result of Iwasawa for the function $L_p(s, \chi, F, h)$ under the situation that F is defined over $\mathbf{Z}_p$ and that $h(X)$ satisfies certain appropriate conditions (Theorems 4.2 and 4.3). For this purpose, we use the method of the logarithmic derivatives developed by Coates and Wiles [1]. Let us give a description of each section.

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In Section 1, we describe the general situation and some important notations in this paper.

In Section 2, we recall the construction of $L_p(s, \chi, F, h)$ in [7].

In Section 3, by the method due to Coates and Wiles [1], we summarize some results on the Galois structure of the local units of the field obtained by adjoining the division points of F to $\mathbf{Q}_p$. Because some complicated situations arise in certain special cases, the proofs will be postponed until the final section.

In Section 4, as a generalization of the cyclotomic or the elliptic units, we define special units by means of the division points of F and study how the functions $L_p(s, \chi, F, h)$ are related to those units. In consequence, we obtain the main results of this paper, which can be regarded as a generalization of the Iwasawa theory of local units.

In Section 5, We give two examples in the case $F = \mathbf{G}_m$; one is related to the generalized Euler numbers ([6],[14]) and the other to the higher-order Dedekind sums ([8]).

Section 6 is the final section, in which we give complete proofs of propositions in Section 3.

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1. Notations.

In the rest of this paper, we assume that p is an odd prime number. As usual, we denote by $\mathbf{Q}$, $\mathbf{Z}$ and $\mathbf{N}$, the rational number field, the ring of integers of $\mathbf{Q}$ and the set of positive integers, respectively, and put $\bar{\mathbf{N}} = \mathbf{N} \cup \{0\}$. Let $|\ |$ be the p-adic valuation of $\mathbf{C}_p$ normalized by $|p| = 1/p$, and for each $x \in \mathbf{C}_p^\times$, let $\mathrm{ord}_p(x)$ denote the value in $\mathbf{Q}$ satisfying $|x| = |p|^{\mathrm{ord}_p(x)}$. We put $\mathrm{ord}_p(0) = \infty$ and regard $\infty > a$ for any $a \in \mathbf{Q}$.

Let π be a prime element of $\mathbf{Z}_p$ and $f(T)$ a Frobenius power series determined by π, namely $f(T)$ is a power series in $\mathbf{Z}_p[[T]]$ satisfying

$$f(T) \equiv \pi T \pmod{\text{degree } 2} \quad \text{and} \quad f(T) \equiv T^p \pmod{\pi \mathbf{Z}_p}.$$

There exsists a unique formal group $F_f(X, Y) \in \mathbf{Z}_p[[X, Y]]$, called the Lubin-Tate formal group associated with f, such that $f(T)$ is the endomorphism $[\pi]_{F_f}(T)$ of F_f ([11]). In what follows, we fix π and $f(T)$, and instead of $F_f(X, Y)$, we also write $F(X, Y)$, F or $X +_F Y$. Let $\lambda_F(T)$ and $e_F(T)$ denote respectively the logarithmic series and the exponential series of $F(X, Y)$ such that $\lambda'_F(0) = 1$ and $(\lambda_F \circ e_F)(T) = (e_F \circ \lambda_F)(T) = T$. We denote the completion of the maximal unramified extension of $\mathbf{Q}_p$ by $\overline{\mathbf{Q}_p^{nr}}$, the ring of integers of $\overline{\mathbf{Q}_p^{nr}}$ by $\mathcal{I}$ and the Frobenius automorphism of $\overline{\mathbf{Q}_p^{nr}}$ over $\mathbf{Q}_p$ by ψ. Then, $p/\pi = \Omega^\psi/\Omega$ holds for some $\Omega \in \mathcal{I}^\times$. Further, there exists a unique isomorphism $\phi_F : \mathbf{G}_m \to F$ over $\mathcal{I}$ such that $\phi'_F(0) = \Omega^{-1}$. Throughout this paper, we take $\Omega = 1$ in the case $\pi = p$, so that we have $\phi_{\mathbf{G}_m}(T) = T$. Note that we have

$$[a]_F(\phi_F(T)) = \phi_F([a]_{\mathbf{G}_m}(T)) = \phi_F((1 + T)^a - 1) \quad \text{for any } a \in \mathbf{Z}_p, \tag{1.1}$$

$$\phi_F(e^Z - 1) = e_F(\Omega^{-1} Z), \tag{1.2}$$

$$(\lambda_F \circ \phi_F)(T) = \Omega^{-1} \log(1+T)\,. \tag{1.3}$$

For each $n \in \bar{\mathbf{N}}$, we put

$$\mathcal{T}_n = \{\tau \in \mathbf{C}_p \mid [\pi]_F^{n+1}(\tau) = 0\},\ K_n = \mathbf{Q}_p(\mathcal{T}_n)\,, \quad G_n = \mathrm{Gal}(K_n/\mathbf{Q}_p)\,.$$

We put further

$$K_\infty = \bigcup_{n=0}^{\infty} K_n\,, \quad G_\infty = \mathrm{Gal}(K_\infty/\mathbf{Q}_p)\,, \quad \Gamma = \mathrm{Gal}(K_\infty/K_0).$$

Then, there exist canonical isomorphisms

$$\kappa : G_\infty \xrightarrow{\sim} \mathbf{Z}_p^\times \quad \text{and} \quad \kappa\,|_\Gamma : \Gamma \xrightarrow{\sim} 1 + p\mathbf{Z}_p$$

such that

$$\sigma\alpha = [\kappa(\sigma)]_F(\alpha) \quad \text{for any } \sigma \in G_\infty \text{ and } \alpha \in \bigcup_{n=0}^{\infty} \mathcal{T}_n\,. \tag{1.4}$$

Let Δ denote the torsion subgroup of G_∞, which is a cyclic group of order $p-1$ and isomorphic to G_0. If A is a $\mathbf{Z}_p[\Delta]$-module, we put

$$A^{(i)} = \{a \in A \mid \delta a = \kappa^i(\delta)a \text{ for any } \delta \in \Delta\}$$

for each integer i modulo $p-1$. Here and in the rest of this paper, we always use the letter i to mean an integer modulo $p-1$. We have a canonical decomposition

$$A = \bigoplus_{i=0}^{p-2} A^{(i)}\,.$$

For any $a \in A$, putting

$$a^{(i)} = \frac{1}{p-1} \sum_{\delta\in\Delta} \kappa^{-i}(\delta)\delta a\,, \tag{1.5}$$

we have $a^{(i)} \in A^{(i)}$ and $a = \sum_{i=0}^{p-2} a^{(i)}$.

Put $\Lambda = \mathbf{Z}_p[[T]]$. Fix a topological generator γ of Γ and put $c_\gamma = \kappa(\gamma)$. Then, any compact $\mathbf{Z}_p$-module B on which Γ acts continuously admits a structure of Λ-module such that $(T+1)b = \gamma b$ for any $b \in B$.

For each $n \in \bar{\mathbf{N}}$, let U_n denote the group of units of K_n which are congruent to 1 modulo the maximal ideal and put

$$U_\infty = \operatorname{proj\,lim} U_n\,,$$

where the projective limit is taken relative to the norm maps on the K_n. The ring $\mathbf{Z}_p[G_\infty]$ operates naturally on U_∞, and so, U_∞ is a compact Γ-module and also a Λ-module.

Finally, we denote by ω the Teichmüller character for p and regard it also as a homomorphism from $\mathcal{O}^\times$ to the group of roots of unity such that, for each $x \in \mathcal{O}^\times$, we decompose $x = \omega(x)\langle x\rangle$ with $|\langle x\rangle - 1| < 1$ and $\omega(x)^k = 1$ for some integer k prime to p.

2. The function $L_p(s, \chi, F, H)$.

Let χ be a primitive Dirichlet character with conductor a power of p and let $H(T) \in \mathcal{O}((T))^\times$. In this section, we summarize the construction of the function $L_p(s, \chi, F, H)$ in [7].

In general, each $g(T)$ in $\mathcal{O}[[T]]$ defines a $\mathcal{O}$-valued measure μ_g on $\mathbf{Z}_p$ such that

$$g(T) = \int_{\mathbf{Z}_p} (1+T)^x \, d\mu_g(x) .$$

Let $\ell : \mathbf{Z}_p^\times \to \mathbf{Z}_p$ be the homomorphism defined by $\langle x \rangle = c_\gamma{}^{\ell(x)}$ for each $x \in \mathbf{Z}_p^\times$. For each $g(T) \in \mathcal{O}[[T]]$, we put

$$g^{(i)}(T) = \int_{\mathbf{Z}_p^\times} (1+T)^{\ell(x)} \omega^i(x) \, d\mu_g(x) \in \mathcal{O}[[T]] .$$

Then, we have

$$g^{(i)}(c_\gamma{}^s - 1) = \int_{\mathbf{Z}_p^\times} \langle x \rangle^s \omega^i(x) \, d\mu_g(x) . \tag{2.1}$$

More precisely, for any primitive Dirichlet character φ of the second kind for p, we have

$$g^{(i)}(\varphi(c_\gamma) c_\gamma{}^s - 1) = \int_{\mathbf{Z}_p^\times} \langle x \rangle^s \varphi\omega^i(x) \, d\mu_g(x) . \tag{2.2}$$

Let D denote the operator $(1+T)(d/dT)$. We put

$$g_H(T) = D\{\log(H \circ \phi_F)(T)\} \tag{2.3}$$
$$\left(= \frac{(1+T)(H \circ \phi_F)'(T)}{(H \circ \phi_F)(T)} = \frac{(H' \circ \phi_F)(T)}{\Omega(\lambda_F' \circ \phi_F)(T)(H \circ \phi_F)(T)} \right),$$

which is in $(1/T)\mathcal{O}[[T]]$. For each $c \in \mathbf{Z}_p^\times$, we put

$$g_H^c(T) = c g_H\big((1+T)^c - 1\big) - g_H(T) \, (\in \mathcal{O}[[T]]) . \tag{2.4}$$

If c is selected such that $\omega^i(c)(1+T)^{\ell(c)} - 1 \neq 0$, we set

$$\mathcal{G}_H^{(i)}(T) = (g_H^c)^{(i-1)}(c_\gamma^{-1}(1+T) - 1)/(\omega^i(c)(1+T)^{\ell(c)} - 1) , \tag{2.5}$$

which is independent of c ([7, Proposition 3.1]). Now, we decompose χ in the form

$$\chi = \varphi_\chi \omega^{i_\chi} ,$$

where φ_χ is a primitive Dirichlet character of the second kind for p and i_χ is an integer modulo $p-1$. Then, the function $L_p(s, \chi, F, H)$ is defined by

$$L_p(s, \chi, F, H) = -\mathcal{G}_H^{(i_\chi)}(\varphi_\chi(c_\gamma) c_\gamma^{1-s} - 1) . \tag{2.6}$$

In the case where $H(T)$ is in $\mathcal{O}[[T]]^\times$, we also have $g_H(T) \in \mathcal{O}[[T]]$ and

$$\mathcal{G}_H^{(i)}(T) = (g_H)^{(i-1)}(c_\gamma^{-1}(1+T) - 1) . \tag{2.7}$$

Hence, the function $L_p(s, \chi, F, H)$ is also expressed as

$$L_p(s, \chi, F, H) = -\int_{\mathbf{Z}_p^\times} \langle x\rangle^{-s} \chi\omega^{-1}(x)\, d\mu_{gH}(x)\,.$$

Let us describe the interpolation property of $L_p(s, \chi, F, H)$. Let p^{m_χ} be the conductor of χ. We choose a primitive p^{m_χ}-th root of unity ζ_χ arbitrarily, put $\tau(\chi^{-1}, \zeta_\chi) = \sum_{a=1}^{p^{m_\chi}} \chi^{-1}(a)\zeta_\chi^a$ and define the numbers $B_{n,\chi}(F, H)$ by

$$\frac{1}{\tau(\chi^{-1}, \zeta_\chi)} \sum_{a=1}^{p^{m_\chi}} \frac{\chi^{-1}(a) Z \dfrac{d}{dZ}\{H(\phi_F(\zeta_\chi^a - 1) +_F e_F(Z))\}}{H(\phi_F(\zeta_\chi^a - 1) +_F e_F(Z))} = \sum_{n=0}^{\infty} B_{n,\chi}(F, H)\frac{Z^n}{n!}\,.$$

This definition is independent of the choice of ζ_χ. For each $n \in \mathbf{Z}$, we put $\chi_n = \chi\omega^{-n}$. Let $\mathcal{N}_F$ denote Coleman's norm operator, so that we have

$$\mathcal{N}_F H(T) \in \mathcal{O}((T))^\times\,, \quad \mathcal{N}_F H([\pi]_F(T)) = \prod_{\tau \in \mathcal{T}_0} H(T +_F \tau)\,.$$

Then, for each $n \in \mathbf{N}$, we have

$$L_p(1-n, \chi, F, H) = -\Omega^{-n}\left\{\frac{B_{n,\chi_n}(F, H)}{n} - \frac{\chi_n(p)\pi^n}{p} \cdot \frac{B_{n,\chi_n}(F, \mathcal{N}_F H)}{n}\right\} \tag{2.8}$$

Let us give examples used later. We have $B_{n,\chi}(\mathbf{G}_m, T) = B_{n,\chi}(\mathbf{G}_m, \mathcal{N}_{\mathbf{G}_m} T)$ $= B_{n,\chi}$ (the n-th generalized Bernoulli number) and $L_p(s, \chi, \mathbf{G}_m, T) = L_p(s, \chi)$. On the other hand, for a linear polynomial $aT + b$ with $a \in \mathcal{O}$ and $a \in \mathcal{O}^\times$, put $u = 1 - (b/a)$ and define the generalized Euler numbers $H_\chi^n(u)$ by

$$\sum_{a=0}^{p^{m_\chi}-1} \frac{(1 - u^{p^{m_\chi}})\chi(a)e^{at}u^{p^{m_\chi}-a-1}}{e^{p^{m_\chi}t} - u^{p^{m_\chi}}} = \sum_{n=0}^{\infty} H_\chi^n(u)\frac{t^n}{n!} \tag{2.9}$$

([14]). Then, we have

$$\frac{u}{1-u^{p^{m_\chi}}} H_\chi^n(u) = \frac{B_{n+1,\chi}(\mathbf{G}_m, aT + b)}{n+1} \quad \text{for any } n \in \bar{\mathbf{N}}\,, \tag{2.10}$$

except for the case where $n = 0$ and $\chi = 1$. The function $-L_p(s, \chi, \mathbf{G}_m, aT + b)$ is equal to the function $l_p(u, s, \chi\omega^{-1})$ in [14], namely the p-adic interpolating function for the generalized Euler numbers.

3. The local units and Coleman power series.

Let $(\tau_n)_{n\geq 0}$ be a basis of the Tate module $\mathcal{T}_F = \operatorname{proj\,lim} \mathcal{T}_n$, where the projective limit is taken relative to the projection maps given by multiplication by powers of π. Then, for each $u = (u_n)_{n\geq 0} \in U_\infty$, there exists a unique power series $R_u(T) \in \Lambda^\times$ such that

$$u_n = R_u(\tau_n) \quad \text{for all } n \in \bar{\mathbf{N}} \tag{3.1}$$

([2], [5]). The power series $R_u(T)$ is called the Coleman power series attached to u. It also satisfies $R_u(0) \equiv 1 \pmod{p\mathbf{Z}_p}$ and the functional equation

$$R_u\big([\pi]_F(T)\big) = \prod_{\tau \in \mathcal{T}_0} R_u(T +_F \tau), \quad \text{namely} \quad \mathcal{N}_F R_u(T) = R_u(T). \tag{3.2}$$

The map $u \mapsto R_u(T)$ gives a bicontinuous isomorphism from U_∞ to the multiplicative group $\{R(T) \in \Lambda^\times \mid R(0) \equiv 1 \pmod{p\mathbf{Z}_p}, \mathcal{N}_F R(T) = R(T)\}$.

Let $u \in U_\infty$. In connection with (3.3) and (3.7), we put

$$g_u(T) = g_{R_u(T)}(T) = D\{\log(R_u \circ \phi_F)(T)\} \quad (\in \mathcal{I}[[T]]), \tag{3.3}$$

$$\mathcal{G}_u^{(i)}(T) = \mathcal{G}_{R_u}^{(i)}(T) = g_u^{(i-1)}(c_\gamma^{-1}(1+T) - 1). \tag{3.4}$$

As described in the Introduction, the method of using the logarithmic derivative $g_u(T)$ of the Coleman power series is due to [1]. It is useful in studying the structure of $U_\infty^{(i)}$. In this section, we summarize some main results on the structure of $U_\infty^{(i)}$, mainly in terms of the map $u \mapsto \mathcal{G}_u^{(i)}(T)$. The proofs can be given in almost the same way as in the cyclotomic case or in the elliptic case ([1], [9], [15]). However, some complicated situations arise in the case of $i \equiv 0, 1 \pmod{p-1}$, and they are mostly omitted in the above references. So, in this paper, we give complete proofs including the case of $i \equiv 0, 1 \pmod{p-1}$ in the final section.

We introduce a Δ-action on $\mathcal{I}[[T]]$ by defining $(\delta f)(T) = \kappa^i(\delta) f(T)$ for any $\delta \in \Delta$ and $f(T) \in \mathcal{I}[[T]]$, and denote by $\mathcal{I}[[T]]_{(i)}$ the $\Lambda[\Delta]$-module $\mathcal{I}[[T]]$ equipped with such a Δ-action.

For each $n \in \mathbf{N}$, we denote by $\boldsymbol{u}_n$ the group of n-th roots of unity. Note that if $\pi = p$, we have $K_n = \mathbf{Q}_p(\boldsymbol{u}_{p^{n+1}})$ for all $n \in \bar{\mathbf{N}}$ and $U_\infty^{(1)} \supset \operatorname{proj\,lim} \boldsymbol{u}_{p^{n+1}}$, where the projective limit is taken with respect to the p-th power map.

PROPOSITION 3.1. *The map* $u \mapsto \mathcal{G}_u^{(i)}(T)$ *gives a continuous* $\Lambda[\Delta]$-*homomorphism from* U_∞ *to* $\mathcal{I}[[T]]_{(i)}$. *It is the* 0-*map on* $U_\infty^{(j)}$ *for any* $j \not\equiv i \pmod{p-1}$, *and except for the case where* $\pi = p$ *and* $i \equiv 1 \pmod{p-1}$, *it is an injection on* $U_\infty^{(i)}$. *In the case* $\pi = p$, *the kernel of the map* $u \mapsto \mathcal{G}_u^{(1)}(T)$ *for* $u \in U_\infty^{(1)}$ *coincides with* $\operatorname{proj\,lim} \boldsymbol{u}_{p^{n+1}}$.

For each $n \in \bar{\mathbf{N}}$, let V_n denote the image of the projection map from U_∞ to U_n. Denote by N_n the norm map from K_n to $\mathbf{Q}_p$. Then, by local class field theory, we have $V_n = \{u \in U_n \mid N_n(u) = 1\}$. We put $w_n(T) = (1+T)^{p^n} - 1$ and $\nu_\pi = \operatorname{ord}_p((\pi/p) - 1)$.

PROPOSITION 3.2. (1) *If* $i \not\equiv 0 \pmod{p-1}$, *we have*

$$V_n^{(i)} = U_n^{(i)} \cong U_\infty^{(i)}/w_n(T)U_\infty^{(i)} \quad \text{for any } n \in \bar{\mathbf{N}}.$$

(2) *We have* $V_n^{(0)} \cong U_\infty^{(0)} \Big/ \dfrac{w_n(T)}{T} U_\infty^{(0)}$ for any $n \in \bar{\mathbf{N}}$.

PROPOSITION 3.3. *We set* $\mathcal{K}^{(i)} = \{\mathcal{G}_u^{(i)}(T) \mid u \in U_\infty^{(i)}\}$. *If* $i \not\equiv 1 \pmod{p-1}$, *or if* $\nu_\pi = 0$ *or* $\nu_\pi = \infty$, *we have*

$$\mathcal{K}^{(i)} = \mathcal{G}_{v^{(i)}}^{(i)}(T)\Lambda \quad \text{for some } v \in U_\infty \text{ independent of } i.$$

Further, if $i \not\equiv 1 \pmod{p-1}$ *or* $\nu_\pi = 0$, *then* $\mathcal{G}^{(i)}_{v^{(i)}}(T) \in \mathcal{I}[[T]]^\times$. *If* $\nu_\pi = \infty$, *then* $\mathcal{G}^{(1)}_{v^{(1)}}(T)/(T+1-c_\gamma) \in \mathcal{I}[[T]]^\times$. *In the remaining case of* $i \equiv 1 \pmod{p-1}$ *and* $0 < \nu_\pi < \infty$, *we have*

$$\mathcal{K}^{(1)} = \{(T+1-c_\gamma)\Lambda + p^{\nu_\pi}\Lambda\}P(T) \tag{3.5}$$

for some $P(T) \in \mathcal{I}[[T]]^\times$.

In what follows, we fix an element $v \in U_\infty$ which is independent of i and satisfies the conditions in Proposition 3.3. If $0 < \nu_\pi < \infty$, we also fix a power series $P(T) \in \mathcal{I}[[T]]^\times$ satisfying (3.5). Then, we can define a Λ-homomorphism

$$W^{(i)} : U^{(i)}_\infty \to \Lambda$$

by

$$W^{(i)}(u) = \mathcal{G}^{(i)}_u(T)/\mathcal{G}^{(i)}_{v^{(i)}}(T) \quad \text{if } i \not\equiv 1 \pmod{p-1} \text{ or } \nu_\pi = 0 \text{ or } \nu_\pi = \infty,$$

and by

$$W^{(1)}(u) = \mathcal{G}^{(1)}_u(T)/P(T) \quad \text{if } i \equiv 1 \pmod{p-1} \text{ and } 0 < \nu_\pi < \infty.$$

As will be seen in the proofs of above propositions in Section 6, we have $U^{(i)}_\infty \cong \Lambda$ in most cases. However, by combining Propositions 3.1 and 3.3, we also see that the map $W^{(i)}$ gives a concrete description of the structure of $U^{(i)}_\infty$ as in the following

THEOREM 3.4. *The map* $W^{(i)}$ *gives a* Λ*-isomorphism*

$$W^{(i)} : U^{(i)}_\infty \xrightarrow{\sim} \Lambda$$

except for the case where $i \equiv 1 \pmod{p-1}$ *and* $0 < \nu_\pi \leq \infty$. *If* $\nu_\pi = \infty$, $W^{(1)}$ *is surjective with* $\operatorname{Ker} W^{(1)} = \operatorname{proj\,lim} \boldsymbol{u}_{p^{n+1}}$. *If* $0 < \nu_\pi < \infty$, $W^{(1)}$ *is injective with* $\operatorname{Im} W^{(1)} = (T+1-c_\gamma)\Lambda + p^{\nu_\pi}\Lambda$.

4. Λ-modules related to the function $L_p(s, \chi, F, H)$.

In this section, we suppose that $H(T)$ is in $\mathbf{Z}_p((T))^\times$ and satisfies

$$\mathcal{N}_F H(T) = c_0 H(T) \quad \text{for some } c_0 \in \mathbf{Z}_p. \tag{4.1}$$

For each $c \in \mathbf{Z}_p^\times$ and $n \in \bar{\mathbf{N}}$, we put $v_{H,n}(c) = \langle H([c]_F(\tau_n))/H(\tau_n)\rangle$ and $v_H(c) = (v_{H,n}(c))_{n\geq 0}$. By virtue of (4.1), we have $v_H(c) \in U_\infty$. If $H(T) = \sum_{n\geq n_0} a_n T^n$ with $a_{n_0} \neq 0$, the Coleman power series attached to $v_H(c)$ is

$$R_{v_H(c)}(T) = \omega^{-n_0}(c) H([c]_F(T))/H(T).$$

Hence, by (1.1), (2.3), (2.4) and (3.3), we deduce

$$g_{v_H(c)}(T) = D\{\log(H \circ \phi_F)((1+T)^c - 1) - \log(H \circ \phi_F)(T)\} = g^c_H(T). \tag{4.2}$$

Let $C_{H,n}$ denote the subgroup of U_n generated by $\{v_{H,n}(c) \mid c \in \mathbf{Z}_p^\times\}$, and $\bar{C}_{H,n}$ the closure of $C_{H,n}$ in U_n. Then, $\bar{C}_{H,n}$ is also the Λ-submodule of V_n generated by $\{v_{H,n}(c) \mid c \in \mathbf{Z}_p^\times\}$. We define $\bar{C}_{H,\infty} = \operatorname{proj\,lim} \bar{C}_{H,n}$. Note that it coincides with the Λ-submodule of U_∞ generated by $\{v_H(c) \mid c \in \mathbf{Z}_p^\times\}$. If c_2 is a primitive root modulo p^2, then $\bar{C}_{H,n}$ (resp. $\bar{C}_{H,\infty}$) also coincides with the Λ-module generated by $v_{H,n}(c_2)$ (resp. $v_H(c_2)$), namely $\bar{C}_{H,n} = \Lambda v_{H,n}(c_2)$ (resp. $\bar{C}_{H,\infty} = \Lambda v_H(c_2)$). In addition, we define $Y_{H,n} = V_n/\bar{C}_{H,n}$ or $V_n/\bar{C}_{H,n}\boldsymbol{u}_{p^{n+1}}$ according as $\pi \neq p$ or $\pi = p$, and $Y_{H,\infty} = \text{proj lim}\, Y_{H,n}$, where the projective limit is taken relative to the norm maps. Then, we have

$$Y_{H,\infty}^{(i)} \cong U_\infty^{(i)}/\bar{C}_{H,\infty}^{(i)}$$

except for the case where $i \equiv 1 \pmod{p-1}$ and $\pi = p$, and in this case we have

$$Y_{H,\infty}^{(1)} \cong U_\infty^{(1)}/(\bar{C}_{H,\infty}^{(1)} \cdot \text{proj limit}\, \boldsymbol{u}_{p^{n+1}})\,.$$

Similarly, for each $u = (u_n)_{n\geq 0} \in U_\infty$ and $n \in \bar{\mathbf{N}}$, let $\bar{D}_{u,n}$ be the Λ- submodule of U_n generated by u_n and define $\bar{D}_{u,\infty} = \text{proj lim}\, \bar{D}_{u,n}$, $Z_{u,n} = V_n/\bar{D}_{u,n}$ or $V_n/\bar{D}_{u,n}\boldsymbol{u}_{p^{n+1}}$ according as $\pi \neq p$ or $\pi = p$ and $Z_{u,\infty} = \text{proj lim}\, Z_{u,n}$.

By the definition (2.6), we have

$$\mathcal{G}_H^{(i)}(\varphi(c_\gamma)c_\gamma^{1-s} - 1) = -L_p(s, \varphi\omega^i, F, H) \tag{4.3}$$

for any primitive Dirichlet character φ of the second kind for p. Further, since $\phi_F(T)$ is in $\mathcal{I}[[T]]$ and $H(T)$ is assumed to be in $\mathbf{Z}_p((T))^\times$, we see from (2.5) that $\mathcal{G}_{F,H}^{(i)}(T)$ is in $\mathcal{I}[[T]]$ or in $(1/T)\mathcal{I}[[T]]$ according as $i \not\equiv 0 \pmod{p-1}$ or $i \equiv 0 \pmod{p-1}$.

To study the Λ-structures of $Y_{H,\infty}$, $Y_{H,n}$, $Z_{u,\infty}$ and $Z_{u,n}$ is the main purpose of this paper. For this, we use the map $W^{(i)}$ defined in the previous section. In order to describe the image of $\bar{C}_{H,\infty}^{(i)}$ by $W^{(i)}$, we put

$$\tilde{\mathcal{G}}_H^{(i)}(T) = \begin{cases} \mathcal{G}_H^{(0)}(T)T/\mathcal{G}_{v^{(0)}}^{(0)}(T) & \text{if } i \equiv 0 \pmod{p-1} \\ \mathcal{G}_H^{(1)}(T)/P(T) & \text{if } i \equiv 1 \pmod{p-1} \text{ and } 0 < v_\pi < \infty \\ \mathcal{G}_H^{(i)}(T)/\mathcal{G}_{v^{(i)}}^{(i)}(T) & \text{otherwise}. \end{cases}$$

Then, in particular we have

$$W^{(i)}(u) = \begin{cases} \tilde{\mathcal{G}}_{R_u}^{(i)}(T) & \text{if } i \not\equiv 0 \pmod{p-1} \\ \tilde{\mathcal{G}}_{R_u}^{(0)}(T)/T & \text{if } i \equiv 0 \pmod{p-1} \end{cases} \tag{4.4}$$

for any $u \in U_\infty^{(i)}$.

PROPOSITION 4.1. *We have $\tilde{\mathcal{G}}_H^{(i)}(T) \in \Lambda$ and $W^{(i)}(\bar{C}_{H,\infty}^{(i)}) = \tilde{\mathcal{G}}_H^{(i)}(T)\Lambda$.*

PROOF. Let $c_2 \in \mathbf{Z}_p$ be a primitive root modulo p^2, so that we have $\bar{C}_{H,\infty} = \Lambda v_H(c_2)$ and $\bar{C}_{H,\infty}^{(i)} = \Lambda v_H(c_2)^{(i)}$. It follows that $W^{(i)}(\bar{C}_{H,\infty}^{(i)}) = \Lambda W^{(i)}(v_H(c_2)^{(i)})$.

Now, we have

$$\begin{aligned} \mathcal{G}_{v_H(c_2)^{(i)}}^{(i)}(T) &= g_{v_H(c_2)}^{(i-1)}(c_\gamma^{-1}(1+T) - 1)(\text{ by (3.4) and Proposition 3.1}) \\ &= (g_H^{c_2})^{(i-1)}(c_\gamma^{-1}(1+T) - 1)\ (\text{ by (4.2)}) \\ &= (\omega^i(c_2)(1+T)^{\ell(c_2)} - 1)\mathcal{G}_H^{(i)}(T)(\text{by (2.5)})\,. \end{aligned}$$

Note that we have

$$(\omega^i(c_2)(1+T)^{\ell(c_2)}-1)/T \in \Lambda^\times \quad \text{or} \quad \omega^i(c_2)(1+T)^{\ell(c_2)}-1 \in \Lambda^\times$$

according as $i \equiv 0 \pmod{p-1}$ or $i \not\equiv 0 \pmod{p-1}$. Hence, by Proposition 3.3 and the definitions of $W^{(i)}$ and $\tilde{\mathcal{G}}_H^{(i)}(T)$, we see that $\tilde{\mathcal{G}}_H^{(i)}(T) \in \Lambda$ and that $W^{(i)}(\bar{C}_{H,\infty}^{(i)}) = W^{(i)}(v_H(c_2)^{(i)})\Lambda = \tilde{\mathcal{G}}_H^{(i)}(T)\Lambda$. This completes the proof.

We note that by the definition of $\tilde{\mathcal{G}}_H^{(i)}(T)$ and Proposition 3.3, we have

$$\tilde{\mathcal{G}}_H^{(i)}(T)\mathcal{I}[[T]] = \begin{cases} \mathcal{G}_H^{(0)}(T)T\mathcal{I}[[T]] & \text{if } i \equiv 0 \pmod{p-1} \\ \dfrac{\mathcal{G}_H^{(1)}(T)}{T+1-c_\gamma}\mathcal{I}[[T]] & \text{if } i \equiv 1 \pmod{p-1} \text{ and } \pi = p \\ \mathcal{G}_H^{(i)}(T)\mathcal{I}[[T]] & \text{otherwise}. \end{cases} \tag{4.5}$$

In the case of $0 < \nu_\pi < \infty$, we put $\mathcal{H}_\pi = (T+1-c_\gamma)\Lambda + p^{\nu_\pi}\Lambda$.

By Theorem 3.4 and Proposition 4.1, we obtain the following main theorem, which is a generalization of Theorem 5.2 of Chapter 7 of [9] (or equivalently [15, Theorem 13.56.1]) and also of Theorem 1 of [1] (namely, the Iwasawa theory of local units in the cyclotomic or elliptic extensions of $\mathbf{Q}_p$).

THEOREM 4.2. *Except for the case where* $i \equiv 1 \pmod{p-1}$ *and* $0 < \nu_\pi < \infty$, *we have*

$$Y_{H,\infty}^{(i)} \cong \Lambda/\tilde{\mathcal{G}}_H^{(i)}(T)\Lambda.$$

If $0 < \nu_\pi < \infty$, *we have*

$$Y_{H,\infty}^{(1)} \cong \mathcal{H}_\pi/\tilde{\mathcal{G}}_H^{(1)}(T)\Lambda.$$

On account of (4.4), we also obtain the following

COROLLARY 4.2.1. *Let* $u \in U_\infty^{(i)}$. *If* $i \not\equiv 0 \pmod{p-1}$, *we have*

$$Z_{u,\infty}^{(i)} = Y_{R_u,\infty}^{(i)}.$$

If $i \equiv 0 \pmod{p-1}$, *we have*

$$\tilde{\mathcal{G}}_{R_u}^{(0)}(T) \in T\Lambda \quad \text{and} \quad Z_{u,\infty}^{(0)} \cong \Lambda \Big/ \frac{\tilde{\mathcal{G}}_{R_u}^{(0)}(T)}{T}\Lambda.$$

Applying Proposition 3.2, we also see the structures of $Y_{H,n}^{(i)}$ and $Z_{u,n}^{(i)}$ as follows:

THEOREM 4.3. *Let* $n \in \mathbf{N}$.

(1) *If* $i \not\equiv 0 \pmod{p-1}$, *and if* $i \not\equiv 1 \pmod{p-1}$ *or* $\nu_\pi = 0$ *or* $\nu_\pi = \infty$, *we have*

$$Y_{H,n}^{(i)} \cong \Lambda/(\tilde{\mathcal{G}}_H^{(i)}(T)\Lambda + w_n(T)\Lambda).$$

(2) *If* $0 < \nu_\pi < \infty$, *we have*

$$Y_{H,n}^{(1)} \cong \mathcal{H}_\pi/(\tilde{\mathcal{G}}_H^{(1)}(T)\Lambda + w_n(T)\mathcal{H}_\pi).$$

(3) *We have*

$$Y_{H,n}^{(0)} \cong \Lambda \Big/ \left(\tilde{\mathcal{G}}_H^{(0)}(T)\Lambda + \frac{w_n(T)}{T}\Lambda \right).$$

COROLLARY 4.3.1. *Let $u \in U_\infty^{(i)}$ and $n \in \bar{\mathbf{N}}$. If $i \not\equiv 0 \pmod{p-1}$, then*

$$Z_{u,n}^{(i)} = Y_{R_u,n}^{(i)}.$$

If $i \equiv 0 \pmod{p-1}$, then we have

$$Z_{u,n}^{(0)} \cong \Lambda \Big/ \frac{1}{T}(\tilde{\mathcal{G}}_{R_u}^{(0)}(T)\Lambda + w_n(T)\Lambda).$$

For each $n \in \bar{\mathbf{N}}$, let Φ_n denote the group of primitive Dirichlet characters of the second kind for p of which the conductors divide p^{n+1}. Concerning the orders of the torsion subgroups of $Y_{H,n}^{(i)}$ and $Z_{u,n}^{(i)}$, we have the following

THEOREM 4.4. (1) *If $i \not\equiv 0 \pmod{p-1}$, and if $i \not\equiv 1 \pmod{p-1}$ or $\nu_\pi = 0$, then we have*

$$\mathrm{ord}_p\{\sharp(Y_{H,n}^{(i)})_{tors}\} = \mathrm{ord}_p\left\{ {\prod_{\varphi\in\Phi_n}}' L_p(1, \varphi\omega^i, F, H) \right\}.$$

Here and throughout this theorem, we use the symbol $\prod'$ to mean the product with respect to the non-vanishing values.

(2) *We have*

$$\mathrm{ord}_p\{\sharp(Y_{H,n}^{(0)})_{tors}\} = \mathrm{ord}_p\left\{ {\prod_{\varphi\in\Phi_n}}' (\varphi(c_\gamma) - 1) L_p(1, \varphi, F, H) \right\}.$$

(3) *If $\pi = p$, then we have*

$$\mathrm{ord}_p\{\sharp(Y_{H,n}^{(1)})_{tors}\} = \mathrm{ord}_p\left\{ {\prod_{\varphi\in\Phi_n}}' L_p(1, \varphi\omega, F, H) / (\varphi(c_\gamma) - c_\gamma) \right\}.$$

(4) *If $0 < \nu_\pi < \infty$, put*

$$d_n(T) = \prod_{\substack{\varphi\in\Phi_n \\ L_p(1,\varphi\omega,F,H)\neq 0}} (T + 1 - \varphi(c_\gamma)),$$

$\delta_n = \mathrm{ord}_p(d_n(c_\gamma - 1))$ and $\eta_n = \mathrm{ord}_p(\tilde{\mathcal{G}}_H^{(1)}(c_\gamma - 1)) - \delta_n$. Then,

$$\mathrm{ord}_p\{\sharp(Y_{H,n}^{(1)})_{tors}\} = \mathrm{ord}_p\left\{ {\prod_{\varphi\in\Phi_n}}' L_p(1, \varphi\omega, F, H) \right\} + \rho_n,$$

where

$$\rho_n = \begin{cases} \mathrm{ord}_p(\tilde{\mathcal{G}}_H^{(1)}(c_\gamma - 1)) - \nu_\pi & \text{if } \nu_\pi \geq \delta_n,\ \eta_n \\ \delta_n & \text{if } \delta_n \leq \nu_\pi < \eta_n \\ \eta_n & \text{if } \eta_n \leq \nu_\pi < \delta_n \\ \nu_\pi & \text{if } \nu_\pi < \delta_n,\ \eta_n\,. \end{cases}$$

(5) *For any* $u \in U_\infty^{(i)}$, *we have*

$$\sharp(Z_{u,n}^{(i)})_{tors} = \sharp(Y_{R_u,n}^{(i)})_{tors} \quad or \quad \sharp(Z_{u,n}^{(i)})_{tors} = \mathrm{ord}_p\Bigg\{\prod_{\varphi\in\Phi_n-\{1\}}{}' L_p(1,\varphi,F,R_u)\Bigg\}$$

according as $i \not\equiv 0 \pmod{p-1}$ *or* $i \equiv 0 \pmod{p-1}$.

PROOF. In general, for any $g(T) \in \Lambda$, we have

$$\mathrm{ord}_p\{\sharp(\Lambda/(g(T)\Lambda + w_n(T)\Lambda))_{tors}\} = \mathrm{ord}_p\Bigg(\prod_{\zeta\in \boldsymbol{u}_{p^n}}{}' g(\zeta - 1)\Bigg),$$

$$\mathrm{ord}_p\{\sharp(\Lambda/(g(T)\Lambda + (w_n(T)/T)\Lambda))\} = \mathrm{ord}_p\Bigg(\prod_{\zeta\in \boldsymbol{u}_{p^n}-\{1\}}{}' g(\zeta - 1)\Bigg).$$

For each $\varphi \in \Phi_n$, we have $\varphi(c_\gamma) \in \boldsymbol{u}_{p^n}$. The map $\varphi \mapsto \varphi(c_\gamma)$ gives an isomorphism from Φ_n to $\boldsymbol{u}_{p^n}$. By (4.3), we have

$$\mathrm{ord}_p(\mathcal{G}_H^{(i)}(\varphi(c_\gamma) - 1)) = \mathrm{ord}_p(L_p(1,\varphi\omega^i,F,H)).$$

Hence the statements (1), (2) and (3) follow immediately from (4.5) and Theorem 4.3.

Suppose that $0 < \nu_\pi < \infty$. Note that $w_n(T) = \prod_{\varphi\in\Phi_n}(T + 1 - \varphi(c_\gamma))$ and that for any $\varphi \in \Phi_n$, $\mathcal{G}_H^{(1)}(\varphi(c_\gamma) - 1) = 0$ holds if and only if $\varphi \notin \Phi_n^{(1)}(H)$. Hence, $d_n(T)$ is the greatest common divisor of $\mathcal{G}_H^{(1)}(T)$ and $w_n(T)$. It follows from (4.5) and Theorem 4.3 that the torsion subgroup of $Y_{H,n}^{(1)}$ is isomorphic to the group $(\mathcal{H}_\pi \cap d_n(T)\Lambda)/(\tilde{\mathcal{G}}_H^{(1)}(T)\Lambda + w_n(T)\mathcal{H}_\pi)$. Put

$$\mathcal{A}_1 = d_n(T)\Lambda/(\tilde{\mathcal{G}}_H^{(1)}(T)\Lambda + w_n(T)\Lambda)\,,$$
$$\mathcal{A}_2 = (\tilde{\mathcal{G}}_H^{(1)}(T)\Lambda + w_n(T)\Lambda)/(\tilde{\mathcal{G}}_H^{(1)}(T)\Lambda + w_n(T)\mathcal{H}_\pi)\,,$$
$$\mathcal{A}_3 = d_n(T)\Lambda/\mathcal{H}_\pi \cap d_n(T)\Lambda\,.$$

Then, we have the following canonical diagram in which the horizontal and vertical sequences are exact.

$$\begin{array}{c} 0 \\ \downarrow \\ (\mathcal{H}_\pi \cap d_n(T)\Lambda) \,/\, (\tilde{\mathcal{G}}_H^{(1)}(T)\Lambda + w_n(T)\mathcal{H}_\pi) \\ \downarrow \\ 0 \to \mathcal{A}_2 \to d_n(T)\Lambda \,/\, (\tilde{\mathcal{G}}_H^{(1)}(T)\Lambda + w_n(T)\mathcal{H}_\pi) \to \mathcal{A}_1 \to 0 \\ \downarrow \\ \mathcal{A}_3 \\ \downarrow \\ 0 \end{array}$$

It follows that

$$\mathrm{ord}_p\{\sharp(Y_{H,n}^{(i)})_{tors}\} = \sum_{j=1}^{3} \mathrm{ord}_p(\mathcal{A}_j)\,. \tag{4.6}$$

Since $\mathcal{A}_1$ is the torsion subgroup of $\Lambda/(\tilde{\mathcal{G}}_H^{(1)}(T)\Lambda + w_n(T)\Lambda)$, in the same way that we have shown the statement (1), we obtain

$$\mathrm{ord}_p(\mathcal{A}_1) = \mathrm{ord}_p\biggl\{\prod_{\varphi\in\Phi_n^{(1)}} \tilde{\mathcal{G}}_H^{(1)}(\varphi(c_\gamma)-1)\biggr\} = \mathrm{ord}_p\biggl\{\prod_{\varphi\in\Phi_n^{(1)}} L_p(1, \varphi\omega, F, H)\biggr\}.$$

As for $\mathcal{A}_2$, we have

$$\begin{aligned} \mathcal{A}_2 &\cong w_n(T)\Lambda/\{w_n(T)\Lambda \cap (\tilde{\mathcal{G}}_H^{(1)}(T)\Lambda + w_n(T)\mathcal{H}_\pi)\} \\ &\cong w_n(T)\Lambda\Big/ w_n(T)\biggl(\frac{\tilde{\mathcal{G}}_H^{(1)}(T)}{d_n(T)}\Lambda + \mathcal{H}_\pi\biggr). \end{aligned}$$

The map $A(T) \mapsto A(c_\gamma - 1)$ of Λ onto $\mathbf{Z}_p$ induces a $\mathbf{Z}_p$-isomorphism

$$\Lambda\Big/\biggl(\frac{\tilde{\mathcal{G}}_H^{(1)}(T)}{d_n(T)}\Lambda + \mathcal{H}_\pi\biggr) \cong \mathbf{Z}_p\Big/\biggl(\frac{\tilde{\mathcal{G}}_H^{(1)}(c_\gamma - 1)}{d_n(c_\gamma - 1)}\mathbf{Z}_p + p^{\nu_\pi}\mathbf{Z}_p\biggr).$$

Thus, we obtain $\mathrm{ord}_p(\mathcal{A}_2) = \min\{\eta_n, \nu_\pi\}$. Similarly, we have

$$\mathcal{A}_3 \cong (d_n(T)\Lambda + \mathcal{H}_\pi)/\mathcal{H}_\pi \cong (d_n(c_\gamma - 1)\mathbf{Z}_p + p^{\nu_\pi}\mathbf{Z}_p)/p^{\nu_\pi}\mathbf{Z}_p\,,$$

and obtain $\mathrm{ord}_p(\mathcal{A}_3) = \max\{\nu_\pi - \delta_n, 0\}$. Therefore, the statement (4) follows from (4.6).

The statement (5) follows immediately from Corollary 4.3.1.

5. Examples.

In this section, as stated in the Introduction, we give two examples in the case $F = \mathbf{G}_m$.

(I) Let $\lambda \in \boldsymbol{u}_{p-1} - \{1\}$. Then, as described at the end of Section 3, we have

$$l_p(\lambda, s, \chi\omega^{-1}) = -L_p(s, \chi, \mathbf{G}_m, T + 1 - \lambda)\,.$$

We also have

$$\mathcal{N}_{\mathbf{G}_m}(T+1-\lambda) = T+1-\lambda\,. \tag{5.1}$$

A basis $(\tau_n)_{n\geq 0}$ of the Tate module $\mathcal{T}_{\mathbf{G}_m}$ is written as $(\tau_n)_{n\geq 0} = (\zeta_{p^{n+1}} - 1)_{n\geq 0}$, where for each $n \in \bar{\mathbf{N}}$, $\zeta_{p^{n+1}}$ is a primitive p^{n+1}-th root of unity and satisfies $(\zeta_{p^{n+2}})^p = \zeta_{p^{n+1}}$. We put

$$u(\lambda) = (\langle\zeta_{p^{n+1}} - \lambda\rangle)_{n\geq 0} = (\omega^{-1}(1-\lambda)(\zeta_{p^{n+1}} - \lambda))_{n\geq 0}\,,$$

which is in U_∞ by virtue of (5.1). Then,

$$R_{u(\lambda)}(T) = \omega^{-1}(1-\lambda)(T+1-\lambda)\,,$$

$$\mathcal{G}^{(i)}_{u(\lambda)}(c_\gamma^{1-s} - 1) = l_p(\lambda, s, \omega^{i-1})\,.$$

We also have $\mathcal{G}^{(i)}_{u(\lambda)}(T) \in \Lambda$, and by (4.5) and Corollary 4.2.1, we obtain

$$Z^{(i)}_{u(\lambda),\infty} = \begin{cases} \Lambda/\mathcal{G}^{(i)}_{u(\lambda)}(T)\Lambda & \text{if } i \not\equiv 1 \pmod{p-1} \\ \Lambda\Big/\dfrac{\mathcal{G}^{(1)}_{u(\lambda)}(T)}{T+1-c_\gamma}\Lambda & \text{if } i \equiv 1 \pmod{p-1}\,. \end{cases}$$

By slightly modifying the power series $\mathcal{G}^{(i)}_{u(\lambda)}(T)$ for $\lambda \in \boldsymbol{u}_{p-1} - \{1\}$, we can relate them to the p-adic L-function $L_p(s, \omega^i)$ as in the following

PROPOSITION 5.1. *Let $c \in \mathbf{N} - \{1\}$ with $c \mid p-1$ and suppose that $\omega^i(c) \neq 1$ or $\langle c\rangle \not\equiv 1 \pmod{p^2}$ according as $i \not\equiv 0 \pmod{p-1}$ or $i \equiv 0 \pmod{p-1}$. Then, there exists a set $\{\tilde{A}^{(i)}(T,\lambda) \in \Lambda \mid \lambda \in \boldsymbol{u}_c - \{1\}\}$ such that*

$$\tilde{A}^{(i)}(T,\lambda)\Lambda = \mathcal{G}^{(i)}_{u(\lambda)}(T)\Lambda$$

for each $\lambda \in \boldsymbol{u}_c - \{1\}$ and

$$L_p(s,\omega^i) = \begin{cases} \displaystyle\sum_{\lambda\in\boldsymbol{u}_c-\{1\}} \tilde{A}^{(i)}(c_\gamma^{1-s} - 1, \lambda) & \text{if } i \not\equiv 0 \pmod{p-1} \\ \displaystyle\sum_{\lambda\in\boldsymbol{u}_c-\{1\}} \tilde{A}^{(0)}(c_\gamma^{1-s} - 1, \lambda)/(c_\gamma^{1-s} - 1) & \text{if } i \equiv 0 \pmod{p-1}\,. \end{cases}$$

PROOF. The statement is essentially contained in what is described at the end of Section 2 of [14]. Indeed, we have

$$(1 - \langle c\rangle^{1-s}\omega^i(c))L_p(s,\omega^i) = \sum_{\lambda\in\boldsymbol{u}_c-\{1\}} l_p(\lambda, s, \omega^{i-1})\,.$$

So, it is sufficient to put

$$\tilde{A}^{(i)}(T,\lambda) = \begin{cases} (1 - \omega^i(c)(1+T)^{\ell(c)})^{-1}\mathcal{G}^{(i)}_{u(\lambda)}(T) & \text{if } i \not\equiv 0 \pmod{p-1} \\ (1 - (1+T)^{\ell(c)})^{-1}T\mathcal{G}^{(0)}_{u(\lambda)}(T) & \text{if } i \equiv 0 \pmod{p-1}\,. \end{cases}$$

(II) Let $B_n(X)$ be the n-th Bernoulli polynomial defined by

$$\frac{te^{tX}}{e^t-1}=\sum_{n=0}^{\infty}B_n(X)\frac{t^n}{n!}\,.$$

Put $B_n=B_n(0)$, which is the n-th Bernoulli number. For each root of unity ρ and $n\in\bar{\mathbf{N}}$, we define the number $E_n(\rho)$ by

$$\frac{\rho}{e^t-\rho}=\begin{cases}\displaystyle\sum_{n=0}^{\infty}E_n(\rho)\frac{t^n}{n!} & \text{if } \rho\neq 1\\ \displaystyle\frac{1}{t}+\sum_{n=0}^{\infty}E_n(1)\frac{t^n}{n!} & \text{if } \rho=1\,.\end{cases}\tag{5.2}$$

For each $x\in\mathbf{Q}$, we denote by $\langle\!\langle x\rangle\!\rangle$ the unique element in $\mathbf{Q}$ satisfying $0\leq\langle\!\langle x\rangle\!\rangle<1$ and $x-\langle\!\langle x\rangle\!\rangle\in\mathbf{Z}$. Let $m,\ r,\ h,\ k\in\mathbf{N}$ with $1\leq r\leq m$. For simplicity, we assume that $m\equiv 1 \pmod 2$, $m\geq 3$ and $(h,k)=1$. Then, as shown in [8], the higher-order Dedekind sum $S_{m+1}^{(r)}(h,k)$ is expressed as

$$\begin{aligned}S_{m+1}^{(r)}(h,k)&=\sum_{a=0}^{k-1}B_{m+1-r}\left(\frac{a}{k}\right)B_r\left(\left\langle\!\!\left\langle\frac{ha}{k}\right\rangle\!\!\right\rangle\right)\\&=k^{-m}(m+1-r)r\sum_{\zeta^k=1}E_{m-r}(\zeta^h)E_{r-1}(\zeta^{-1})\,.\end{aligned}\tag{5.3}$$

For each even integer α modulo $p-1$, Kudo constructed a p-adic analytic function $S_{p,\alpha}(s;r,h,k)$ on $\mathbf{Z}_p$ such that

$$S_{p,\alpha}(m;r,h,k)=k^mS_{m+1}^{(r)}(h,k)-p^{m-r}k^mS_{m+1}^{(r)}(ph,k)\tag{5.4}$$

holds for any $m\in\mathbf{N}$ with $m+1\equiv\alpha \pmod{p-1}$ and $m\geq r$ ([8]). In what follows, we also assume that $(k,p)=1$. Let us show that under this assumption, $S_{p,\alpha}(s;r,h,k)$ is also expressed as

$$S_{p,\alpha}(s;r,h,k)=-(s+1-r)r\sum_{\zeta^k=1}L_p(r-s,\omega^{\alpha-r},\mathbf{G}_m,T+1-\zeta^h)E_{r-1}(\zeta^{-1})\,.\tag{5.5}$$

It is sufficient to show that for any $m\in\mathbf{N}$ with $m\geq r+1$, $m\geq 3$ and $m+1\equiv\alpha \pmod{p-1}$, (5.5) holds for $s=m$. In order to see this, we recall equation (2.8), which shows

$$\begin{aligned}&-L_p(r-m,\omega^{\alpha-r},\mathbf{G}_m,T+1-\zeta^h)\\&=\frac{B_{m-r+1,1}(\mathbf{G}_m,T+1-\zeta^h)}{m-r+1}-p^{m-r}\cdot\frac{B_{m-r+1,1}(\mathbf{G}_m,\mathcal{N}_{\mathbf{G}_m}(T+1-\zeta^h))}{m-r+1}\end{aligned}$$

for any $\zeta\in\boldsymbol{u}_k$. Considering the case $\chi=1$ for (2.9) and (2.10), and taking (5.2) into consideration, we see that

$$\frac{B_{m-r+1,1}(\mathbf{G}_m,T+1-\zeta^h)}{m-r+1}=\frac{\zeta^h}{1-\zeta^h}H_1^{m-r}(\zeta^h)=E_{m-r}(\zeta^h)$$

for any $\zeta \in \boldsymbol{u}_k - \{1\}$. If $\zeta = 1$, we also have $B_{m-r+1,1}(\mathbf{G}_m, T)/(m-r+1) = E_{m-r}(1) = B_{m-r+1}/(m-r+1)$. Further, it is easy to see that $\mathcal{N}_{\mathbf{G}_m}(T+1-\zeta^h) = T+1-\zeta^{ph}$ for any $\zeta \in \boldsymbol{u}_k$. Hence, we obtain

$$-L_p(r-m, \omega^{\alpha-r}, \mathbf{G}_m, T+1-\zeta^h) = E_{m-r}(\zeta^h) - p^{m-r}E_{m-r}(\zeta^{ph})$$

for any $\zeta \in \boldsymbol{u}_k$, and by virtue of (5.3), we see that the value of the right hand side of (5.5) at $s = m$ is equal to (5.4). This proves equation (5.5).

Now, take an integer $b_1 \in \mathbf{N}$ such that $p^{b_1} \equiv 1 \pmod{k}$. Put $i \equiv \alpha - r \pmod{p-1}$. Then by (5.5), we have

$$\begin{aligned}\sum_{j=0}^{b_1-1} \frac{S_{p,i+r}(r-s; r, p^j h, k)}{(s-1)r} &= \sum_{j=0}^{b_1-1} \sum_{\zeta^k=1} L_p(s, \omega^i, \mathbf{G}_m, T+1-\zeta^{p^j h})E_{r-1}(\zeta^{-1}) \\ &= \sum_{\zeta^k=1} L_p(s, \omega^i, \mathbf{G}_m, T+1-\zeta^h) \sum_{j=0}^{b_1-1} E_{r-1}(\zeta^{-p^j}).\end{aligned}$$

If $\zeta \in \boldsymbol{u}_k - \{1\}$, we have $\sum_{j=0}^{b_1-1} E_{r-1}(\zeta^{-p^j}) \in \mathbf{Z}_p$. However, we have $\sum_{j=0}^{b_1-1} E_{r-1}(1) = b_1 B_r/r$, which is not necessarily in $\mathbf{Z}_p$. Taking this into consideration, we take an integer $b_2 \in \mathbf{N}$ such that $b_1 b_2 B_r/r \in \mathbf{Z}$. We put $E(\zeta, r-1, b_1) = \sum_{j=0}^{b_1-1} E_{r-1}(\zeta^{-p^j})$ for each $\zeta \in \boldsymbol{u}_k$ and define

$$H_{r,h,k}(T) = \left\{ \prod_{\zeta \in \boldsymbol{u}_k - \{1\}} \{\omega^{-1}(1-\zeta^h)(1-\zeta^h+T)\}^{b_2 E(\zeta, r-1, b_1)} \right\} \cdot T^{b_2 E(1, r-1, b_1)}.$$

Then, we have $\mathcal{N}_{\mathbf{G}_m} H_{r,h,k}(T) = H_{r,h,k}(T) \in \Lambda$,

$$\mathcal{G}^{(i)}_{H_{r,h,k}}(c_\gamma^{1-s} - 1) = -\frac{b_2}{(s-1)r} \sum_{j=0}^{b_1-1} S_{p,i+r}(r-s; r, p^j h, k),$$

$$Y^{(i)}_{H_{r,h,k},\infty} = \begin{cases} \Lambda/\mathcal{G}^{(i)}_{H_{r,h,k}}(T)\Lambda & \text{if } i \not\equiv 0,\ 1 \pmod{p-1} \\ \Lambda/\dfrac{\mathcal{G}^{(1)}_{H_{r,h,k}}(T)}{T+1-c_\gamma}\Lambda & \text{if } i \equiv 1 \pmod{p-1} \\ \Lambda/\mathcal{G}^{(0)}_{H_{r,h,k}}(T)T\Lambda & \text{if } i \equiv 0 \pmod{p-1}. \end{cases}$$

REMARK. In the case that α is odd, by defining $S^{(r)}_{m+1}(h, k)$ and $S_{p,\alpha}(s; r, h, k)$ by (5.3) and (5.5) respectively, the above argument is valid as well. However, we should note that in this case, we have

$$S_{p,\alpha}(s; r, h, k) = \begin{cases} 0 & \text{if } r \geq 2 \\ \dfrac{s}{2}\langle k\rangle^s \omega^{\alpha-1}(k) L_p(1-s, \omega^{\alpha-1}) & \text{if } r = 1. \end{cases} \tag{5.6}$$

In order to verify this, we put

$$F(t_1, t_2) = \sum_{\zeta^k=1} \frac{\zeta^h}{e^{t_1} - \zeta^h} \cdot \frac{\zeta^{-1}}{e^{t_2} - \zeta^{-1}} .$$

Then,

$$\begin{aligned} F(-t_1, -t_2) &= \sum_{\zeta^k=1} \frac{\zeta^h}{e^{-t_1} - \zeta^h} \cdot \frac{\zeta^{-1}}{e^{-t_2} - \zeta^{-1}} = \sum_{\zeta^k=1} \frac{e^{t_1}}{\zeta^{-h} - e^{t_1}} \cdot \frac{e^{t_2}}{\zeta - e^{t_2}} \\ &= \sum_{\zeta^k=1} \left(1 + \frac{\zeta^{-h}}{e^{t_1} - \zeta^{-h}}\right)\left(1 + \frac{\zeta}{e^{t_2} - \zeta}\right) \\ &= k + \frac{k}{e^{kt_1} - 1} + \frac{k}{e^{kt_2} - 1} + F(t_1, t_2) . \end{aligned}$$

Expanding above equation as power series of t_1 and t_2, and comparing the coefficients of $t_1^{m-r} t_2^{r-1}$ for m and r in $\mathbf{N}$ with $m \geq r + 1$, we see from (5.2) and (5.3) that

$$(-1)^{m-1} S_{m+1}^{(r)}(h, k) = \begin{cases} S_{m+1}^{(r)}(h, k) & \text{if } r \geq 2 \\ S_{m+1}^{(1)}(h, k) + B_m & \text{if } r = 1 . \end{cases}$$

Hence, if $\alpha \equiv 1 \pmod 2$ and $m + 1 \equiv \alpha \pmod{p - 1}$, we have $m \equiv 0 \pmod 2$ and $S_{m+1}^{(r)}(h, k) = 0$ or $(-1/2)B_m$ according as $r \geq 2$ or $r = 1$. The same result also holds for $S_{m+1}^{(r)}(ph, k)$. Therefore, (5.6) follows from (5.4).

6. Proofs of propositions in Section 3.

(I) Proof of Proposition 3.1.

We first prove the following

LEMMA 6.1. *Let $u \in U_\infty$ and put $\tilde{g}_u(T) = g_u(T) - (\pi/p) g_u((1 + T)^\pi - 1)$. Then for any $n \in \mathbf{N}$ with $n \equiv i \pmod{p - 1}$, we have*

$$\mathcal{G}_u^{(i)}(c_\gamma^n - 1) = \frac{d^{n-1}}{dZ^{n-1}} \tilde{g}_u(e^Z - 1)\bigg|_{Z=0} = \left(1 - \frac{\pi^n}{p}\right) \frac{d^{n-1}}{dZ^{n-1}} g_u(e^Z - 1)\bigg|_{Z=0} .$$

PROOF. By (2.1) and (3.4), we have

$$\mathcal{G}_u^{(i)}(c_\gamma^n - 1) = \int_{\mathbf{Z}_p^\times} x^{n-1}\, d\mu_{g_u}(x)$$

for any $n \in \mathbf{N}$ with $n \equiv i \pmod{p - 1}$. In general, for any $g(T) \in \mathcal{O}[[T]]$, putting $\tilde{g}(T) = g(T) - (1/p) \sum_{\zeta^p=1} g(\zeta(1 + T) - 1)$, we have

$$\int_{\mathbf{Z}_p^\times} x^n\, d\mu_g(x) = \frac{d^n}{dZ^n} \tilde{g}(e^Z - 1)\bigg|_{Z=0}$$

for any $n \in \bar{\mathbf{N}}$. By (3.2), (3.3) and (1.1), it is easy to deduce that

$$\sum_{\zeta^p=1} g_u\big(\zeta(1+T)-1\big) = \pi\, g_u\big((1+T)^\pi - 1\big).$$

Hence, our assertion follows immediately.

Let us prove Proposition 3.1.

By (1.4) and (3.1), we have $R_{\sigma u}(T) = R_u([\kappa(\sigma)]_F(T))$ for any $\sigma \in G_\infty$ and $u \in U_\infty$. Hence, by (1.1) and (3.3), we deduce

$$g_{\sigma u}(T) = D\{\log(R_u \circ \phi_F)((1+T)^{\kappa(\sigma)} - 1)\} = \kappa(\sigma) g_u\big((1+T)^{\kappa(\sigma)} - 1\big).$$

Then, we see from Lemma 6.1 that

$$\mathcal{G}^{(i)}_{\sigma u}(c_\gamma{}^n - 1) = \kappa(\sigma)^n \mathcal{G}^{(i)}_u(c_\gamma{}^n - 1)$$

holds for any $n \in \mathbf{N}$ with $n \equiv i \pmod{p-1}$. It follows that

$$\mathcal{G}^{(i)}_{\sigma u}(T) = \omega^i(\kappa(\sigma))(1+T)^{\ell(\kappa(\sigma))}\mathcal{G}^{(i)}_u(T).$$

In particular, we have

$$\mathcal{G}^{(i)}_{\delta u}(T) = \kappa^i(\delta)\mathcal{G}^{(i)}_u(T) \tag{6.1}$$

for any $\delta \in \Delta$ and

$$\mathcal{G}^{(i)}_{\gamma u}(T) = (1+T)\mathcal{G}^{(i)}_u(T).$$

The first statement follows immediately from these equations.

Let j be an integer modulo $p-1$. Then, for any $u \in U_\infty$ and $\delta \in \Delta$ we see from (1.5) and (6.1) that

$$\mathcal{G}^{(i)}_{u(j)}(T) = \frac{1}{p-1}\sum_{\delta \in \Delta} \kappa^{i-j}(\delta)\mathcal{G}^{(i)}_u(T).$$

Hence, if $j \not\equiv i \pmod{p-1}$, we have $\mathcal{G}^{(i)}_{u(j)}(T) = 0$, namely the map $u \mapsto \mathcal{G}^{(i)}_{u(j)}(T)$ is the 0-map on $U^{(j)}_\infty$.

Let $u = (u_n)_{n\geq 0} \in U^{(i)}_\infty$ and suppose that $\mathcal{G}^{(i)}_u(T) = 0$. Then, from what we have just shown above, we have $\mathcal{G}^{(j)}_u(T) = 0$ for any integer j modulo $p-1$. It follows from Lemma 6.1 that

$$\left.\frac{d^{n-1}}{dZ^{n-1}} g_u(e^Z - 1)\right|_{Z=0} = 0$$

holds for any $n \in \mathbf{N} - \{1\}$. This shows that $g_u(T)$ is constant, namely $g_u(T) = g_u(0)$.

In the first place, suppose that $\pi \neq p$. Then, Lemma 6.1 also shows that $g_u(0) = 0$, and consequently, $g_u(T) = 0$. It follows from (3.3) that $R_u(T)$ is constant, that is, $u_n = R_u(\tau_n) = R_u(0) \in \mathbf{Z}_p$ holds for any $n \in \bar{\mathbf{N}}$. By taking the norm from K_n to K_{n-1}, this yields $R_u(0)^p = u_n{}^p = u_{n-1} = R_u(0)$. Since $R_u(0) \equiv 1 \pmod{p\mathbf{Z}_p}$, we obtain $R_u(0) = 1$, namely, $u = 1$.

Next, suppose that $\pi = p$. Then, Lemma 6.1 shows that $\tilde{g}_u(T) = g_u(T) - g_u((1+T)^p - 1) = 0$. It follows from (1.1) and (3.3) that

$$\begin{aligned}\frac{d}{dT}\{\log(R_u \circ \phi_F)(T)\} &= \frac{1}{p}\frac{d}{dT}\{\log(R_u \circ \phi_F)((1+T)^p - 1)\} \\ &= \frac{1}{p}\frac{d}{dT}\{\log(R_u \circ [p]_F \circ \phi_F)(T)\}.\end{aligned}$$

Hence, $R_u(T)^p/(R_u \circ [p]_F)(T)$ is constant, namely

$$R_u(T)^p/(R_u \circ [p]_F)(T) = R_u(0)^{p-1} \in \mathbf{Z}_p.$$

Substituting τ_n for T, we see $u_n{}^p/u_{n-1} = R_u(0)^{p-1}$. Taking norm from K_n to K_{n-1}, we obtain

$$R_u(0)^{p(p-1)} = u_{n-1}^p/u_{n-1}^p = 1.$$

Thus, we obtain $R_u(0) = 1$, and hence $R_u(T)^p = (R_u \circ [p]_F)(T)$. This implies $u \in \operatorname{proj}\lim \boldsymbol{u}_{p^{n+1}}$. As we noted just before Proposition 3.1, $\operatorname{proj}\lim \boldsymbol{u}_{p^{n+1}}$ is contained in $U_\infty^{(1)}$. Hence, in order to complete the proof, it is sufficient to show that $\mathcal{G}_u^{(1)}(T) = 0$ holds for any $u \in \operatorname{proj}\lim \boldsymbol{u}_{p^{n+1}}$. Since equation $R_u(T)^p = (R_u \circ [p]_F)(T)$ yields $\tilde{g}_u(T) = 0$, the required equation $\mathcal{G}_u^{(1)}(T) = 0$ follows immediately from Lemma 6.1. This completes the proof of Proposition 3.1.

(II) Proof of Proposition 3.2.

(1) As in Section 3, let N_n be the norm map from K_n to $\mathbf{Q}_p$. In order to prove $V_n^{(i)} = U_n^{(i)}$, it is enough to show that $N_n(u) = 1$ holds for any $u \in U_n^{(i)}$. Let $N_{n,0}$ denote the norm map from K_n to K_0. Since $\sum_{\delta\in\Delta}\kappa^i(\delta) = 0$ holds for any $i \not\equiv 0 \pmod{p-1}$, we see that

$$N_n(u) = \prod_{\delta\in\Delta} N_{n,0}(u)^\delta = \prod_{\delta\in\Delta} N_{n,0}(u)^{\kappa^i(\delta)} = 1$$

holds for any $u \in U_n^{(i)}$.

Now, let M_n denote the maximal p-abelian p-ramified extension of K_n. We put $M_\infty = \bigcup_{n=0}^\infty M_n$, $\mathcal{X}_n = \mathrm{Gal}(M_n/K_n)$ and $\mathcal{X}_\infty = \mathrm{Gal}(M_\infty/K_\infty)$. The Galois group G_∞ operates on $\mathcal{X}_n$ and $\mathcal{X}_\infty$ by conjugation. We regard G_∞ as operating trivially on $\mathbf{Z}_p$. Then, $\mathcal{X}_n$, $\mathcal{X}_\infty$ and $\mathbf{Z}_p$ become compact Γ-modules and also Λ-modules. As in the proof of Lemma 2 of [1], we make use of the relations among the structures of U_∞, U_n, $\mathcal{X}_\infty$ and $\mathcal{X}_n$, which are obtained by local class field theory. We have the following canonical commutative diagram of Λ-modules:

$$\begin{array}{ccccccccc} 1 & \to & U_\infty & \to & \mathcal{X}_\infty & \to & \mathbf{Z}_p & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & U_n & \to & \mathcal{X}_n & \to & \mathbf{Z}_p & \to & 1, \end{array} \tag{6.2}$$

where the horizontal sequences are exact. Since $\mathbf{Z}_p^{(i)} = \{0\}$ for $i \not\equiv 0 \pmod{p-1}$, (6.2) induces the following commutative diagram:

$$\begin{array}{ccc} U_\infty^{(i)} & \xrightarrow{\sim} & \mathcal{X}_\infty^{(i)} \\ \downarrow & & \downarrow \\ U_n^{(i)} & \xrightarrow{\sim} & \mathcal{X}_n^{(i)}. \end{array} \tag{6.3}$$

Since we have $V_n^{(i)} = U_n^{(i)}$, the vertical homomorphisms in (6.3) are surjective. Further, the kernel of the homomorphism $\mathcal{X}_\infty \to \mathcal{X}_n$ in (6.2) is $\mathrm{Gal}(M_\infty/M_n)$, which is the commutative subgroup of $\mathrm{Gal}(M_\infty/K_n)$ and coincides with $\mathcal{X}_\infty^{\gamma^{p^n}-1} = w_n(T)\mathcal{X}_\infty$. Hence, we see from (6.3) that

$$U_n^{(i)} \cong U_\infty^{(i)}/w_n(T)U_\infty^{(i)}.$$

(2) The diagram (6.2) induces the following commutative one:

$$\begin{array}{ccccccccc} 1 & \to & U_\infty^{(0)} & \to & \mathcal{X}_\infty^{(0)} & \to & \mathbf{Z}_p & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & U_0^{(0)} & \to & \mathcal{X}_0^{(0)} & \to & \mathbf{Z}_p & \to & 1, \end{array} \tag{6.4}$$

where the horizontal sequences are exact.

In the first place, let us show that $\mathcal{X}_\infty^{(0)} \cong \Lambda$. Since we have $U_0^{(0)} = U_0 \cap \mathbf{Q}_p = 1 + p\mathbf{Z}_p$, we see from the lower sequence in (6.4) that $\mathcal{X}_0^{(0)}$ is isomorphic to $\mathbf{Z}_p^2$ as $\mathbf{Z}_p$-modules. Further, since $\mathcal{X}_0^{(0)}/\mathrm{Gal}(M_0/K_\infty)^{(0)} \cong \Gamma^{(0)} = \Gamma \cong \mathbf{Z}_p$, we also see that $\mathrm{Gal}(M_0/K_\infty)^{(0)} \cong \mathbf{Z}_p$. It follows that

$$\mathrm{Gal}(M_0/K_\infty)^{(0)} \cong \mathcal{X}_\infty^{(0)}/\mathrm{Gal}(M_\infty/M_0)^{(0)} \cong \mathcal{X}_\infty^{(0)}/T\mathcal{X}_\infty^{(0)} \cong \mathbf{Z}_p.$$

Then, Nakayama's lemma ([15, Lemma 13.16]) shows that $\mathcal{X}_\infty^{(0)}$ is generated over Λ by a single element. Moreover, (6.4) shows that $\mathcal{X}_\infty^{(0)}$ has a submodule isomorphic to $U_\infty^{(0)}$, which has no Λ-torsion as a result of Proposition 3.1. Hence, we conclude that

$$\mathcal{X}_\infty^{(0)} \cong \Lambda. \tag{6.5}$$

Next, note that the image of the projection $U_\infty^{(0)} \to U_0^{(0)}$ in (6.4) is

$$V_0^{(0)} = \{u \in U_0^{(0)} \mid N_0(u) = 1\} = \{u \in 1 + p\mathbf{Z}_p \mid u^{p-1} = 1\} = \{1\}.$$

From this, it is easy to see that the maps in (6.4) satisfy

$$\mathrm{Im}(U_\infty^{(0)} \to \mathcal{X}_\infty^{(0)}) = \mathrm{Ker}(\mathcal{X}_\infty^{(0)} \to \mathcal{X}_0^{(0)}) = T\mathcal{X}_\infty^{(0)}. \tag{6.6}$$

Identifying $U_\infty^{(0)}$ with $\mathrm{Im}(U_\infty^{(0)} \to \mathcal{X}_\infty^{(0)})$, we see from (6.2) that

$$\begin{aligned} V_n^{(0)} &\cong U_\infty^{(0)}/(\mathrm{Gal}(M_\infty/M_n)^{(0)} \cap U_\infty^{(0)}) = U_\infty^{(0)}/(w_n(T)\mathcal{X}_\infty^{(0)} \cap T\mathcal{X}_\infty^{(0)}) \\ &= U_\infty^{(0)}/w_n(T)\mathcal{X}_\infty^{(0)} = U_\infty^{(0)} \Big/ \frac{w_n(T)}{T} U_\infty^{(0)}. \end{aligned}$$

This completes the proof.

(III) Proof of Proposition 3.3.

We first prove some preliminary lemmas.

LEMMA 6.2. *Let $n \in \bar{\mathbf{N}}$. If $n+1 \leq \nu_\pi$, then $K_n = \mathbf{Q}_p(\boldsymbol{u}_{p^{n+1}})$. Otherwise, the group of p-power roots of unity in K_n is $\boldsymbol{u}_{p^{\nu_\pi}}$.*

PROOF. The norm groups of the extensions $K_n/\mathbf{Q}_p$ and $\mathbf{Q}_p(\boldsymbol{u}_{p^{n+1}})/\mathbf{Q}_p$ are

$$(1+p^{n+1}\mathbf{Z}_p) \times \{\pi^m | m \in \mathbf{Z}\} \quad \text{and} \quad (1+p^{n+1}\mathbf{Z}_p) \times \{p^m | m \in \mathbf{Z}\},$$

respectively. Both of them coincide if and only if $n+1 \leq \nu_\pi$. Hence, the statements follow immediately by local class field theory.

LEMMA 6.3. *Let $n \in \bar{\mathbf{N}}$ and put $\nu(\pi, n+1) = \min\{\nu_\pi, n+1\}$. Then, as $\mathbf{Z}_p$-modules, we have $U_n^{(i)} \cong \mathbf{Z}_p^{p^n}$ or $U_n^{(i)} \cong \mathbf{Z}_p^{p^n} \times \boldsymbol{u}_{p^{\nu(\pi,n+1)}}$ according as $i \not\equiv 1 \pmod{p-1}$ or $i \equiv 1 \pmod{p-1}$.*

PROOF. In the same way as in the proof of Lemma 1 of Chapter 7 in [9], we see that $\operatorname{rank}_{\mathbf{Z}_p} U_n^{(i)} = p^n$. By Lemma 6.2, the torsion subgroup of U_n is $\boldsymbol{u}_{p^{\nu(\pi,n+1)}}$ and in fact, it is contained in $U_n^{(1)}$. Hence, the statement is obvious.

Let $\Omega \in \mathcal{I}^\times$ be as defined in Section 1.

LEMMA 6.4. *There exists an element $v \in U_\infty$ which is independent of i and satisfies the following:*

(1) *If $i \not\equiv 1 \pmod{p-1}$ or $\nu_\pi = 0$, then $\mathcal{G}_v^{(i)}(T) \in \mathcal{I}[[T]]^\times$.*

(2) *If $\nu_\pi > 0$, then $\{\Omega\mathcal{G}_v^{(1)}(T) - (1 - \frac{\pi}{p})\}/(T+1-c_\gamma) \in \mathcal{I}[[T]]^\times$.*

PROOF. Let $\mathcal{F}$ denote the basic Lubin-Tate formal group belonging to π, namely, the formal group such that $[\pi]_\mathcal{F}(T) = T^p + \pi T$. There exists an isomorphism $\eta : F \to \mathcal{F}$ over $\mathbf{Z}_p$ such that $\eta'(0) = 1$. Let $\varepsilon \in 1 + p\mathbf{Z}_p$ be a $(p-1)$-st root of $1-\pi$ and put $v_\varepsilon = (\varepsilon - \eta(\tau_n))_{n \geq 0}$. Let us show that v_ε is in U_∞. Indeed, for each $n \in \mathbf{N}$, the minimal equation for $\varepsilon - \eta(\tau_n)$ over K_{n-1} is $[\pi]_\mathcal{F}(\varepsilon - T) - \eta(\tau_{n-1}) = 0$, and so, the image of $\varepsilon - \eta(\tau_n)$ by the norm map from K_n to K_{n-1} is $[\pi]_\mathcal{F}(\varepsilon) - \eta(\tau_{n-1}) = \varepsilon^p + \pi\varepsilon - \eta(\tau_{n-1})$, which is also equal to $\varepsilon - \eta(\tau_{n-1})$ by virtue of $\varepsilon^{p-1} = 1 - \pi$. It follows that $v_\varepsilon \in U_\infty$.

Now, it is obvious that $R_{v_\varepsilon}(T) = \varepsilon - \eta(T)$. Hence, (3.3) yields

$$\begin{aligned} g_{v_\varepsilon}(T) &= -D\left\{\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{1}{\varepsilon}(\eta \circ \phi_F)(T)\right)^k\right\} \\ &= -(1+T)\sum_{k=0}^{\infty}\left(\frac{1}{\varepsilon}(\eta \circ \phi_F)(T)\right)^k \cdot \frac{1}{\varepsilon}(\eta \circ \phi_F)'(T). \end{aligned}$$

Note that we have $(\eta \circ \phi_F)(e^Z - 1) = e_{\mathcal{F}}(\Omega^{-1}Z)$, which implies $(\eta \circ \phi_F)'(e^Z - 1) \cdot e^Z = \Omega^{-1}e'_{\mathcal{F}}(\Omega^{-1}Z)$. Hence, putting $v = v_\varepsilon^{-\varepsilon}$, we have

$$g_v(e^Z - 1) = \Omega^{-1}e'_{\mathcal{F}}(\Omega^{-1}Z)\sum_{k=0}^{\infty}\left(\frac{1}{\varepsilon}e_{\mathcal{F}}(\Omega^{-1}Z)\right)^k . \tag{6.7}$$

Further, the equation $[\pi]_{\mathcal{F}}(T) = T^p + \pi T$ implies $e_{\mathcal{F}}(\pi Z) = e_{\mathcal{F}}(Z)^p + \pi e_{\mathcal{F}}(Z)$. From this, we deduce

$$e_{\mathcal{F}}(Z) \equiv Z + \frac{1}{\pi^p - \pi}Z^p \pmod{\text{degree } p+1},$$

and by (6.7) and by virtue of $\varepsilon^{p-1} = 1 - \pi$, we obtain

$$g_v(e^Z - 1) \equiv \Omega^{-1}\left(1 + \frac{p\Omega^{1-p}}{\pi^p - \pi}Z^{p-1}\right)\sum_{k=0}^{p-1}\varepsilon^{-k}\Omega^{-k}Z^k \pmod{\text{degree } p}$$

$$\equiv \sum_{k=0}^{p-2}\varepsilon^{-k}\Omega^{-k-1}Z^k + \Omega^{-p}\left(\frac{p}{\pi(\pi^{p-1}-1)} + \frac{1}{1-\pi}\right)Z^{p-1} \pmod{\text{degree } p}. \tag{6.8}$$

Hence, if $i \not\equiv 1 \pmod{p-1}$ or $v_\pi = 0$, then for a unique integer k satisfying $k \equiv i \pmod{p-1}$ and $1 \le k \le p-1$, Lemma 6.1 shows that $|\mathcal{G}_v^{(i)}(c_\gamma{}^k - 1)| = 1$. The statement (1) follows immediately from this.

Let us prove the statement (2). Suppose that $v_\pi > 0$. By (6.8) and Lemma 6.1, we have

$$\Omega\mathcal{G}_v^{(1)}(c_\gamma - 1) = 1 - \frac{\pi}{p},$$

$$\Omega\mathcal{G}_v^{(1)}(c_\gamma{}^p - 1) = \Omega^{1-p}\left(1 - \frac{\pi^p}{p}\right)(p-1)!\left(\frac{p}{\pi(\pi^{p-1}-1)} + \frac{1}{1-\pi}\right).$$

Hence, we can write

$$\Omega\mathcal{G}_v^{(1)}(T) = 1 - \frac{\pi}{p} + (T + 1 - c_\gamma)A(T)$$

for some $A(T) \in \mathcal{I}[[T]]$ such that

$$1 - \frac{\pi}{p} + (c_\gamma{}^p - c_\gamma)A(c_\gamma{}^p - 1) = \Omega^{1-p}\left(1 - \frac{\pi^p}{p}\right)(p-1)!\left(\frac{p}{\pi(\pi^{p-1}-1)} + \frac{1}{1-\pi}\right). \tag{6.9}$$

In order to complete the proof, it is sufficient to show that $A(c_\gamma{}^p - 1) \not\equiv 0 \pmod{p\mathcal{I}}$. Recall that $\Omega^\psi/\Omega = p/\pi$, which implies $\Omega^{p-1} \equiv p/\pi \pmod{p\mathcal{I}}$. By the assumption $v_\pi > 0$, we have $1 - (\pi/p) \equiv 0 \pmod{p\mathbf{Z}_p}$. Hence, we see from (6.9) that

$$1 - \frac{\pi}{p} + (c_\gamma{}^p - c_\gamma)A(c_\gamma{}^p - 1) \equiv \Omega^{1-p}(p-1)!\left(-\frac{p}{\pi} + 1 + \pi\right) \pmod{p^2\mathcal{I}}$$

$$\equiv \frac{\pi}{p}\cdot(-1)\left(-\frac{p}{\pi} + 1 + \pi\right) \pmod{p^2\mathcal{I}}$$

$$\equiv 1 - \frac{\pi}{p} - \frac{\pi^2}{p} \pmod{p^2\mathcal{I}}.$$

Since $\mathrm{ord}_p({c_\gamma}^p - c_\gamma) = 1$, it follows that

$$A({c_\gamma}^p - 1) \equiv -\frac{\pi^2}{({c_\gamma}^p - c_\gamma)p} \not\equiv 0 \pmod{p\mathcal{I}}.$$

This completes the proof of Lemma 6.4.

Let us proceed to prove Proposition 3.3.

We note that, in general if a rank one Λ-submodule $\mathcal{K}$ of $\mathcal{I}[[T]]$ contains an element $\mathcal{G} \in \mathcal{I}[[T]]^\times$, then we have

$$\mathcal{K} = \mathcal{G}\Lambda.$$

In the case $i \equiv 0 \pmod{p-1}$, we see from (6.5) and (6.6) that

$$\mathcal{K}^0 \cong U_\infty^{(0)} \cong \Lambda.$$

Let $v \in U_\infty$ be as in Lemma 6.4. Then, we have

$$\mathcal{G}_v^{(0)}(T) = \mathcal{G}_{v^{(0)}}^{(0)}(T) \in \mathcal{I}[[T]]^\times,$$

and consequently

$$\mathcal{K}^{(0)} = \mathcal{G}_{v^{(0)}}^{(0)}(T)\Lambda.$$

In the case where $i \not\equiv 0 \pmod{p-1}$ and either $i \not\equiv 1 \pmod{p-1}$ or $\nu_\pi = 0$, Lemma 6.3 shows that $U_0^{(i)} \cong \mathbf{Z}_p$. Then, by Proposition 3.2 (1), we have $U_\infty^{(i)}/TU_\infty^{(i)} \cong \mathbf{Z}_p$. It follows from Nakayama's lemma that $U_\infty^{(i)}$ is generated over Λ by a single element. Further, by Proposition 3.1, $U_\infty^{(i)}$ has no Λ-torsion. Hence, we see that

$$U_\infty^{(i)} \cong \Lambda,$$

and in the same way as in the previous case, we obtain $\mathcal{G}_{v^{(i)}}^{(i)}(T) \in \mathcal{I}[[T]]^\times$ and

$$\mathcal{K}^{(i)} = \mathcal{G}_{v^{(i)}}^{(i)}(T)\Lambda.$$

In the case where $i \equiv 1 \pmod{p-1}$ and $\pi = p$, Lemma 6.3 together with Proposition 3.2 (1) shows that there exists an element $u = (u_n)_{n\geq 0} \in U_\infty^{(1)}$ such that $U_0^{(1)}$ is generated over $\mathbf{Z}_p$ by u_0 and $\boldsymbol{u}_p$. Then, by Nakayama's lemma and by the fact that $\mathrm{proj\,lim}\,\boldsymbol{u}_{p^{n+1}}$ is in $U_\infty^{(1)}$, we have

$$U_\infty^{(1)} = \Lambda u \cdot \mathrm{proj\,lim}\,\boldsymbol{u}_{p^{n+1}}.$$

On the other hand, Lemma 6.1 shows that $\mathcal{G}_u^{(1)}(c_\gamma - 1) = 0$ holds for any $u \in U_\infty^{(1)}$. It follows that $\mathcal{K}^{(1)} \subset (T + 1 - c_\gamma)\mathcal{I}[[T]]$. Hence, by Proposition 3.1 and Lemma 6.4 (2), we see that $\mathcal{G}_{v^{(1)}}^{(1)}(T)/(T + 1 - c_\gamma) \in \mathcal{I}[[T]]^\times$ and

$$\mathcal{K}^{(1)} = \mathcal{G}_{v^{(1)}}^{(1)}(T)\Lambda.$$

It remains to prove the statement in the case where $i \equiv 1 \pmod{p-1}$ and $0 < \nu_\pi < \infty$. For each $n \in \bar{\mathbf{N}}$, let $\zeta_{p^{n+1}}$ denote the primitive p^{n+1}-th root of unity such that $\phi_F(\zeta_{p^{n+1}} - 1) = \tau_n$. If $n + 1 \leq \nu_\pi$, Lemma 6.2 shows that $\zeta_{p^{n+1}} \in U_n$. More precisely, we have $\zeta_{p^{n+1}} \in U_n^{(1)}$

and $\zeta_{p^{n+1}}^{p} = \zeta_{p^n}$. By Lemma 6.3, there exist two elements $u' = (u'_n)_{n\geq 0}$ and $v' = (v'_n)_{n\geq 0}$ in $U_\infty^{(1)}$ such that $u'_n = \zeta_{p^{n+1}}$ for all $n \in \bar{\mathbf{N}}$ with $n + 1 \leq \nu_\pi$ and that $U_0^{(1)}$ is generated over $\mathbf{Z}_p$ by ζ_p and v'_0. Then, by Proposition 3.2 and Nakayama's lemma, $U_\infty^{(1)}$ is generated over Λ by u' and v'. It follows from Proposition 3.1 that

$$U_\infty^{(1)} \cong \mathcal{K}^{(1)} = \mathcal{G}_{u'}^{(1)}(T)\Lambda + \mathcal{G}_{v'}^{(1)}(T)\Lambda\,. \tag{6.10}$$

Since $(u'_0)^p = (u'_{\nu_\pi - 1})^{p^{\nu_\pi}} = 1$, Proposition 3.2 (1) shows that $(u')^p \in TU_\infty^{(1)}$ and $(u')^{p^{\nu_\pi}} \in w_{\nu_\pi - 1}(T)U_\infty^{(1)}$. Then, by (6.10), we can write

$$p\mathcal{G}_{u'}^{(1)}(T) = T\{\mathcal{G}_{u'}^{(1)}(T)A_1(T) + \mathcal{G}_{v'}^{(1)}(T)A_2(T)\}\,, \tag{6.11}$$

$$p^{\nu_\pi}\mathcal{G}_{u'}^{(1)}(T) = w_{\nu_\pi - 1}(T)\{\mathcal{G}_{u'}^{(1)}(T)B_1(T) + \mathcal{G}_{v'}^{(1)}(T)B_2(T)\} \tag{6.12}$$

for some $A_1(T)$, $A_2(T)$, $B_1(T)$ and $B_2(T)$ in Λ. Applying p-adic Weierstrass Preparation Theorem ([15, Theorem 7.3]) to the power series $\mathcal{G}_{u'}^{(1)}(T)$, we see from (6.12) that $\mathcal{G}_{u'}^{(1)}(T)$ is divided by $w_{\nu_\pi - 1}(T)$ in $\mathcal{I}[[T]]$, namely, we can write

$$\mathcal{G}_{u'}^{(1)}(T) = w_{\nu_\pi - 1}(T)P(T) \tag{6.13}$$

with $P(T) = p^{-\nu_\pi}\{\mathcal{G}_{u'}^{(1)}(T)B_1(T) + \mathcal{G}_{v'}^{(1)}(T)B_2(T)\} \in \mathcal{I}[[T]]$. Let us show that in fact, $P(T)$ is in $\mathcal{I}[[T]]^\times$. Equation $(R_{u'} \circ \phi_F)(\zeta_p - 1) = R_{u'}(\tau_0) = \zeta_p$ implies

$$(R_{u'} \circ \phi_F)(T) \equiv T + 1 \left(\mathrm{mod}\frac{w_1(T)}{T}\mathcal{I}[[T]]\right).$$

Then, by (3.3), we have $|g_{u'}(0)| = 1$, and by Lemma 6.1, we have further

$$\mathrm{ord}_p(\mathcal{G}_{u'}^{(1)}(c_\gamma - 1)) = \mathrm{ord}_p((1 - (\pi/p))g_{u'}(0)) = \nu_\pi\,.$$

It follows from (6.13) that

$$\mathrm{ord}_p(P(c_\gamma - 1)) = \mathrm{ord}_p\left(\frac{\mathcal{G}_{u'}^{(1)}(c_\gamma - 1)}{w_{\nu_\pi - 1}(c_\gamma - 1)}\right) = \nu_\pi - \nu_\pi = 0\,.$$

Thus, we see that $P(T) \in \mathcal{I}[[T]]^\times$.

If $\nu_\pi = 1$, then (6.13) becomes $\mathcal{G}_{u'}^{(1)}(T) = TP(T)$. It follows from (6.11) that

$$(p - TA_1(T))P(T) = \mathcal{G}_{v'}^{(1)}(T)A_2(T)\,.$$

Substituting 0 for T, we see that either $\mathrm{ord}_p(\mathcal{G}_{v'}^{(1)}(0)) = 1$ or $\mathrm{ord}_p(A_2(0)) = 1$ holds. If the latter held, then we would have $\mathcal{G}_{v'}^{(1)}(T) \in \mathcal{I}[[T]]^\times$, which contradicts the result of Lemma 6.1 for $n = 1$. Hence, we must have $\mathrm{ord}_p(\mathcal{G}_{v'}^{(1)}(0)) = 1$, and consequently $A_2(T) \in \Lambda^\times$. Then, we see from (6.10) that

$$\mathcal{K}^{(1)} = TP(T)\Lambda + (p - TA_1(T))P(T)\Lambda = (T\Lambda + p\Lambda)P(T)\,,$$

and obtain the assertion in the case $\nu_\pi = 1$.

Suppose that $\nu_\pi > 1$. Combining (6.11) and (6.13), we deduce

$$(p - TA_1(T))w_{\nu_\pi - 1}(T)P(T)/T = \mathcal{G}_{v'}^{(1)}(T)A_2(T)\,. \tag{6.14}$$

Let us show that $A_2(T)$ is in $(w_{\nu_\pi-1}(T)/T)\Lambda$. For any $n \in \mathbf{N}$ with $0 < n < \nu_\pi$, we see from (6.14) that $\mathcal{G}_{v'}^{(1)}(\zeta_{p^n}-1)A_2(\zeta_{p^n}-1)=0$. If $\mathcal{G}_{v'}^{(1)}(\zeta_{p^n}-1)=0$ held, then by (6.10) and (6.13), we would have $\mathcal{G}_u^{(1)}(\zeta_{p^n}-1)=\mathcal{G}_{u^{(1)}}^{(1)}(\zeta_{p^n}-1)=0$ for all $u \in U_\infty$, which contradicts Lemma 6.4 (2). Hence, we must have $A_2(\zeta_{p^n}-1)=0$ and consequently, $A_2(T) \in (w_{\nu_\pi-1}(T)/T)\Lambda$. Write $A_2(T)=w_{\nu_\pi-1}(T)A_3(T)/T$ with $A_3(T) \in \Lambda$. Then, equation (6.14) becomes

$$(p-TA_1(T))P(T)=\mathcal{G}_{v'}^{(1)}(T)A_3(T)\,. \tag{6.15}$$

In the same way that we have shown $A_2(T) \in \Lambda^\times$ in the case $\nu_\pi = 1$, we see that $A_3(T) \in \Lambda^\times$. Further, we also have $A_1(T) \in \Lambda^\times$. Indeed, by (6.15) and Lemma 6.1, we have

$$\mathrm{ord}_p(p-(c_\gamma-1)A_1(c_\gamma-1))=\mathrm{ord}_p(\mathcal{G}_{v'}^{(1)}(c_\gamma-1)) \geq \nu_\pi\,,$$

and by the assumption $\nu_\pi > 1$, we see that $\mathrm{ord}_p((c_\gamma-1)A_1(c_\gamma-1))=1$, and hence $A_1(T) \in \Lambda^\times$. Then, applying p-adic Weierstrass Preparation Theorem to the power series $p-TA_1(T)$, we have

$$p-TA_1(T) \in (T-\pi')\Lambda^\times$$

for some prime element π' of $\mathbf{Z}_p$. It follows from (6.10), (6.13) and (6.15) that

$$\mathcal{K}^{(1)}=(w_{\nu_\pi-1}(T)\Lambda+(T-\pi')\Lambda)P(T)\,.$$

By remainder theorem, we have further

$$\begin{aligned}\mathcal{K}^{(1)}&=(w_{\nu_\pi-1}(\pi')\Lambda+(T-\pi')\Lambda)P(T)\\&=(p^{\nu_\pi}\Lambda+(T-\pi')\Lambda)P(T)\,.\end{aligned}$$

In order to complete the proof, it is sufficient to show that $\pi' \equiv c_\gamma - 1 \pmod{p^{\nu_\pi}\mathbf{Z}_p}$. Let v be as in Lemma 6.4, and write

$$\begin{aligned}\Omega\mathcal{G}_{v^{(1)}}^{(1)}(T)&=1-(\pi/p)+(T+1-c_\gamma)Q(T)\\&=(p^{\nu_\pi}C_1(T)+(T-\pi')C_2(T))P(T)\Omega\,,\end{aligned} \tag{6.16}$$

with $Q(T) \in \mathcal{I}[[T]]^\times$ and $C_1(T),\ C_2(T) \in \Lambda$. Substituting 0 for T in (6.16), we see

$$(1-c_\gamma)Q(0) \equiv -\pi' C_2(0)P(0)\Omega \pmod{p^{\nu_\pi}\mathcal{I}}\,.$$

By the assumption $\nu_\pi > 1$, this implies $C_2(T) \in \Lambda^\times$. Then, substituting $c_\gamma - 1$ for T in (6.16), we obtain

$$(c_\gamma-1-\pi')C_2(c_\gamma-1)P(c_\gamma-1)\Omega \equiv \Omega\mathcal{G}_{v^{(1)}}^{(1)}(c_\gamma-1) \equiv 0 \pmod{p^{\nu_\pi}\mathcal{I}}$$

and consequently

$$\pi' \equiv c_\gamma - 1 \pmod{p^{\nu_\pi}\mathbf{Z}_p}\,.$$

This completes the proof of Proposition 3.3.

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Present Addresses:
KAZUHITO KOZUKA
DEPARTMENT OF MATHEMATICS,
MIYAKONOJO NATIONAL COLLEGE OF TECHNOLOGY,
MIYAKONOJO, MIYAZAKI, 885–8567 JAPAN.

HIROFUMI TSUMURA
DEPARTMENT OF MANAGEMENT, TOKYO METROPOLITAN COLLEGE,
AKISHIMA, TOKYO, 196–8540 JAPAN.