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An Algorithm for Acylindrical Surfaces in 3-manifolds

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Abstract. An algorithm to decide if an orientable atoroidal 3-manifold contains closed incompressible acylindrical surfaces, and construct closed incompressible acylindrical surfaces is given. Mainly, the normal surface theory is used. To assure that the algorithm stops after finite steps, we show that each acylindrical surface is isotopic to some "edge surface" which is constructible.

Introduction.

We give an algorithm to construct certain embedded surfaces in a compact orientable 3-manifold M which are important to study the structure of M. A 3-manifold M is said to be *irreducible* if every embedded 2-sphere bounds a 3-ball in M. A surface F embedded in a 3-manifold M is said to be *injective in* M if the inclusion $i : F \to M$ induces the injection $i_* : \pi_1(F) \to \pi_1(M)$. A 3-manifold M is *atoroidal* if each injective torus is isotopic to ∂M. An annulus A properly embedded in a 3-manifold M is *essential* if it is injective and not isotopic to ∂M (rel. ∂A). A closed surface F embedded in M is said to be *acylindrical* if F is injective, not homeomorphic to S^2 and each component of $M - \mathring{N}(F)$ contains no essential annuli. Acylindrical surfaces are important for the 3-dimensional hyperbolic geometry. For example, a totally geodesic surface in a hyperbolic 3-manifold is acylindrical, and an acylindrical surface is quasi-Fuchsian, that is, the limit set of the subgroup of $\mathrm{Isom}^+(\mathbf{H}^3)$ corresponding to $i_*(\pi_1(F))$ is a topological embedded circle in S^2_∞. In [5], Hass showed that if a hyperbolic 3-manifold M contains an acylindrical surface with high genus, then M has a large volume. In [2], Agol gave an explicit lower bound to the hyperbolic volume of M with the genera of acylindrical surfaces embedded in M.

In general, a 3-manifold M may contain infinitely many injective surfaces, up to isotopy. Neumann showed that if M is a closed surface bundle over the circle and the first Betti number $\beta_1(M; \mathbf{Z}) \geq 2$, then M contains infinitely many surfaces which are fibers of some fibration over S^1 ([13]). But in [5], [14] and in [15], it was shown that M contains only finitely many acylindrical surfaces, up to isotopy. In this paper, we will show that if a triangulation of an orientable, irreducible, atoroidal 3-manifold M is given, then one can decide if M contains

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an orientable acylindrical surface, and one can construct all acylindrical surfaces contained in M.

In Section 1, we recall Haken's normal surface theory from the viewpoint of triangulations, and finiteness property of acylindrical surfaces. According to the theory as Haken developed it, given the presentation of the manifold M, there exists a finite number of "fundamental surfaces" such that all incompressible surfaces in M are generated by "cut-and-paste" operations from the finite number of fundamental surfaces. In [4], Haken developed the theories of normal surfaces, and gave an algorithm to decide if a bounded 3-manifold M contains a properly embedded disk D such that ∂D is not contractible in ∂M. This algorithm is known as the algorithm of Haken, and this algorithm is applied to decide if a knot is knotted or not. Our claim is that all acylindrical surfaces are isotopic to surfaces in some subset of the fundamental surfaces which are called "edge surfaces". This claim assures the finiteness of acylindrical surfaces. The number of edge surfaces is strictly fewer than that of fundamental surfaces. In [9], Jaco and Oertel gave an algorithm to decide if M contains an injective surface by applying the algorithm of Haken to finite number of the edge surfaces. In Section 2, we give an algorithm to construct acylindrical surfaces in orientable, irreducible, atoroidal 3-manifolds.

We have some motivation to study acylindrical surfaces in 3-manifolds. A knot K in the 3-sphere S^3 is an embedded circle in S^3. A knot K is said to be *hyperbolic* if the complement $S^3 - K$ admits a complete Riemannian metric with the constant sectional curvature -1. It is conjectured that, for any hyperbolic knot K in S^3, the complement $S^3 - K$ contains no closed embedded totally geodesic surface ([12]). An example of a knot which contains a quasi-Fuchsian surface in its complement is given in [1]. In fact, the knot indicated in [1, Figure 10] contains a genus two acylindrical surface in its complement. However it was shown in [1] that the acylindrical surface is not totally geodesic. It was stated that the knot is one of the first explicit examples which contain closed embedded acylindrical surfaces in their complements. A characterization of knots which contain closed acylindrical surfaces in their complements still seems to be open. But one can decide whether $S^3 - K$ contains a closed acylindrical surface or not, using our algorithm.

1. Normal surfaces and acylindrical surfaces.

The aim in this section is to show that each acylindrical surface embedded in an irreducible 3-manifold is isotopic to an "edge surface", which will be defined later. To explain our methods, we give a sketch of Haken's normal surface theory developed in [4], from the viewpoint of the triangulation. See [10, Sections 1, 2] for details. A review of the normal surface theory based on handle-decompositions is given in [9, Section 1]. If X is a topological space, then we denote the number of component by $|X|$. If Y is a subset of X, then $\mathring{Y}$ denotes the topological interior of Y in the topology of X. Definitions not stated in this paper are found in [7] and [8].

Let $\mathcal{T}$ be a triangulation of a compact 3-manifold M. Let $\mathcal{T}^{(i)}$ be the set of i-simplexes of $\mathcal{T}$ and let $\mathcal{T}^i$ be the i-skeleton. An embedded surface S is *normal with respect to* $\mathcal{T}$ if S is in general position with respect to $\mathcal{T}$ and for any 3-simplex τ in $\mathcal{T}^{(3)}$, $S \cap \tau$ is a union of squares and triangles whose vertices are contained in separate edges of τ. These squares and triangles are called *normal disks* in τ. Two normal disk Δ_1 and Δ_2 in a 3-simplex τ are *equivalent* if these are parallel in τ. The equivalent classes are called *disk types* in τ.

If a surface S is in general position with respect to $\mathcal{T}^1$ and $S \cap \mathcal{T}^0 = \emptyset$, then we call the number of points of the intersection $S \cap \mathcal{T}^1$ *complexity* of S, and we denote it by $\gamma(S)$. A surface S is called a *minimal complexity surface* (or a *minimal surface, minimal*) if $\gamma(S)$ is minimal among all surfaces in the isotopy class of S.

A surface S properly embedded in M is *incompressible in M* if either: (a) S is homeomorphic to a 2-sphere which does not bound a 3-ball in M;

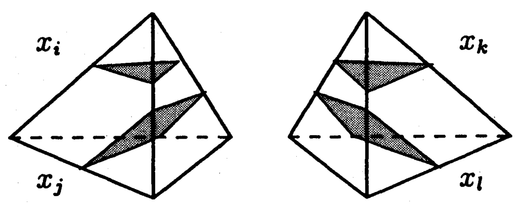
(b) S is homeomorphic to a 2-disk such that ∂S does not bound a disk in ∂M, or if ∂S bounds a disk E in ∂M, then the sphere $S \cup E$ does not bound a 3-ball in M;

(c) S is homeomorphic neither to a 2-sphere nor to a 2-disk and for any disk D with $S \cap D = \partial D$, ∂D bounds a disk in S.

It is known that every incompressible surface in M is isotopic to a minimal normal surface ([4]). A surface S properly embedded in M is said to be *two-sided in M* if a regular neighborhood $N(S)$ is homeomorphic to a product $S \times I$ where $I = [0, 1]$, otherwise S is said to be *one-sided.* For a two-sided surface S, we regard it as $S \times \{1/2\} \subset N(S) = S \times I$.

There are seven disk types in a 3-simplex (four triangles and three squares). Once a triangulation $\mathcal{T}$ of M and a normal surface S are given, a $7t$-tuple $x = (x_1, \cdots, x_{7t})$ corresponds to S, where t is the cardinality of $\mathcal{T}^{(3)}$ and x_i is the number of normal disks in the corresponding disk types.

Let σ be a simplex in $\mathcal{T}^{(2)}$. Let τ_1 and τ_2 be simplexes in $\mathcal{T}^{(3)}$ such that $\tau_1 \cap \tau_2 = \sigma$. A *matching equation* is the equation on the number of disks between adjacent normal disk types (Figure 1). It was observed by Haken that there is a finite set of integral solutions to the matching equations and inequalities $x_i \geq 0$, so that any non-negative integral solution is a non-negative integral linear combination of this finite set of solutions. These solutions are called the *fundamental solutions.* See [6, Sections 8, 9] for fundamental solutions. A fundamental integer solution $x = (x_1, \cdots, x_{7t})$ can be characterized by the property of not having a decomposition $x = y + z$ where y and z are non-negative integer solutions to the matching equations.



$$x_i + x_j = x_k + x_l$$

FIGURE 1. Matching equation.

Now we consider how to construct embedded surfaces from a $7t$-tuple. There exist three disk types which are squares in a tetrahedron, but only one type, of the three, normal disks can exist at once. If a non-negative integral solution x satisfies the following property:

($\star$) If x_i, x_j and x_k corresponding to the distinct disk types which are squares in the same tetrahedron, then two of the three is equal to zero.

Then one can construct a normal surface F corresponds to the solution x. A normal surface corresponds to a fundamental solution is called a *fundamental surface for* $\mathcal{T}$.

Hereafter, unless stated otherwise, all 3-manifolds are assumed to be orientable. If F is a normal surface corresponds to the $7t$-tuple $(x_1, \cdots, x_{7t})$, we denote the normal surface corresponds to $(nx_1, \cdots, nx_{7t})$ by nF. Two normal surfaces F_1 and F_2 are said to be *compatible* if there is no 3-simplex τ in $\mathcal{T}^{(3)}$ such that F_1 and F_2 intersect τ with squares of distinct types. An isotopy of M is said to be *normal* if it preserves the triangulation $\mathcal{T}$. If F_1 and F_2 are compatible normal surfaces, then there exist surfaces F_1' and F_2' such that each F_i is normally isotopic to F_i' and for each component Δ_i of $F_i' \cap \tau$, the intersection $\Delta_1 \cap \Delta_2$ is empty or a single embedded arc with ends in the distinct faces of τ for each $\tau \in \mathcal{T}^{(3)}$. Hence if F_1 and F_2 are compatible, we may assume each F_i is normally isotoped so that they satisfy the above condition. So, there is a "geometric addition" of F_1 and F_2 achieved by removing a small regular neighborhood of a component a of $F_1 \cap F_2$ and pasting annuli as shown in Figure 2. We call the cores of the pasted annuli *trace curves*. Clearly along any curve a of $F_1 \cap F_2$, there are two ways to switch, but we perform a *regular switch* which yields a surface $F = F_1 + F_2$ actually in normal position without further isotopy (Figure 2(B)), the other switch is called an *irregular switch* (Figure 2(C)). This geometric addition of F_1 and F_2 agrees with the natural linear algebra coming from the solution space of the matching equations, if F_i corresponds to the $7t$-tuple $(x_1^i, \cdots, x_{7t}^i)$ for $i = 1, 2$, then $F_1 + F_2$ corresponds to the $7t$-tuple $(x_1^1 + x_1^2, \cdots, x_{7t}^1 + x_{7t}^2)$. It follows that each normal surface in $\mathcal{T}$ is an integral linear combination of the finite set of fundamental surfaces for $\mathcal{T}$.

The *projective solution space* for $\mathcal{T}$ is the set of solutions to matching equations $x_i + x_j = x_k + x_l$, inequalities $x_i \geq 0$ and *normal equations* $x_1 + \cdots + x_{7t} = 1$. The projective solution space, which is denoted by $\mathcal{P}$, is a compact, convex, linear cell. If v is a vertex of $\mathcal{P}$, then it has rational coordinates. If k is the smallest non-negative integer such that kv is integral, then kv is called an *edge solution*. The surface corresponding to the edge solution kv is called *edge*

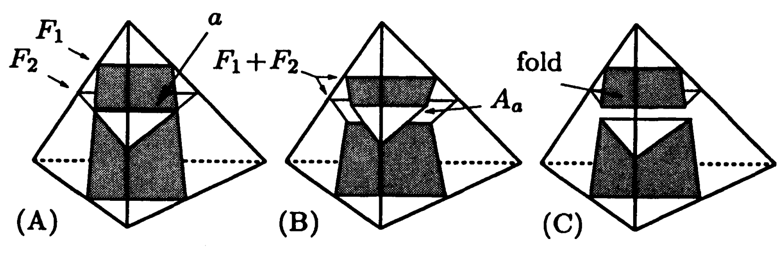


FIGURE 2. Regular and irregular switch.

surface. Notice that lattice points in $\mathcal{P}$ are fundamental solutions. It is remarked that for any normal surface F, there exists an integer n such that $nF = n_1E_1 + \cdots + n_kE_k$ where E_i is an edge surface and n_i is a non-negative integer. Edge surfaces are connected and necessarily fundamental surfaces.

The sum $F = F_1 + F_2$ is in *reduced form* if F cannot be written as $F = F_1' + F_2'$, where F_i' is a normal surface isotopic to F_i and $F_1' \cap F_2'$ has fewer components than $F_1 \cap F_2$. For a surface $F = F_1 + F_2$, the components of $F_1 \cup F_2 - \dot{N}(F_1 \cap F_2)$ are called *patches of* $F_1 + F_2$.

Now we prove some lemmas needed later.

LEMMA 1.1. *Let M be an orientable irreducible* 3*-manifold and let F be an incompressible minimal normal surface in M. Suppose nF is represented as a sum $nF = n_1F_1 + G$ and there does not exist a normal surface G' such that:*

(1) *G' is isotopic to G,*
(2) $\gamma(G') = \gamma(G)$,
(3) $|F_1 \cap G'| < |F_1 \cap G|$.
(4) *$n_1F_1 + G'$ is isotopic to nF.*

Then no patch of $n_1F_1 + G$ is a disk.

PROOF. We assume the contrary. Since F is minimal, the surface nF is also minimal by the argument same as in [9, Lemma 3.1]. If e is a disk patch of the sum $nF = n_1F_1 + G$ and a is the component of $n_1F_1 \cap G$ which determines the patch e, the curve a must be two-sided in both component since a bounds a disk. So the curve a contributes two trace curves in nF, say α (bounding e) and α'. Clearly, the trace curve α' bounds a disk e_0 in M such that $e_0 \cap F = \alpha'$ (e_0 is a parallel copy of e). Hence α' bounds a disk e' in nF by the incompressibility of nF (Figure 3). First notice that the disk e' is such that if an irregular switch is made at the curve a, then e and e' are joined to form a 2-sphere. For otherwise, if e' does not contain e in its interior, then the 2-sphere $\Sigma = e' \cup e_0$ separates nF. Since M is irreducible, a component of nF would be in a 3-cell bounded by Σ, which is impossible since nF is incompressible. If e' contains e in its interior, then the surface nF would be isotopic to the surface $F' = (nF - e') \cup e_0$. Since F' has a "fold" coming from ∂e_0 or $\gamma(e' - e) > 0$ (see [17] for details), the surface F' is isotopic to a normal surface F'' with $\gamma(F'') < \gamma(nF)$. This is a contradiction to the minimality of $\gamma(nF)$.

Since nF is a minimal surface, there exists no compressible torus or Klein bottle T such that $nF = S + T$ where S is a surface isotopic to nF and $\gamma(T) \neq 0$. Hence if some patch of $n_1F_1 + G$ is a disk, then by the above argument and by the same argument as the proof of [9, Lemma 2.1], we obtain two disks $D_1 \subset n_1F_1$ and $D_2 \subset G$ which are patches of $n_1F_1 + G$ determined by the same component β of $n_1F_1 \cap G$. If $\gamma(D_1) \neq \gamma(D_2)$, then nF would be isotopic to a surface F^* such that $\gamma(F^*) < \gamma(nF)$ since the sphere $D_1 \cup D_2$ bounds a 3-ball on the side not containing the incompressible surface nF. Now we have $\gamma(D_1) = \gamma(D_2)$ since nF is minimal.

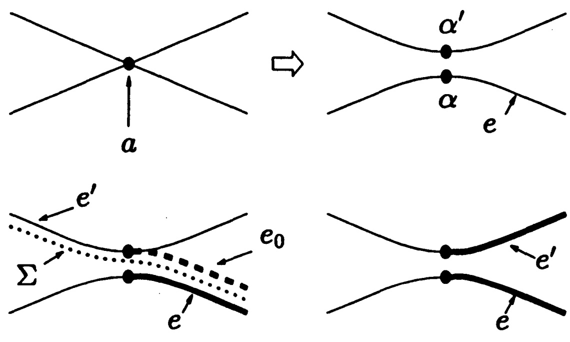


FIGURE 3.

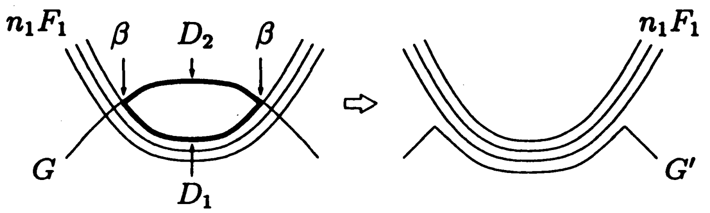


FIGURE 4.

If we set the surface $G' = (G - N(D_2)) \cup$ (a parallel copy of D_1) as shown in Figure 4, then $n_1F_1 + G'$ is isotopic to nF, $\gamma(G') = \gamma(G)$ and $F_1 \cap G'$ has fewer components. This contradicts the assumption. □

As a corollary, for a two-sided incompressible minimal normal surface $F = F_1 + F_2$ in a compact irreducible orientable 3-manifold, if the sum $F_1 + F_2$ is in reduced form, then it has no disk patch (cf. [9, Lemma 2.1] and [10, Lemma 6.6]).

LEMMA 1.2. *Let M be an orientable irreducible 3-manifold, and let S be a two-sided acylindrical surface in M. Then each component of $M - \mathring{N}(S)$ contains no properly embedded Möbius band.*

PROOF. We assume that A is a properly embedded Möbius band in $M - \mathring{N}(S)$. Since M is orientable, the frontier $A' = \partial N(A; M - \mathring{N}(S))$ is an annulus properly embedded in $M - \mathring{N}(S)$. If A' is compressible in $M - \mathring{N}(S)$, then it follows that for a compressing disk D', $\partial D'$ bounds a Möbius band A'' in $N(A; M - \mathring{N}(S))$ and $D' \cup A''$ forms a projective plane. By the irreducibility of M, it follows that M is homeomorphic to P^3. However this contradicts the assumption that S is two-sided acylindrical since P^3 does not contain two-sided incompressible surfaces. So, the annulus A' is incompressible in $M - \mathring{N}(S)$. Since S is acylindrical, the annulus A' is ∂-parallel to $\partial N(S)$. As the parallelism solid torus cannot be contained in $N(A; M - \mathring{N}(S))$, it follows that S is isotopic to a torus which bounds a solid torus. This contradicts the incompressibility of S. □

LEMMA 1.3. *Let M be an orientable irreducible 3-manifold and let S be an incompressible minimal normal surface. If the sum $S = F_1 + F_2$ has no disk patch, then for each* *component a of $F_1 \cap F_2$ and for the annulus or Möbius band A_a bounded by the trace curves or curve contributed by a, the frontier $A = \partial N(A_a; M_0)$ is essential in M_0 where M_0 denotes the closure of $M - S$.*

PROOF. By an innermost argument on a compressing disk for A, the annulus A is incompressible in M_0 since the sum $S = F_1 + F_2$ has no disk patch. If A is ∂-parallel to an annulus $B \subset \partial M_0$, then S is isotopic to a surface $S' = (S - B) \cup A$. But S' has a "fold" coming from the curve a or $\gamma(B) > 0$ (see [17] for details). So the surface S' is isotoped to a normal surface S'' with $\gamma(S'') < \gamma(S)$. This contradicts the minimality of $\gamma(S)$. □

THEOREM 1.4 ("Acylindrical" implies "edge"). *Let M be an orientable, irreducible 3-manifold. Each acylindrical surface is isotopic to an edge surface for any triangulation of M.*

PROOF. Let F be an acylindrical surface which is a minimal and normal surface. For some positive integer n, we have $nF = n_1E_1 + \cdots + n_kE_k$ where E_i is an edge surface and n_i is some non-negative integer. By [10, Theorem 6.5], each edge surface E_i is incompressible in M.

If $n = 1$, then we are done by Lemma 1.3. So, we assume $n \geq 2$. We can take a regular neighborhood N of F such that N is homeomorphic to $F \times I$ and nF is embedded in N so that the first parallel copy and the n-th parallel copy of F form ∂N.

CLAIM 1.5. For each i, E_i is isotopic into N.

Since E_i is incompressible in M, it is also incompressible in N. It is well-known that closed incompressible surface in the product $F \times I$ is unique up to isotopy ([18]). So, by this claim we can conclude that E_i is isotopic to F in N for each i, since N is a product $F \times I$. Thus F is isotopic to an edge surface E_i. □

PROOF OF CLAIM 1.5. Without loss of generality, it is sufficient to show that E_1 is contained in N. We set $G = n_2E_2 + \cdots + n_kE_k$ and apply Lemma 1.1 to the sum $nF = n_1E_1 + G$. Furthermore, we may assume that the sum $nF = n_1E_1 + G$ satisfies the condition in Lemma 1.1. So, no patch of $n_1E_1 + G$ is a disk.

By the definition of a regular switch, for each component a of $E_1 \cap G$, we have $a \subset \dot{N}$ or $a \subset M - N$. Hence if E_1 is not contained in N, the closure of $E_1 - N$ is a union of annuli and Möbius bands properly embedded in $M - \dot{N}$.

By Lemma 1.2, each component A of the closure of $E_1 - N$ is homeomorphic to the annulus. By an innermost argument on a compressing disk for A, the annulus A is incompressible in $M - \dot{N}$ since no patch of $n_1E_1 + G$ is a disk. Since F is acylindrical, it follows that A is ∂-parallel in $M - \dot{N}$. Thus there exists an annulus B in ∂N with $A \cap B = \partial A = \partial B$ such that A is isotopic to B (rel. ∂A). It follows that E_1 is isotopic into N. One can also conclude this by the fact that $\gamma(B) > 0$. □

Now we can give a proof of the following finiteness theorem.

THEOREM 1.6 (Finiteness, cf. [5], [14], [15]). *There are only finitely many acylindrical surfaces, up to isotopy, in a compact* 3*-manifold.*

PROOF. Recall that a properly embedded annulus is said to be *essential* if it is incompressible and not ∂-parallel in the 3-manifold. It is known that if M is not irreducible, then M does not contain acylindrical surfaces ([15]). So, we may assume M is irreducible. By Lemma 1.4, a closed acylindrical surface in a compact irreducible 3-manifold is isotopic to some edge surface. Since the number of edge surfaces is finite, the conclusion holds. □

If properly embedded annuli in a 3-manifold which are incompressible and ∂-incompressible are defined to be essential, then one can construct a reducible 3-manifold M such that M contains infinitely many acylindrical surfaces, up to isotopy. An example of such a 3-manifold is given in [15]. Notice that these two definitions of essentiality are equivalent for irreducible and ∂-irreducible 3-manifolds.

In [5], it was remarked that D. Gabai had pointed out that Theorem 1.6 can be obtained by the arguments of branched surface theory. In [15], the finiteness properties of acylindrical surface are also studied using branched surfaces.

Here we give another finiteness result obtained by the normal surface theory. A homotopy H between two closed curves in a surface S embedded in M can be decomposed into essential homotopies in $M - \dot{N}(S)$. The number of these essential subhomotopies is called the *length of* H. An incompressible surface S embedded in M is said to be *k-acylindrical* if no homotopy between closed curves in S has length greater than k. It is known that if M is closed hyperbolic and S is not a fiber, that is, each component of $M - \dot{N}(S)$ is an I-bundle, then S is acylindrical or k-acylindrical for some k (see [16]).

THEOREM 1.7 (Finiteness, [16]). *Let k be any positive integer. A compact, irreducible* 3*-manifold M contains only finitely many isotopy classes of closed k-acylindrical surfaces.*

PROOF. Let S be a k-acylindrical minimal normal surface in M which is written as a sum $S = a_1F_1 + \cdots + a_nF_n$ where F_i is a fundamental surface. Set $G = a_2F_2 + \cdots + a_nF_n$ and by Lemma 1.1, we may assume that no patch of $S = a_1F_1 + G$ is a disk. By Lemma 1.3, there exists an embedding of an annulus $f : A \to M$ "near" the intersection $a_1F_1 \cap G$, so that $f(\partial A) \subset S$, $f^{-1}(S)$ consists of a_1 components and the closure of each component of $f(A) - S$ is essential in the closure of $M - S$. The annulus $f(A)$ forms an essential homotopy of length a_1 between two curves $f(\partial A) \subset S$. Since S is k-acylindrical, we have $a_i \leq k$ for each i. This means that the number of choice of the coefficients $\{a_1, \cdots, a_n\}$ is finite and completes the proof. □

REMARK 1.8. In [14], Sela studied a much larger class of groups than 3-manifold groups, and obtained the same k-acylindrical finiteness result for a simple 3-manifold ([14, Theorem 4.5]).

In [2], Agol showed that if a hyperbolic 3-manifold M contains an acylindrical surface F, the volume $Vol(M) \geq -2V_1\chi(F)$ where V_1 is the volume of a regular ideal tetrahedron, approximately equal to 1.01494. The *canonical genus* of a knot K in S^3 is the minimum among all genera of Seifert surfaces built by "Seifert's algorithm" on a diagram of the knot, denoted by $g_c(K)$. Brittenham ([3]) showed that the volume of the complement of a hyperbolic knot K with canonical genus $g_c(K)$ is less than $120 g_c(K) V_1$. Brittenham's results and Agol's results prove the following theorem.

THEOREM 1.9. *Let K be a hyperbolic knot in S^3. If F is a closed acylindrical surface embedded in $S^3 - K$, then $g(F) \leq 30 g_c(K)$.*

The following result is obtained in [12] by arguments of amalgamated free products along malnormal subgroups. In fact, an embedded surface S is acylindrical in an irreducible 3-manifold M if and only if $i_*(\pi_1(S))$ is malnormal in $\pi_1(M)$.

THEOREM 1.10 (Two generator 3-manifolds, [12, Theorem 4]). *Let M be an orientable 3-manifold with $\pi_1(M)$ two-generated. Then M contains no separating acylindrical surface.*

By Theorem 1.10, if M admits a genus two Heegaard splitting, that is, M is obtained by gluing two genus two handlebodies with their boundaries, then M does not contain separating acylindrical surfaces. But there exists a 3-manifold which admits a genus two Heegaard splitting and contains a non-separating acylindrical torus. For example, let M' be the exterior of the Whitehead link L, and let M be the closed manifold obtained by identifying two boundaries of M' so that a meridian of one component of $\partial N(L)$ is identified with that of the other component. See [11] for 3-manifolds which contain incompressible tori and have genus two Heegaard splittings.

According to Theorems 1.9 and 1.10, it seems that a 3-manifold M which contains an acylindrical surface is somewhat complicated. But using the algorithm which will be obtained in Section 2, one can decompose M into "simpler" parts each of which contains no acylindrical surfaces.

2. Algorithms.

At first, we give an algorithm to judge if M is atoroidal or not (cf. [10, Corollary 6.8 and Section 8]). Next, we give an algorithm to decide if an orientable, irreducible, atoroidal, 3-manifold contains an acylindrical surface or not, and constructing acylindrical surfaces. Hereafter, we assume all 3-manifolds are triangulated.

The proof of the following lemma is given in [17].

LEMMA 2.1 ([17], cf. [10, Corollary 6.8]). *Let M be an orientable irreducible 3-manifold. If M contains an essential torus, then some edge surface E_i is an essential torus.*

Now one can test the atoroidality of M as follows:

LEMMA 2.2. *Let M be an orientable, irreducible, 3-manifold. There exists an algorithm to decide whether M is atoroidal or not.*

PROOF. Let $\mathcal{T}$ be a triangulation of M. By solving the matching and normal equations, one can find the finite number of edge surfaces $\{E_1, \cdots, E_k\}$. By the data of $\mathcal{T}$, one can calculate the Euler characteristic $\chi(E_i)$ for each i ([10, Algorithm 9.1]). If E_i is one-sided, then we replace it by $2E_i$. Using the Haken's algorithm and [10, Algorithm 9.7], one can check the essentiality of the edge surface E_i. By Lemma 2.1, if each edge surface is not an essential torus, then it follows that M is atoroidal. □

LEMMA 2.3. *Let M be an orientable, irreducible, atoroidal 3-manifold. An incompressible surface S is acylindrical if and only if the double of each component of $M - \dot{N}(S)$ is atoroidal.*

PROOF. If S is acylindrical, each component of $\partial N(S)$ is also acylindrical in the double of $M - \dot{N}(S)$ along the boundary component corresponding to S. We denote it by $\mathcal{D}(M - \dot{N}(S))$. An essential torus T in $\mathcal{D}(M - \dot{N}(S))$ is isotoped off of acylindrical surfaces. This means that T is contained in M, which contradicts the hypothesis that M is atoroidal. If S is not acylindrical, there exists an essential annulus A in $M - \dot{N}(S)$. The double $\mathcal{D}A$ is an essential torus in $\mathcal{D}(M - \dot{N}(S))$. □

By Lemma 2.2 and Lemma 2.3, one can decide if an incompressible surface S is acylindrical or not.

THEOREM 2.4 (The algorithm). *There exists an algorithm to decide if a closed, orientable, irreducible 3-manifold is toroidal, or contains a closed acylindrical surface. Furthermore, all acylindrical surfaces are constructed by the algorithm.*

PROOF. We shall prove this theorem by producing an explicit algorithm.

We are given a triangulation $\mathcal{T}$ of an orientable, irreducible 3-manifold M. By solving matching and normal equations, it is possible to find the finite number of edge surfaces $\mathcal{E} = \{E_1, \cdots, E_k\}$.

If E_i is one-sided, then we replace it by $2E_i$ which is two-sided. Apply the algorithm of Haken to each closed edge surfaces. If none of closed edge surface which is not homeomorphic to S^2 in $\mathcal{E}$ is injective, M contains no closed incompressible surface by the argument same as in [9]. Using the algorithm obtained by Lemma 2.2, one can test the atoroidality of M. If M is atoroidal, then we go to the next step.

At this stage, M is assumed to be orientable, irreducible and atoroidal. In Lemma 1.4, we have seen that an acylindrical surface is isotopic to an edge surface. By Lemmas 2.2 and 2.3, one can decide if an edge surface E_i is acylindrical or not.

The edge surface E_i is constructed by pasting normal disks corresponding to the edge solution, so acylindrical surfaces are constructible. □

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