On the Complex WKB Method for a Secondary Turning Point Problem

Minoru NAKANO

Keio University

1. Introduction.

1.1. We consider the following n-th order linear ordinary differential equation containing a small parameter ε

(1.1)
$$\varepsilon^{nh} y^{(n)} = \sum_{k=1}^{n} \varepsilon^{(n-k)h} p_k \cdot (x^m - \varepsilon^l)^k y^{(n-k)} \quad (x, y \in \mathbf{C}, |x| \le x_0, 0 < \varepsilon \le \varepsilon_0, ':= d/dx),$$

where h, x_0 and ε_0 are positive constants, and m and l are positive integers. The characteristic equation for (1.1) is defined by

(1.2)
$$L(x, \lambda) := \lambda^n - \sum_{k=1}^{n} p_k \cdot x^{km} \lambda^{n-k} = 0$$

and we suppose that it is factored as follows:

(1.3)
$$L(x, \lambda) = \prod_{k=1}^{n} (\lambda - a_k \cdot x^m) = 0,$$

where a_k's are real constants such that

(1.4)
$$a_1 < a_2 < \cdots < a_n; \quad \forall a_k \ne 0.$$

Then the characteristic roots $\lambda_k := a_k \cdot x^m$ coincide at the origin $x = 0$, which is, by definition, *a turning point* of (1.1). Furthermore we suppose that the following equality among three constants h, m and l is valid:

(1.5)
$$h = \frac{l(m+1)}{m} + 1.$$

We call (1.5) *the singular perturbation condition*, by which *a stretched equation* (2.2) in §2 below becomes a singular perturbation type. The stretched equation possesses its own turning points and they are called *secondary turning points* of (1.1). Thus, we call our analysis *a secondary turning point problem*.

1.2. In analyzing our secondary turning point problem, we need solutions of the stretched equation in order to know an asymptotics in the neighborhood of $x = 0$. Thus, we adopt the so-called *stretching-matching method* (Nakano [7], Nishimoto [13], Wasow [18]) for (1.1) by dividing the domain $|x| \leq x_0$ into two subdomains. In them two reduced equations are obtained. One of them is a stretched equation. They are separately analyzed in each subdomain, and their solutions should be related linearly by *a matching matrix*.

Secondary turning point problems are treated in Fedoryuk [3], Nakano [8]–[10], Nakano-Nishimoto [11], Roos [15], [16] and Wasow [20], etc. Other contributions can be seen in the references of these papers and books.

The contents of this paper are as follows: In §2 we reduce (1.1) to two equations, one of which is the stretched equation, in respective divided subdomains near the turning point. In §3 and §4 we will introduce WKB solutions for n-th order differential equations that can be regarded as generalization of the classical WKB solutions for the one-dimensional Schrödinger equation. In §5 we construct a canonical domain, i.e., the maximal existence region of the true solutions whose asymptotic expansions are WKB solutions for the stretched equation. In §6 we show existence theorems of the solutions for the reduced differential equations, and we compute a matching matrix in §7.

We have analyzed in Nakano [10] the differential equation

$$(1.1)_r \qquad \varepsilon^{nh} y^{(n)} = \sum_{k=1}^{n} \varepsilon^{(n-k)h} p_k \cdot (x^m - \varepsilon^l/x^r)^k y^{(n-k)},$$

where r is a positive integer. This differential equation has a singular point at the origin, which is a turning point at the same time. In this paper we study the case where r is zero in $(1.1)_r$.

2. Reduction of (1.1) and the problem.

2.1. We can rewrite (1.1) as follows:

$$\varepsilon^{nh} y^{(n)} = \sum_{k=1}^{n} (x^{-m/l} \varepsilon)^{(n-k)h} \cdot x^{m(n-k)h/l+mk} \cdot p_k \cdot \{1 + O((x^{-m/l}\varepsilon)^l)\} y^{(n-k)}$$

for (x, ε) such that $x^{-m/l}\varepsilon \leq 1/K'$, where K' is a sufficiently large constant. We can neglect the term $O((x^{-m/l}\varepsilon)^l)$ as a regular perturbation term if $x^{-m/l}\varepsilon \to 0$ (Iwano-Sibuya [6]). Then, if $x^{-m/l}\varepsilon \to 0$, the leading term of the solution of (1.1) is derived from the reduced differential equation

$$(2.1) \qquad \varepsilon^{nh} y^{(n)} = \sum_{k=1}^{n} \varepsilon^{(n-k)h} p_k \cdot x^{km} y^{(n-k)},$$

whose solutions should be got in (a subdomain of) the domain $D^{out} := \{x : K\varepsilon^{l/m} \leq |x| \leq x_0\}$, where K is a sufficiently large constant. The equation (2.1) is called *the outer equation* of (1.1), and the x-domain D^{out} is called *the outer domain* for (1.1).

2.2. For x and ε not belonging to D^{out}, we introduce a new variable $t := x\varepsilon^{-1/m}$ (*a stretching transformation*) and apply (1.5) to (1.1). Then we get *a stretched equation* of (1.1)

$$(2.2) \qquad \varepsilon^n y^{(n)} = \sum_{k=1}^{n} \varepsilon^{n-k} p_k \cdot (t^m - 1)^k y^{(n-k)} \qquad (t := x\varepsilon^{-1/m}, ' := d/dt)$$

in the domain $\{t : |t| \leq K\}$, which is the complement of D^{out} in the domain $|x| \leq x_0$ but the boundary $|t| = K$ is common. We call (2.2) *the inner equation* of (1.1).

Since a characteristic equation of (2.2) is defined by

$$(2.3) \qquad \prod_{k=1}^{n} \{\lambda - a_k \cdot (t^m - 1)\} = 0,$$

its characteristic roots λ's coincide at zeros of $t^m - 1$. The zeros of $t^m - 1$ are turning points of (2.2) and they are called, by definition, *secondary turning points* of (1.1). The turning point $x = 0$ corresponds to $t = 0$ that is not a secondary turning point. Thus, if we can get solutions of (2.2) in (a subdomain containing $t = 0$ of) the domain $\{t : |t| \leq K\}$, we can know some information of asymptotics about the turning point $x = 0$. Furthermore, if we can get soutions of (2.2) in (an unbounded subdomain containing $t = 0$ of) $D^{in} := \{t : |t| < \infty\}$, we can use the matching method and we can match, i.e., connect two sets of solutions of the outer and the inner equations because D^{out} and D^{in} have common interior points. We call the t-domain D^{in} *the inner domain* for (1.1).

Thus, our main problems are

(i)   to get solutions of (2.1) and (2.2) (§§4, 6), and

(ii) to match them at a common interior point of D^{out} and D^{in} (§7).

3. WKB solutions and the double asymptotic property.

3.1 We recall several known results to obtain the formal solutions of (2.1) and (2.2). Both equations have a common form except exponents of ε, i.e., they are of n-th order singular perturbation type with polynomial coefficients. Therefore they may have similar type of solutions.

Thus we consider here the following linear ordinary differential equation with polynomial coefficients $q_k(x)$'s

$$(3.1) \qquad L[y] := \sum_{k=0}^{n} \varepsilon^{n-k} q_k(x) y^{(n-k)} = 0 \qquad (':= d/dx, q_0(x) \equiv 1, 0 < \varepsilon \leq \varepsilon_0).$$

The characteristic equation of (3.1) is given by

$$(3.2) \qquad L(x, \lambda) := \sum_{k=0}^{n} q_k(x) \lambda^{n-k} = 0,$$

where the characteristic roots $\lambda_j(x)$'s are supposed to satisfy the condition $\lambda_j(x) \neq \lambda_k(x)$ ($j \neq k$). The point x_0 is, by definition, *a turning point* of (3.1) if x_0 satisfies $\lambda_j(x_0) = \lambda_k(x_0)$

$(j \neq k)$. We define *WKB solutions* $\tilde{y}_j(x, \varepsilon)$'s $(j = 1, 2, \cdots, n)$ of (3.1) by

$$(3.3) \quad \tilde{y}_j(x, \varepsilon) := \exp\left(\frac{1}{\varepsilon} \int_a^x \lambda_j(t)dt + \int_a^x \lambda_j^*(t)dt\right), \quad \lambda_j^*(x) := \sum_{k \neq j} \frac{\lambda_j'(x)}{\lambda_k(x) - \lambda_j(x)},$$

where a is a constant. The WKB solutions (3.3) are the truncated formal series solutions of (3.1). How to get (3.3) is outlined in Nakano [10]. The formula (3.3) is originally led in Федорюк [2]. They are considered as generalization of the classical WKB solutions

$$(3.3)' \quad \frac{1}{\sqrt[4]{Q(x)}} \exp\left(\pm\frac{1}{\varepsilon} \int^x \sqrt{Q(x)}dx\right)$$

for the one-dimensional Schrödinger equation $\varepsilon^2 y'' = Q(x)y$.

3.2. Let D_j be a simply connected and unbounded region in the complex x-plane containing no turning points of (3.1). Fixing j $(= 1, 2, \cdots, n)$, we introduce symbols

$$(3.4) \quad \xi_j(a, x) := \int_a^x \lambda_j(s)ds,$$

$$(3.5) \quad \xi_{jk}(a, x) := \xi_j(a, x) - \xi_k(a, x) \quad (k = 1, 2, \cdots, n; k \neq j).$$

Let $\gamma_{jk}(x)$ $(k \neq j)$ be a curve joining x and ∞ in D_j. We call $\gamma_{jk}(x)$ a (j, k)-*canonical path* for (3.1) if the function $\Re \xi_{jk}(x, t)$ of t decreases when t varies from the fixed x to ∞ along $\gamma_{jk}(x)$. A set of paths $\Gamma_j(x) := (\gamma_{j1}(x), \cdots, \gamma_{j,j-1}(x), \gamma_{j,j+1}(x), \cdots, \gamma_{jn}(x))$ is called a j-*canonical vector path* for (3.1). If D_j is a maximal region such that there exists a j-canonical vector path $\Gamma_j(x)$ for each $x \in D_j$, then D_j is called a j-*admissible region* for (3.1). Then we can show the following existence theorem of the solutions for (3.1).

LEMMA 3.1. *Let* D_j $(j = 1, 2, \cdots, n)$ *be a* j-*admissible region for* (3.1). *Then* (3.1) *has a true solution* $y_j(x, \varepsilon)$ *with the asymptotic property*:

$$(3.6) \quad y_j(x, \varepsilon) = \tilde{y}_j(x, \varepsilon)\{1 + \varepsilon\delta_j(x, \varepsilon)\},$$

where $\tilde{y}_j(x, \varepsilon)$ *is the WKB solution of* (3.1) *and* $\delta_j(x, \varepsilon)$ *satisfies the condition*

$$(3.7) \quad |\delta_j(x, \varepsilon)| \leq \nu_j(x) \quad (0 < \varepsilon \leq \varepsilon_0), \quad \lim_{D_j \ni x \to \infty} \nu_j(x) = 0.$$

Moreover, D_j *is the maximal existence region of the true solution* $y_j(x, \varepsilon)$ *with the property* (3.6).

The proof is very similar to that of the result for the lower order differential equation given in Nakano et al. [12] and it is omitted here.

When D_j is the j-admissible region for (3.1), we call the intersection $\bigcap_{j=1}^n D_j$ a *canonical domain* for (3.1). Then, from the above lemma and the definition of the canonical domain for (3.1), we get the following

LEMMA 3.2. *Let D be the canonical domain of (3.1). Then, in D, there exist n linearly independent true solutions $y_j(x, \varepsilon)$'s $(j = 1, 2, \cdots, n)$ of (3.1) such that*

$$(3.8) \qquad y_j(x, \varepsilon) \sim \tilde{y}_j(x, \varepsilon) \begin{cases} as \ \varepsilon \to 0, x \in D; \\ as \ x \to \infty \ in \ D, 0 < \varepsilon \leq \varepsilon_0. \end{cases}$$

The property (3.8) is called *the double asymptotic property* of solutions of (3.1) (Fedoryuk [3]). We will construct a canonical domain for (2.2) in §5, and Lemma 3.2 plays an important role in §6.

4. The outer and the inner WKB solutions.

4.1. Solutions of (2.1) are called *outer solutions* of (1.1), and WKB solution of (2.1) are called *outer WKB solutions* of (1.1), which are given below.

THEOREM 4.1. *There exist n outer WKB solutions $\tilde{y}_j^{out}(x, \varepsilon)$'s $(j = 1, 2, \cdots, n)$ of* (1.1) :

$$(4.1) \qquad \tilde{y}_j^{out}(x, \varepsilon) := x^{-m\mu_j} \exp\left(\frac{a_j}{\varepsilon^h} \frac{x^{m+1}}{m+1}\right) \quad \left(\mu_j := \sum_{k \neq j} \frac{a_j}{a_j - a_k}\right).$$

PROOF. Putting

$$(4.2) \qquad L[y] := \varepsilon^{nh} y^{(n)} - \sum_{k=1}^{n} \varepsilon^{(n-k)h} p_k \cdot x^{km} y^{(n-k)} = 0,$$

we get its characteristic equation

$$L(x, \lambda) := \lambda^n - \sum_{k=1}^{n} p_k \cdot x^{km} \lambda^{n-k} = \prod_{j=1}^{n} (\lambda - a_j x^m) = 0$$

with characteristic roots

$$\lambda = \lambda_j(x) := a_j x^m \quad (j = 1, 2, \cdots, n),$$

from which we get

$$\lambda_j^*(x) = -\frac{m\mu_j}{x}.$$

By substituting them into (3.3), we can obtain (4.1). Q.E.D.

4.2. Solutions of the inner equation (2.2) are called *inner solutions* of (1.1), and WKB solutions of (2.2) are called *inner WKB solutions* of (1.1), which are given below.

THEOREM 4.2. *There exist n inner WKB solutions $\tilde{y}_j^{in}(t, \varepsilon)$'s $(j = 1, 2, \cdots, n)$ of* (1.1) :

$$(4.3) \quad \tilde{y}_j^{in}(t, \varepsilon) := (t^m - 1)^{-\mu_j} \exp\left\{\frac{a_j}{\varepsilon}\left(\frac{t^{m+1}}{m+1} - t\right)\right\} \left(t := x\varepsilon^{-1/m}, \mu_j := \sum_{k \neq j} \frac{a_j}{a_j - a_k}\right).$$

The proof is very similar to that of Theorem 4.1 and is omitted.

5. The canonical domains for (2.2).

5.1.   In this subsection we give a brief sketch of the Fedoryuk's theory about the canonical domain for (3.1) to apply it to (2.2). Fedoryuk's theory is explained in, say, Evgrafov-Fedoryuk [1], Fedoryuk [3] and Wasow [20]. We use the notation introduced in §3.2 again:

$$(5.1) \qquad\qquad \xi_j(a, x) := \int_a^x \lambda_j(s)ds \,,$$

$$(5.2) \qquad \xi_{jk}(a, x) := \xi_j(a, x) - \xi_k(a, x) = \int_a^x \{\lambda_j(s) - \lambda_k(s)\}ds \quad (j \neq k) \,,$$

where $\lambda_j(t)$'s are the characteristic roots of (3.1) and a is a constant. The curves in the x-plane defined by the equations

$$(5.3) \qquad \Re\xi_{jk}(x_0, x) = 0 \,, \quad \Im\xi_{jk}(x_0, x) = 0 \quad (\lambda_j(x_0) = \lambda_k(x_0), j \neq k)$$

are respectively called *Stokes curves* and *anti-Stokes curves* of (3.1), which emerge from the turning point x_0. A *Stokes domain* is, by definition, a simply connected region in the x-plane bounded by several Stokes curves without any Stokes curves in it.

Since (3.1) has polynomial coefficients, it is known that there are two types of Stokes domains: one is *a half-plane type* and the other is *a strip type*. A Stokes domain of half-plane type is mapped conformally by the mapping $\xi = \xi_{jk}(a, x)$, which is defined by (5.2), onto a right half plane $\Re\xi > C$ or a left half plane $\Re\xi < C$ ($C =$ const.) in the ξ-plane ($\Re\xi, \Im\xi$). A Stokes domain of strip type is mapped conformally onto a strip $C_1 < \Re\xi < C_2$ ($C_1, C_2 =$ const.) in the ξ-plane. In a Stokes domain of half-plane type there exists an anti-Stokes curve along which $\Re\xi_{jk} \rightarrow +\infty$ (or $-\infty$) as $x \rightarrow \infty$ if $\Re\xi_{jk} > 0$ (or $\Re\xi_{jk} < 0$) in it. The function $\Re\xi_{jk}(x, t)$ is a decreasing function of t when t varies from a fixed x to ∞ along a (j, k)-canonical path $\gamma_{jk}(x)$ in a j-admissible region D_j due to the definition of a (j, k)-canonical path (§3.2).

If a simply connected x-region is mapped one-to-one conformally onto the whole ξ-plane except for slits emerging from the images of the turning points, then it is a canonical domain for (3.1), because a canonical domain is the intersection of all j-admissible regions. A canonical domain consists of two half-plane type Stokes domains with or without strip type ones. In the following subsection we will construct canonical domains for (2.2).

5.2.   In order to get a canonical domain for (2.2) we need the topology of Stokes curves of (2.2). Since the characteristic roots of (2.2) are $a_k \cdot (t^m - 1)$ with $a_j \neq a_k$ ($j \neq k$), the Stokes and the anti-Stokes curves of (2.2) can be derived from the integral

$$(5.4) \qquad\qquad \xi = \xi(t_0, t) := \int_{t_0}^t (s^m - 1)ds \,,$$

where t_0 is a secondary turning point of (1.1), i.e., a zero of $t^m - 1$. Both Stokes and anti-Stokes curves are essentially the same as thoes for the third order differential equation studied in Nakano [9] and we state here only the information of the Stokes curves of (2.2).

THEOREM 5.1.   *The Stokes curves of* (2.2) *have the following properties.*

(a)   *Every Stokes curve emerges from a secondary turning point and it tends either to the point at infinity or to one of the other secondary turning points. Two Stokes curves from a secondary turning point do not tend to the infinity in the same direction.*

(b)   *Every Stokes curve emerging from a secondary turning point does not cross any Stokes curves emerging from any other secondary turning points except secondary turning points.*

(c)   *There exist no closed Stokes curves homotopic to a circle.*

PROOF.   The Stokes curves are determined by the equation $\mathfrak{R}\xi = 0$ where ξ is given by (5.4). The integrand $\sqrt{(t^m - 1)^2}$ of ξ is a square root of a polynomial and we get the result similar to the case of second order differential equations (Evgrafov-Fedoryuk [1]).   Q.E.D.

THEOREM 5.2.   *The Stokes curves of (2.2) have the following properties.*

(a)   *Four Stokes curves of (2.2) emerge from a secondary turning point t_0 in the four directions of arguments*

$$(5.5) \qquad \pm\frac{\pi}{4} - \frac{m-1}{2}\arg t_0, \quad \pm\frac{3\pi}{4} - \frac{m-1}{2}\arg t_0,$$

and Stokes curves tend to the point at infinity in the $2m + 2$ directions of arguments

$$(5.6) \qquad \pm\frac{\pi}{2(m+1)}, \pm\frac{3\pi}{2(m+1)}, \pm\frac{5\pi}{2(m+1)}, \cdots, \pm\frac{(2m+1)\pi}{2(m+1)}.$$

(b)   *The Stokes curve configuration is symmetric with respect to the real axis. There are no Stokes curves connecting two secondary turning points t_0 and t_0' if $\mathfrak{R}t_0 \neq \mathfrak{R}t_0'$.*

For a secondary turning point t_0 such as $t_0 \neq \pm 1$, there exists only one Stokes curve connecting t_0 and $\overline{t_0}$. For a secondary turning point t_0 ($\neq \pm 1$) such as $\mathfrak{R}t_0 \neq 0$, there exists only one Stokes curve connecting t_0 and $\overline{t_0}$ not passing through the origin. Here, the notation $\overline{t_0}$ means a complex conjugate of t_0.

(c)   *When $m = 4p$ ($p \in \mathbf{N}$), the imaginary axis is a Stokes curve.*

PROOF.   (a) By integrating (5.4), we get $\xi = (m/2)t_0^{m-1}\tau^2 + O(\tau^3)$ ($\tau := t - t_0$) for small τ. Putting $\tau := re^{i\theta}$ and $\mathfrak{R}\xi = 0$, we obtain arguments θ of (5.5) near t_0. For large t we see $\xi = t^{m+1}/(m+1) + O(t)$. Then putting $t := Re^{i\Theta}$ and $\mathfrak{R}\xi = 0$, we obtain arguments Θ of (5.6) near the point at infinity.

(b)   The equation

$$(5.7) \qquad \overline{\int_\alpha^\beta (s^m - 1)ds} = \int_{\bar\alpha}^{\bar\beta} (s^m - 1)ds \quad (\forall \alpha, \forall \beta \in \mathbf{C})$$

is valid and then holds the equality

$$\mathfrak{R}\int_\alpha^\beta (s^m - 1)ds = \mathfrak{R}\int_{\bar\alpha}^{\bar\beta} (s^m - 1)ds,$$

which shows the desired symmetry of the Stokes curve configuration.

Let t_0 and t_0' be secondary turning points. Since $\tau^{m+1} = \tau$ for any secondary turning point τ, we get the integral

$$\int_{t_0}^{t_0'} (s^m - 1)ds = \frac{m}{m+1}(t_0 - t_0'),$$

whose real part does not vanish if $\Re t_0 \neq \Re t_0'$, and so there are no Stokes curves connecting t_0 and t_0' if $\Re t_0 \neq \Re t_0'$.

If t_0 is a secondary turning point, then its complex conjugate $\overline{t_0}$ is also a secondary turning point, because the equation $t^m = 1$ induces the equation $\overline{t}^m = 1$. Then, from the relation

$$\Re \int_{t_0}^{\overline{t_0}} (s^m - 1)ds = \frac{m}{m+1}\Re(t_0 - \overline{t_0}) = 0,$$

we see that there exists a Stokes curve connecting t_0 and $\overline{t_0}$. There exists only one such Stokes curve because there exist no closed Stokes curves homotopic to a circle from Theorem 5.1.

If the point t_0 is a secondary turning point such that $0 < \Re t_0 < 1$, we get the following inequality:

$$\left(\Re \int_{t_0}^{0} (s^m - 1)ds\right) \cdot \left(\Re \int_{t_0}^{1} (s^m - 1)ds\right) = \left(\frac{m}{m+1}\right)^2 \Re t_0 \cdot \Re(t_0 - 1) < 0.$$

From this and the monotonicity of the integral $\Re \int_{t_0}^{t} (s^m - 1)ds$, we can see that there exists only one $t^* \in \mathbf{R}$ such that $0 < t^* < 1$ and $\Re \int_{t_0}^{t^*} (s^m - 1)ds = 0$. Hence, the Stokes curve above does not pass through the origin. Similarly, we get the result for the case: $-1 < \Re t_0 < 0$.

(c) There are two secondary turning points $\pm i$ for $m = 4p$ ($p \in \mathbf{N}$). We can see that the whole imaginary axis is a Stokes curve emerging from $\pm i$ by the direct computation of the integral $\int_{\pm i}^{ia} (s^m - 1)ds$ ($a \in \mathbf{R}$), which takes only the purely imaginary values.   Q.E.D.

We must obtain a canonical domain containing the origin $t = 0$ as its interior point because we want to know an asymptotic property of solutions of (1.1) near $x = 0$ (cf. §2.2). Thus, by Theorem 5.2, we should suppose $m \neq 4p$ ($p \in \mathbf{N}$).

To get the accurate Stokes curve configuration by the analytic means is extremely difficult. We should use a computer program such as Uchiyama-Nakano [17].

5.3. We suppose that $m \neq 4p$ ($p \in \mathbf{N}$) and we will construct a canonical domain for (2.2) containing the origin as its interior point.

We first assume $m \geq 5$. Let t_0 be a secondary turning point situating nearest to $t = i$ and $\Re t_0 > 0$, and let t_1 be a secondary turning point situating nearest to $t = i$ and $\Re t_1 < 0$. There exist two Stokes curves $l_0^{(1)}$, $l_1^{(1)}$ connecting t_0 and $\overline{t_0}$, t_1 and $\overline{t_1}$ respectively by Theorem 5.2 (Fig. 5-1). $L_0^{(1)}$, $L_0^{(2)}$ and L_1 are anti-Stokes curves and they do not cross each other, because they are Stokes curves induced from the integral $i\xi$ (cf. Theorem 5.1). Let D_0 be a Stokes domain bounded by two Stokes curves $l_0^{(2)}$ and $l_0^{(3)}$. Let D_1 be a Stokes domain bounded by two Stokes curves $l_1^{(2)}$ and $l_1^{(3)}$. Let D_2 be a Stokes domain bounded by six Stokes curves $l_0^{(1)}$,
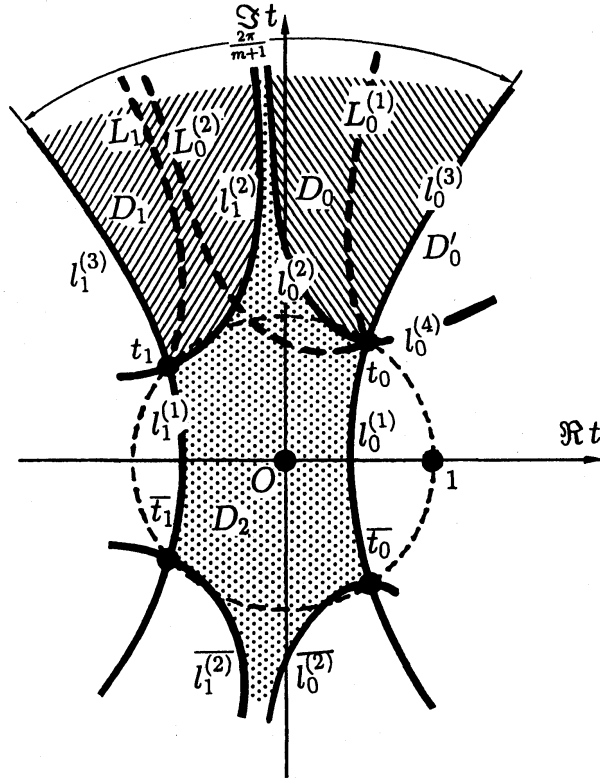
FIGURE. 5-1.   The canonical region D^{can}.

$l_0^{(2)}, \overline{l_0^{(2)}}, l_1^{(1)}, l_1^{(2)}$ and $\overline{l_1^{(2)}}$. Here, $l_j^{(*)}$'s $(j = 0, 1)$ are Stokes curves emerging from t_j with the properties:

$$\lim_{l_0^{(2)} \ni t \to \infty} \arg t = \lim_{l_1^{(2)} \ni t \to \infty} \arg t = \begin{cases} \dfrac{m+2}{2(m+1)}\pi & (m = \text{odd}), \\ \dfrac{\pi}{2} & (m = \text{even}), \end{cases}$$

$$\lim_{l_0^{(3)} \ni t \to \infty} \arg t = \begin{cases} \dfrac{m}{2(m+1)}\pi & (m = \text{odd}), \\ \dfrac{m-1}{2(m+1)}\pi & (m = \text{even}), \end{cases}$$

$$\lim_{l_1^{(3)} \ni t \to \infty} \arg t = \begin{cases} \dfrac{m+4}{2(m+1)}\pi & (m = \text{odd}), \\ \dfrac{m+3}{2(m+1)}\pi & (m = \text{even}), \end{cases}$$

$$\lim_{l_0^{(2)} \ni t \to \infty} \arg t = \lim_{l_1^{(2)} \ni t \to \infty} \arg t = \begin{cases} -\dfrac{m+2}{2(m+1)}\pi & (m = \text{odd}) , \\[2mm] -\dfrac{\pi}{2} & (m = \text{even}) . \end{cases}$$

Put

$$\xi_0 := \int_{t_0}^{t_1} (s^m - 1)ds ,$$

then $\Re\xi_0 > 0$ because $\xi_0 = (m/(m + 1))(t_0 - t_1)$ and $\Re t_0 > \Re t_1$. Therefore the Stokes domain D_2 is mapped by the mapping $\xi = \xi(t_0, t)$, which is defined by (5.4), onto a strip region $0 < \Re\xi < \Re\xi_0$ on the ξ-plane, and the Stokes domains D_0 and D_1 are mapped by $\xi = \xi(t_0, t)$ onto the left half plane $\Re\xi < 0$ and the right half plane $\Re\xi > \Re\xi_0$, respectively. D_0 and D_1 are Stokes domains of half-plane type, and D_2 is a Stokes domain of strip type.

If $m = 3$ then $t_0 = \overline{t_0} = 1$, and so $l_0^{(1)}$ does not exist. If $m = 2$, then $t_0 = \overline{t_0} = 1$, $t_1 = \overline{t_1} = -1$ and $l_0^{(1)}, l_1^{(1)}$ do not exist. If $m = 1$, we set $t_0 = t_1 = 1$. In these cases, D_2 is empty, $D_0 = \{(\Re t, \Im t) : \Im t > |\Re t - 1|\}$, $D_1 = \{(\Re t, \Im t) : |\Im t| < 1 - \Re t\}$ and $l_0^{(2)} = l_1^{(2)}$.

The set, shaded in Fig. 5-1,

(5.8) $D_0^{can} := D_0 \cup l_0^{(2)} \cup D_2 \cup l_1^{(2)} \cup D_1$

is a canonical domain which contains the origin as its interior point. The region D_0^{can} is mapped by $\xi = \xi(t_0, t)$, defined by (5.4), onto the ξ-plane with one slit ($m = 1$) or with two slits ($m \geq 2$) along upper half lines $\Re\xi = 0$ and $\Re\xi = \Re\xi_0$ emerging from the images of
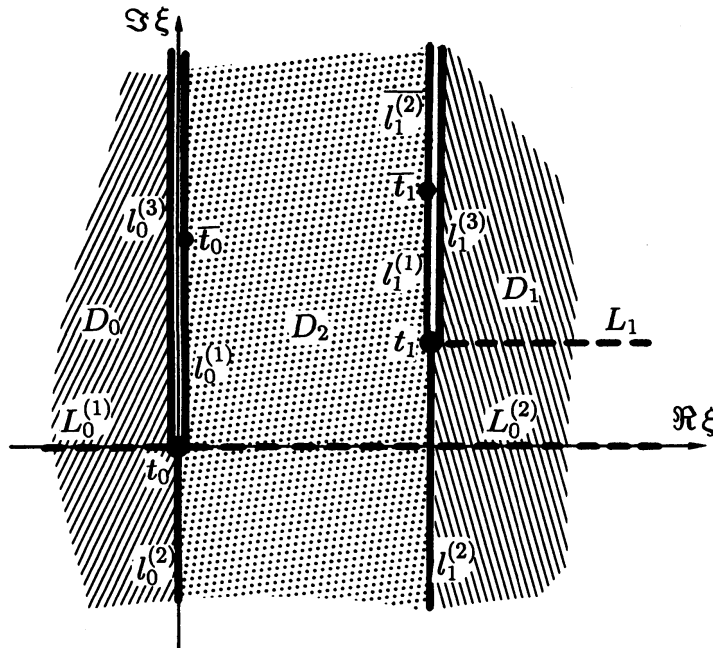

FIGURE. 5-2.   The image of D^{can} by (5.4). The same letters are used for images.

t_0 and t_1, respectively. (See Fig. 5-2. Notice that the same letters are used for the images for convenience's sake.)

There may exist several canonical domains. The set $D' := D_0 \cup l_0^{(3)} \cup D_0'$ (Fig. 5-1), for instance, is also a canonical domain. However, it is not suitable for our analysis, because it does not contain the origin. Thus we adopt D_0^{can} as our canonical domain.

We notice the angle relations (Fig. 5-1)

$$(5.9) \qquad \lim_{l_1^{(3)} \ni t \to \infty} \arg t - \lim_{l_0^{(3)} \ni t \to \infty} \arg t = \lim_{l_0^{(2)} \ni t \to \infty} \arg t - \lim_{l_0^{(4)} \ni t \to \infty} \arg t = \frac{2\pi}{m+1}.$$

Then the angles of D_0^{can} and D' near $t = \infty$ are respectively $2\pi/(m+1)$, which is the maximum angle of the existence regions of true solutions near an irregular singular point $t = \infty$ (Hukuhara [5], Wasow [19]).

6. The existence theorems.

6.1. We can derive two existence theorems from Lemma 3.2 by interpretating it appropriately as follows. First, we show the existence theorem with respect to the true inner solutions of (1.1).

THEOREM 6.1. *Suppose that* $m \neq 4p$ $(p \in \mathbf{N})$ *and that* D_0^{can} *is the canonical domain of (2.2) given by (5.8). Then there exist the true inner solutions* $y_j^{in}(t, \varepsilon)$*'s* $(j = 1, 2, \cdots, n)$ *of (1.1) with the double asymptotic property in* D_0^{can}:

$$(6.1) \qquad y_j^{in}(t, \varepsilon) \sim \tilde{y}_j^{in}(t, \varepsilon) \begin{cases} as\ \varepsilon \to 0, & t \in D_0^{can}, \\ as\ t \to \infty\ in\ D_0^{can}, & 0 < \varepsilon \le \varepsilon_0, \end{cases}$$

where $\tilde{y}_j^{in}(t, \varepsilon)$ *is the inner WKB solution of (1.1).*

PROOF. It is sufficient to show the existence of a (j, k)-canonical path $\gamma_{jk}(t)$ emerging from any $t \in D_0^{can}$. We recall that $\gamma_{jk}(t)$ for a fixed j in D_0^{can} is an integral path of

$$\xi_{jk}(t, \tau) := \int_t^\tau \{\lambda_j(s) - \lambda_k(s)\} ds.$$

Fist we consider the case where $a_j > a_k$ for any k $(\neq j)$. Let t^* and τ^* designate the images of t and τ by (5.4), respectively. The origin of the ξ-plane is the image of t_0. If we consider τ^* on the ξ-plane, the integral path $\gamma_{jk}(t)$ can be easily constructed as the inverse image of it (Fig. 5-2). There exist three possibilities of moving of τ.

(i) Let t belong to a subregion of D_0^{can} which is mapped onto the half-plane $\Re \xi < 0$ by (5.4), namely, we suppose that t^* belongs to the image of D_0. When τ^* starts from t^* and moves upward or downward along a curve $\Re \xi = $ const. passing through t^*, then it can cross an anti-Stokes curve, say, $L_0^{(1)}$. Furthermore, τ^* can continue along this anti-Stokes curve to the left such that $\Re \xi \to -\infty$. The trace of τ constructs $\gamma_{jk}(t)$.

(ii) Let t belong to $D_1 \cup l_1^{(2)} \cup D_2$ such that $\Im\xi < 0$. When τ^* starts from t^* and moves to the left along a curve $\Im\xi = \text{const.}$ passing through t^*, then τ^* can enter the image of D_0. Repeating the case (i), all the trace of τ constructs $\gamma_{jk}(t)$.

(iii) Let t belong to $D_1 \cup l_1^{(2)} \cup D_2$ such that $\Im\xi \geq 0$. If τ^* starts from t^* and moves downward along the curve $\Re\xi = \text{const.}$ passing through t^*, then τ^* can enter the image of $D_1 \cup l_1^{(2)} \cup D_2$ such that $\Im\xi < 0$. Repeating the case (ii), all the trace of τ constructs $\gamma_{jk}(t)$.

In the same way, we can obtain a (j, k)-canonical path for the other case $a_j < a_k$ such that $\Re\xi \to +\infty$ along it.   Q.E.D.

6.2. Similarly, we can get the existence theorem of the true outer solutions of (1.1) by applying Lemma 3.2 (cf. Fukuhara [4], Wasow [19]).

THEOREM 6.2. *Let D_0^{out} be a sector-like domain with an angle $2\pi/(m+1)$ in the x-plane given by*

$$(6.2) \qquad D_0^{out} := \left\{ x \left| \frac{m}{2(m+1)}\pi < \arg x < \frac{m+4}{2(m+1)}\pi, \, K\varepsilon^{1/m} \leq |x| \leq x_0, m = odd \right. \right\}$$

or

$$(6.2)' \qquad D_0^{out} := \left\{ x \left| \frac{m-1}{2(m+1)}\pi < \arg x < \frac{m+3}{2(m+1)}\pi, \, K\varepsilon^{1/m} \leq |x| \leq x_0, m = even \right. \right\}.$$

Then there exist the true outer solutions $y_j^{out}(x, \varepsilon)$'s $(j = 1, 2, \cdots, n)$ of (1.1) such that

$$(6.3) \qquad y_j^{out}(x, \varepsilon) \sim \tilde{y}_j^{out}(x, \varepsilon) \quad as \quad \varepsilon \to 0, \quad x \in D_0^{out},$$

where $\tilde{y}_j^{out}(x, \varepsilon)$ is the outer WKB solution of (1.1).

PROOF. The origin $x = 0$ can be considered as the turning point of the outer equation because two characteristic roots coincide there and we get curves defined by the equations

$$(6.4) \qquad \Re\xi_{jk}(0, x) = 0 \quad and \quad \Im\xi_{jk}(0, x) = 0,$$

where

$$\xi_{jk}(0, x) := \int_0^x \{\lambda_j(s) - \lambda_k(s)\}ds, \quad \lambda_j(s) = a_j s^m \quad (j = 1, 2 \cdots, n).$$

These curves emerge from the origin $x = 0$ and tend to the boundary $|x| = x_0$, and their parts are contained outside of the outer domain. Then we adopt only parts of curves (6.4) contained in the outer domain as Stokes and anti-Stokes curves for the outer equation. They can be considered respectively as the continuation from Stokes and anti-Stokes curves in the t-plane as $t \to \infty$ because $\arg\varepsilon = 0$ and $\arg(x^{m+1}/(m+1)) = \arg\{\lim_{t\to\infty}(t^{m+1}/(m+1) - t)\}$. Thus, we can adopt, say for $m = $ odd, three lines with arguments of $m\pi/2(m+1)$, $(m+2)\pi/2(m+1)$, and $(m+4)\pi/2(m+1)$ as Stokes curves of the outer equation, two of them are boundaries D_0^{out}. The equation (3.1) and the outer equation (2.1) have the same form and the domain D_0^{out} can read a canonical domain D in Lemma 3.2. Hence, we can apply Lemma 3.2 to (2.1).   Q.E.D.

We notice here that the boundary lines of the x-domain D_0^{out} in Theorem 6.2 correspond to those of the inner domain D_0^{can} with the same arguments as, say in the case where $t \to \infty$ for $m =$ odd, $\arg t = \arg x = m\pi/2(m+1)$ and $(m+4)\pi/2(m+1)$ (cf. (5.9)).

7. The matching matrix.

7.1. The outer domain D_0^{out} and the inner domain D_0^{can} overlap each other for small ε (cf. §2.2), and we could show the existence of the true outer and the true inner solutions in the preceding section. Now, we can compute a matching matrix, i.e., a linear relation between the outer and the inner solutions. By using n independent outer solutions $y_j^{out}(x, \varepsilon)$'s, we put

$$(7.1) \qquad Y^{out} := {}^t(y_1^{out}(x, \varepsilon), y_2^{out}(x, \varepsilon), \cdots, y_n^{out}(x, \varepsilon)),$$

which is called an outer vector solution of (1.1). Similarly, we get an inner vector solution of (1.1)

$$(7.2) \qquad Y^{in} := {}^t(y_1^{in}(t, \varepsilon), y_2^{in}(t, \varepsilon), \cdots, y_n^{in}(t, \varepsilon)) \quad (t = x\varepsilon^{-l/m}).$$

The matching matrix $M = (m_{ij})$ is, by definition, an n-by-n matrix such that

$$(7.3) \qquad\qquad\qquad M Y^{out} = Y^{in}.$$

7.2. By using the outer and the inner WKB solutions we can compute the matching matrix as follows.

THEOREM 7.1. *The asymptotic representation* M_0 *of* M *is given by*

$$(7.4) \qquad M \sim M_0, \quad M_0 := \varepsilon^{l \operatorname{diag}(\mu_1, \mu_2, \cdots, \mu_n)}, \quad \mu_j := \sum_{k \neq j} \frac{a_j}{a_j - a_k} \quad (\varepsilon \to 0).$$

PROOF. The matching relation (7.3) is asymptotically represented by

$$(7.5) \qquad\qquad\qquad M \tilde{Y}^{out} \sim \tilde{Y}^{in} \quad (\varepsilon \to 0),$$

where $\tilde{Y}^{out}$ (resp. $\tilde{Y}^{in}$) is an outer (resp. inner) vector WKB solution which is obtained by substituting the outer WKB solutions $\tilde{y}_j^{out}$'s (resp. $\tilde{y}_j^{in}$'s) for y_j^{out}'s of (7.1) (resp. y_j^{in}'s of (7.2)) (cf. §4).

The elements of the matrix equation (7.5) are written as

$$(7.6) \qquad \sum_{j'=1}^n m_{jj'} \tilde{y}_{j'}^{out} \sim \tilde{y}_j^{in} \quad (\varepsilon \to 0; j = 1, 2, \cdots, n),$$

i.e.,

$$(7.6)' \qquad \sum_{j'=1}^n m_{jj'} \frac{\tilde{y}_{j'}^{out}}{\tilde{y}_j^{in}} \sim 1 \quad (\varepsilon \to 0; j = 1, 2, \cdots, n).$$

Put

$$(7.7) \qquad x := \eta\rho, \quad t := \eta\rho^{-1} \quad (|\eta| = 1), \quad \rho := \varepsilon^{l/2m}.$$

The point x can belong to the outer domain $(K\rho^2 \leq |x| \leq x_0)$ for small ρ and the point t can tend to ∞ in the inner domain $(|t| < \infty)$ as ρ tends to 0. A new complex parameter η will be defined soon later. By substituting $x = \eta\rho$ and $\varepsilon = \rho^{2m/l}$ into the outer WKB solutions (4.1), we get

$$(7.8) \qquad \tilde{y}_j^{out} = (\eta\rho)^{-m\mu_j} \exp\left(\frac{a_j\eta^{m+1}}{m+1}\rho^{-m-1-2m/l}\right).$$

In the same way, from (7.7) and (4.3) we get

$$(7.9) \qquad \tilde{y}_j^{in} \sim (\eta\rho^{-1})^{-m\mu_j} \exp\left(\frac{a_j\eta^{m+1}}{m+1}\rho^{-m-1-2m/l}\right) \qquad (\rho \to 0).$$

From (7.8) and (7.9) we get for small ρ

$$(7.10) \qquad \frac{\tilde{y}_j^{out}}{\tilde{y}_j^{in}} \sim \rho^{-2m\mu_j} \quad (= \varepsilon^{-l\mu_j})$$

and

$$(7.11) \qquad \frac{\tilde{y}_{j'}^{out}}{\tilde{y}_j^{in}} \sim \rho^{m(\mu_j-\mu_{j'})} \exp\left(\frac{a_{j'}-a_j}{m+1}\eta^{m+1}\rho^{-m-1-2m/l}\right) \qquad (j \neq j').$$

If we choose a parameter η such that $\Re\eta^{m+1} > 0$ for $a_{j'} - a_j > 0$ and if we choose η such that $\Re\eta^{m+1} < 0$ for $a_{j'} - a_j < 0$, then the magnitude of the exponential term in (7.11) tends to $+\infty$ as $\rho \to 0$. In fact we can choose a parameter η with this property, because our canonical domain contains routes on which η satisfies the above conditions: Such routes are given by two anti-Stokes curves defined by $\Im\xi = 0$ contained in the canonical domain D_0^{can} (cf. §5). The anti-Stokes curves $L_0^{(1)}$ and $L_0^{(2)}$ in Fig. 5-1 emerge from the secondary turning point t_0, and hold the inequalities $\Re\xi > 0$ on $L_0^{(2)}$ and $\Re\xi < 0$ on $L_0^{(1)}$. From (7.10), (7.11) and (7.6)$'$ we get

$$(7.12) \qquad m_{jj} \sim \varepsilon^{l\mu_j}, \quad m_{jj'} \sim 0 \quad (j \neq j') \quad (\varepsilon \to 0),$$

then follows the matching matrix (7.4).   Q.E.D.

7.3. We remark finally that we have studied in Nakano [10] the differential equation

$$(1.1)_r \qquad \varepsilon^{nh}y^{(n)} = \sum_{k=1}^{n} \varepsilon^{(n-k)h}p_k \cdot (x^m - \varepsilon^l/x^r)^k y^{(n-k)},$$

where l, m and r are positive integers satisfying the inequality

$$(7.13) \qquad h > \frac{m+1}{m+r}l,$$

and we got the matching matrix M_r for $(1.1)_r$:

$$(7.14) \qquad M_r := \varepsilon^{lm/(m+r)\mathrm{diag}(\mu_1,\mu_2,\cdots,\mu_n)},$$

where μ_j's are the same as in (7.4). The matrix M_r is a continuous function of r $(\neq -m)$, and if $r = 0$ in (7.14) it coincides with M_0 of (7.4), although the equation (1.5) is a special

case of (7.13) for $r = 0$ and the canonical domains are quite different from each other (cf. Fig. 6 in Nakano [10]).

ACKNOWLEDGEMENT. The author expresses his thanks to the referee and the editor for their valuable advice.

References

[1] M. A. EVGRAFOV and M. V. FEDORYUK, Asymptotic behavior as $\lambda \to \infty$ of solutions of the equation $w''(z) - p(z, \lambda)w(z) = 0$ in the complex z-plane, Uspekhi Mat. Nauk 21 or Russian Math. Surveys 21 (1966), 1–48.

[2] M. В. Федорюк, Асимптотика решений обыкновенных линейных дифференциальных уравнений n-го порядка, Дифф. Урав. 2 (1966), 492–507.

[3] M. V. FEDORYUK, Asymptotic Analysis, Springer (1993).

[4] M. FUKUHARA, Sur les proprietes asymptotiques des solution d'un systeme d'equations differentielles linéaires contenant un parametre, Mem. Fac. Eng. Kyushu Imp. Univ. 8 (1937), 249–280.

[5] M. HUKUHARA, Sur les points singulieres des équations différentielles linéaires III, Mém. Fac. Sci. Kyushu Univ. 2 (1941), 125–137.

[6] M. IWANO and Y. SIBUYA, Reduction of the order of a linear ordinary differential equation containing a small parameter, Kodai Math. Sem. Rep. 15 (1963), 1–28.

[7] M. NAKANO, On a system of linear ordinary differential equations with a turning point I, II, Kodai Math. Sem. Rep. 21 (1969), 1–15: Hiyoshi Kiyo 16 (1974), 59–74.

[8] M. NAKANO, On a linear ordinary differential equation with a turning point, Kokyuroku of RIMS, Kyoto University 105 (1971), 1–17 (in Japanese).

[9] M. NAKANO, On a third order linear ordinary differential equation with secondary turning points, Proc. of Sixth International Colloquium on Diff. Eqs. at Plovdiv, Bulgaria (1995).

[10] M. NAKANO, On a n-th order linear ordinary differential equation with a turning-singular point, Tokyo J. Math. 21 (1998), 201–215.

[11] M. NAKANO and T. NISHIMOTO, On a secondary turning point problem, Kodai Math. Sem. Rep. 22 (1970), 355–384.

[12] M. NAKANO, M. NAMIKI, and T. NISHIMOTO, On the WKB method for certain third order ordinary differential equations, Kodai Math. J. 14 (1991), 432–462.

[13] T. NISHIMOTO, On matching method for a linear ordinary differential equation containing a parameter I, II, III, Kodai Math. Sem. Rep. 17 (1965), 307–328: 18 (1966), 61–86: 19 (1967), 80–94.

[14] F. W. J. OLVER, Asymptotics and special functions, Academic Press (1974).

[15] H. G. ROOS, Die asymptotische Lösung einer linearen Differentialgleichung zweiter Ordnung mit zweiseg-mentigem charakteristischen Polygon, Beitr. Anal. 7 (1975), 55–63.

[16] H. G. ROOS, Die asymptotische Lösung einer linearen Differentialgleichung mit dreisegmentigem charakter-istischen Polygon, Math. Nachr. 88 (1979), 93–103.

[17] K. UCHIYAMA and M. NAKANO, How to draw Stokes curves for 2nd order differential equations (computer program).

[18] W. WASOW, Turning point problems for systems of linear equations I, II, Comm. Pure Appl. Math. 14 (1961), 657–673: 15 (1962), 173–187.

[19] W. WASOW, Asymptotic expansions for ordinary differential equations, John Wiley (1965).

[20] W. WASOW, Linear turning point theory, Springer (1985).

Present Address:
DEPARTMENT OF MATHEMATICS, KEIO UNIVERSITY,
HIYOSHI, KOHOKU, YOKOHAMA, 223–8522 JAPAN
e-mail: nakano@math.keio.ac.jp