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Kakutani-Fan's Fixed Point Theorem in Hyperspaces

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1. Introduction.

Denote by $\mathbf{K}$ a real Banach space, and by $2^{\mathcal{X}}$ ($\mathcal{X}$ a non-empty set) the set of all non-empty subsets of $\mathcal{X}$.

A version of the classical fixed point theorem of Kakutani [13] and Fan [6] states that, if D is a non-empty convex bounded closed subset of $\mathbf{K}$, and $F : D \to 2^{\mathbf{K}}$ is a multifunction with non-empty convex compact values $F(x) \subset D$, which is h-upper semicontinuous ("h" stands for "in the sense of Pompeiu-Hausdorff") and $\mathbf{K}$-compact, then there exists an $x_0 \in D$ such that $x_0 \in F(x_0)$.

This theorem has been generalized in several directions. The interested reader can consult Hu and Papageorgiou [9], Istrăţescu [11], Joshi and Bose [12], Ma [16], Sehgal, Singh and Watson [23], and the references therein.

In the present paper we consider some variants of Kakutani-Fan's fixed point theorem in hyperspaces. Denote by $\mathcal{K}$ the hyperspace of all non-empty convex bounded closed subsets of $\mathbf{K}$ endowed with the Pompeiu-Hausdorff metric h. Let $\mathcal{D}$ be a non-empty convex bounded closed subset of $\mathcal{K}$. If $F : \mathcal{D} \to 2^{\mathcal{K}}$ is a h-upper semicontinuous and $\mathcal{K}$-compact multifunction, with non-empty convex bounded closed values $F(X) \subset \mathcal{D}$, then we show that there exists an $X_0 \in \mathcal{D}$ such that $X_0 \in F(X_0)$. As a corollary we obtain a hyperspace version of the fixed point theorem of Brouwer and Schauder (see [12]) for h-continuous and $\mathcal{K}$-compact maps $F : \mathcal{D} \to \mathcal{D}$. Actually we shall prove a Kakutani-Fan's type result (see Theorem 1) in a slightly more general setting, which seems more convenient for applications. One of its corollaries is used to establish the existence of solutions to a Cauchy problem, for differential equations with set-valued solutions, under Peano type assumptions.

2. Notation and preliminaries.

Let $(\mathcal{X}, d)$ be a non-empty complete metric space. If $X \subset \mathcal{X}$, the closure of X is denoted

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by $\overline{X}$ or $cl_{\mathcal{X}}X$. Further, $U_{\mathcal{X}}(x, r)$ stands for an open ball in $\mathcal{X}$ with center x and radius r. If X, Y are non-empty bounded subsets of $\mathcal{X}$, we put $e(X, Y) = \sup_{x \in X} \inf_{y \in Y} d(x, y)$. Now set

$$B(\mathcal{X}) = \{X \in 2^{\mathcal{X}} : X \text{ is bounded and closed in } \mathcal{X}\}.$$

The space $B(\mathcal{X})$ is endowed with the Pompeiu-Hausdorff metric h, given by

$$h(X, Y) = \max\{e(X, Y), e(Y, X)\}, \qquad X, Y \in B(\mathcal{X}).$$

As $\mathcal{X}$ is complete, $(B(\mathcal{X}), h)$ is a complete metric space; for instance see [15].

Let M be an arbitrary metric space. In the sequel when a subset, say Z, of M is considered as a metric space it is assumed that Z is given the induced metric.

Let $A(\mathcal{X})$ be a non-empty closed subset of $B(\mathcal{X})$, thus $(A(\mathcal{X}), h)$ is a complete metric space. To emphasize, when necessary, the metric space $A(\mathcal{X})$ we are dealing with, we shall write $e_{A(\mathcal{X})}$, $h_{A(\mathcal{X})}$ in place of e, h, respectively.

Let F be a map which associates to each $x \in M$ a non-empty subset $F(x)$ of $\mathcal{X}$. When, for each $x \in M$, $F(x)$ is a member of a set, say $A(\mathcal{X})$, we write (by abuse of notation) $F : M \to A(\mathcal{X})$ and we call F an $A(\mathcal{X})$-valued multifunction or, simply, a multifunction.

For an $A(\mathcal{X})$-valued multifunction there are two different notions of range, according to whether $F(x)$ is considered as a subset of the underlying space $\mathcal{X}$ or as an element of the space $A(\mathcal{X})$.

DEFINITION 1. Let $F : M \to A(\mathcal{X})$ be a multifunction. The $\mathcal{X}$-range $\mathcal{R}_{\mathcal{X}}(F)$ of F and the $A(\mathcal{X})$-range $\mathcal{R}_{A(\mathcal{X})}(F)$ of F, are given by:

$$\mathcal{R}_{\mathcal{X}}(F) = \{y \in \mathcal{X} : \text{ there exists } x \in M \text{ such that } y \in F(x)\},$$

$$\mathcal{R}_{A(\mathcal{X})}(F) = \{Y \in A(\mathcal{X}) : \text{ there exists } x \in M \text{ such that } Y = F(x)\}.$$

The $\mathcal{X}$-range of a single valued map $F : M \to \mathcal{X}$ is denoted also by $F(M)$.

DEFINITION 2. A multifunction $F : M \to A(\mathcal{X})$ is said to be Pompeiu-Hausdorff upper semicontinuous (resp. lower semicontinuous, continuous) if for every $x_0 \in M$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in U_M(x_0, \delta)$ implies $e(F(x), F(x_0)) < \varepsilon$ (resp. $e(F(x_0), F(x)) < \varepsilon$, $h(F(x), F(x_0)) < \varepsilon$).

Instead of Pompeiu-Hausdorff upper semicontinuous, lower semicontinuous, continuous, we write, respectively, $h_{A(\mathcal{X})}$-upper semicontinuous, $h_{A(\mathcal{X})}$-lower semicontinuous, $h_{A(\mathcal{X})}$-continuous or, for brevity, $h_{A(\mathcal{X})}$-u.s.c., $h_{A(\mathcal{X})}$-l.s.c., $h_{A(\mathcal{X})}$-continuous.

DEFINITION 3. A multifunction $F : M \to A(\mathcal{X})$ is called $\mathcal{X}$-compact, if the set $\mathcal{R}_{\mathcal{X}}(F)$ is precompact in $\mathcal{X}$. Whenever $\mathcal{R}_{A(\mathcal{X})}(F)$ is precompact in $A(\mathcal{X})$, then F is called $A(\mathcal{X})$-compact.

REMARK 1. Let $F : M \to A(\mathcal{X})$ be given. Then F is $\mathcal{X}$-compact if and only if F is $A(\mathcal{X})$-compact and, for every $x \in M$, $F(x)$ is a compact subset of $\mathcal{X}$. Furthermore, an $A(\mathcal{X})$-compact F is not necessarily $\mathcal{X}$-compact.

Throughout $\mathbf{K}$ will denote a real Banach space. Set

$$\mathcal{K} = \{X \in 2^{\mathbf{K}} : X \text{ is convex bounded and closed in } \mathbf{K}\}.$$

We equip $\mathcal{K}$ with the Pompeiu-Hausdorff distance h. Clearly, $(\mathcal{K}, h)$ is a complete metric space.

To avoid possible ambiguities we point out that in the sequel when we say that a multifunction $F : M \to \mathcal{K}$ is $\mathbf{K}$-compact (resp. $\mathcal{K}$-compact), we mean that the $\mathbf{K}$-range (resp. $\mathcal{K}$-range) of F is a precompact subset of $\mathbf{K}$ (resp. $\mathcal{K}$). If $X, Y \in \mathcal{K}$ and $\lambda \geq 0$ the sum $X + Y$ and the product λX are the elements of $\mathcal{K}$ given by:

$$X + Y = \overline{\{x + y : x \in X, y \in Y\}}, \quad \lambda X = \{\lambda x : x \in X\}. \tag{2.1}$$

For arbitrary $X, Y, Z \in \mathcal{K}$ and $\lambda, \mu \geq 0$, denoting by 0 the zero of $\mathbf{K}$, we have:

$$X + \{0\} = X, \quad X + Y = Y + X, \quad X + (Y + Z) = (X + Y) + Z; \tag{2.2}$$

$$1X = X, \quad \lambda(\mu X) = (\lambda\mu)X, \quad \lambda(X + Y) = \lambda X + \lambda Y, \quad (\lambda + \mu)X = \lambda X + \mu Y. \tag{2.3}$$

DEFINITION 4. A subset $\mathcal{A}$ of $\mathcal{K}$ is called convex if for every $X, Y \in \mathcal{A}$ and $\lambda \in [0, 1]$ we have $(1 - \lambda)X + \lambda Y \in \mathcal{A}$.

If $A \subset \mathbf{K}$, by $\overline{co}A$, we mean the closed convex hull of A.

DEFINITION 5. If $\mathcal{A}$ is a subset of $\mathcal{K}$, define

$$conv_{\mathcal{K}}(\mathcal{A}) = \left\{X \in \mathcal{K} : X = \sum_{i=1}^{m} \lambda_i Y_i, \text{ for some } m \in \mathbf{N}, Y_i \in \mathcal{A}, \right.$$
$$\left. \text{and } \lambda_i \geq 0 \text{ with } \sum_{i=1}^{m} \lambda_i = 1\right\}.$$

The sets $conv_{\mathcal{K}}(\mathcal{A})$ and $cl_{\mathcal{K}}conv_{\mathcal{K}}(\mathcal{A})$ are called, respectively, $\mathcal{K}$-convex hull and $\mathcal{K}$-closed convex hull of $\mathcal{A}$.

REMARK 2. The sets $conv_{\mathcal{K}}(\mathcal{A})$ and $cl_{\mathcal{K}}(\mathcal{A})$ are convex.

In the sequel we shall use the Rådström-Hörmander embedding of the space $\mathcal{K}$, endowed with the Pompeiu-Hausdorff metric h and the above defined operations of sum and product, on a positive and closed cone of a real Banach space. Further details can be found in Rådström [20], Hörmander [8], and Schmidt [21].

Let $\mathbf{K}^*$ be the topological dual of $\mathbf{K}$. Following Hörmander [8], define

$$\mathbf{H} = \{q : \mathbf{K}^* \to \mathbf{R} : q \text{ is positively homogeneous and continuous}\}.$$

Here continuity is understood in the norm topology of $\mathbf{K}^*$. $\mathbf{H}$ is equipped with the norm

$$\|q\|_{\mathbf{H}} = \sup_{\|x^*\| \leq 1} \|q(x^*)\|, \quad q \in \mathbf{H},$$

under which $\mathbf{H}$ becomes a real Banach space.

For $X \in \mathcal{K}$, let $q_X : \mathbf{K}^* \to \mathbf{R}$ be the support function of X, that is the function given by

$$q_X(x^*) = \sup_{x \in X} \langle x, x^* \rangle, \quad x^* \in \mathbf{K}^*.$$

Here $\langle \cdot, \cdot \rangle$ stands for the pairing between $\mathbf{K}$ and $\mathbf{K}^*$. By Hörmander [8], q_X is positively homogeneous, convex, and continuous in the norm topology of $\mathbf{K}^*$, whence $q_X \in \mathbf{H}$; furthermore, if $X, Y \in \mathcal{K}$ and $\lambda, \mu \geq 0$ we have:

(i) $q_X = q_Y$ if and only if $X = Y$,

(ii) $q_{\lambda X + \mu Y} = \lambda q_X + \mu q_Y$.

Thus, the function $j : \mathcal{K} \to \mathbf{H}$ defined by

$$j(X) = q_X, \quad X \in \mathcal{K}, \tag{2.4}$$

establishes an isomorphism between $\mathcal{K}$ and the positive convex cone $\mathbf{V} = \{q_X \in \mathbf{H} : X \in \mathcal{K}\}$. More precisely we have the following:

HÖRMANDER'S THEOREM [8]. *The function $j : \mathcal{K} \to \mathbf{H}$ is an isometric isomorphism of $\mathcal{K}$ on the positive convex cone $\mathbf{V} = j(\mathcal{K})$ contained in the real Banach space $\mathbf{H}$, namely, for every $X, Y \in \mathcal{K}$ and $\lambda, \mu \geq 0$ we have:*

(i) $j(\lambda X + \mu Y) = \lambda j(X) + \mu j(Y)$,

(ii) $\|j(X) - j(Y)\|_{\mathbf{H}} = h(X, Y)$.

REMARK 3. Since $(\mathcal{K}, h)$ is complete, the set $\mathbf{V}$ is closed in $\mathbf{H}$. Thus the positive convex closed cone $\mathbf{V}$, equipped with the metric induced by the norm of $\mathbf{H}$, is a complete metric space.

3. Kakutani-Fan's type fixed point theorems.

In this section we prove a fixed point theorem of Kakutani-Fan's type in spaces of multifunctions. A corollary of it will be used later, in section 5, to show the existence of solutions for differential equations with set-valued solutions.

Let M be a non-empty metric space. Given two multifunctions $X, Y : M \to \mathcal{K}$ and $\lambda \geq 0$ the sum $X + Y : M \to \mathcal{K}$ and the product $\lambda X : M \to \mathcal{K}$ are defined by

$$(X + Y)(t) = X(t) + Y(t), \quad (\lambda X)(t) = \lambda X(t), \quad t \in M.$$

Observe, that in view of (2.1), both sets $X(t) + Y(t)$ and $\lambda X(t)$ are in $\mathcal{K}$.

Now put

$$\mathcal{X} = \{X : M \to \mathcal{K} : X \text{ is } h\text{-continuous and bounded}\}, \tag{3.1}$$

and equip $\mathcal{X}$ with the metric

$$\rho_{\mathcal{X}}(X, Y) = \sup_{t \in M} h(X(t), Y(t)), \quad X, Y \in \mathcal{X}. \tag{3.2}$$

As $(\mathcal{K}, h)$ is complete, $(\mathcal{X}, \rho_{\mathcal{X}})$ is a complete metric space.

For every $X, Y \in \mathcal{X}$ and $\lambda \geq 0$ we have $X + Y \in \mathcal{X}$ and $\lambda X \in \mathcal{X}$. Moreover, if $X, Y, Z \in \mathcal{X}$ and $\lambda, \mu \geq 0$, then (2.2) (where 0 stands for the map identically zero on M) and

(2.3) are satisfied. In $\mathcal{X}$ the notions of convex set, $\mathcal{X}$-convex hull and $\mathcal{X}$-closed convex hull are given by Definitions 4 and 5, with $\mathcal{X}$ in place of $\mathcal{K}$.

Next put

$$\mathbf{B} = \{\xi : M \to \mathbf{H} : \xi \text{ is continuous and bounded}\},$$

and equip $\mathbf{B}$ with the norm

$$\|\xi\|_{\mathbf{B}} = \sup_{t \in M} \|\xi(t)\|_{\mathbf{H}}.$$

Clearly $(\mathbf{B}, \|.\|_{\mathbf{B}})$ is a real Banach space.

Now denote by $J : \mathcal{X} \to \mathbf{B}$ the map which associates to each $X \in \mathcal{X}$ the element $J(X)$ of $\mathbf{B}$ defined by

$$(JX)(t) = j(X(t)), \quad t \in M, \tag{3.3}$$

where j is given by (2.4). Observe that $JX \in \mathbf{B}$, by Hörmander's theorem. Set

$$\mathbf{W} = \{\xi \in \mathbf{B} : \text{ there exists } X \in \mathcal{X} \text{ such that } J(X) = \xi\}.$$

$\mathbf{W}$ is a convex cone contained in $\mathbf{B}$. More precisely, by Hörmander's theorem and Remark 3 we have:

PROPOSITION 1. *The map $J : \mathcal{X} \to \mathbf{B}$ given by* (3.3) *is a isometric isomorphism of $\mathcal{X}$ on the positive convex cone $\mathbf{W}$ contained in the real Banach space $\mathbf{B}$, namely, for every $X, Y \in \mathcal{X}$ and $\lambda, \mu \geq 0$ we have*:

(i) $J(\lambda X + \mu Y) = \lambda J(X) + \mu J(Y)$,

(ii) $\|JX - JY\|_{\mathbf{B}} = \rho_{\mathcal{X}}(X, Y)$.

REMARK 4. $\mathbf{W}$ endowed with the metric induced by the norm of $\mathbf{B}$ is a complete metric space. Further, if $\mathcal{A}$ is a convex (resp. bounded, closed) subset of $\mathcal{X}$ then also $J(\mathcal{A})$ is a convex (resp. bounded, closed) subset of $\mathbf{W}$.

Now define

$$C(\mathcal{X}) = \{\mathcal{A} \in 2^{\mathcal{X}} : \mathcal{A} \text{ is convex, bounded and closed in } \mathcal{X}\},$$

$$C(\mathbf{B}) = \{A \in 2^{\mathbf{B}} : A \text{ is convex, bounded and closed in } \mathbf{B}\},$$

$C(\mathcal{X}), C(\mathbf{B})$ are equipped with the Pompeiu-Hausdorff metrics $h_{C(\mathcal{X})}, h_{C(\mathbf{B})}$ respectively.

Observe that $(C(\mathcal{X}), h_{C(\mathcal{X})})$ and $(C(\mathbf{B}), h_{C(\mathbf{B})})$ are complete metric spaces, because the underlying spaces $(\mathcal{X}, \rho_{\mathcal{X}})$ and $(\mathbf{B}, \|.\|_{\mathbf{B}})$ are so.

The following proposition is due to De Blasi and Georgiev [3], Theorem 3. A variant of it has previously been established, by a different approach, by Dawidowicz [2].

PROPOSITION 2. *Let D be a non-empty convex bounded closed subset of* $\mathbf{K}$. *Let $\varphi : D \to \mathcal{K}$ be h-u.s.c. and $\mathcal{K}$-compact. Then for every $\varepsilon > 0$ there exist $x_\varepsilon \in D$ and $y_\varepsilon \in F(x_\varepsilon)$ such that $\|y_\varepsilon - x_\varepsilon\| \leq \inf_{x \in D} \|y_\varepsilon - x\| + \varepsilon$. If, in addition, $F(x) \subset D$ for every $x \in D$, then there exists $x_0 \in D$ such that $x_0 \in F(x_0)$.*

THEOREM 1. *Let $\mathcal{D}$ be a non-empty convex bounded closed subset of $\mathcal{X}$. Let $F : \mathcal{D} \to C(\mathcal{X})$ be a h-u.s.c. and $C(\mathcal{X})$-compact multifunction with values $F(X) \subset \mathcal{D}$, for every $X \in \mathcal{D}$. Then there exists $X_0 \in \mathcal{D}$ such that $X_0 \in F(X_0)$.*

PROOF. Set $\Xi = J(\mathcal{D})$. By Proposition 1 and Remark 4, Ξ is non-empty convex bounded closed subset of $\mathbf{B}$ and $\Xi \subset \mathbf{W}$; moreover Ξ is isometric to $\mathcal{D}$. For $\xi \in \Xi$, set

$$\varphi(\xi) = \{\eta \in \mathbf{B} : \text{ there exists } Y \in F(J^{-1}(\xi)), \text{ such that } J(Y) = \eta\}. \tag{3.4}$$

Moreover $\varphi(\xi) \subset \Xi$ and, by Remark 4, $\varphi(\xi) \in C(\mathbf{B})$. Thus (3.4) defines a multifunction

$$\varphi : \Xi \to C(\mathbf{B}),$$

with values $\varphi(\xi) \subset \Xi$.

φ is $h_{C(\mathbf{B})}$-u.s.c. To show this, let $\xi_0 \in \Xi$ and $\varepsilon > 0$ be given. Let $X_0 \in \mathcal{D}$ be such that $J(X_0) = \xi_0$. Since F is $h_{C(\mathcal{X})}$-u.s.c., there exists $\delta > 0$ such that

$$X \in U_{\mathcal{D}}(X_0, \delta) \quad \text{implies} \quad e_{C(\mathcal{X})}(F(X), F(X_0)) < \varepsilon. \tag{3.5}$$

Let $\xi \in U_{\Xi}(\xi_0, \delta)$ be arbitrary. Hence $X \in U_{\mathcal{D}}(X_0, \delta)$, where $X = J^{-1}(\xi)$. In view of Proposition 1 (ii), we have:

$$\begin{aligned} e_{C(\mathbf{B})}(\varphi(\xi), \varphi(\xi_0)) &= \sup_{\eta \in \varphi(\xi)} \inf_{\eta_0 \in \varphi(\xi_0)} \|\eta - \eta_0\|_{\mathbf{B}} \\ &= \sup_{Y \in F(J^{-1}(\xi))} \inf_{Y_0 \in F(J^{-1}(\xi_0))} \|J(Y) - J(Y_0)\|_{\mathbf{B}} \\ &= \sup_{Y \in F(X)} \inf_{Y_0 \in F(X_0)} \rho_{\mathcal{X}}(Y, Y_0) \\ &= e_{C(\mathcal{X})}(F(X), F(X_0)) < \varepsilon, \end{aligned}$$

where the latter inequality holds by (3.5). Therefore φ is $h_{C(\mathbf{B})}$-u.s.c.

Similarly, for arbitrary $\xi', \xi'' \in \Xi$, if $X', X'' \in \mathcal{D}$ are given by $X' = J^{-1}(\xi')$, $X'' = J^{-1}(\xi'')$, we have $h_{C(\mathbf{B})}(\varphi(\xi'), \varphi(\xi'')) = h_{C(\mathcal{X})}(F(X'), F(X''))$. Since, by hypothesis, F is $C(\mathcal{X})$-compact, it follows that also φ is $C(\mathbf{B})$-compact.

By Proposition 2 (with $\mathbf{K}$, D and $\mathcal{K}$ replaced by $\mathbf{B}$, Ξ, and $C(\mathbf{B})$), there exists $\xi_0 \in \Xi$ such that $\xi_0 \in \varphi(\xi_0)$. Let $X_0 \in \mathcal{D}$ be such that $J(X_0) = \xi_0$. Since $\xi_0 \in \varphi(\xi_0)$, in view of (3.4) there exists $Y_0 \in F(X_0)$ such that $J(Y_0) = \xi_0$. As J is one-to-one, it follows that $Y_0 = X_0$, thus $X_0 \in F(X_0)$, completing the proof.

COROLLARY 1. *Let $\mathcal{D}$ be a non-empty convex bounded closed subset of $\mathcal{X}$. Let $F : \mathcal{D} \to \mathcal{D}$ be a $\mathcal{X}$-continuous and $\mathcal{X}$-compact mapping. Then, there exists $X_0 \in \mathcal{D}$ such that $X_0 = F(X_0)$.*

PROOF. Let $G : \mathcal{D} \to C(\mathcal{X})$ be given by $G(X) = \{F(X)\}$, for every $X \in \mathcal{D}$. The multifunction G is $h_{C(\mathcal{X})}$-continuous and $C(\mathcal{X})$-compact and so, by Theorem 1, there exists $X_0 \in \mathcal{D}$ such that $X_0 \in G(X_0)$, whence $X_0 = F(X_0)$. □

Now set

$$C(\mathcal{K}) = \{\mathcal{A} \in 2^{\mathcal{K}} : \mathcal{A} \text{ is convex, bounded and closed in } \mathcal{K}\}.$$

and equip $C(\mathcal{K})$ with the Pompeiu-Hausdorff metric $h_{C(\mathcal{K})}$. Clearly $(C(\mathcal{K}), h_{C(\mathcal{K})})$ is a complete metric space, for $(\mathcal{K}, h)$ is so.

The following Corollary 2, and Corollary 3, are hyperspace versions of the fixed point theorems of Kakutani-Fan, and of Brouwer-Schauder, respectively.

COROLLARY 2. *Let $\mathcal{D}$ be a non-empty convex bounded closed subset of $\mathcal{K}$. Let $F : \mathcal{D} \to C(\mathcal{K})$ be a $h_{C(\mathcal{K})}$-u.s.c. and $C(\mathcal{K})$-compact multifunction with values $F(X) \subset \mathcal{D}$, for every $X \in \mathcal{D}$. Then there exists $X_0 \in \mathcal{D}$ such that $X_0 \in F(X_0)$.*

PROOF. In the definition of $\mathcal{X}$ take M to be a set consisting of a single point. The statement follows at once from Theorem 1, by the natural identification of $\mathcal{K}$ with $\mathcal{X}$, and of $C(\mathcal{K})$ with $C(\mathcal{X})$. □

A special case of Corollary 2 is the following:

COROLLARY 3. *A h-continuous and $\mathcal{K}$-compact mapping $F : \mathcal{D} \to \mathcal{D}$, where $\mathcal{D}$ is as in Corollary 2, has a fixed point, i.e. there exists $X_0 \in \mathcal{D}$ such that $X_0 = F(X_0)$.*

From Corollary 2 one can derive the classical fixed point theorem of Kakutani-Fan. In fact we have:

COROLLARY 4. *Let D be a non-empty convex bounded closed subset of $\mathcal{K}$. Let $F : D \to \mathcal{K}$ be a h-u.s.c. and **K**-compact multifunction, with non-empty convex compact values $F(x) \subset D$, for every $x \in D$. Then, there exists $x_0 \in D$ such that $x_0 \in F(x_0)$.*

PROOF. Set $\mathcal{D} = \{\{x\} \in \mathcal{K} : x \in D\}$, and define $G : \mathcal{D} \to C(\mathcal{K})$ by $G(\{x\}) = \{\{y\} \in \mathcal{K} : y \in F(x)\}$, $\{x\} \in \mathcal{D}$. By Remark 1, G is $C(\mathcal{K})$-compact. Moreover, G is $h_{C(\mathcal{K})}$-u.s.c. with values $G(\{x\}) \subset \mathcal{D}$, for every $\{x\} \in \mathcal{D}$. Thus, by Corollary 2, there exists $\{x_0\} \in \mathcal{D}$ such that $\{x_0\} \in G(\{x_0\})$, whence $x_0 \in F(x_0)$. □

4. Further results.

In this section we use some of the previous results to establish further fixed point theorems for mapping defined on hyperspaces.

The following is a hyperspace version of Mazur's theorem.

PROPOSITION 3. *Let $\mathcal{A}$ be a compact subset of $\mathcal{K}$. Then the set $\mathcal{C} = cl_{\mathcal{K}}conv_{\mathcal{K}}(\mathcal{A})$ is a convex compact subset of $\mathcal{K}$.*

PROOF. By Remark 2, $\mathcal{C}$ is convex. To prove that $\mathcal{C}$ is compact, it suffices to show that each sequence $\{X_n\} \subset conv_{\mathcal{K}}(\mathcal{A})$ contains a subsequence which converges to some $X_0 \in \mathcal{C}$. From the definition of $conv_{\mathcal{K}}(\mathcal{A})$, for each X_n there exist $p_n \in \mathbf{N}$, $Y_n^i \in \mathcal{A}$, and $\lambda_n^i \geq 0$ with $\sum_{i=1}^{p_n} \lambda_n^i = 1$ such that

$$X_n = \sum_{i=1}^{p_n} \lambda_n^i Y_n^i .$$

Let $j : \mathcal{K} \to \mathbf{H}$ be the isometric isomorphism, given by Hörmander's theorem, of $\mathcal{K}$ on the positive convex closed cone $\mathbf{V} = j(\mathcal{K})$ contained in the Banach space $\mathbf{H}$. The set $\Delta = \{j(X) : X \in \mathcal{A}\}$ is compact, hence by Mazur's theorem [5, p. 416], also $\overline{co}\Delta$ is so, and $\Delta \subset \mathbf{V}$.

For each $n \in \mathbf{N}$ we have

$$j(X_n) = \sum_{i=1}^{p_n} \lambda_n^i j(Y_n^i),$$

thus $\{j(X_n)\} \subset \overline{co}\Delta$. Let $\{j(X_{n_k})\}$ be a subsequence converging to some point $\xi_0 \in \overline{co}\Delta$. As $\xi_0 \in \mathbf{V}$, there is an $X_0 \in \mathcal{K}$ such that $j(X_0) = \xi_0$. Since $h_{\mathcal{K}}(X_{n_k}, X_0) = \|j(X_{n_k}) - j(X_0)\|_{\mathbf{H}}$, $k \in \mathbf{N}$, it follows that $\{X_{n_k}\}$ converges to X_0 and, clearly, $X_0 \in \mathcal{C}$. This completes the proof. □

THEOREM 2. *Let $\mathcal{D}$ be a non-empty convex bounded closed subset of $\mathcal{K}$. Let $F : \mathcal{D} \to \mathcal{D}$ be a h-u.s.c. (resp. h-l.s.c.) and $\mathcal{K}$-compact mapping. Then, there exists $X_0 \in \mathcal{D}$ such that*

$$X_0 \subset F(X_0) \text{ (resp. } X_0 \supset F(X_0)).$$

In particular $X_0 = F(X_0)$, if F is h-continuous.

PROOF. Put $\mathcal{A} = cl_{\mathcal{K}}\mathcal{R}_{\mathcal{K}}(F)$ and $\mathcal{C} = cl_{\mathcal{K}}conv_{\mathcal{K}}(\mathcal{A})$. The set $\mathcal{A} \subset \mathcal{K}$ is compact, since F is $\mathcal{K}$-compact. Hence, by Proposition 3, $\mathcal{C}$ is a convex compact subset of $\mathcal{K}$, and F maps $\mathcal{C}$ into itself.

Suppose F is h-u.s.c. (resp. h-l.s.c.). For $X \in \mathcal{C}$, put

$$G(X) = \{Y \in \mathcal{C} : Y \subset F(X)\} \text{ (resp. } G(X) = \{Y \in \mathcal{C} : Y \supset F(X)\}). \tag{4.1}$$

$G(X)$ is a non-empty convex compact set contained in $\mathcal{C}$. Hence, (4.1) defines a multifunction $G : \mathcal{C} \to C(\mathcal{K})$, with values $G(X) \subset \mathcal{C}$ for every $X \in \mathcal{C}$.

G is $h_{C(\mathcal{K})}$-u.s.c. In the contrary case, there exist $X_0 \in \mathcal{C}$, $\varepsilon > 0$, and a sequence $\{X_n\} \subset \mathcal{C}$ converging to X_0, such that $e_{C(\mathcal{K})}(G(X_n), G(X_0)) > \varepsilon$ for every $n \in \mathbf{N}$. Take $Y_n \in G(X_n)$ satisfying

$$\inf_{Z \in G(X_0)} h(Y_n, Z) > \varepsilon, \quad n \in \mathbf{N}. \tag{4.2}$$

Since $\{Y_n\} \subset \mathcal{C}$, there is a subsequence, say $\{Y_n\}$, which converges to some $Y_0 \in \mathcal{C}$. From (4.1) we have $Y_n \subset F(X_n)$ (resp. $Y_n \supset F(X_n)$), for every $n \in \mathbf{N}$. As F is h-u.s.c. (resp. h-l.s.c.), it follows that $Y_0 \subset F(X_0)$ (resp. $Y_0 \supset F(X_0)$). Consequently $Y_0 \in G(X_0)$ and, by (4.2), $h(Y_n, Y_0) > \varepsilon$ for every $n \in \mathbf{N}$, a contradiction. Hence G is $h_{C(\mathcal{K})}$-u.s.c.

Clearly G is also $C(\mathcal{K})$-compact. Thus, by Corollary 2, there exists an $X_0 \in \mathcal{C}$ such that $X_0 \in G(X_0)$, and so $X_0 \subset F(X_0)$ (resp. $X_0 \supset F(X_0)$). This completes the proof. □

THEOREM 3. *Let $\mathcal{D}$ be a non-empty convex compact subset of $\mathcal{K}$. Let $f : \mathcal{D} \to \mathbf{K}$ be a continuous function satisfying $f(\mathcal{D}) \subset \bigcup_{X \in \mathcal{D}} X$. Then, there exists $X_0 \in \mathcal{D}$ such that $f(X_0) \in X_0$.*

PROOF. For $X \in \mathcal{D}$, put

$$G(X) = \{Y \in \mathcal{D} : f(X) \in Y\}. \tag{4.3}$$

Clearly, the set $G(X) \subset \mathcal{D}$ is non-empty convex and compact. Thus (4.3) defines a multifunction $G : \mathcal{D} \to C(\mathcal{K})$ with values $G(X) \subset \mathcal{D}$, for every $X \in \mathcal{D}$.

G is $h_{C(\mathcal{K})}$-u.s.c. In the contrary case, there exist $X_0 \in \mathcal{D}, \varepsilon > 0$ and a sequence $\{X_n\} \subset \mathcal{D}$ converging to X_0, such that $e_{C(\mathcal{K})}(G(X_n), G(X_0)) > \varepsilon$ for every $n \in \mathbf{N}$. Take $Y_n \in G(X_n)$ satisfying (4.2). Now $\{Y_n\} \subset \mathcal{D}$, a compact set, whence there is a subsequence, say $\{Y_n\}$, which converges to some $Y_0 \in \mathcal{D}$. Since $f(X_n) \in Y_n$, letting $n \to \infty$, one has $f(X_0) \in Y_0$. Therefore $Y_0 \in G(X_0)$ and, by (4.2), it follows that $h(Y_n, Y_0) > \varepsilon$ for every $n \in \mathbf{N}$, a contradiction. Thus G is $h_{C(\mathcal{K})}$-u.s.c.

Clearly G is also $C(\mathcal{K})$-compact. By Corollary 2, there exists $X_0 \in \mathcal{D}$ such that $X_0 \in G(X_0)$ and thus, by (4.3), $f(X_0) \in X_0$, completing the proof. □

Brouwer-Schauder's fixed point theorem follows at once from Theorem 3, as it is shown in the following:

COROLLARY 5. *Let D be a non-empty convex bounded closed subset of* $\mathbf{K}$. *Let $f : D \to D$ be continuous and* $\mathbf{K}$*-compact. Then, there exists $x_0 \in D$ such that $x_0 = f(x_0)$.*

PROOF. Put $C = \overline{co} f(D)$. Clearly, C is a non-empty convex compact subset of D, and f maps C into itself. Now set $\mathcal{D} = \{\{x\} \in \mathcal{K} : x \in C\}$ and define $g : \mathcal{D} \to \mathbf{K}$ by $g(\{x\}) = f(x)$. Since g is continuous and $g(\mathcal{D}) = f(C) \subset C = \bigcup_{\{x\} \in \mathcal{D}} \{x\}$, by Theorem 3 there exists $\{x_0\} \in \mathcal{D}$ such that $g(\{x_0\}) \in \{x_0\}$. Thus, $x_0 \in D$ and $x_0 = f(x_0)$, completing the proof. □

The following is a hyperspace version of a theorem of Fan [7]. The proof runs as in Sehgal [22], and so it is omitted.

THEOREM 4. *Let $\mathcal{D}$ be a non-empty convex compact subset of $\mathcal{K}$, and let $F : \mathcal{D} \to \mathcal{K}$ be a h-continuous mapping. Then there exists $X_0 \in \mathcal{D}$ such that $h(F(X_0), X_0) = \min_{Z \in \mathcal{D}} h(F(X_0), Z)$.*

5. An application to differential equations with set-valued solutions.

In this section we use one of the previous fixed point results in order to establish the existence of solutions to a Cauchy problem, for differential equations of the form:

$$DX(t) = F(t, X(t)), \quad X(0) = A. \tag{5.1}$$

Here, F is a $\mathcal{K}$-valued mapping defined on $I \times \mathcal{K}$, $I = [0, 1]$, $DX(t)$ denotes the Hukuhara derivative at time t of the multifunction $X : I \to \mathcal{K}$, and $A \in \mathcal{K}$.

The study of the above differential equations was started by De Blasi and Iervolino [4]. Developments and applications to some problems of control theory can be found in Artstein [1], Kisielewicz [14], Plotnikov [18], [19], Tolstonogov [24].

DEFINITION 6. Let $X : I \to \mathcal{K}$ and $t_0 \in (0, 1)$ be given. Suppose that there exist two multifunctions $\Delta^+, \Delta^- : (0, \delta) \to \mathcal{K}$, for some $\delta > 0$, and a set $B \in \mathcal{K}$, such that:

(i) $X(t_0 + h) = X(t_0) + \Delta^+(h)$, $X(t_0) = X(t_0 - h) + \Delta^-(h)$
for every $h \in (0, \delta)$,

(ii) $\lim_{h\to 0+} \Delta^+(h)/h = B = \lim_{h\to 0+} \Delta^-(h)/h$.

Then B is called *Hukuhara's derivative* of X at t_0, and denoted by $DX(t_0)$.

When $t_0 = 0, 1$, the modifications are obvious.

REMARK 5. X is h-continuous at each point $t \in I$, in which $DX(t)$ exists.

REMARK 6. If $U : I \to \mathcal{K}$ is h-continuous and $A \in \mathcal{K}$, then the multifunction $X : I \to \mathcal{K}$ given by $X(t) = A + \int_0^t U(s)ds$, $t \in I$, where the integral is in the sense of Hukuhara [10], has Hukuhara's derivative $DX(t) = U(t)$, for every $t \in I$.

For other elementary properties of the Hukuhara derivative see [10] and [4].

PROPOSITION 4. *Let $\{X_n\}$ be an equi-h-continuous sequence of multifunctions $X_n : I \to \mathcal{K}$ such that for every $t \in I$, the set $\{X_n(t) : n \in \mathbf{N}\}$ is precompact in $\mathcal{K}$. Then there exists a subsequence $\{X_{n_k}\}$ which converges uniformly to a continuous multifunction $X_0 : I \to \mathcal{K}$.*

DEFINITION 7. A multifunction $X : I \to \mathcal{K}$, with $X(0) = A$, which has Hukuhara's derivative $DX(t)$ satisfying $DX(t) = F(t, X(t))$, for every $t \in I$, is called solution of the Cauchy problem (5.1).

In the following proposition we prove the existence of solutions to the Cauchy problem (5.1), under Peano type assumptions. In finite dimension a similar result was obtained in [4].

PROPOSITION 5. *Let C be a non-empty closed convex cone contained in $\mathcal{K}$. Let $F : I \times C \to C$ be h-continuous and $\mathcal{K}$-compact. Then, for every $A \in C$, the Cauchy problem (5.1) has a solution $X : I \to C$.*

PROOF. Let $\mathcal{X}$ and $\rho_{\mathcal{X}}$ be given by (3.1) and (3.2), with $M = I$, and observe that $(\mathcal{X}, \rho_{\mathcal{X}})$ is a complete metric space. (In (3.1) the boundedness of X is redundant, as I is compact.)

From the hypothesis, F is bounded by a constant M, say. Define:

$$\mathcal{D} = \{X : I \to C : X \text{ is } h\text{-continuous, and } h(X(t), A) \le M \text{ for every } t \in I\}.$$

$\mathcal{D}$ is a non-empty convex bounded closed subset of $\mathcal{X}$.

For $X \in \mathcal{D}$, set

$$\Phi(X)(t) = A + \int_0^t F(s, X(s))ds, \quad t \in I. \tag{5.2}$$

For every $t \in I$, $\Phi(X)(t)$ is in C, as the positive cone C is convex and closed in $\mathcal{K}$. Moreover, $\Phi(X)$ is h-continuous, by Remarks 5, 6, and satisfies $h(\Phi(X)(t), A) \le M$ for every $t \in I$. Whence $\Phi(X) \in \mathcal{D}$, thus (5.2) defines a mapping $\Phi : \mathcal{D} \to \mathcal{D}$.

Φ is $\mathcal{X}$-compact. It suffices to show that each sequence $\{Y_n\} \subset \mathcal{D}$, where $Y_n = \Phi(X_n)$ for some $X_n \in \mathcal{D}$, contains a subsequence which converges to an $X_0 \in \mathcal{D}$. It is routine to see that $\{Y_n\}$ is equi-h-continuous. From the hypothesis the set $\mathcal{A} = cl_{\mathcal{K}}\mathcal{R}_{\mathcal{K}}(F)$ is compact, whence, by Proposition 3, also $cl_{\mathcal{K}}conv_{\mathcal{K}}(\mathcal{A})$ is so. Moreover, for each fixed $t \in I$ we have

$$Y_n(t) = A + \int_0^t F(s, X_n(s))ds \in A + tcl_{\mathcal{K}}conv_{\mathcal{K}}(\mathcal{A}), \quad n \in \mathbf{N}.$$

By Proposition 4, $\{Y_n\}$ contains a subsequence $\{Y_{n_k}\}$ which converges uniformly, therefore in the metric $\rho_{\mathcal{X}}$ of $\mathcal{X}$, to some $X_0 \in \mathcal{D}$. Thus Φ is $\mathcal{X}$-compact.

It is easy to see that Φ is also $\mathcal{X}$-continuous.

By Corollary 1, there exists $X_0 \in \mathcal{D}$ such that $X_0 = \Phi(X_0)$, and so

$$X_0(t) = A + \int_0^t F(s, X_0(s))ds, \quad t \in I.$$

In view of Remark 6 X_0 is a solution of the Cauchy problem (5.1), completing the proof. □

REMARK 7. If $F : I \times \mathcal{C} \to \mathcal{C}$, where $\mathcal{C}$ is as in Proposition 5, is h-continuous and Lipschitzean in X uniformly in t, i.e. there exists a constant $L \geq 0$ such that $h(F(t, X), F(t, Y)) \leq Lh(X, Y)$ for every $(t, X), (t, Y) \in I \times \mathcal{C}$, then one can prove that the Cauchy problem (5.1) has a unique solution

REMARK 8. Suppose that $\mathbf{K}$, the underlying space of $\mathcal{K}$, is $\mathbf{R}^n$. Define

$$J = \{X \in \mathcal{K} : X = [a_1, b_1] \times \cdots \times [a_n, b_n]\}$$

where $a_i \leq b_i$, $i = 1, ..., n$. J is a positive convex closed cone contained in $\mathcal{K}$ thus Proposition 5 holds (with J in place of C).

The cone J is useful in approximation theory and interval analysis. Further applications and developments in other directions can be found in Nickel [17] and Schmidt [21].

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