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Satake Diagrams and Restricted Root Systems of Semisimple Pseudo-Riemannian Symmetric Spaces

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Abstract. In this paper, we complete the lists of Satake diagrams and restricted root systems (including signatures of roots) for all classical semisimple pseudo-Riemannian symmetric spaces, which were classified by M. Berger. We also complete the list of the cohomogeneities of the linear isotropy representations of the spaces.

1. Introduction

Let $(\mathfrak{g}, \mathfrak{h})$ be a semisimple symmetric pair and σ be an involution of $\mathfrak{g}$ such that the set of all fixed points of σ coincides with $\mathfrak{h}$. If we put $\mathfrak{q} := \{X \in \mathfrak{g} \mid \sigma(X) = -X\}$, we have a direct decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$. The pair $(\mathfrak{g}, \mathfrak{h})$ is said to be *classical* (resp. *exceptional*) if the Lie algebra $\mathfrak{g}$ is classical (resp. exceptional). The main purpose of this paper is to complete the lists of Satake diagrams and restricted root systems of all classical semisimple symmetric pairs. According to Berger's classification [1], there exist 54 classical symmetric pairs and 104 exceptional symmetric pairs. The theory of restricted root systems for semisimple symmetric spaces is developed by W. Rossmann [6], T. Oshima and J. Sekiguchi [5].

Let θ be a Cartan involution of $\mathfrak{g}$ commuting with σ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the direct decomposition of $\mathfrak{g}$ corresponding to θ. Let $\mathfrak{a}$ be a maximal split abelian subspace of $\mathfrak{q}$, i.e., $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{q}$ which consists of only hyperbolic elements or only elliptic elements. Let $\mathfrak{a}_{\mathfrak{q}}$ be a maximal abelian subspace of $\mathfrak{q}$ containing $\mathfrak{a}$. It is known that if $(\mathfrak{g}, \mathfrak{k})$ and $(\mathfrak{g}', \mathfrak{k}')$ are semisimple Riemannian symmetric pairs whose restricted root systems coincide including the multiplicities of the roots, $(\mathfrak{g}, \mathfrak{k})$ and $(\mathfrak{g}', \mathfrak{k}')$ are isomorphic. But the analogous statement does not hold for a general semisimple symmetric pair. The Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$ (resp. $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_{\mathfrak{q}})$) is constructed from the Dynkin diagram of the complexification $\mathfrak{g}^{\mathbf{C}}$ of $\mathfrak{g}$ by indicating how simple roots of $\mathfrak{g}^{\mathbf{C}}$ are restricted to $\mathfrak{a}$ (resp. $\mathfrak{a}_{\mathfrak{q}}$). To characterize $(\mathfrak{g}, \mathfrak{h})$, it is important to describe the Satake diagrams of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$ and $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_{\mathfrak{q}})$ and to calculate the signatures of restricted roots. In Table 1, we complete the list of the Satake diagrams of all classical semisimple symmetric pairs. We also complete the list of the restricted root systems of the pairs in Table 2. The codimension of the principal orbit of s-representation through a

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point of $\mathfrak{p} \cap \mathfrak{q}$ is not necessarily equal to that of the principal orbit through a point of $\mathfrak{k} \cap \mathfrak{q}$. Therefore the representation possesses two kinds of cohomogeneities. In this paper, we call the codimension of the principal orbit through a point of $\mathfrak{p}\cap\mathfrak{q}$ (resp. $\mathfrak{k}\cap\mathfrak{q}$) the *$v$-cohomogeneity* (resp. *$t$-cohomogeneity*). In Table 3, we complete the list of the v-cohomogeneities and the t-cohomogeneities of s-representations of classical semisimple symmetric spaces. In the case of a Riemannian symmetric pair, the cohomogeneity of the s-representation is equal to the rank of the Riemannian symmetric space. But the v-cohomogeneity and the t-cohomogeneity are greater than or equal to the rank of a general semisimple symmetric space.

RELATED RESEARCH. We plan to complete the lists of Satake diagrams and restricted root systems of all exceptional semisimple symmetric pairs. Using the results of this paper, we will study the geometry of s-representations of semisimple symmetric spaces. We will investigate the local orbit types of s-representations in a subsequent paper. In the case of Riemannian symmetric spaces, the local orbit types of s-representations were investigated by H. Tamaru [7] and K. Kondo [3].

2. Preliminaries

Let (G, H) be a semisimple symmetric pair, $(\mathfrak{g}, \mathfrak{h})$ be its infinitesimal pair and σ be an involution of $\mathfrak{g}$ such that the set of all fixed points of σ coincides with $\mathfrak{h}$. If we put $\mathfrak{q} := \{X \in \mathfrak{g} \mid \sigma(X) = -X\}$, we have an orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ with respect to the Killing form B of $\mathfrak{g}$. We denote by Ad_G (resp. $\mathrm{ad}_{\mathfrak{g}}$) the adjoint representation of G (resp. $\mathfrak{g}$). Then B restricted to $\mathfrak{q} \times \mathfrak{q}$ is nondegenerate and $\mathrm{Ad}_G(H)$-invariant. Since $\mathfrak{q}$ is identified with the tangent space of G/H at eH, the bilinear form on $\mathfrak{q} \times \mathfrak{q}$ determines a G-invariant nondegenerate metric on G/H, where e is the identity element of G. It follows from Lemma 10.2 of [1] that there exists a Cartan involution θ of $\mathfrak{g}$ commuting with σ. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition corresponding to θ, where $\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}$ and $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$. Since $\sigma \circ \theta = \theta \circ \sigma$ holds, we have an orthogonal decomposition $\mathfrak{g} = \mathfrak{k}\cap\mathfrak{h} + \mathfrak{p}\cap\mathfrak{h} + \mathfrak{k}\cap\mathfrak{q} + \mathfrak{p}\cap\mathfrak{q}$. If we put $\mathfrak{h}^a := \{X \in \mathfrak{g} \mid \theta\circ\sigma(X) = X\}$ and $\mathfrak{q}^a := \{X \in \mathfrak{g} \mid \theta \circ \sigma(X) = -X\}$, we have an orthogonal decomposition $\mathfrak{g} = \mathfrak{h}^a + \mathfrak{q}^a$. Also we have $\mathfrak{h}^a = \mathfrak{k}\cap\mathfrak{h} + \mathfrak{p}\cap\mathfrak{q}$ and $\mathfrak{q}^a = \mathfrak{k}\cap\mathfrak{q} + \mathfrak{p}\cap\mathfrak{h}$. We denote by $\mathfrak{g}^{\mathbf{C}}$ the complexification of $\mathfrak{g}$. We extend σ and θ to $\mathfrak{g}^{\mathbf{C}}$ as $\mathbf{C}$-linear involutions, which are also denoted by the same symbols σ and θ, respectively. Then

$$\mathfrak{g}^d := \mathfrak{k}\cap\mathfrak{h} + \sqrt{-1}(\mathfrak{p}\cap\mathfrak{h}) + \sqrt{-1}(\mathfrak{k}\cap\mathfrak{q}) + \mathfrak{p}\cap\mathfrak{q}$$

is another real form of $\mathfrak{g}^{\mathbf{C}}$. We also denote by σ and θ the restrictions of σ and θ to $\mathfrak{g}^d$, respectively. Then σ is a Cartan involution of $\mathfrak{g}^d$. If we put $\mathfrak{h}^d := \{X \in \mathfrak{g}^d \mid \theta(X) = X\}$ and $\mathfrak{q}^d := \{X \in \mathfrak{g}^d \mid \theta(X) = -X\}$, we have an orthogonal decomposition $\mathfrak{g}^d = \mathfrak{h}^d + \mathfrak{q}^d$ with respect to the Killing form of $\mathfrak{g}^d$. The pair $(\mathfrak{g}, \mathfrak{h}^a)$ (resp. $(\mathfrak{g}^d, \mathfrak{h}^d)$) is called the *associated* (resp. *dual*) symmetric pair of $(\mathfrak{g}, \mathfrak{h})$. For simplicity, we write $(\mathfrak{g}, \mathfrak{h})^a$ and $(\mathfrak{g}, \mathfrak{h})^d$ instead of $(\mathfrak{g}, \mathfrak{h}^a)$ and $(\mathfrak{g}^d, \mathfrak{h}^d)$, respectively. Then it is clear that $(\mathfrak{g}, \mathfrak{h})^{aa}$ and $(\mathfrak{g}, \mathfrak{h})^{dd}$ are isomorphic to $(\mathfrak{g}, \mathfrak{h})$. Moreover, $(\mathfrak{g}, \mathfrak{h})^{ada} = (\mathfrak{g}, \mathfrak{h})^{dad}$ holds.

We recall that an element $X \in \mathfrak{q}$ is said to be *semisimple* if the complexification $\mathrm{ad}_{\mathfrak{g}}(X)^{\mathbf{C}}$ of the endomorphism $\mathrm{ad}_{\mathfrak{g}}(X)$ of $\mathfrak{g}$ is diagonalizable. A semisimple element $X \in \mathfrak{q}$ is said to be *hyperbolic* (resp. *elliptic*) if any eigenvalue of $\mathrm{ad}_{\mathfrak{g}}(X)^{\mathbf{C}}$ is real (resp. pure imaginary). Let $\mathfrak{a}$ be a maximal split abelian subspace of $\mathfrak{q}$, i.e., $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{q}$ which consists of only hyperbolic elements or only elliptic elements. Let $\mathfrak{a}_{\mathfrak{q}}$ be a maximal abelian subspace of $\mathfrak{q}$ containing $\mathfrak{a}$. Then $\mathfrak{a}_{\mathfrak{q}}$ consists of only semisimple elements of $\mathfrak{g}$ (cf. Lemma 2.2 of [5]). The dimension of $\mathfrak{a}$ and $\mathfrak{a}_{\mathfrak{q}}$ are called the *split rank* and the *rank* of $(\mathfrak{g}, \mathfrak{h})$, respectively. We call $\mathfrak{a}$ *vector-type* (resp. *toroidal-type*) if there exists a Cartan involution ρ of $\mathfrak{g}$ such that $\mathfrak{a}$ is contained in the (-1)-eigenspace (resp. $(+1)$-eigenspace) of ρ. Let Δ be the restricted root system of $(\mathfrak{g}, \mathfrak{h})$ with respect to $\mathfrak{a}$. Then if $\mathfrak{a}$ is vector-type (resp. toroidal-type), Δ coincides with the restricted root system of $(\mathfrak{g}^d, \mathfrak{h}^d)$ (resp. $(\mathfrak{g}^a, \mathfrak{h}^a)$) with respect to $\mathfrak{a}$ (cf. Lemma 2.15.1 of [5]). Note that the restricted root system of $(\mathfrak{g}, \mathfrak{h})$ with respect to a toroidal-type maximal split abelian subspace coincides with that of $(\mathfrak{g}, \mathfrak{h})^{ada}$ with respect to a vector-type maximal split abelian subspace. From this fact, if $(\mathfrak{g}, \mathfrak{h})$ is anti-Kaehlerian, the restricted root systems with respect to a vector-type maximal split abelian subspace and a toroidal-type maximal split abelian subspace coincide. It follows from Theorem 2.11 of [5] that Δ is a root system. In Table I and V of [5], T. Oshima and J. Sekiguchi gave the restricted root systems of some irreducible semisimple symmetric pairs.

In the sequel, we assume that $\mathfrak{a}$ is contained in $\mathfrak{p}$. If we put $\mathfrak{g}_\lambda := \{X \in \mathfrak{g} \mid [A, X] = \lambda(A)X, \forall A \in \mathfrak{a}\}$ for any $\lambda \in \mathfrak{a}^*$, we have the restricted root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\lambda \in \Delta} \mathfrak{g}_\lambda .$$

For any $\lambda, \mu \in \Delta \cup \{0\}$, we have $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$, $\sigma(\mathfrak{g}_\lambda) = \mathfrak{g}_{-\lambda}$ and $\theta(\mathfrak{g}_\lambda) = \mathfrak{g}_{-\lambda}$. We denote by $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{a})$ (resp. $\mathfrak{z}_{\mathfrak{q}}(\mathfrak{a})$) the centralizer of $\mathfrak{a}$ in $\mathfrak{h}$ (resp. $\mathfrak{q}$). Note that $\mathfrak{a}$ coincides with $\mathfrak{z}_{\mathfrak{q}}(\mathfrak{a})$, if $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{q}$ (for example, in the case where G/H is a Riemannian symmetric space). We put $\mathfrak{h}_\lambda := (\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda}) \cap \mathfrak{h}$ and $\mathfrak{q}_\lambda := (\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda}) \cap \mathfrak{q}$. In this paper, we call the dimension of $\mathfrak{q}_\lambda$ and the pair $(\dim(\mathfrak{p} \cap \mathfrak{q}_\lambda), \dim(\mathfrak{k} \cap \mathfrak{q}_\lambda))$ the *multiplicity* and the *signature* of λ, which are different from Definition 2.14 of [5], respectively.

LEMMA 2.1. *Let Δ_+ be the positive root system of Δ with respect to some lexicographic ordering of $\mathfrak{a}^*$. Then $\mathfrak{h}$ and $\mathfrak{q}$ are orthogonally decomposed as*

$$\mathfrak{h} = \mathfrak{z}_{\mathfrak{h}}(\mathfrak{a}) + \sum_{\lambda \in \Delta_+} \mathfrak{h}_\lambda ,$$

and

$$\mathfrak{q} = \mathfrak{z}_{\mathfrak{q}}(\mathfrak{a}) + \sum_{\lambda \in \Delta_+} \mathfrak{q}_\lambda ,$$

respectively.

PROOF. Since $\mathfrak{g}_0$ is invariant under σ, we have $\mathfrak{g}_0 = \mathfrak{z}_{\mathfrak{h}}(\mathfrak{a}) + \mathfrak{z}_{\mathfrak{q}}(\mathfrak{a})$. Similarly, $\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda} = \mathfrak{h}_\lambda + \mathfrak{q}_\lambda$ holds for any $\lambda \in \Delta_+$. Hence we obtain the orthogonal decompositions of $\mathfrak{h}$ and $\mathfrak{q}$ as in the statement. □

Let $\mathrm{Ad}_{\mathfrak{q}} : H \to GL(\mathfrak{q})$ be defined by $\mathrm{Ad}_{\mathfrak{q}}(h) := \mathrm{Ad}_G(h)|_{\mathfrak{q}}$ for all $h \in H$. Then we call the representation $\mathrm{Ad}_{\mathfrak{q}}$ the *s-representation* of G/H. Let A be a regular point of $\mathfrak{a}$. It follows from Lemma 2.1 that the normal space of the $\mathrm{Ad}_{\mathfrak{q}}(H)$-orbit through A coincides with $\mathfrak{z}_{\mathfrak{q}}(\mathfrak{a})$. We call the dimension of $\mathfrak{z}_{\mathfrak{q}}(\mathfrak{a})$ the *v-cohomogeneity* (resp. *t-cohomogeneity*) if $\mathfrak{a}$ is vector-type (resp. toroidal-type).

Let $\mathfrak{a}_{\mathfrak{p}}$ be a maximal abelian subspace of $\mathfrak{p}$. Then the Satake diagram of $(\mathfrak{g}, \mathfrak{k}, \mathfrak{a}_{\mathfrak{p}})$ is defined (see p. 531 of [2]). By imitating the definition of the Satake diagram of $(\mathfrak{g}, \mathfrak{k}, \mathfrak{a}_{\mathfrak{p}})$, we shall define the Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$ as follows. Let $\mathfrak{c}$ be a Cartan subalgebra of $\mathfrak{g}^{\mathbf{C}}$ containing $\mathfrak{a}$. Denote by R the root system of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\mathfrak{c}$ and by A_α ($\alpha \in R$) the vector of $\mathfrak{c}$ defined by $B_{\mathfrak{g}^{\mathbf{C}}}(A, A_\alpha) = \alpha(A)$ for all $A \in \mathfrak{c}$, where $B_{\mathfrak{g}^{\mathbf{C}}}$ is the Killing form of $\mathfrak{g}^{\mathbf{C}}$. Set $\mathfrak{c}_{\mathbf{R}} := \mathrm{Span}_{\mathbf{R}}\{A_\alpha \mid \alpha \in R\}$. We take compatible orderings in the dual spaces of $\mathfrak{a}$ and $\mathfrak{c}_{\mathbf{R}}$, respectively. Denote by R_+ the positive root system of R with respect to the ordering of $(\mathfrak{c}_{\mathbf{R}})^*$ and by $\Psi(R)$ the simple root system of R contained in R_+. Set $\Psi(R)_0 := \{\alpha \in \Psi(R) \mid \alpha|_{\mathfrak{a}} = 0\}$. By imitating the definition of the Satake diagram of $(\mathfrak{g}, \mathfrak{k}, \mathfrak{a}_{\mathfrak{p}})$ the Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$ is defined by using $\Psi(R)$ and $\Psi(R)_0$. Similarly, we can define the Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_{\mathfrak{q}})$.

3. Determination of Satake diagrams and restricted root systems

In this section, we describe how to determine Satake diagrams and the restricted root systems of classical semisimple symmetric pairs. Let $(\mathfrak{g}, \mathfrak{h})$ be a classical semisimple symmetric pair. Following to Algorithms 1–6, we can determine the Satake diagrams of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$ and $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_{\mathfrak{q}})$, and the restricted root system of $(\mathfrak{g}, \mathfrak{h})$ with respect to $\mathfrak{a}$, where $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$ and $\mathfrak{a}_{\mathfrak{q}}$ is a maximal abelian subspace of $\mathfrak{q}$ containing $\mathfrak{a}$.

(Algorithm 1) First, we take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p} \cap \mathfrak{q}$ and a maximal abelian subspace $\mathfrak{a}_{\mathfrak{q}}$ (resp. $\mathfrak{a}_{\mathfrak{p}}$) of $\mathfrak{q}$ (resp. $\mathfrak{p}$) which containing $\mathfrak{a}$. In the case where $\mathfrak{g}^{\mathbf{C}}$ is a simple complex Lie algebra, we take the Cartan subalgebras of simple complex Lie algebras described in Chapter III, §8 of [2] as a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}^{\mathbf{C}}$. Also in the case where $\mathfrak{g}^{\mathbf{C}}$ is the direct sum of two simple complex Lie algebras, we take the direct sum of their Cartan subalgebras as a Cartan subalgebra of $\mathfrak{g}^{\mathbf{C}}$. We find an element τ of $\mathrm{Int}(\mathfrak{g}^{\mathbf{C}})$ such that $\tau(\mathfrak{a}_{\mathfrak{p}} + \mathfrak{a}_{\mathfrak{q}})$ is contained in $\mathfrak{c}$, where $\mathrm{Int}(\mathfrak{g}^{\mathbf{C}})$ is the adjoint group of $\mathfrak{g}^{\mathbf{C}}$.

(Algorithm 2) Let $\mathfrak{g}^{\mathbf{C}} = \mathfrak{c} + \sum_{\alpha \in R} \mathfrak{g}^{\mathbf{C}}_\alpha$ be the root space decomposition of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\mathfrak{c}$, where R is the root system of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\mathfrak{c}$. We find a basis $(A_1, \ldots, A_r)$ of $\mathfrak{c}_{\mathbf{R}}$ such that $(A_1, \ldots, A_m)$ is a basis of $\tau(\mathfrak{a})$ and $(A_{m+1}, \ldots, A_n)$ is a basis of $\tau(\sqrt{-1}(\mathfrak{a}_{\mathfrak{q}} \cap \mathfrak{k}))$, where r (resp. n) is the rank of $\mathfrak{g}^{\mathbf{C}}$ (resp. $(\mathfrak{g}, \mathfrak{h})$) and m is the split rank of $(\mathfrak{g}, \mathfrak{h})$. Then the lexicographic orderings of $(\tau(\mathfrak{a}))^*$, $(\tau(\mathfrak{a} + \sqrt{-1}(\mathfrak{a}_{\mathfrak{q}} \cap \mathfrak{k})))^*$ and $(\mathfrak{c}_{\mathbf{R}})^*$ with respect to these bases are compatible. Let R_+ be the positive root system of R with respect to the lexicographic ordering of $(\mathfrak{c}_{\mathbf{R}})^*$ and $\Psi(R)$ be the simple root system of R which is contained in R_+.

(Algorithm 3) We put $\Psi(R)_0 := \{\alpha \in \Psi(R) \mid \bar{\alpha} = 0\}$, where $\bar{\cdot}$ denotes the restriction to $\tau(\mathfrak{a})$. Then the Satake diagram of $(\tau(\mathfrak{g}), \tau(\mathfrak{h}), \tau(\mathfrak{a}))$ is described as follows. In the Dynkin diagram of $\mathfrak{g}^{\mathbf{C}}$, every root of $\Psi(R)_0$ is denoted by a black circle $\bullet$ and every root of $\Psi(R) \setminus \Psi(R)_0$ by a white circle $\circ$. If $\alpha, \beta \in \Psi(R) \setminus \Psi(R)_0$ satisfies $\bar{\alpha} = \bar{\beta}$, α and β are joined by a curved arrow.

(Algorithm 4) We put $\Psi(R)'_0 := \{\alpha \in \Psi(R) \mid \hat{\alpha} = 0\}$, where $\hat{\cdot}$ denotes the restriction to $\tau(\mathfrak{a}_{\mathfrak{q}})$. Then the Satake diagram of $(\tau(\mathfrak{g}), \tau(\mathfrak{h}), \tau(\mathfrak{a}_{\mathfrak{q}}))$ is described as follows. In the Dynkin diagram of $\mathfrak{g}^{\mathbf{C}}$, every root of $\Psi(R)'_0$ is denoted by a black circle $\bullet$ and every root of $\Psi(R) \setminus \Psi(R)'_0$ by a white circle $\circ$. If $\alpha, \beta \in \Psi(R) \setminus \Psi(R)'_0$ satisfies $\hat{\alpha} = \hat{\beta}$, α and β are joined by a curved arrow.

(Algorithm 5) The set $\Delta := \{\bar{\alpha} \mid \alpha \in R \text{ such that } \bar{\alpha} \neq 0\}$ is the restricted root system with respect to $\tau(\mathfrak{a})$. Then we can determine the positive root system Δ_+ of Δ with respect to the above lexicographic ordering of $(\tau(\mathfrak{a}))^*$ and the simple root system of Δ which is contained in Δ_+. Let $m(\lambda)$ and $(m^+(\lambda), m^-(\lambda))$ be the multiplicity and the signature of $\lambda \in \Delta$, respectively. We investigate the cardinality of $\{\alpha \in R \mid \bar{\alpha} = \lambda\}$ and the dimension of

$$\mathfrak{k} \cap \mathfrak{q} \cap \tau^{-1}\left(\sum_{\alpha \in R \text{ such that } \bar{\alpha} = \lambda} (\mathfrak{g}^{\mathbf{C}}_{\alpha} + \mathfrak{g}^{\mathbf{C}}_{-\alpha})\right).$$

As its result, we obtain $m(\lambda)$, $m^-(\lambda)$ and $m^+(\lambda)(= m(\lambda) - m^-(\lambda))$.

(Algorithm 6) We calculate $\dim \mathfrak{q} - \sum_{\lambda \in \Delta_+} m(\lambda)$. As its result, we obtain the v-cohomogeneity of the s-representation associated with $(\mathfrak{g}, \mathfrak{h})$.

(Algorithm 7) The t-cohomogeneity of the s-representation associated with $(\mathfrak{g}, \mathfrak{h})$ coincides with the v-cohomogeneity of the s-representation associated with $(\mathfrak{g}^{ad}, \mathfrak{h})$. We obtain the t-cohomogeneity of the s-representation associated with $(\mathfrak{g}, \mathfrak{h})$ by calculating the v-cohomogeneity of the s-representation associated with $(\mathfrak{g}^{ad}, \mathfrak{h})$.

By Algorithms 1–7, we complete the list of the Satake diagrams of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$ and $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_{\mathfrak{q}})$ for all classical semisimple symmetric spaces $(\mathfrak{g}, \mathfrak{h})$ in Table 1. In Table 1, α_i $(1 \leq i \leq m)$ and β_j $(1 \leq j \leq n)$ are elements of $\Psi(R) \setminus \Psi(R)_0$ and $\Psi(R) \setminus \Psi(R)'_0$, respectively. Note that the cardinalities of $\Psi(R) \setminus \Psi(R)_0$ and $\Psi(R) \setminus \Psi(R)'_0$ coincide with the split rank and the rank of $(\mathfrak{g}, \mathfrak{h})$, respectively. Moreover, we complete the list of the Dynkin diagrams of the restricted root systems of $(\mathfrak{g}, \mathfrak{h})$ with respect to $\mathfrak{a}$ and the signatures of the roots in Table 2. We denote by $\{\lambda_1, \ldots, \lambda_m\}$ the simple root system of Δ such that $\lambda_i = \bar{\alpha_i}$ for all $1 \leq i \leq m$. We write the signatures of λ_i and $2\lambda_i$ as in the third column of Table 2. In Table 3, we complete the list of the v-cohomogeneities, the t-cohomogeneities, the dimensions of $\mathfrak{q}$ and the indices of the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{q} \times \mathfrak{q}$.

REMARK. (1) Let $\mathfrak{c}$ (resp. τ) be the Cartan subalgebra of $\mathfrak{g}^{\mathbf{C}}$ (resp. the element of $\mathrm{Int}(\mathfrak{g}^{\mathbf{C}})$) as in Algorithm 1. Then $\tau^{-1}(\mathfrak{c})(:= \tilde{\mathfrak{c}})$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbf{C}}$ containing $\mathfrak{a}_{\mathfrak{p}} + \mathfrak{a}_{\mathfrak{q}}$.

Denote by $\tilde{R}$ the root system of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\tilde{\mathfrak{c}}$. Then $\{(\alpha \circ \tau)|_{\tilde{\mathfrak{c}}} \mid \alpha \in \Psi(R)\}(:= \tilde{\Psi}(R))$ is a simple root system of $\tilde{R}$ and, for each $\tilde{\alpha} \in \tilde{\Psi}(R)$, the restriction of $\tilde{\alpha}$ to $\mathfrak{a}$ is equal to zero if and only $(\tilde{\alpha} \circ \tau^{-1})|_{\mathfrak{c}}$ is an element of $\Psi(R)_0$. Hence the Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$ coincides with that of $(\tau(\mathfrak{g}), \tau(\mathfrak{h}), \tau(\mathfrak{a}))$. Similarly, it is shown that the Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_{\mathfrak{q}})$ coincides with that of $(\tau(\mathfrak{g}), \tau(\mathfrak{h}), \tau(\mathfrak{a}_{\mathfrak{q}}))$.

(2) The Satake diagrams of semisimple Riemannian symmetric pairs are well known (for example, Table 9 of [4]). In [4], O. Loos pointed out that, for classical semisimple Riemannian symmetric pairs $(\mathfrak{g}, \mathfrak{k})$, their Satake diagrams follow in most cases from the structure of $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}_{\mathfrak{p}})$, where $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}_{\mathfrak{p}})$ denotes the centralizer of a maximal abelian subspace $\mathfrak{a}_{\mathfrak{p}}$ of $\mathfrak{p}$ in $\mathfrak{k}$. In general, for classical semisimple symmetric pairs $(\mathfrak{g}, \mathfrak{h})$, it is difficult to determine the Satake diagrams of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$ and $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_{\mathfrak{q}})$ from the structures of $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ and $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{a}_{\mathfrak{q}})$, where $\mathfrak{a}$ is a maximal split abelian subspace of $\mathfrak{q}$ and $\mathfrak{a}_{\mathfrak{q}}$ is a maximal abelian subspace of $\mathfrak{q}$ containing $\mathfrak{a}$. Therefore we shall determine the Satake diagrams by Algorithms 1–4.

4. Example

In this section, we give Satake diagrams and restricted root systems of some semisimple symmetric pairs by Algorithms 1–7. For convenience, we use the following diagram.

$$\begin{array}{ccccc}
(\mathfrak{g}, \mathfrak{h}) & \xleftrightarrow{\textit{associated}} & (\mathfrak{g}, \mathfrak{h}^a) & \xleftrightarrow{\textit{dual}} & (\mathfrak{g}^{ad}, \mathfrak{h}^d) \\
\updownarrow \textit{dual} & & & & \updownarrow \textit{associated} \\
(\mathfrak{g}^d, \mathfrak{h}^d) & \xleftrightarrow{\textit{associated}} & (\mathfrak{g}^d, \mathfrak{h}^a) & \xleftrightarrow{\textit{dual}} & (\mathfrak{g}^{ad}, \mathfrak{h})
\end{array}$$

We consider the case of $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{sl}(2n, \mathbf{R}), \mathfrak{sp}(n, \mathbf{R}))$, i.e.,

$$\begin{array}{ccccc}
(\mathfrak{sl}(2n, \mathbf{R}), \mathfrak{sp}(n, \mathbf{R})) & \xleftrightarrow{\textit{associated}} & (\mathfrak{sl}(2n, \mathbf{R}), \mathfrak{sl}(n, \mathbf{C}) + \mathfrak{so}(2)) & \xleftrightarrow{\textit{dual}} & (\mathfrak{su}(n, n), \mathfrak{so}^*(2n)) \\
\updownarrow \textit{dual} & & & & \updownarrow \textit{associated} \\
(\mathfrak{su}^*(2n), \mathfrak{so}^*(2n)) & \xleftrightarrow{\textit{associated}} & (\mathfrak{su}^*(2n), \mathfrak{sl}(n, \mathbf{C}) + \mathfrak{so}(2)) & \xleftrightarrow{\textit{dual}} & (\mathfrak{su}(n, n), \mathfrak{sp}(n, \mathbf{R}))
\end{array}$$

Let I_k denote the unit matrix of order $k \in \mathbf{N}$ and E_{ij} denote the $2n \times 2n$ matrix with entry 1 where the i-th row and the j-th column meet, all other entries being 0. For simplicity, we write E_k instead of E_{kk}. We define a involution σ of $\mathfrak{g}$ by

$$\sigma(X) := J_n{}^t X J_n$$

for all $X \in \mathfrak{g}$, where $J_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Then the set of all fixed points of σ is isomorphic to $\mathfrak{h}$. We define a Cartan involution θ of $\mathfrak{g}$ by $\theta(X) := -{}^t X$ for all $X \in \mathfrak{g}$. We put $\mathfrak{k} := \{X \in \mathfrak{g} \mid \theta(X) = X\}$ and $\mathfrak{p} := \{X \in \mathfrak{g} \mid \theta(X) = -X\}$. Then $\mathfrak{k} = \mathfrak{so}(2n)$ holds. Since $\sigma \circ \theta = \theta \circ \sigma$ holds, we have an orthogonal decomposition $\mathfrak{g} = \mathfrak{k} \cap \mathfrak{h} + \mathfrak{p} \cap \mathfrak{h} + \mathfrak{k} \cap \mathfrak{q} + \mathfrak{p} \cap \mathfrak{q}$, where

$$\mathfrak{k} \cap \mathfrak{h} = \left\{ \begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix} \;\middle|\; \begin{array}{l} X_1 \in \mathfrak{so}(n), \\ X_2 : n \times n \text{ symmetric} \end{array} \right\},$$

$$\mathfrak{p} \cap \mathfrak{h} = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} \;\middle|\; \begin{array}{l} X_1 : n \times n \text{ symmetric}, \\ X_2 : n \times n \text{ symmetric} \end{array} \right\},$$

$$\mathfrak{k} \cap \mathfrak{q} = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} \;\middle|\; X_1, X_2 \in \mathfrak{so}(n) \right\},$$

$$\mathfrak{p} \cap \mathfrak{q} = \left\{ \begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix} \;\middle|\; \begin{array}{l} X_1 : n \times n \text{ symmetric}, \\ X_2 \in \mathfrak{so}(n), \quad \mathrm{Tr} X_1 = 0 \end{array} \right\}.$$

(1) $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{sl}(2n, \mathbf{R}), \mathfrak{sp}(n, \mathbf{R}))$. We take a maximal abelian subspace

$$\mathfrak{a} := \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \;\middle|\; \begin{array}{l} A = \mathrm{diag}(a_1, \ldots, a_n), \\ a_i \in \mathbf{R}, \quad \mathrm{Tr} A = 0 \end{array} \right\}$$

of $\mathfrak{p} \cap \mathfrak{q}$, where $\mathrm{diag}(a_1, \ldots, a_n)$ denotes

$$\begin{pmatrix} a_1 & & & \\ & a_2 & & 0 \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix}.$$

Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{q}$ and its dimension is equal to $n-1$. We take a maximal abelian subspace

$$\mathfrak{a}_{\mathfrak{p}} := \mathfrak{a} + \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \;\middle|\; A = \mathrm{diag}(a_1, \ldots, a_n), a_i \in \mathbf{R} \right\}$$

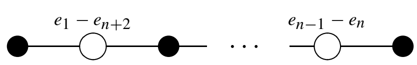
of $\mathfrak{p}$ containing $\mathfrak{a}$. We choose $\mathfrak{c} := \sum_{1 \leq i \leq 2n-1} \mathbf{C}(E_i - E_{i+1})$ as a Cartan subalgebra of $\mathfrak{g}^{\mathbf{C}}$. Then $\mathfrak{a}$ and $\mathfrak{a}_{\mathfrak{p}}$ are contained in $\mathfrak{c}$. We denote by $R := \{e_i - e_j \mid 1 \leq i \neq j \leq 2n\}$ the root system of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\mathfrak{c}$, where $e_i \in \mathfrak{c}^*$ is defined by $e_i(E_j - E_{j+1}) := \delta_{ij} - \delta_{i(j+1)}$ for all $1 \leq j \leq 2n-1$. We choose

$$(E_1 - E_n + E_{n+1} - E_{2n}, \ldots, E_{n-1} - E_n + E_{2n-1} - E_{2n}, E_n - E_{2n}, \ldots, E_{2n-1} - E_{2n})$$

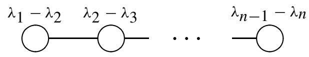
as a basis of $\mathfrak{c}_{\mathbf{R}}$. Then

$$\{e_i - e_{n+i+1} \mid 1 \leq i \leq n-2\} \cup \{e_{n+i} - e_i \mid 1 \leq i \leq n-1\} \cup \{e_{n-1} - e_n, e_n - e_{2n}\}$$

is the simple root system of R for the lexicographic ordering of $(\mathfrak{c}_{\mathbf{R}})^*$ with respect to the above basis, which is contained in the positive root system. Since $e_i(A) = e_{n+i}(A)$ holds for all $A \in \mathfrak{a}$, $e_{n+i} - e_i$ is denoted by a black circle. Therefore the Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$ is described as follows.



If λ_i $(1 \leq i \leq n)$ is the linear form on $\mathfrak{a}$ defined by $\lambda_i(A) := e_i(A)$ for all $A \in \mathfrak{a}$, $\Delta := \{\lambda_i - \lambda_j \mid 1 \leq i \neq j \leq n\}$ is the restricted root system with respect to $\mathfrak{a}$. Then we have $\Delta_+ = \{\lambda_i - \lambda_j \mid 1 \leq i < j \leq n\}$ and $\Psi(\Delta) = \{\lambda_i - \lambda_{i+1} \mid 1 \leq i \leq n-1\}$. Therefore we have the Dynkin diagram of Δ as follows.



Moreover, we obtain $(m^+(\lambda_i - \lambda_j), m^-(\lambda_i - \lambda_j)) = (2, 2)$ for any $1 \leq i < j \leq n$, and v-cohom $= n - 1$.

(2) $(\mathfrak{g}, \mathfrak{h})^d = (\mathfrak{su}^*(2n), \mathfrak{so}^*(2n))$. We take the same $\mathfrak{a}$ as a maximal abelian subspace of $\mathfrak{p}^d \cap \mathfrak{q}^d$. Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}^d$. We take a maximal abelian subspace

$$\mathfrak{a}_{\mathfrak{q}^d} := \mathfrak{a} + \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \middle| A = \sqrt{-1}\mathrm{diag}(a_1, \ldots, a_n), a_i \in \mathbf{R} \right\}$$

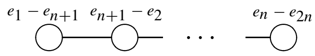
of $\mathfrak{q}^d$ containing $\mathfrak{a}$. Then $\mathfrak{a}_{\mathfrak{q}^d}$ is contained in $\mathfrak{c}$. We choose

$$(E_1 - E_n + E_{n+1} - E_{2n}, \ldots, E_{n-1} - E_n + E_{2n-1} - E_{2n}, E_1 - E_{n+1}, \ldots, E_n - E_{2n})$$

as a basis of $\mathfrak{c}_{\mathbf{R}}$. Then

$$\{e_i - e_{n+i} \mid 1 \leq i \leq n\} \cup \{e_{n+i} - e_{i+1} \mid 1 \leq i \leq n-1\}$$

is the simple root system of R for the lexicographic ordering of $(\mathfrak{c}_{\mathbf{R}})^*$ with respect to the above basis, which is contained in the positive root system. Then the Satake diagram of $(\mathfrak{g}^d, \mathfrak{h}^d, \mathfrak{a})$ coincides with that of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$ and the Satake diagram of $(\mathfrak{g}^d, \mathfrak{h}^d, \mathfrak{a}_{\mathfrak{q}^d})$ is described as follows.



Note that the restricted root system of $(\mathfrak{g}^d, \mathfrak{h}^d)$ with respect to $\mathfrak{a}$ coincides with that of $(\mathfrak{g}, \mathfrak{h})$ including their signatures of restricted roots. We obtain v-cohom $= 3n - 1$.

(3) $(\mathfrak{g}, \mathfrak{h})^a = (\mathfrak{sl}(2n, \mathbf{R}), \mathfrak{sl}(n, \mathbf{C}) + \mathfrak{so}(2))$. We take a maximal abelian subspace

$$\mathfrak{a}^a := \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \middle| A = \mathrm{diag}(a_1, \ldots, a_n), a_i \in \mathbf{R} \right\}$$

of $\mathfrak{p} \cap \mathfrak{q}^a$. Then $\mathfrak{a}^a$ is a maximal abelian subspace of $\mathfrak{q}^a$ and its dimension is equal to n. We take a maximal abelian subspace

$$\mathfrak{a}^a_{\mathfrak{p}} := \mathfrak{a}^a + \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \middle| \begin{array}{l} A = \mathrm{diag}(a_1, \ldots, a_n), \\ a_i \in \mathbf{R}, \quad \mathrm{Tr}A = 0 \end{array} \right\}$$

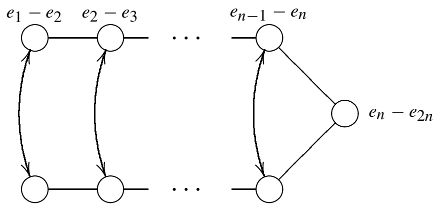
of $\mathfrak{p}$. Then $\mathfrak{a}^a$ and $\mathfrak{a}^a_{\mathfrak{p}}$ are contained in $\mathfrak{c}$. We choose

$$(E_1 - E_{n+1}, \ldots, E_n - E_{2n}, E_{n+1} - E_{2n}, \ldots, E_{2n-1} - E_{2n})$$

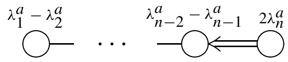
as a basis of $\mathfrak{c}_{\mathbf{R}}$. Then

$$\{e_i - e_{i+1} \mid 1 \leq i \leq n-1\} \cup \{e_{n+i+1} - e_{n+i} \mid 1 \leq i \leq n-1\} \cup \{e_n - e_{2n}\}$$

is the simple root system of R for the lexicographic ordering of $(\mathfrak{c}_{\mathbf{R}})^*$ with respect to the above basis, which is contained in the positive root system. Since $e_i(A) = -e_{n+i}(A)$ holds for all $A \in \mathfrak{a}^a$, $e_i - e_{i+1}$ and $e_{n+i+1} - e_{n+i}$ are joined by a curved arrow. Therefore the Satake diagram of $(\mathfrak{g}, \mathfrak{h}^a, \mathfrak{a}^a)$ is described as follows.



If λ_i^a $(1 \leq i \leq n)$ is the linear form on $\mathfrak{a}^a$ defined by $\lambda_i^a(A) := e_i(A)$ for all $A \in \mathfrak{a}^a$, $\Delta^a := \{\pm\lambda_i^a \pm \lambda_j^a \mid 1 \leq i < j \leq n\} \cup \{\pm 2\lambda_i^a \mid 1 \leq i \leq n\}$ is the restricted root system with respect to $\mathfrak{a}^a$. Then we have $\Delta_+^a = \{\lambda_i^a \pm \lambda_j^a \mid 1 \leq i < j \leq n\} \cup \{2\lambda_i^a \mid 1 \leq i \leq n\}$ and $\Psi(\Delta^a) = \{\lambda_i^a - \lambda_{i+1}^a \mid 1 \leq i \leq n-1\} \cup \{2\lambda_n^a\}$. Therefore we have the Dynkin diagram of Δ^a as follows.



Moreover, we obtain $(m^+(\lambda_i^a \pm \lambda_j^a), m^-(\lambda_i^a \pm \lambda_j^a)) = (1, 1)$, $(m^+(2\lambda_i^a), m^-(2\lambda_i^a)) = (1, 0)$ for any $1 \leq i < j \leq n$, and υ-cohom $= n$.

(4) $(\mathfrak{g}, \mathfrak{h})^{ad} = (\mathfrak{su}(n, n), \mathfrak{so}^*(2n))$. We take the same $\mathfrak{a}^a$ as a maximal abelian subspace of $\mathfrak{p}^{ad} \cap \mathfrak{q}^{ad}$. Then $\mathfrak{a}^a$ is a maximal abelian subspace of $\mathfrak{p}^{ad}$. We take a maximal abelian subspace

$$\mathfrak{a}^a_{\mathfrak{q}^{ad}} := \mathfrak{a}^a + \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \,\middle|\, \begin{array}{l} A = \sqrt{-1}\mathrm{diag}(a_1, \ldots, a_n), \\ a_i \in \mathbf{R}, \quad \mathrm{Tr}A = 0 \end{array} \right\}$$

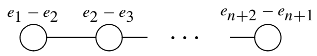
of $\mathfrak{q}^{ad}$ containing $\mathfrak{a}^a$. Then $\mathfrak{a}^a_{\mathfrak{q}^{ad}}$ is contained in $\mathfrak{c}$. We choose

$$(E_1 - E_{n+1}, \ldots, E_n - E_{2n}, E_1 - E_n + E_{n+1} - E_{2n}, \ldots, E_{n-1} - E_n + E_{2n-1} - E_{2n})$$

as a basis of $\mathfrak{c}_{\mathbf{R}}$. Then

$$\{e_i - e_{i+1} \mid 1 \leq i \leq n-1\} \cup \{e_{n+i+1} - e_{n+i} \mid 1 \leq i \leq n-1\} \cup \{e_n - e_{2n}\}$$

is the simple root system of R for the lexicographic ordering of $(\mathfrak{c}_{\mathbf{R}})^*$ with respect to the above basis, which is contained in the positive root system. Then the Satake diagram of $(\mathfrak{g}^{ad}, \mathfrak{h}^{ad}, \mathfrak{a}^a)$ coincides with that of $(\mathfrak{g}^a, \mathfrak{h}^a, \mathfrak{a}^a)$ and the Satake diagram of $(\mathfrak{g}^{ad}, \mathfrak{h}^{ad}, \mathfrak{a}^a_{\mathfrak{q}^{ad}})$ is described as follows.



Note that the restricted root system of $(\mathfrak{g}^{ad}, \mathfrak{h}^{ad})$ with respect to $\mathfrak{a}^a$ coincides with that of $(\mathfrak{g}^a, \mathfrak{h}^a)$ including their signatures of restricted roots. We obtain υ-cohom $= 2n - 1$.

(5) $(\mathfrak{g}, \mathfrak{h})^{da} = (\mathfrak{su}^*(2n), \mathfrak{sl}(n, \mathbf{C}) + \mathfrak{so}(2))$. First, we consider the case where $n(= 2m)$ is even. We take a maximal abelian subspace

$$\mathfrak{a}^{da} := \left\{ \sum_{1\le i\le m} a_i (E_{i(n+1-i)} - E_{(n+1-i)i} - E_{(n+i)(2n+1-i)} + E_{(2n+1-i)(n+i)}) \;\middle|\; a_i \in \sqrt{-1}\mathbf{R} \right\}$$

of $\mathfrak{p}^{da} \cap \mathfrak{q}^{da}$. We also take maximal abelian subspaces

$$\mathfrak{a}^{da}_{\mathfrak{q}^{da}} := \mathfrak{a}^{da} + \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \;\middle|\; A = \sqrt{-1}\mathrm{diag}(a_1, \dots, a_m, a_m, \dots, a_1), a_i \in \mathbf{R} \right\} \quad \text{and}$$

$$\mathfrak{a}^{da}_{\mathfrak{p}^{da}} := \mathfrak{a}^{da} + \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \;\middle|\; \begin{array}{l} A = \mathrm{diag}(a_1, \dots, a_m, a_m, \dots, a_1), \\ a_i \in \mathbf{R}, \quad \mathrm{Tr}A = 0 \end{array} \right\}$$

of $\mathfrak{q}^{da}$ and $\mathfrak{p}^{da}$ containing $\mathfrak{a}^{da}$, respectively. If we put

$$\tau := \mathrm{Ad}\left(\exp \frac{\sqrt{-1}\pi}{4} \sum_{1\le i\le m} \left(E_{i(n+1-i)} + E_{(n+1-i)i} + E_{(n+i)(2n+1-i)} + E_{(2n+1-i)(n+i)} \right) \right),$$

we have

$$\tau(\mathfrak{a}^{da}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \;\middle|\; A = \mathrm{diag}(a_1, \dots, a_m, -a_m, \dots, -a_1), a_i \in \mathbf{R} \right\},$$

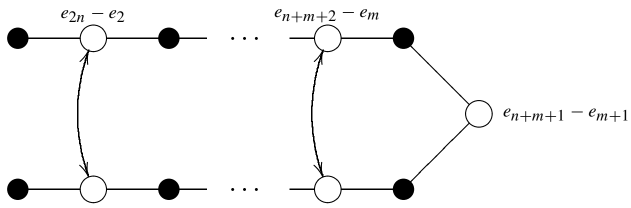
where Ad is the adjoint representation of $SL(2n, \mathbf{C})$ and exp is the matrix exponential function. Moreover, τ fixes each vector of $\mathfrak{k}^{da} \cap \mathfrak{a}^{da}_{\mathfrak{q}^{da}}$ and $\mathfrak{h}^{da} \cap \mathfrak{a}^{da}_{\mathfrak{p}^{da}}$. Hence $\tau(\mathfrak{a}^{da}_{\mathfrak{q}^{da}})$ and $\tau(\mathfrak{a}^{da}_{\mathfrak{p}^{da}})$ are contained in $\mathfrak{c}$. We choose

$$\begin{aligned} (E_1 - E_n - E_{n+1} + E_{2n}, \dots, E_m - E_{n+1-m} - E_{n+m} + E_{2n+1-m}, \\ E_1 + E_n - E_{n+1} - E_{2n}, \dots, E_m + E_{n+1-m} - E_{n+m} - E_{2n+1-m}, \\ E_{n+1} - E_{2n}, \dots, E_{2n-1} - E_{2n}) \end{aligned}$$

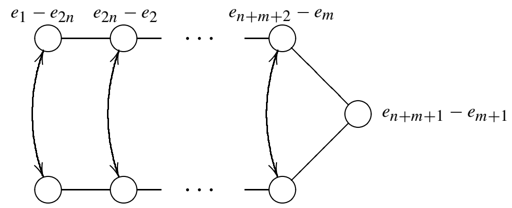
as a basis of $\mathfrak{c}_{\mathbf{R}}$. Then

$$\{e_i - e_{2n+1-i} \mid 1 \le i \le n\} \cup \{e_{n+i+1} - e_{n+1-i} \mid 1 \le i \le n-1\}$$

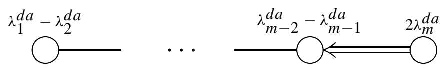
is the simple root system of R for the lexicographic ordering of $(\mathfrak{c}_{\mathbf{R}})^*$ with respect to the above basis, which is contained in the positive root system. Since $e_i(A) = -e_{n+1-i}(A) = -e_{n+i}(A) = e_{2n+1-i}(A)$ holds for all $A \in \tau(\mathfrak{a}^{da})$, $e_i - e_{2n+1-i}$ is denoted by a black circle and $e_{2n+1-i} - e_{i+1}$ and $e_{n+i+1} - e_{n+1-i}$ are joined by a curved arrow. Therefore the Satake diagram of $(\tau(\mathfrak{g}^{da}), \tau(\mathfrak{h}^{da}), \tau(\mathfrak{a}^{da}))$ is described as follows.



Since $e_i(A) = -e_{n+i}(A)$ for all $A \in \tau(\mathfrak{a}^{da}_{\mathfrak{q}^{da}})$, the Satake diagram of $(\tau(\mathfrak{g}^{da}), \tau(\mathfrak{h}^{da}), \tau(\mathfrak{a}^{da}_{\mathfrak{q}^{da}}))$ is described as follows.



If λ^{da}_i $(1 \le i \le m)$ is the linear form on $\tau(\mathfrak{a}^{da})$ defined by $\lambda^{da}_i(A) := e_i(A)$ for all $A \in \tau(\mathfrak{a}^{da})$, $\Delta^{da} := \{\pm\lambda^{da}_i \pm \lambda^{da}_j \mid 1 \le i < j \le m\} \cup \{\pm 2\lambda^{da}_i \mid 1 \le i \le m\}$ is the restricted root system with respect to $\tau(\mathfrak{a}^{da})$. Then we have $\Delta^{da}_+ = \{\lambda^{da}_i \pm \lambda^{da}_j \mid 1 \le i < j \le m\} \cup \{2\lambda^{da}_i \mid 1 \le i \le m\}$ and $\Psi(\Delta^{da}) = \{\lambda^{da}_i - \lambda^{da}_{i+1} \mid 1 \le i \le m-1\} \cup \{2\lambda^{da}_m\}$. Therefore we have the Dynkin diagram of Δ^{da} as follows.



We also have $(m^+(\lambda^{da}_i \pm \lambda^{da}_j), m^-(\lambda^{da}_i \pm \lambda^{da}_j)) = (4,4)$, $(m^+(2\lambda^{da}_i), m^-(2\lambda^{da}_i)) = (1,3)$ for any $1 \le i < j \le m$, and υ-cohom $= 2n-1$.

Next, we consider the case where $n(=2l+1)$ is odd. We take a maximal abelian subspace

$$\mathfrak{b}^{da} := \left\{ \sum_{1\le i\le l} a_i(E_{i(n+1-i)} - E_{(n+1-i)i} - E_{(n+i)(2n+1-i)} + E_{(2n+1-i)(n+i)}) \,\middle|\, a_i \in \sqrt{-1}\mathbf{R} \right\}$$

of $\mathfrak{p}^{da} \cap \mathfrak{q}^{da}$. We also take maximal abelian subspaces

$$\mathfrak{b}^{da}_{\mathfrak{q}^{da}} := \mathfrak{b}^{da} + \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \,\middle|\, A = \sqrt{-1}\mathrm{diag}(a_1, \dots, a_l, a_0, a_l, \dots, a_1), a_i \in \mathbf{R} \right\} \quad \text{and}$$

$$\mathfrak{b}^{da}_{\mathfrak{p}^{da}} := \mathfrak{b}^{da} + \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \,\middle|\, \begin{array}{l} A = \mathrm{diag}(a_1, \dots, a_l, a_0, a_l, \dots, a_1), \\ a_i \in \mathbf{R}, \quad \mathrm{Tr}A = 0 \end{array} \right\}$$

of $\mathfrak{q}^{da}$ and $\mathfrak{p}^{da}$ containing $\mathfrak{b}^{da}$, respectively. If we put

$$\rho := \mathrm{Ad}\left(\exp \frac{\sqrt{-1}\pi}{4} \sum_{1 \le i \le l} \left(E_{i(n+1-i)} + E_{(n+1-i)i} + E_{(n+i)(2n+1-i)} + E_{(2n+1-i)(n+i)} \right) \right),$$

we have

$$\rho(\mathfrak{b}^{da}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \;\middle|\; A = \mathrm{diag}(a_1, \ldots, a_l, 0, -a_l, \ldots, -a_1), a_i \in \mathbf{R} \right\}.$$

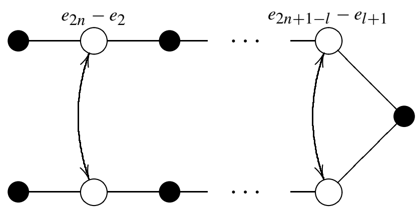
Moreover, ρ fixes each vector of $\mathfrak{k}^{da} \cap \mathfrak{b}^{da}_{\mathfrak{q}^{da}}$ and $\mathfrak{h}^{da} \cap \mathfrak{b}^{da}_{\mathfrak{p}^{da}}$. Hence $\rho(\mathfrak{b}^{da}_{\mathfrak{q}^{da}})$ and $\rho(\mathfrak{b}^{da}_{\mathfrak{p}^{da}})$ are contained in $\mathfrak{c}$. We choose

$$\begin{aligned}(E_1 - E_n - E_{n+1} + E_{2n}, \ldots, E_l - E_{n+1-l} - E_{n+l} + E_{2n+1-l},\\ E_1 + E_n - E_{n+1} - E_{2n}, \ldots, E_l + E_{n+1-l} - E_{n+l} - E_{2n+1-l}, E_{l+1} - E_{l+1+n},\\ E_{n+1} - E_{2n}, \ldots, E_{2n-1} - E_{2n})\end{aligned}$$

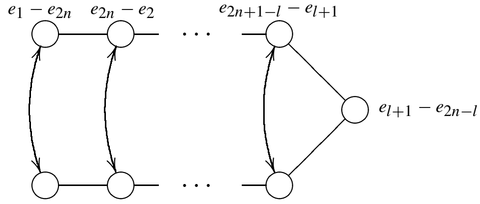
as a basis of $\mathfrak{c}_{\mathbf{R}}$. Then

$$\{e_i - e_{2n+1-i} \mid 1 \le i \le n\} \cup \{e_{n+i+1} - e_{n+1-i} \mid 1 \le i \le n-1\}$$

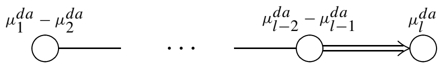
is the simple root system of R for the lexicographic ordering of $(\mathfrak{c}_{\mathbf{R}})^*$ with respect to the above basis, which is contained in the positive root system. Since $e_i(A) = -e_{n+1-i}(A) = -e_{n+i}(A) = e_{2n+1-i}(A)$ holds for all $A \in \rho(\mathfrak{b}^{da})$, $e_i - e_{2n+1-i}$ is denoted by a black circle and $e_{2n+1-i} - e_{i+1}$ and $e_{n+i+1} - e_{n+1-i}$ are joined by a curved arrow. Therefore the Satake diagram of $(\rho(\mathfrak{g}^{da}), \rho(\mathfrak{h}^{da}), \rho(\mathfrak{b}^{da}))$ is described as follows.



Since $e_i(A) = -e_{n+i}(A)$ for all $A \in \rho(\mathfrak{b}^{da}_{\mathfrak{q}^{da}})$, the Satake diagram of $(\rho(\mathfrak{g}^{da}), \rho(\mathfrak{h}^{da}), \rho(\mathfrak{b}^{da}_{\mathfrak{q}^{da}}))$ is described as follows.



If μ_i^{da} $(1 \le i \le l)$ is the linear form on $\rho(\mathfrak{b}^{da})$ defined by $\mu_i^{da}(A) := e_i(A)$ for all $A \in \rho(\mathfrak{b}^{da})$, $\Gamma^{da} := \{\pm\mu_i^{da} \pm \mu_j^{da} \mid 1 \le i < j \le l\} \cup \{\pm\mu_i^{da}, \pm 2\mu_i^{da} \mid 1 \le i \le l\}$ is the restricted root system with respect to $\rho(\mathfrak{b}^{da})$. Then we have $\Gamma_+^{da} = \{\mu_i^{da} \pm \mu_j^{da} \mid 1 \le i < j \le l\} \cup \{\mu_i^{da}, 2\mu_i^{da} \mid 1 \le i \le l\}$ and $\Psi(\Gamma^{da}) = \{\mu_i^{da} - \mu_{i+1}^{da} \mid 1 \le i \le l-1\} \cup \{\mu_l^{da}\}$. Therefore we have the Dynkin diagram of Γ^{da} as follows.



We also have

$$(m^+(\mu_i^{da} \pm \mu_j^{da}), m^-(\mu_i^{da} \pm \mu_j^{da})) = (m^+(\mu_i^{da}), m^-(\mu_i^{da})) = (4,4),$$
$$(m^+(2\mu_i^{da}), m^-(2\mu_i^{da})) = (1,3)$$

for any $1 \le i < j \le l$, and v-cohom $= 2n$.

(6) $(\mathfrak{g}, \mathfrak{h})^{dad} = (\mathfrak{su}(n,n), \mathfrak{sp}(n, \mathbf{R}))$. First, we consider the case where n is even. We take the same $\mathfrak{a}^{da}$ as a maximal abelian subspace of $\mathfrak{p}^{dad} \cap \mathfrak{q}^{dad}$. We also take maximal abelian subspaces

$$\mathfrak{a}^{dad}_{\mathfrak{q}^{dad}} := \mathfrak{a}^{da} + \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \middle| \begin{array}{l} A = \sqrt{-1}\mathrm{diag}(a_1, \ldots, a_m, a_m, \ldots, a_1), \\ a_i \in \mathbf{R}, \quad \mathrm{Tr} A = 0 \end{array} \right\} \quad \text{and}$$
$$\mathfrak{a}^{dad}_{\mathfrak{p}^{dad}} := \mathfrak{a}^{da} + \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \middle| A = \mathrm{diag}(a_1, \ldots, a_m, a_m, \ldots, a_1), a_i \in \mathbf{R} \right\}$$

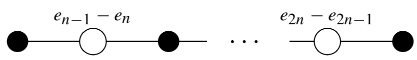
of $\mathfrak{q}^{dad}$ and $\mathfrak{p}^{dad}$ containing $\mathfrak{a}^{da}$, respectively. Then $\tau(\mathfrak{a}^{dad}_{\mathfrak{q}^{dad}})$ and $\tau(\mathfrak{a}^{dad}_{\mathfrak{p}^{dad}})$ are contained in $\mathfrak{c}$. We choose

$$\begin{aligned}(E_1 - E_n - E_{n+1} + E_{2n}, \ldots, E_m - E_{n+1-m} - E_{n+m} + E_{2n+1-m},\\ E_1 - E_m - E_{m+1} + E_n + E_{n+1} - E_{n+m} - E_{n+m+1} + E_{2n}, \ldots,\\ E_1 - E_m - E_{m+1} + E_n + E_{n+1} - E_{n+m} - E_{n+m+1} + E_{2n},\\ E_n - E_{2n}, \ldots, E_{2n-1} - E_{2n})\end{aligned}$$

as a basis of $\mathfrak{c}_{\mathbf{R}}$. Then

$$\{e_{i+1} - e_{2n-i-1} \mid 1 \le i \le 2m-3\} \cup \{e_{n+i+1} - e_{n-i} \mid 1 \le i \le 2m-2\}$$
$$\cup \{e_1 - e_{2n}, e_{n-1} - e_n, e_n - e_{n+1}, e_{2n} - e_{2n-1}\}$$

is the simple root system of R for the lexicographic ordering of $(\mathfrak{c}_{\mathbf{R}})^*$ with respect to the above basis, which is contained in the positive root system. Then the Satake diagram of $(\tau(\mathfrak{g}^{dad}), \tau(\mathfrak{h}^{dad}), \tau(\mathfrak{a}^{da}))$ coincides with that of $(\tau(\mathfrak{g}^{da}), \tau(\mathfrak{h}^{da}), \tau(\mathfrak{a}^{da}))$ and the Satake diagram of $(\tau(\mathfrak{g}^{dad}), \tau(\mathfrak{h}^{dad}), \tau(\mathfrak{a}^{dad}_{\mathfrak{q}^{dad}}))$ is described as follows.



Note that the restricted root system of $(\tau(\mathfrak{g}^{dad}), \tau(\mathfrak{h}^{dad}))$ with respect to $\tau(\mathfrak{a}^{da})$ coincides with that of $(\tau(\mathfrak{g}^{da}), \tau(\mathfrak{h}^{da}))$ including their signatures of restricted roots. We obtain υ-cohom $= n - 1$.

Next, we consider the case where n is odd. We take the same $\mathfrak{b}^{da}$ as a maximal abelian subspace of $\mathfrak{p}^{dad} \cap \mathfrak{q}^{dad}$. We also take maximal abelian subspaces

$$\mathfrak{b}^{dad}_{\mathfrak{q}^{dad}} := \mathfrak{b}^{da} + \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \,\middle|\, \begin{array}{l} A = \sqrt{-1}\mathrm{diag}(a_1, \dots, a_l, a_0, a_l, \dots, a_1), \\ a_i \in \mathbf{R}, \quad \mathrm{Tr}A = 0 \end{array} \right\} \quad \text{and}$$

$$\mathfrak{b}^{dad}_{\mathfrak{p}^{dad}} := \mathfrak{b}^{da} + \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \,\middle|\, A = \mathrm{diag}(a_1, \dots, a_l, a_0, a_l, \dots, a_1), a_i \in \mathbf{R} \right\}$$

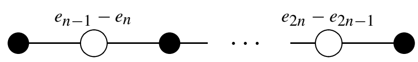
of $\mathfrak{q}^{dad}$ and $\mathfrak{p}^{dad}$ containing $\mathfrak{b}^{da}$, respectively. Then $\rho(\mathfrak{b}^{dad}_{\mathfrak{q}^{dad}})$ and $\rho(\mathfrak{b}^{dad}_{\mathfrak{p}^{dad}})$ are contained in $\mathfrak{c}$. We choose

$$\begin{aligned}(&E_1 - E_n - E_{n+1} + E_{2n}, \dots, E_l - E_{n+1-l} - E_{n+l} + E_{2n+1-l}, \\ &E_1 - 2E_{l+1} + E_n + E_{n+1} - 2E_{n+l+1} - E_{2n}, \dots, E_l - 2E_{l+1} + E_{l+2} + E_{n+l} - 2E_{n+l+1} - E_{n+l+2}, \\ &\qquad E_{l+1} - E_{2n}, E_n - E_{2n}, \dots, E_{2n-1} - E_{2n})\end{aligned}$$

as a basis of $\mathfrak{c}_{\mathbf{R}}$. Then

$$\begin{aligned}&\{e_{i+1} - e_{2n-i-1} \mid 1 \le i \le 2l-2\} \cup \{e_{n+i+1} - e_{n-i} \mid 1 \le i \le 2l-1\} \\ &\qquad \cup \{e_1 - e_{2n}, e_{n-1} - e_n, e_n - e_{n+1}, e_{2n} - e_{2n-1}\}\end{aligned}$$

is the simple root system of R for the lexicographic ordering of $(\mathfrak{c}_{\mathbf{R}})^*$ with respect to the above basis, which is contained in the positive root system. Then the Satake diagram of $(\rho(\mathfrak{g}^{dad}), \rho(\mathfrak{h}^{dad}), \rho(\mathfrak{b}^{da}))$ coincides with that of $(\rho(\mathfrak{g}^{da}), \rho(\mathfrak{h}^{da}), \rho(\mathfrak{b}^{da}))$ and the Satake diagram of $(\rho(\mathfrak{g}^{dad}), \rho(\mathfrak{h}^{dad}), \rho(\mathfrak{b}^{dad}_{\mathfrak{q}^{dad}}))$ is described as follows.



Note that the restricted root system of $(\rho(\mathfrak{g}^{dad}), \rho(\mathfrak{h}^{dad}))$ with respect to $\rho(\mathfrak{b}^{da})$ coincides with that of $(\rho(\mathfrak{g}^{da}), \rho(\mathfrak{h}^{da}))$ including their signatures of restricted roots. We obtain υ-cohom $= n - 1$.

By the results of (1) – (6), we give the Satake diagrams and the restricted root systems with respect to toroidal-type maximal split abelian subspaces. For example, by the result of $(\mathfrak{g}, \mathfrak{h})^{dad}$, we give the Satake diagram and the restricted root system of $(\mathfrak{g}, \mathfrak{h})$ with respect to toroidal-type maximal split abelian subspace as follows.

(1)′ $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{sl}(2n, \mathbf{R}), \mathfrak{sp}(n, \mathbf{R}))$. Recall that we have

$$\begin{aligned}&\mathfrak{k} \cap \mathfrak{h} = \mathfrak{k}^{dad} \cap \mathfrak{h}^{dad}, \quad \mathfrak{p} \cap \mathfrak{h} = \mathfrak{p}^{dad} \cap \mathfrak{h}^{dad}, \\ &\mathfrak{k} \cap \mathfrak{q} = \sqrt{-1}(\mathfrak{p}^{dad} \cap \mathfrak{q}^{dad}), \quad \mathfrak{p} \cap \mathfrak{q} = \sqrt{-1}(\mathfrak{k}^{dad} \cap \mathfrak{q}^{dad}).\end{aligned}$$

First, we consider the case where n $(= 2m)$ is even. We take $\sqrt{-1}\mathfrak{a}^{da}$ as a maximal abelian subspace of $\mathfrak{k} \cap \mathfrak{q}$. We also take maximal abelian subspaces $\sqrt{-1}\mathfrak{a}^{dad}_{\mathfrak{q}^{dad}}$ and $\sqrt{-1}\mathfrak{a}^{da} + \mathfrak{a}^{dad}_{\mathfrak{p}^{dad}} \cap \mathfrak{h}^{dad}$ of $\mathfrak{q}$ and $\mathfrak{p}$ containing $\sqrt{-1}\mathfrak{a}^{da}$, respectively. Therefore the Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \sqrt{-1}\mathfrak{a}^{da})$ (resp. $(\mathfrak{g}, \mathfrak{h}, \sqrt{-1}\mathfrak{a}^{dad}_{\mathfrak{q}^{dad}})$) coincides with that of $(\tau(\mathfrak{g}^{dad}), \tau(\mathfrak{h}^{dad}), \tau(\mathfrak{a}^{da}))$ (resp. $(\tau(\mathfrak{g}^{dad}), \tau(\mathfrak{h}^{dad}), \tau(\mathfrak{a}^{dad}_{\mathfrak{q}^{dad}}))$). If $\hat{\lambda}_i$ $(1 \le i \le m)$ is the linear form on $\tau(\sqrt{-1}\mathfrak{a}^{da})$ defined by $\hat{\lambda}_i(A) := \sqrt{-1}e_i(A)$ for all $A \in \tau(\sqrt{-1}\mathfrak{a}^{da})$, $\hat{\Delta} := \{\pm\hat{\lambda}_i \pm \hat{\lambda}_j \mid 1 \le i \le m\} \cup \{\pm 2\hat{\lambda}_i \mid 1 \le i \le m\}$ is the restricted root system with respect to $\tau(\sqrt{-1}\mathfrak{a}^{da})$. Then $\hat{\Delta}$ is isomorphic to Δ^{da}. Moreover, we have

$$(m^+(\hat{\lambda}_i \pm \hat{\lambda}_j), m^-(\hat{\lambda}_i \pm \hat{\lambda}_j)) = (m^-(\lambda^{da}_i \pm \lambda^{da}_j), m^+(\lambda^{da}_i \pm \lambda^{da}_j)) = (4, 4)\,,$$
$$(m^+(2\hat{\lambda}_i), m^-(2\hat{\lambda}_i)) = (m^-(2\lambda^{da}_i), m^+(2\lambda^{da}_i)) = (3, 1)\,.$$

Since the t-cohomogeneity of $(\mathfrak{g}, \mathfrak{h})$ is equal to the v-cohomogeneity of $(\mathfrak{g}, \mathfrak{h})^{dad}$, we have $t\text{-cohom} = n - 1$.

Next, we consider the case where n $(= 2l + 1)$ is odd. We take $\sqrt{-1}\mathfrak{b}^{da}$ as a maximal abelian subspace of $\mathfrak{k} \cap \mathfrak{q}$. We also take maximal abelian subspaces $\sqrt{-1}\mathfrak{b}^{dad}_{\mathfrak{q}^{dad}}$ and $\sqrt{-1}\mathfrak{b}^{da} + \mathfrak{b}^{dad}_{\mathfrak{p}^{dad}} \cap \mathfrak{h}^{dad}$ of $\mathfrak{q}$ and $\mathfrak{p}$ containing $\sqrt{-1}\mathfrak{b}^{da}$. Therefore the Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \sqrt{-1}\mathfrak{b}^{da})$ (resp. $(\mathfrak{g}, \mathfrak{h}, \sqrt{-1}\mathfrak{b}^{dad}_{\mathfrak{q}^{dad}})$) coincides with that of $(\tau(\mathfrak{g}^{dad}), \tau(\mathfrak{h}^{dad}), \tau(\mathfrak{b}^{da}))$ (resp. $(\tau(\mathfrak{g}^{dad}), \tau(\mathfrak{h}^{dad}), \tau(\mathfrak{b}^{dad}_{\mathfrak{q}^{dad}}))$). If $\hat{\mu}_i$ $(1 \le i \le l)$ is the linear form on $\tau(\sqrt{-1}\mathfrak{b}^{da})$ defined by $\hat{\mu}_i(A) := \sqrt{-1}e_i(A)$ for all $A \in \tau(\sqrt{-1}\mathfrak{b}^{da})$, $\hat{\Gamma} := \{\pm\hat{\mu}_i \pm \hat{\mu}_j \mid 1 \le i \le l\} \cup \{\pm\hat{\mu}_i, \pm 2\hat{\mu}_i \mid 1 \le i \le l\}$ is the restricted root system with respect to $\tau(\sqrt{-1}\mathfrak{b}^{da})$. Then $\hat{\Gamma}$ is isomorphic to Γ^{da}. Moreover, we have

$$(m^+(\hat{\mu}_i \pm \hat{\mu}_j), m^-(\hat{\mu}_i \pm \hat{\mu}_j)) = (m^-(\mu^{da}_i \pm \mu^{da}_j), m^+(\mu^{da}_i \pm \mu^{da}_j)) = (4, 4)\,,$$
$$(m^+(\hat{\mu}_i), m^-(\hat{\mu}_i)) = (m^-(\mu^{da}_i), m^+(\mu^{da}_i)) = (4, 4)\,,$$
$$(m^+(2\hat{\mu}_i), m^-(2\hat{\mu}_i)) = (m^-(2\mu^{da}_i), m^+(2\mu^{da}_i)) = (3, 1)$$

and $t\text{-cohom} = n - 1$.

TABLE 1. Satake diagrams

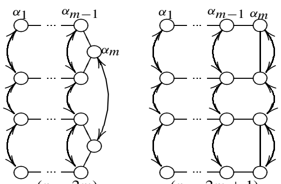
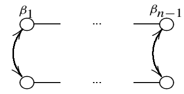
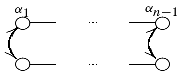
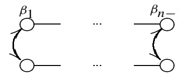
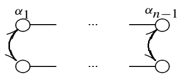
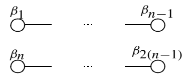
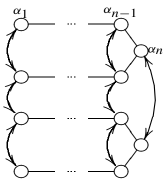
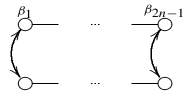
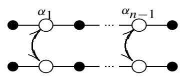
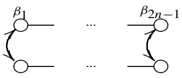
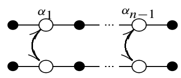
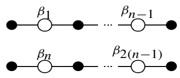
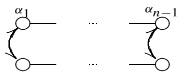
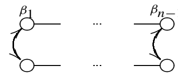
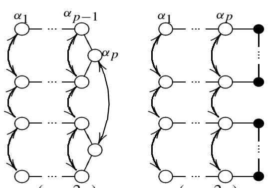
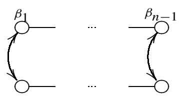
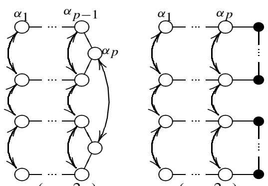
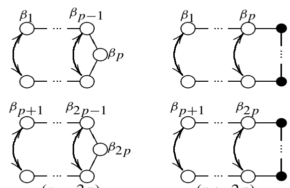
$(\mathfrak{g}, \mathfrak{h})$	Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$	Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_{\mathfrak{q}})$
$(\mathfrak{sl}(n, \mathbf{C}), \mathfrak{sl}(n, \mathbf{R}))$	 $(n = 2m)$ $(n = 2m + 1)$	
$(\mathfrak{sl}(n, \mathbf{R}) + \mathfrak{sl}(n, \mathbf{R}), \mathfrak{sl}(n, \mathbf{R}))$		
$(\mathfrak{sl}(n, \mathbf{C}), \mathfrak{so}(n, \mathbf{C}))$		
$(\mathfrak{sl}(2n, \mathbf{C}), \mathfrak{su}^*(2n))$		
$(\mathfrak{su}^*(2n) + \mathfrak{su}^*(2n), \mathfrak{su}^*(2n))$		
$(\mathfrak{sl}(2n, \mathbf{C}), \mathfrak{sp}(n, \mathbf{C}))$		
$(\mathfrak{sl}(n, \mathbf{C}), \mathfrak{su}(p, n-p))$		
$(\mathfrak{su}(p, n-p) + \mathfrak{su}(p, n-p), \mathfrak{su}(p, n-p))$	 $(n = 2p)$ $(n > 2p)$	
$(\mathfrak{sl}(n, \mathbf{C}), \mathfrak{sl}(p, \mathbf{C}) + \mathfrak{sl}(n-p, \mathbf{C}) + \mathbf{C})$	 $(n = 2p)$ $(n > 2p)$	 $(n = 2p)$ $(n > 2p)$

TABLE 1. (continued)

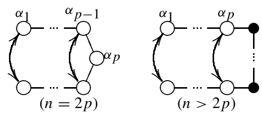
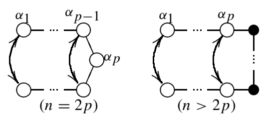
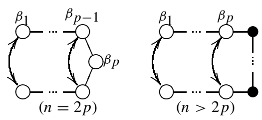
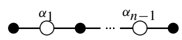
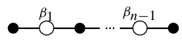
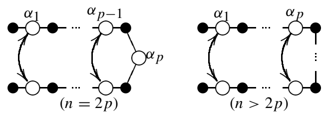
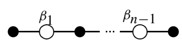
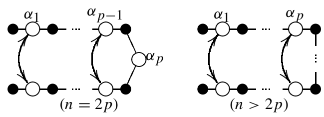
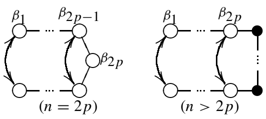
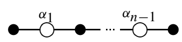
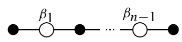
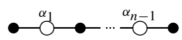
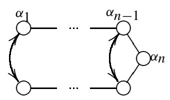
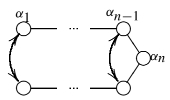
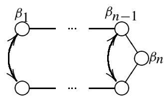
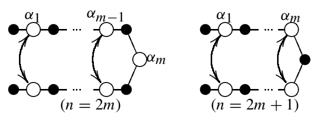
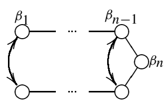
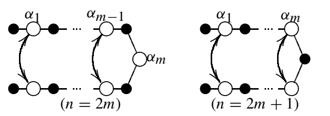
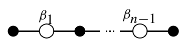
$(\mathfrak{g}, \mathfrak{h})$	Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$	Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_\mathfrak{q})$
$(\mathfrak{sl}(n, \mathbf{R}), \mathfrak{so}(p, n-p))$		
$(\mathfrak{su}(p, n-p), \mathfrak{so}(p, n-p))$		
$(\mathfrak{sl}(n, \mathbf{R}), \mathfrak{sl}(p, \mathbf{R}) + \mathfrak{sl}(n-p, \mathbf{R}) + \mathbf{R})$		
$(\mathfrak{su}^*(2n), \mathfrak{sp}(p, n-p))$		
$(\mathfrak{su}(2p, 2(n-p)), \mathfrak{sp}(p, n-p))$		
$(\mathfrak{su}^*(2n), \mathfrak{su}^*(2p) + \mathfrak{su}^*(2(n-p)) + \mathbf{R})$		
$(\mathfrak{sl}(2n, \mathbf{R}), \mathfrak{sp}(n, \mathbf{R}))$		
$(\mathfrak{su}^*(2n), \mathfrak{so}^*(2n))$		
$(\mathfrak{su}(n, n), \mathfrak{so}^*(2n))$		
$(\mathfrak{sl}(2n, \mathbf{R}), \mathfrak{sl}(n, \mathbf{C}) + \mathfrak{so}(2))$		
$(\mathfrak{su}^*(2n), \mathfrak{sl}(n, \mathbf{C}) + \mathfrak{so}(2))$		
$(\mathfrak{su}(n, n), \mathfrak{sp}(n, \mathbf{R}))$		

TABLE 1. (continued)

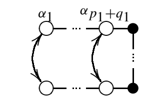
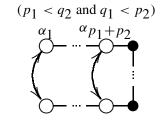
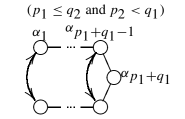
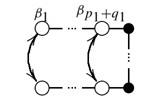
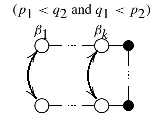
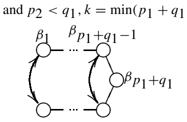
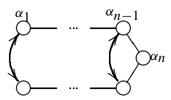
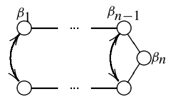
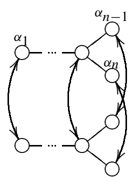
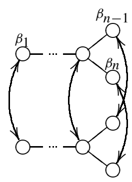
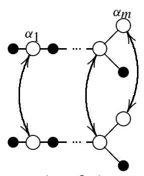
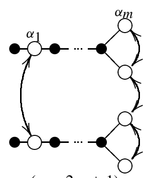
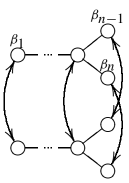
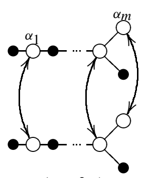
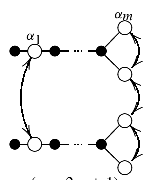
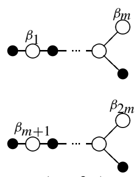
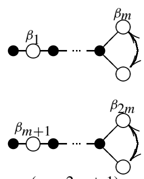
$(\mathfrak{g}, \mathfrak{h})$	Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$	Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_{\mathfrak{q}})$
$(\mathfrak{su}(p_1+p_2, q_1+q_2),$ $\mathfrak{su}(p_1, q_1)+\mathfrak{su}(p_2, q_2)+\mathfrak{so}(2))$	 $(p_1 < q_2$ and $q_1 < p_2)$  $(p_1 \leq q_2$ and $p_2 < q_1)$  $(p_1 = q_2$ and $p_2 = q_1)$	 $(p_1 < q_2$ and $q_1 < p_2)$  $(p_1 \leq q_2$ and $p_2 < q_1, k = \min(p_1+q_1, p_2+q_2))$  $(p_1 = q_2$ and $p_2 = q_1)$
$(\mathfrak{su}(n, n), \mathfrak{sl}(n, \mathbf{C})+\mathbf{R})$		
$(\mathfrak{so}(2n, \mathbf{C}), \mathfrak{so}^*(2n))$		
$(\mathfrak{so}^*(2n)+\mathfrak{so}^*(2n), \mathfrak{so}^*(2n))$	 $(n = 2m)$  $(n = 2m+1)$	
$(\mathfrak{so}(2n, \mathbf{C}), \mathfrak{sl}(n, \mathbf{C})+\mathbf{C})$	 $(n = 2m)$  $(n = 2m+1)$	 $(n = 2m)$  $(n = 2m+1)$

TABLE 1. (continued)

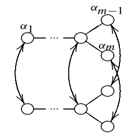
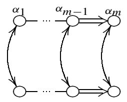
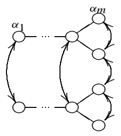
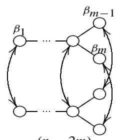
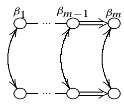
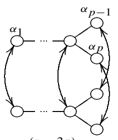
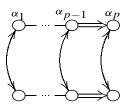
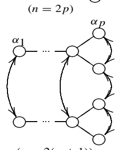
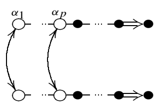
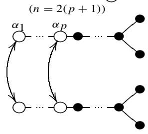
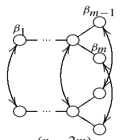
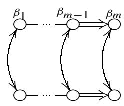
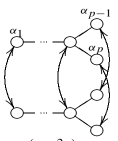
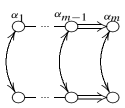
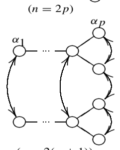
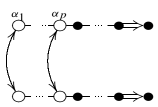
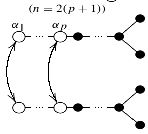
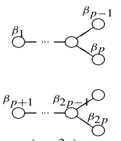
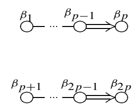
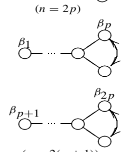
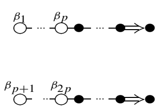
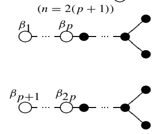
$(\mathfrak{g}, \mathfrak{h})$	Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$	Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_\mathfrak{q})$
$(\mathfrak{so}(n, \mathbf{C}), \mathfrak{so}(p, n-p))$	 $\begin{pmatrix} p, n-p : \text{even} \\ n = 2m \end{pmatrix}$  $\begin{pmatrix} p : \text{even} \\ n-p : \text{odd} \\ n = 2m+1 \end{pmatrix}$  $\begin{pmatrix} p, n-p : \text{odd} \\ n = 2(m+1) \end{pmatrix}$	 $(n = 2m)$  $(n = 2m+1)$
$(\mathfrak{so}(p, n-p) + \mathfrak{so}(p, n-p), \mathfrak{so}(p, n-p))$	 $(n = 2p)$  $(n = 2p+1)$  $(n = 2(p+1))$  $(n = 2(p+k)+1)$  $(n = 2(p+k), k > 1)$	 $(n = 2m)$  $(n = 2m+1)$
$(\mathfrak{so}(n, \mathbf{C}), \mathfrak{so}(p, \mathbf{C}) + \mathfrak{so}(n-p, \mathbf{C}))$	 $(n = 2p)$  $(n = 2p+1)$  $(n = 2(p+1))$  $(n = 2(p+k)+1)$  $(n = 2(p+k), k > 1)$	 $(n = 2p)$  $(n = 2p+1)$  $(n = 2(p+1))$  $(n = 2(p+k)+1)$  $(n = 2(p+k), k > 1)$

TABLE 1. (continued)

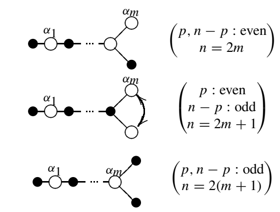
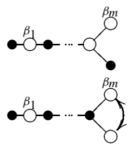
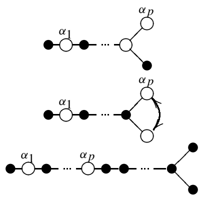
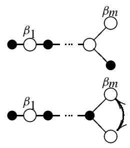
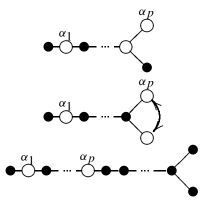
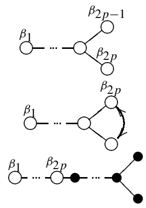
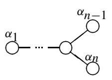
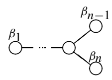
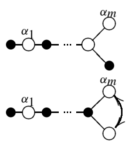
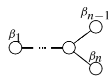
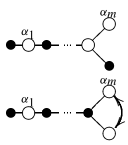
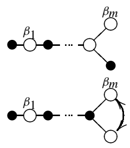
$(\mathfrak{g}, \mathfrak{h})$	Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$	Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_{\mathfrak{q}})$
$(\mathfrak{so}^*(2n), \mathfrak{su}(p, n-p) + \mathfrak{so}(2))$	 $\begin{pmatrix} p, n-p : \text{even} \\ n = 2m \end{pmatrix}$ $\begin{pmatrix} p : \text{even} \\ n-p : \text{odd} \\ n = 2m+1 \end{pmatrix}$ $\begin{pmatrix} p, n-p : \text{odd} \\ n = 2(m+1) \end{pmatrix}$	 $(n = 2m)$ $(n = 2m+1)$
$(\mathfrak{so}(2p, 2(n-p)), \mathfrak{su}(p, n-p) + \mathfrak{so}(2))$	 $(n = 2p)$ $(n = 2p+1)$ $(n > 2p+1)$	 $(n = 2m)$ $(n = 2m+1)$
$(\mathfrak{so}^*(2n), \mathfrak{so}^*(2p) + \mathfrak{so}^*(2(n-p)))$	 $(n = 2p)$ $(n = 2p+1)$ $(n > 2p+1)$	 $(n = 2p)$ $(n = 2p+1)$ $(n > 2p+1)$
$(\mathfrak{so}(n, n), \mathfrak{so}(n, \mathbf{C}))$		
$(\mathfrak{so}^*(2n), \mathfrak{so}(n, \mathbf{C}))$	 $(n = 2m)$ $(n = 2m+1)$	
$(\mathfrak{so}(n, n), \mathfrak{sl}(n, \mathbf{R}) + \mathbf{R})$	 $(n = 2m)$ $(n = 2m+1)$	 $(n = 2m)$ $(n = 2m+1)$

TABLE 1. (continued)

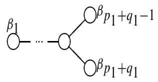
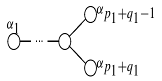
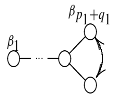
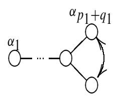
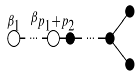
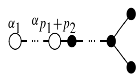
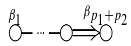
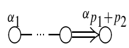
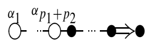
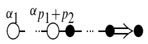
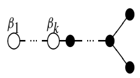
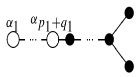
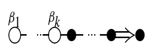
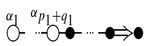
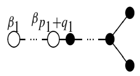
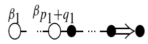
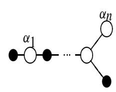
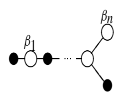
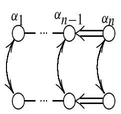
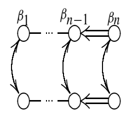
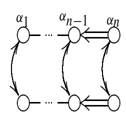
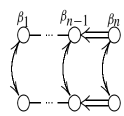
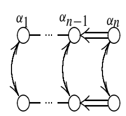
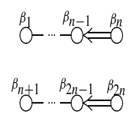
$(\mathfrak{g}, \mathfrak{h})$	Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$	Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_{\mathfrak{q}})$
$(\mathfrak{so}(p_1+p_2, q_1+q_2),$ $\mathfrak{so}(p_1, q_1)+\mathfrak{so}(p_2, q_2))$		 $(p_1 = q_2$ and $p_2 = q_1)$
	 $(p_1 = q_2$ and $p_2 = q_1)$	 $\begin{pmatrix} p_1 = q_2 \text{ and } p_2 < q_1 \\ q_1+q_2 = p_1+p_2+2 \end{pmatrix}$
	 $\begin{pmatrix} p_1 = q_2 \text{ and } p_2 < q_1 \\ q_1+q_2 = p_1+p_2+2 \end{pmatrix}$	 $\begin{pmatrix} p_1 = q_2 \text{ and } p_2 < q_1 \\ p_1+p_2+q_1+q_2 : \text{even} \end{pmatrix}$
	 $\begin{pmatrix} p_1 \le q_2 \text{ and } p_2 < q_1 \\ p_1+p_2+q_1+q_2 : \text{even} \end{pmatrix}$	 $\begin{pmatrix} p_1 = q_2 \text{ and } p_2 < q_1 \\ q_1+q_2 = p_1+p_2+1 \end{pmatrix}$
	 $\begin{pmatrix} p_1 = q_2 \text{ and } p_2 < q_1 \\ q_1+q_2 = p_1+p_2+1 \end{pmatrix}$	 $\begin{pmatrix} p_1 = q_2 \text{ and } p_2 < q_1 \\ p_1+p_2+q_1+q_2 : \text{odd} \end{pmatrix}$
	 $\begin{pmatrix} p_1 \le q_2 \text{ and } p_2 < q_1 \\ p_1+p_2+q_1+q_2 : \text{odd} \end{pmatrix}$	 $\begin{pmatrix} k = \min(p_1+q_1, p_1+q_1) \\ p_1 < q_2 \text{ and } p_2 < q_1 \\ p_1+p_2+q_1+q_2 : \text{even} \end{pmatrix}$
	 $\begin{pmatrix} p_1 < q_2 \text{ and } q_1 < p_2 \\ p_1+p_2+q_1+q_2 : \text{even} \end{pmatrix}$	 $\begin{pmatrix} k = \min(p_1+q_1, p_1+q_1) \\ p_1 < q_2 \text{ and } p_2 < q_1 \\ p_1+p_2+q_1+q_2 : \text{odd} \end{pmatrix}$
	 $\begin{pmatrix} p_1 < q_2 \text{ and } q_1 < p_2 \\ p_1+p_2+q_1+q_2 : \text{odd} \end{pmatrix}$	 $\begin{pmatrix} p_1 < q_2 \text{ and } q_1 < p_2 \\ p_1+p_2+q_1+q_2 : \text{even} \end{pmatrix}$
		 $\begin{pmatrix} p_1 < q_2 \text{ and } q_1 < p_2 \\ p_1+p_2+q_1+q_2 : \text{odd} \end{pmatrix}$
$(\mathfrak{so}^*(4n), \mathfrak{su}^*(2n)+\mathbf{R})$		
$(\mathfrak{sp}(n, \mathbf{C}), \mathfrak{sp}(n, \mathbf{R}))$		
$(\mathfrak{sp}(n, \mathbf{R})+\mathfrak{sp}(n, \mathbf{R}), \mathfrak{sp}(n, \mathbf{R}))$		
$(\mathfrak{sp}(n, \mathbf{C}), \mathfrak{sl}(n, \mathbf{C})+\mathbf{C})$		

TABLE 1. (continued)

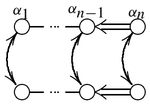
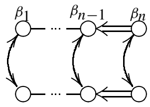
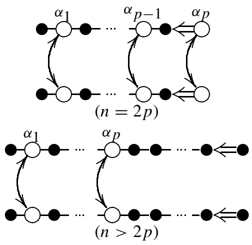
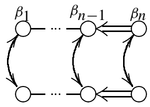
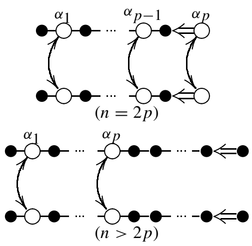
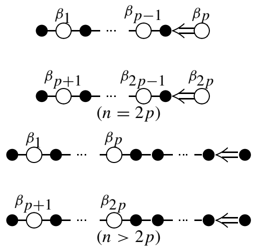
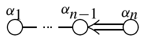
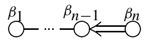
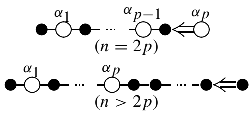
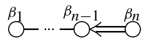
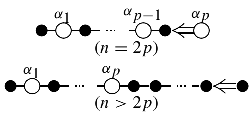
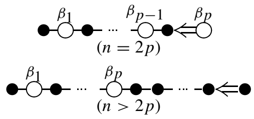
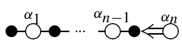
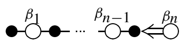
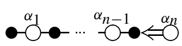
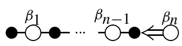
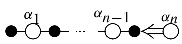
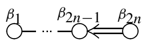
$(\mathfrak{g}, \mathfrak{h})$	Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$	Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_{\mathfrak{q}})$
$(\mathfrak{sp}(n, \mathbf{C}), \mathfrak{sp}(p, n-p))$		
$(\mathfrak{sp}(p, n-p) + \mathfrak{sp}(p, n-p), \mathfrak{sp}(p, n-p))$		
$(\mathfrak{sp}(n, \mathbf{C}), \mathfrak{sp}(p, \mathbf{C}) + \mathfrak{sp}(n-p, \mathbf{C}))$		
$(\mathfrak{sp}(n, \mathbf{R}), \mathfrak{su}(p, n-p) + \mathfrak{so}(2))$		
$(\mathfrak{sp}(p, n-p), \mathfrak{su}(p, n-p) + \mathfrak{so}(2))$		
$(\mathfrak{sp}(n, \mathbf{R}), \mathfrak{sp}(p, \mathbf{R}) + \mathfrak{sp}(n-p, \mathbf{R}))$		
$(\mathfrak{sp}(n, n), \mathfrak{sp}(n, \mathbf{C}))$		
$(\mathfrak{sp}(2n, \mathbf{R}), \mathfrak{sp}(n, \mathbf{C}))$		
$(\mathfrak{sp}(n, n), \mathfrak{su}^*(2n) + \mathbf{R})$		

TABLE 1. (continued)

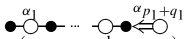
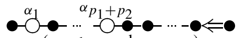
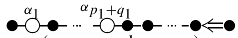
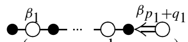
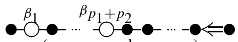
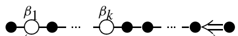
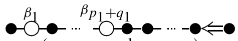
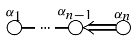
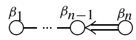
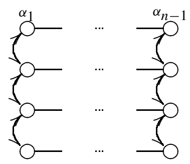
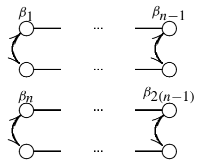
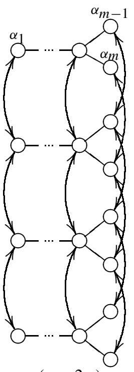
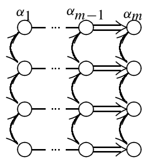
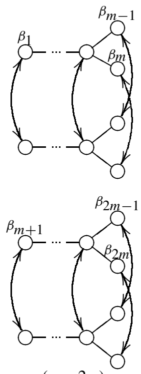
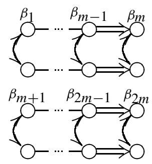
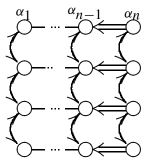
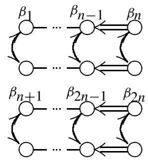
$(\mathfrak{g}, \mathfrak{h})$	Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$	Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_{\mathfrak{q}})$
$(\mathfrak{sp}(p_1+p_2, q_1+q_2), \mathfrak{sp}(p_1, q_1)+\mathfrak{sp}(p_2, q_2))$	 $(p_1 = q_2$ and $p_2 = q_1)$  $(p_1 \le q_2$ and $p_2 < q_1)$  $(p_1 < q_2$ and $q_1 < p_2)$	 $(p_1 = q_2$ and $p_2 = q_1)$  $(p_1 = q_2$ and $p_2 < q_1)$  $\begin{pmatrix} k = \min(p_1+q_1, p_2+q_2) \\ p_1 < q_2 \text{ and } p_2 < q_1 \end{pmatrix}$  $(p_1 < q_2$ and $q_1 < p_2)$
$(\mathfrak{sp}(n, \mathbf{R}), \mathfrak{sl}(n, \mathbf{R})+\mathbf{R})$		
$(\mathfrak{sl}(n, \mathbf{C})+\mathfrak{sl}(n, \mathbf{C}), \mathfrak{sl}(n, \mathbf{C}))$		
$(\mathfrak{so}(n, \mathbf{C})+\mathfrak{so}(n, \mathbf{C}), \mathfrak{so}(n, \mathbf{C}))$	 $(n = 2m)$  $(n = 2m+1)$	 $(n = 2m)$  $(n = 2m+1)$
$(\mathfrak{sp}(n, \mathbf{C})+\mathfrak{sp}(n, \mathbf{C}), \mathfrak{sp}(n, \mathbf{C}))$		

TABLE 2. Dynkin diagrams of restricted root systems

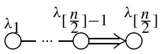
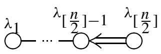
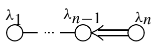
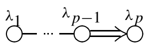
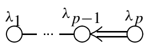
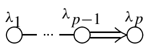
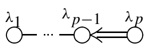
$(\mathfrak{g}, \mathfrak{h})$ $(\mathfrak{g}^d, \mathfrak{h}^d)$	$\Psi(\mathfrak{a})$	$\begin{pmatrix} m^+(\lambda_i) & m^+(2\lambda_i) \\ m^-(\lambda_i) & m^-(2\lambda_i) \end{pmatrix}$
$(\mathfrak{sl}(n, \mathbf{C}), \mathfrak{sl}(n, \mathbf{R}))$ (self dual)	 $(n$:odd$)$	$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $(1 \le i \le [n/2] - 1)$ $\begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}$ $(i = [n/2])$
	 $(n$:even$)$	$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $(1 \le i \le [n/2] - 1)$ $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ $(i = [n/2])$
$(\mathfrak{sl}(n, \mathbf{R}) + \mathfrak{sl}(n, \mathbf{R}), \mathfrak{sl}(n, \mathbf{R}))$ $(\mathfrak{sl}(n, \mathbf{C}), \mathfrak{so}(n, \mathbf{C}))$		$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $(1 \le i \le n - 1)$
$(\mathfrak{sl}(2n, \mathbf{C}), \mathfrak{su}^*(2n))$ (self dual)		$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $(1 \le i \le n - 1)$ $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $(i = n)$
$(\mathfrak{su}^*(2n) + \mathfrak{su}^*(2n), \mathfrak{su}^*(2n))$ $(\mathfrak{sl}(2n, \mathbf{C}), \mathfrak{sp}(n, \mathbf{C}))$		$\begin{pmatrix} 4 & 0 \\ 4 & 0 \end{pmatrix}$ $(1 \le i \le n - 1)$
$(\mathfrak{sl}(n, \mathbf{C}), \mathfrak{su}(p, n - p))$ (self dual)		$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \neq p)$ $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ $(i = p)$
$(\mathfrak{su}(p, n - p) + \mathfrak{su}(p, n - p), \mathfrak{su}(p, n - p))$ $(\mathfrak{sl}(n, \mathbf{C}), \mathfrak{sl}(p, \mathbf{C}) + \mathfrak{sl}(n - p, \mathbf{C}) + \mathbf{C})$	 $(n > 2p)$	$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $(1 \le i \le p - 1)$ $\begin{pmatrix} 2(n - 2p) & 1 \\ 2(n - 2p) & 1 \end{pmatrix}$ $(i = p)$
	 $(n = 2p)$	$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $(1 \le i \le p - 1)$ $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $(i = p)$
$(\mathfrak{sl}(n, \mathbf{R}), \mathfrak{so}(p, n - p))$ (self dual)		$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \neq p)$ $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $(i = p)$
$(\mathfrak{su}(p, n - p), \mathfrak{so}(p, n - p))$ $(\mathfrak{sl}(n, \mathbf{R}), \mathfrak{sl}(p, \mathbf{R}) + \mathfrak{sl}(n - p, \mathbf{R}) + \mathbf{R})$	 $(n > 2p)$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $(1 \le i \le p - 1)$ $\begin{pmatrix} n - 2p & 0 \\ n - 2p & 1 \end{pmatrix}$ $(i = p)$
	 $(n = 2p)$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $(1 \le i \le p - 1)$ $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $(i = p)$
$(\mathfrak{su}^*(2n), \mathfrak{sp}(p, n - p))$ (self dual)		$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \neq p)$ $\begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}$ $(i = p)$

TABLE 2. (continued)

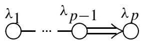
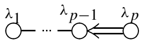
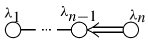
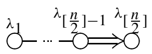
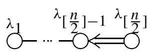
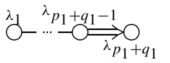
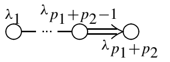
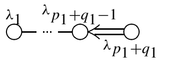
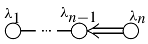
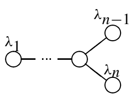
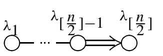
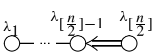
$(\mathfrak{g}, \mathfrak{h})$ $(\mathfrak{g}^d, \mathfrak{h}^d)$	$\Psi(\mathfrak{a})$	$\begin{pmatrix} m^+(\lambda_i) & m^+(2\lambda_i) \\ m^-(\lambda_i) & m^-(2\lambda_i) \end{pmatrix}$
$(\mathfrak{su}(2p, 2(n-p)), \mathfrak{sp}(p, n-p))$ $(\mathfrak{su}^*(2n),$ $\mathfrak{su}^*(2p) + \mathfrak{su}^*(2(n-p)) + \mathbf{R})$	 $(n > 2p)$	$\begin{pmatrix} 4 & 0 \\ 4 & 0 \end{pmatrix}$ $(1 \le i \le p-1)$; $\begin{pmatrix} 4(n-2p) & 3 \\ 4(n-2p) & 1 \end{pmatrix}$ $(i = p)$
	 $(n = 2p)$	$\begin{pmatrix} 4 & 0 \\ 4 & 0 \end{pmatrix}$ $(1 \le i \le p-1)$; $\begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix}$ $(i = p)$
$(\mathfrak{sl}(2n, \mathbf{R}), \mathfrak{sp}(n, \mathbf{R}))$ $(\mathfrak{su}^*(2n), \mathfrak{so}^*(2n))$		$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $(1 \le i \le n-1)$
$(\mathfrak{su}(n, n), \mathfrak{so}^*(2n))$ $(\mathfrak{sl}(2n, \mathbf{R}), \mathfrak{sl}(n, \mathbf{C}) + \mathfrak{so}(2))$		$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $(1 \le i \le n-1)$; $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $(i = n)$
$(\mathfrak{su}^*(2n), \mathfrak{sl}(n, \mathbf{C}) + \mathfrak{so}(2))$ $(\mathfrak{su}(n, n), \mathfrak{sp}(n, \mathbf{R}))$	 $(n\text{:odd})$	$\begin{pmatrix} 4 & 0 \\ 4 & 0 \end{pmatrix}$ $(1 \le i \le [n/2]-1)$; $\begin{pmatrix} 4 & 1 \\ 4 & 3 \end{pmatrix}$ $(i = [n/2])$
	 $(n\text{:even})$	$\begin{pmatrix} 4 & 0 \\ 4 & 0 \end{pmatrix}$ $(1 \le i \le [n/2]-1)$; $\begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$ $(i = [n/2])$
$(\mathfrak{su}(p_1 + p_2, q_1 + q_2),$ $\mathfrak{su}(p_1, q_1) + \mathfrak{su}(p_2, q_2) + \mathfrak{so}(2))$ $(\mathfrak{su}(p_1 + q_1, p_2 + q_2),$ $\mathfrak{su}(p_1, p_2) + \mathfrak{su}(q_1, q_2) + \mathfrak{so}(2))$	 $(p_1 < q_2$ and $q_1 < p_2)$	$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \ne p_1, p_1 + q_1)$; $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ $(i = p_1)$; $\begin{pmatrix} 2(p_2 - q_1) & 1 \\ 2(q_2 - p_1) & 0 \end{pmatrix}$ $(i = p_1 + q_1)$
	 $(p_1 \le q_2$ and $p_2 < q_1)$	$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \ne p_1, p_1 + p_2)$; $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ $(i = p_1)$; $\begin{pmatrix} 2(q_1 - p_2) & 1 \\ 2(q_2 - p_1) & 0 \end{pmatrix}$ $(i = p_1 + p_2)$
	 $(p_1 = q_2$ and $p_2 = q_1)$	$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \ne p_1, p_1 + q_1)$; $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ $(i = p_1)$; $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $(i = p_1 + q_1)$
$(\mathfrak{su}(n, n), \mathfrak{sl}(n, \mathbf{C}) + \mathbf{R})$ (self dual)		$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $(1 \le i \le n-1)$; $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $(i = n)$
$(\mathfrak{so}(2n, \mathbf{C}), \mathfrak{so}^*(2n))$ (self dual)		$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \ne n)$; $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ $(i = n)$
$(\mathfrak{so}^*(2n) + \mathfrak{so}^*(2n), \mathfrak{so}^*(2n))$ $(\mathfrak{so}(2n, \mathbf{C}), \mathfrak{sl}(n, \mathbf{C}) + \mathbf{C})$	 $(n\text{:odd})$	$\begin{pmatrix} 4 & 0 \\ 4 & 0 \end{pmatrix}$ $(1 \le i \le [n/2]-1)$; $\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}$ $(i = [n/2])$
	 $(n\text{:even})$	$\begin{pmatrix} 4 & 0 \\ 4 & 0 \end{pmatrix}$ $(1 \le i \le [n/2]-1)$; $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $(i = [n/2])$

TABLE 2. (continued)

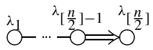
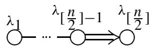
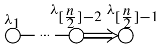
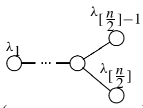
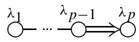
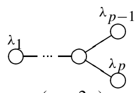
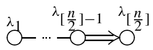
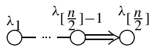
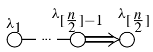
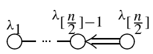
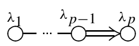
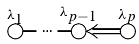
$(\mathfrak{g}, \mathfrak{h})$ $(\mathfrak{g}^d, \mathfrak{h}^d)$	$\Psi(\mathfrak{a})$	$\begin{pmatrix} m^+(\lambda_i) & m^+(2\lambda_i) \\ m^-(\lambda_i) & m^-(2\lambda_i) \end{pmatrix}$
$(\mathfrak{so}(n,\mathbf{C}), \mathfrak{so}(p, n-p))$ (self dual)	 (p:even, $n-p$:odd)	$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \neq [p/2], [n/2])$ $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ $(i = [p/2], [n/2])$
	 (p:odd, $n-p$:even)	$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \neq [p/2])$ $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ $(i = [p/2])$
	 (p:odd, $n-p$:odd)	$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \neq [p/2], [n/2]-1)$ $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ $(i = [p/2])$ $\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $(i = [n/2]-1)$
	 (p:even, $n-p$:even)	$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \neq [p/2])$ $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ $(i = [p/2])$
$(\mathfrak{so}(p, n-p) + \mathfrak{so}(p, n-p),$ $\mathfrak{so}(p, n-p))$ $(\mathfrak{so}(n,\mathbf{C}),$ $\mathfrak{so}(p,\mathbf{C}) + \mathfrak{so}(n-p,\mathbf{C})$	 $(n > 2p)$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $(1 \leq i \leq p-1)$ $\begin{pmatrix} n-2p & 0 \\ n-2p & 0 \end{pmatrix}$ $(i = p)$
	 $(n = 2p)$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $(1 \leq i \leq p)$
$(\mathfrak{so}^*(2n), \mathfrak{su}(p, n-p) + \mathfrak{so}(2))$ (self dual)	 (p:even, $n-p$:odd)	$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \neq p, [n/2])$ $\begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}$ $(i = p)$ $\begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix}$ $(i = [n/2])$
	 (p:odd, $n-p$:even)	$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \neq p, [n/2])$ $\begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}$ $(i = p)$ $\begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$ $(i = [n/2])$
	 (p:odd, $n-p$:odd)	$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \neq p, [n/2])$ $\begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}$ $(i = p)$ $\begin{pmatrix} 4 & 1 \\ 4 & 0 \end{pmatrix}$ $(i = [n/2])$
	 (p:even, $n-p$:even)	$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \neq p, [n/2])$ $\begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}$ $(i = p)$ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $(i = [n/2])$
$(\mathfrak{so}(2p, 2(n-p)), \mathfrak{su}(p, n-p) + \mathfrak{so}(2))$ $(\mathfrak{so}^*(2n), \mathfrak{so}^*(2p) + \mathfrak{so}^*(2(n-p)))$	 $(n > 2p)$	$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $(1 \leq i \leq p-1)$ $\begin{pmatrix} 2(n-2p) & 1 \\ 2(n-2p) & 0 \end{pmatrix}$ $(i = p)$
	 $(n = 2p)$	$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $(1 \leq i \leq p-1)$ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $(i = p)$

TABLE 2. (continued)

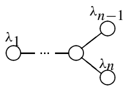
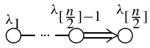
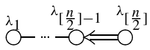
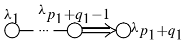
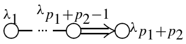
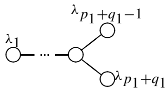
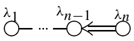
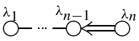
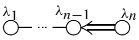
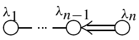
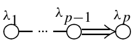
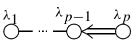
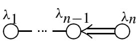
$(\mathfrak{g}, \mathfrak{h})$ $(\mathfrak{g}^d, \mathfrak{h}^d)$	$\Psi(\mathfrak{a})$	$\begin{pmatrix} m^+(\lambda_i) & m^+(2\lambda_i) \\ m^-(\lambda_i) & m^-(2\lambda_i) \end{pmatrix}$
$(\mathfrak{so}(n, n), \mathfrak{so}(n, \mathbf{C}))$ (self dual)		$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \neq n)$ $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $(i = n)$ (n:odd) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \neq [n/2], n)$ $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $(i = [n/2], n)$ (n:even)
$(\mathfrak{so}^*(2n), \mathfrak{so}(n, \mathbf{C}))$ $(\mathfrak{so}(n, n), \mathfrak{sl}(n, \mathbf{R}) + \mathbf{R})$	 (n:odd)  (n:even)	$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $(1 \leq i \leq [n/2] - 1)$ $\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$ $(i = [n/2])$ $\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $(1 \leq i \leq [n/2] - 1)$ $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $(i = [n/2])$
$(\mathfrak{so}(p_1 + p_2, q_1 + q_2),$ $\mathfrak{so}(p_1, q_1) + \mathfrak{so}(p_2, q_2))$ $(\mathfrak{so}(p_1 + q_1, p_2 + q_2),$ $\mathfrak{so}(p_1, p_2) + \mathfrak{so}(q_1, q_2))$	 $(p_1 < q_2$ and $q_1 < p_2)$  $(p_1 \leq q_2$ and $p_2 < q_1)$  $(p_1 = q_2$ and $p_2 = q_1)$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \neq p_1, p_1 + q_1)$ $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $(i = p_1)$ $\begin{pmatrix} p_2 - q_1 & 1 \\ q_2 - p_1 & 0 \end{pmatrix}$ $(i = p_1 + q_1)$ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \neq p_1, p_1 + p_2)$ $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $(i = p_1)$ $\begin{pmatrix} q_1 - p_2 & 1 \\ q_2 - p_1 & 0 \end{pmatrix}$ $(i = p_1 + p_2)$ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \neq p_1)$ $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $(i = p_1)$
$(\mathfrak{so}^*(4n), \mathfrak{su}^*(2n) + \mathbf{R})$ (self dual)		$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$ $(1 \leq i \leq n - 1)$ $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $(i = n)$
$(\mathfrak{sp}(n, \mathbf{C}), \mathfrak{sp}(n, \mathbf{R}))$ (self dual)		$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \leq i \leq n - 1)$ $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ $(i = n)$
$(\mathfrak{sp}(n, \mathbf{R}) + \mathfrak{sp}(n, \mathbf{R}), \mathfrak{sp}(n, \mathbf{R}))$ $(\mathfrak{sp}(n, \mathbf{C}), \mathfrak{sl}(n, \mathbf{C}) + \mathbf{C})$		$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $(i \leq i \leq n)$
$(\mathfrak{sp}(n, \mathbf{C}), \mathfrak{sp}(p, n - p))$ (self dual)		$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \neq p)$ $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ $(i = p)$
$(\mathfrak{sp}(p, n - p) + \mathfrak{sp}(p, n - p), \mathfrak{sp}(p, n - p))$ $(\mathfrak{sp}(n, \mathbf{C}), \mathfrak{sp}(p, \mathbf{C}) + \mathfrak{sp}(n - p, \mathbf{C}))$	 $(n > 2p)$  $(n = 2p)$	$\begin{pmatrix} 4 & 0 \\ 4 & 0 \end{pmatrix}$ $(1 \leq i \leq p - 1)$ $\begin{pmatrix} 4(n - 2p) & 3 \\ 4(n - 2p) & 3 \end{pmatrix}$ $(i = p)$ $\begin{pmatrix} 4 & 0 \\ 4 & 0 \end{pmatrix}$ $(1 \leq i \leq p - 1)$ $\begin{pmatrix} 3 & 0 \\ 3 & 0 \end{pmatrix}$ $(i = p)$
$(\mathfrak{sp}(n, \mathbf{R}), \mathfrak{su}(p, n - p) + \mathfrak{so}(2))$ (self dual)		$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \neq p)$ $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $(i = p)$

TABLE 2. (continued)

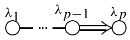
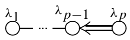
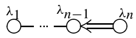
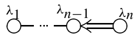
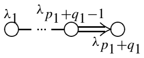
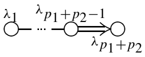
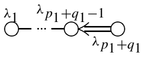
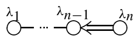
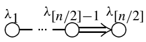
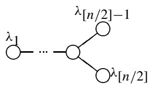
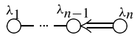
$(\mathfrak{g}, \mathfrak{h})$ $(\mathfrak{g}^d, \mathfrak{h}^d)$	$\Psi(\mathfrak{a})$	$\begin{pmatrix} m^+(\lambda_i) & m^+(2\lambda_i) \\ m^-(\lambda_i) & m^-(2\lambda_i) \end{pmatrix}$
$(\mathfrak{sp}(p, n-p), \mathfrak{su}(p, n-p) + \mathfrak{so}(2))$ $(\mathfrak{sp}(n, \mathbf{R}), \mathfrak{sp}(p, \mathbf{R}) + \mathfrak{sp}(n-p, \mathbf{R}))$	 $(n > 2p)$	$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $(1 \le i \le p-1)$ $\begin{pmatrix} 2(n-2p) & 1 \\ 2(n-2p) & 2 \end{pmatrix}$ $(i = p)$
	 $(n = 2p)$	$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $(1 \le i \le p-1)$ $\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$ $(i = p)$
$(\mathfrak{sp}(n, n), \mathfrak{sp}(n, \mathbf{C}))$ (self dual)		$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$ $(1 \le i \le n-1)$ $\begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}$ $(i = n)$
$(\mathfrak{sp}(2n, \mathbf{R}), \mathfrak{sp}(n, \mathbf{C}))$ $(\mathfrak{sp}(n, n), \mathfrak{su}^*(2n) + \mathbf{R})$		$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $(1 \le i \le n)$ $\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}$ $(i = n)$
$(\mathfrak{sp}(p_1 + p_2, q_1 + q_2),$ $\mathfrak{sp}(p_1, q_1) + \mathfrak{sp}(p_2, q_2))$ $(\mathfrak{sp}(p_1 + q_1, p_2 + q_2),$ $\mathfrak{sp}(p_1, p_2) + \mathfrak{sp}(q_1, q_2))$	 $(p_1 < q_2$ and $q_1 < p_2)$	$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \ne p_1, p_1 + q_1)$ $\begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}$ $(i = p_1)$ $\begin{pmatrix} 4(p_2 - q_1) & 3 \\ 4(q_2 - p_1) & 0 \end{pmatrix}$ $(i = p_1 + q_1)$
	 $(p_1 \le q_2$ and $p_2 < q_1)$	$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \ne p_1, p_1 + p_2)$ $\begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}$ $(i = p_1)$ $\begin{pmatrix} 4(q_1 - p_2) & 3 \\ 4(q_2 - p_1) & 0 \end{pmatrix}$ $(i = p_1 + p_2)$
	 $(p_1 = q_2$ and $p_2 = q_1)$	$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$ $(i \ne p_1, p_1 + p_2)$ $\begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}$ $(i = p_1)$ $\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$ $(i = p_1 + q_1)$
$(\mathfrak{sp}(n, \mathbf{R}), \mathfrak{sl}(n, \mathbf{R}) + \mathbf{R})$ (self dual)		$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $(1 \le i \le n)$ $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $(i = n)$
$(\mathfrak{sl}(n, \mathbf{C}) + \mathfrak{sl}(n, \mathbf{C}), \mathfrak{sl}(n, \mathbf{C}))$		$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $(1 \le i \le n-1)$
$(\mathfrak{so}(n, \mathbf{C}) + \mathfrak{so}(n, \mathbf{C}), \mathfrak{so}(n, \mathbf{C}))$	 (*n*:odd)  (*n*:even)	$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $(1 \le i \le [n/2])$
$(\mathfrak{sp}(n, \mathbf{C}) + \mathfrak{sp}(n, \mathbf{C}), \mathfrak{sp}(n, \mathbf{C}))$		$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ $(1 \le i \le [n/2])$

TABLE 3. v-cohomogeneities, t-cohomogeneities, dimensions and indices

$(\mathfrak{g}, \mathfrak{h})$	v-cohomogeneity	t-cohomogeneity	dimension	index
$(\mathfrak{sl}(n,\mathbf{C}), \mathfrak{sl}(n,\mathbf{R}))$	$n-1$	$n-1$	$(n-1)(n+1)$	$(n-1)(n+2)/2$
$(\mathfrak{sl}(n,\mathbf{R})+\mathfrak{sl}(n,\mathbf{R}), \mathfrak{sl}(n,\mathbf{R}))$	$n-1$	$n-1$	$(n-1)(n+1)$	$n(n-1)/2$
$(\mathfrak{sl}(n,\mathbf{C}), \mathfrak{so}(n,\mathbf{C}))$	$2(n-1)$	$2(n-1)$	$(n-1)(n+2)$	$(n-1)(n+2)/2$
$(\mathfrak{sl}(2n,\mathbf{C}), \mathfrak{su}^*(2n))$	$2n-1$	$4n-1$	$(2n-1)(2n+1)$	$(n-1)(2n+1)$
$(\mathfrak{su}^*(2n)+\mathfrak{su}^*(2n), \mathfrak{su}^*(2n))$	$4n-1$	$2n-1$	$(2n-1)(2n+1)$	$n(2n+1)$
$(\mathfrak{sl}(2n,\mathbf{C}), \mathfrak{sp}(n,\mathbf{C}))$	$2(n-1)$	$2(n-1)$	$2(n-1)(2n+1)$	$(n-1)(2n+1)$
$(\mathfrak{sl}(n,\mathbf{C}), \mathfrak{su}(p,n-p))$	$n-1$	$(n-2p)^2+2p-1$	$(n-1)(n+1)$	$2p(n-p)$
$(\mathfrak{su}(p,n-p)+\mathfrak{su}(p,n-p), \mathfrak{su}(p,n-p))$	$(n-2p)^2+2p-1$	$n-1$	$(n-1)(n+1)$	$p^2+(n-p)^2-1$
$(\mathfrak{sl}(n,\mathbf{C}), \mathfrak{sl}(p,\mathbf{C})+\mathfrak{sl}(n-p,\mathbf{C})+\mathbf{C})$	$2p$	$2p$	$4p(n-p)$	$2p(n-p)$
$(\mathfrak{sl}(n,\mathbf{R}), \mathfrak{so}(p,n-p))$	$n-1$	$n-1+(n-2p)(n-2p-1)/2$	$(n-1)(n+2)/2$	$p(n-p)$
$(\mathfrak{su}(p,n-p), \mathfrak{so}(p,n-p))$	$n-1+(n-2p)(n-2p-1)/2$	$n-1$	$(n-1)(n+2)/2$	$p^2/2+(n-p)^2/2+n/2-1$
$(\mathfrak{sl}(n,\mathbf{R}), \mathfrak{sl}(p,\mathbf{R})+\mathfrak{sl}(n-p,\mathbf{R}))$	p	p	$2p(n-p)$	$p(n-p)$
$(\mathfrak{su}^*(2n), \mathfrak{sp}(p,n-p))$	$n-1$	$2(n-2p)^2+2(n-2p)+n-1$	$(n-1)(2n+1)$	$4p(n-p)$
$(\mathfrak{su}(2p,2(n-p)), \mathfrak{sp}(p,n-p))$	$2(n-2p)^2+2(n-2p)+n-1$	$n-1$	$(n-1)(2n+1)$	$2p^2+2(n-p)^2-n-1$
$(\mathfrak{su}^*(2n), \mathfrak{sp}(p,n-p))$	$4p$	$4p$	$8p(n-p)$	$4p(n-p)$
$(\mathfrak{sl}(2n,\mathbf{R}), \mathfrak{sp}(n,\mathbf{R}))$	$n-1$	$n-1$	$(n-1)(2n+1)$	$n(n-1)$
$(\mathfrak{su}^*(2n), \mathfrak{so}^*(2n))$	$3n-1$	$2n-1$	$(2n-1)(n+1)$	$n(n+1)$
$(\mathfrak{su}(n,n), \mathfrak{so}^*(2n))$	$2n-1$	$3n-1$	$(n-1)(2n+1)$	$(n-1)(n+1)$
$(\mathfrak{sl}(2n,\mathbf{R}), \mathfrak{sl}(n,\mathbf{C})+\mathfrak{so}(2))$	n	$2n$	$2n^2$	$n(n-1)$
$(\mathfrak{su}^*(2n), \mathfrak{sl}(n,\mathbf{C})+\mathfrak{so}(2))$	$2n$	n	$2n^2$	$n(n+1)$
$(\mathfrak{su}(n,n), \mathfrak{sp}(n,\mathbf{R}))$	$n-1$	$n-1$	$(2n-1)(n+1)$	$(n-1)(n+1)$
$(\mathfrak{su}(p_1+p_2,q_1+q_2),$ $\mathfrak{su}(p_1,q_1)+\mathfrak{su}(p_2,q_2)+\mathfrak{so}(2))$	p_1+q_1 ($p_1 \le q_2$ and $q_1 \le p_2$) $p_1+p_2+2(q_1-p_2)(q_2-p_1)$ ($p_1 \le q_2$ and $p_2 < q_1$)	p_1+q_1 ($p_1 \le q_2$ and $q_1 \le p_2$) $p_1+p_2+2(q_1-p_2)(q_2-p_1)$ ($p_1 \le q_2$ and $p_2 < q_1$)	$2(p_1+q_1)(p_2+q_2)$	$2(p_1p_2+q_1q_2)$

TABLE 3. (continued)

$(\mathfrak{g}, \mathfrak{h})$	v-cohomogeneity	t-cohomogeneity	dimension	index
$(\mathfrak{su}(n,n), \mathfrak{sl}(n,\mathbf{C})+\mathbf{R})$	n	n	$2n^2$	n^2
$(\mathfrak{so}(2n,\mathbf{C}), \mathfrak{so}^*(2n))$	n	$2n-1$ (n:odd) $2n$ (n:even)	$n(2n-1)$	$n(n-1)$
$(\mathfrak{so}^*(2n)+\mathfrak{so}^*(2n), \mathfrak{so}^*(2n))$	$2n-1$ (n:odd) $2n$ (n:even)	n	$n(2n-1)$	n^2
$(\mathfrak{so}(2n,\mathbf{C}), \mathfrak{sl}(n,\mathbf{C})+\mathbf{C})$	$n-1$ (n:odd) n (n:even)	$n-1$ (n:odd) n (n:even)	$2n(n-1)$	$n(n-1)$
$(\mathfrak{so}(n,\mathbf{C}), \mathfrak{so}(p,n-p))$	$[\frac{p}{2}]+[\frac{n-p}{2}]+1$ ($p, n-p$:odd) $[\frac{p}{2}]+[\frac{n-p}{2}]$ (others)	$2k(k+1)+p$ ($n=2(p+k)+1$) $k(2k-1)+p$ ($n=2(p+k)$)	$n(n-1)/2$	$p(n-p)$
$(\mathfrak{so}(p,n-p)+\mathfrak{so}(p,n-p),$ $\mathfrak{so}(p,n-p))$	$2k(k+1)+p$ ($n=2(p+k)+1$) $k(2k-1)+p$ ($n=2(p+k)$)	$[\frac{p}{2}]+[\frac{n-p}{2}]+1$ ($p, n-p$:odd) $[\frac{p}{2}]+[\frac{n-p}{2}]$ (others)	$n(n-1)/2$	$p^2/2+(n-p)^2/2-n/2$
$(\mathfrak{so}(n,\mathbf{C}), \mathfrak{so}(p,\mathbf{C})+\mathfrak{so}(n-p,\mathbf{C}))$	$2p$	$2p$	$2p(n-p)$	$p(n-p)$
$(\mathfrak{so}(2p,2(n-p)), \mathfrak{su}(p,n-p)+\mathfrak{so}(2))$	$(n-2p)^2-n+3p$	$[\frac{p}{2}]+[(\frac{n-p}{2}]+2$ ($p, n-p$:odd) $[\frac{p}{2}]+[\frac{n-p}{2}]$ (others)	$n(n-1)$	$p^2+(n-p)^2-n$
$(\mathfrak{so}^*(2n), \mathfrak{so}^*(2p)+\mathfrak{so}^*(2(n-p)))$	$3p$	$3p$	$4p(n-p)$	$2p(n-p)$
$(\mathfrak{so}^*(2n), \mathfrak{su}(p,n-p)+\mathfrak{so}(2))$	$[\frac{p}{2}]+[\frac{n-p}{2}]+2$ ($p, n-p$:odd) $[\frac{p}{2}]+[\frac{n-p}{2}]$ (others)	$(n-2p)^2-n+3p$	$n(n-1)$	$2p(n-p)$
$(\mathfrak{so}(n,n), \mathfrak{so}(n,\mathbf{C}))$	n	$3[\frac{n}{2}]+1$ (n:odd) $3[\frac{n}{2}]$ (n:even)	n^2	$n(n-1)/2$
$(\mathfrak{so}^*(2n), \mathfrak{so}(n,\mathbf{C}))$	$3[\frac{n}{2}]+1$ (n:odd) $3[\frac{n}{2}]$ (n:even)	n	n^2	$n(n+1)/2$
$(\mathfrak{so}(n,n), \mathfrak{sl}(n,\mathbf{R})+\mathbf{R})$	$[\frac{n}{2}]$	$[\frac{n}{2}]$	$n(n-1)$	$n(n-1)/2$
$(\mathfrak{so}(p_1+p_2, q_1+q_2),$ $\mathfrak{so}(p_1,q_1)+\mathfrak{so}(p_2,q_2))$	p_1+q_1 ($p_1 \le q_2$ and $q_1 \le p_2$) $p_1+p_2+(q_1-p_2)(q_2-p_1)$ ($p_1 \le q_2$ and $p_2 < q_1$)	p_1+q_1 ($p_1 \le q_2$ and $q_1 \le p_2$) $p_1+p_2+(q_1-p_2)(q_2-p_1)$ ($p_1 \le q_2$ and $p_2 < q_1$)	$(p_1+q_1)(p_2+q_2)$	$p_1p_2+q_1q_2$

TABLE 3. (continued)

$(\mathfrak{g}, \mathfrak{h})$	v-cohomogeneity	t-cohomogeneity	dimension	index
$(\mathfrak{so}^*(4n), \mathfrak{su}^*(2n) + \mathbf{R})$	n	n	$2n(2n-1)$	$n(2n-1)$
$(\mathfrak{sp}(n, \mathbf{C}), \mathfrak{sp}(n, \mathbf{R}))$	n	n	$n(2n+1)$	$n(n+1)$
$(\mathfrak{sp}(n, \mathbf{R}) + \mathfrak{sp}(n, \mathbf{R}), \mathfrak{sp}(n, \mathbf{R}))$	n	n	$n(2n+1)$	n^2
$(\mathfrak{sp}(n, \mathbf{C}), \mathfrak{sl}(n, \mathbf{C}) + \mathbf{C})$	$2n$	$2n$	$2n(n+1)$	$n(n+1)$
$(\mathfrak{sp}(n, \mathbf{C}), \mathfrak{sp}(p, n-p))$	n	$2(n-2p)^2 + n + 2p$	$n(2n+1)$	$4p(n-p)$
$(\mathfrak{sp}(p, n-p) + \mathfrak{sp}(p, n-p), \mathfrak{sp}(p, n-p))$	$2(n-2p)^2 + n + 2p$	n	$n(2n+1)$	$2p^2 + 2(n-p)^2 + n$
$(\mathfrak{sp}(n, \mathbf{C}), \mathfrak{sp}(p, \mathbf{C}) + \mathfrak{sp}(n-p, \mathbf{C}))$	$2p$	$2p$	$8p(n-p)$	$4p(n-p)$
$(\mathfrak{sp}(n, \mathbf{R}), \mathfrak{su}(p, n-p) + \mathfrak{so}(2))$	n	$(n-2p)^2 + n + p$	$n(n+1)$	$2p(n-p)$
$(\mathfrak{sp}(p, n-p), \mathfrak{su}(p, n-p) + \mathfrak{so}(2))$	$(n-2p)^2 + n + p$	n	$n(n+1)$	$p^2 + (n-p)^2 + n$
$(\mathfrak{sp}(n, \mathbf{R}), \mathfrak{sp}(p, \mathbf{R}) + \mathfrak{sp}(n-p, \mathbf{R}))$	p	p	$4p(n-p)$	$2p(n-p)$
$(\mathfrak{sp}(n, n), \mathfrak{sp}(n, \mathbf{C}))$	n	n	$4n^2$	$n(2n+1)$
$(\mathfrak{sp}(2n, \mathbf{R}), \mathfrak{sp}(n, \mathbf{C}))$	n	n	$4n^2$	$n(2n-1)$
$(\mathfrak{sp}(n, n), \mathfrak{su}^*(2n) + \mathbf{R})$	$3n$	$3n$	$2n(2n+1)$	$n(2n+1)$
$(\mathfrak{sp}(p_1 + p_2, q_1 + q_2), \mathfrak{sp}(p_1, q_1) + \mathfrak{sp}(p_2, q_2))$	$p_1 + q_1$ ($p_1 \le q_2$ and $q_1 \le p_2$) $p_1 + p_2 + 4(q_1 - p_2)(q_2 - p_1)$ ($p_1 \le q_2$ and $p_2 < q_1$)	$p_1 + q_1$ ($p_1 \le q_2$ and $q_1 \le p_2$) $p_1 + p_2 + 4(q_1 - p_2)(q_2 - p_1)$ ($p_1 \le q_2$ and $p_2 < q_1$)	$4(p_1 + q_1)(p_2 + q_2)$	$4(p_1 p_2 + q_1 q_2)$
$(\mathfrak{sp}(n, \mathbf{R}), \mathfrak{sl}(n, \mathbf{R}) + \mathbf{R})$	n	n	$n(n+1)$	$n(n+1)/2$
$(\mathfrak{sl}(n, \mathbf{C}) + \mathfrak{sl}(n, \mathbf{C}), \mathfrak{sl}(n, \mathbf{C}))$	$2(n-1)$	$2(n-1)$	$2(n-1)(n+1)$	$(n-1)(n-1)/2$
$(\mathfrak{so}(n, \mathbf{C}) + \mathfrak{so}(n, \mathbf{C}), \mathfrak{so}(n, \mathbf{C}))$	$2[\frac{n}{2}]$	$2[\frac{n}{2}]$	$n(n-1)$	$n(n-1)/2$
$(\mathfrak{sp}(n, \mathbf{C}) + \mathfrak{sp}(n, \mathbf{C}), \mathfrak{sp}(n, \mathbf{C}))$	$2n$	$2n$	$2n(2n+1)$	$n(2n+1)$

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