TOKYO J. MATH.
VOL. 25, NO. 1, 2002

On the Optimal Relaxation Parameters to the Improved SOR Method with Orderings

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(Communicated by K. Komatsu)

Abstract. To solve non-symmetric linear equations, we have proposed a generalized SOR method, named the improved SOR method with orderings, and for an $n \times n$ tridiagonal matrix, we have given n selections of the multiple relaxation parameters which satisfy $\rho(\mathcal{L}_\Phi) = 0$ and correspond to the reciprocal numbers of the pivots of Gaussian elimination, where $\mathcal{L}_\Phi$ is the $n \times n$ iterative matrix of this method.

In this paper, using the "essential dimensions-reductions for error vectors", we investigate the numbers of all conditions for the multiple relaxation parameters which satisfy $\rho(\mathcal{L}_\Phi) = 0$. As a result, adding to n known selections of the multiple relaxation parameters, we find another type of selections of the multiple relaxation parameters and we conclude that such numbers of conditions are totally 2^{n-1} cases for an $n \times n$ tridiagonal matrix. Examples of such selections of multiple relaxation parameters are also contained. For an $n \times n$ Hessenberg matrix, we also obtain the similar results.

1. Introduction.

Recently, Ehrlich [1] has proposed special selections of the local relaxation parameters in SOR-like methods for two-dimensional problems of a discrete convection-diffusion equation and Elman and Chernesky [2] studied the effect of partitioning and orderings of the unknowns on the convergence of the Gauss-Seidel iterations, for one-dimensional problems.

In previous papers [3–6], the improved SOR method with orderings has been proposed as a generalization of the SOR method for practical use, and clarified some fundamental properties of this method. In particular, in [5], it was given n selections of the multiple relaxation parameters such that the spectral radii of the corresponding improved SOR matrices for a tridiagonal matrix are 0, and also given practical algorithms for the improved SOR method with orderings which require in practice, about 18% fewer operation counts than the iterative refinement method for the special class of tridiagonal H-matrices. In [6], for tridiagonal matrices, error estimates for special n selections of the multiple relaxation parameters and two kinds of orderings were given, and for block tridiagonal matrices, the adaptive improved block SOR method with orderings were proposed.

Received February 8, 2001
This work was partially supported by Waseda University Grant for Special Research Projects 2001 A-571 and Grant-in-Aid for Encouragement of Young Scientists from the Ministry Education, Science, Sports and Culture of Japan.
Key words and phrases. dimensions-reduction, improved SOR method with orderings, multiple relaxation parameters.

In this paper, using the "essential dimensions-reductions of types I, II and III for error vectors", we investigate all conditions for multiple relaxation parameters such that the spectral radii of the corresponding improved SOR matrices for an $n \times n$ tridiagonal matrix are 0. As a result, adding to n selections of multiple relaxation parameters proposed in [5], we find another type of selections of the multiple relaxation parameters and totally we count 2^{n-1} conditions of such selections of the multiple relaxation parameters (see Theorem 2.1). Some examples of such selections for the multiple relaxation parameters are presented (see Examples 2.3–2.5). For $n \times n$ Hessenberg matrices, we also obtain the similar results (see Section 3).

2. Dimensions-reductions for error vectors.

Let us consider $A\boldsymbol{x} = \boldsymbol{b}$, where A is an $n \times n$ nonsingular tridiagonal matrix $A = [-l_i, p_i, -u_i]$, $l_{i+1}u_i \neq 0$, $i = 1, 2, \cdots, n-1$, $p_i \neq 0$, $i = 1, 2, \cdots, n$.

For a proper starting vector $\boldsymbol{x}^{(0)} = [x_i^{(0)}]$, the improved SOR method with natural ordering is expressed as follows (see [4]).

$$\begin{cases} x_i^{(m+1)} = \omega_i l_i x_{i-1}^{(m+1)} + (1-\omega_i p_i)x_i^{(m)} + \omega_i u_i x_{i+1}^{(m)} + \omega_i b_i\,, \quad i = 1, 2, \cdots, n \\ x_0^{(m+1)} = 0\,, \quad x_{n+1}^{(m)} = 0\,, \quad m = 0, 1, 2, \cdots . \end{cases}$$

Let $\hat{\boldsymbol{x}} = [\hat{x}_i]$ be the exact solution of $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{e}^{(m)} = \boldsymbol{x}^{(m)} - \hat{\boldsymbol{x}} \equiv [e_i^{(m)}]$ be the error vectors of $\boldsymbol{x}^{(m)} = [x_i^{(m)}]$, $m = 0, 1, 2, \cdots$. Then $\boldsymbol{e}^{(m+1)} = \mathcal{L}_\Phi \boldsymbol{e}^{(m)}$ where $\mathcal{L}_\Phi$ is the improved SOR matrix for A, that is,

$$\begin{cases} e_i^{(m+1)} - \omega_i l_i e_{i-1}^{(m+1)} = (1-\omega_i p_i)e_i^{(m)} + \omega_i u_i e_{i+1}^{(m)}\,, \quad i = 1, 2, \cdots, n \\ e_0^{(m+1)} = 0\,, \quad e_{n+1}^{(m)} = 0\,, \quad m = 0, 1, 2, \cdots . \end{cases}$$

The error vector $\boldsymbol{e}^{(m+1)} = [e_i^{(m+1)}]$ at the $(m+1)$-th iteration is expressed as follows.

$$\begin{cases} e_1^{(m+1)} = (1-\omega_1 p_1)e_1^{(m)} + \omega_1 u_1 e_2^{(m)} \\ e_2^{(m+1)} = \omega_2 l_2 (1-\omega_1 p_1)e_1^{(m)} + \{\omega_2 l_2 \omega_1 u_1 + (1-\omega_2 p_2)\}e_2^{(m)} + \omega_2 u_2 e_3^{(m)} \\ e_3^{(m+1)} = \omega_3 l_3 \omega_2 l_2 (1-\omega_1 p_1)e_1^{(m)} + \omega_3 l_3 \{\omega_2 l_2 \omega_1 u_1 + (1-\omega_2 p_2)\}e_2^{(m)} \\ \qquad\qquad + \{\omega_3 l_3 \omega_2 u_2 + (1-\omega_3 p_3)\}e_3^{(m)} + \omega_3 u_3 e_4^{(m)} \\ \qquad\vdots \\ e_i^{(m+1)} = \omega_i l_i \cdots \omega_2 l_2 (1-\omega_1 p_1)e_1^{(m)} + \omega_i l_i \cdots \omega_3 l_3 \{\omega_2 l_2 \omega_1 u_1 + (1-\omega_2 p_2)\}e_2^{(m)} \\ \qquad\qquad + \cdots + \{\omega_i l_i \omega_{i-1} u_{i-1} + (1-\omega_i p_i)\}e_i^{(m)} + \omega_i u_i e_{i+1}^{(m)} \\ \qquad\vdots \\ e_n^{(m+1)} = \omega_n l_n \cdots \omega_2 l_2 (1-\omega_1 p_1)e_1^{(m)} + \omega_n l_n \cdots \omega_3 l_3 \{\omega_2 l_2 \omega_1 u_1 + (1-\omega_2 p_2)\}e_2^{(m)} \\ \qquad\qquad + \cdots + \{\omega_n l_n \omega_{n-1} u_{n-1} + (1-\omega_n p_n)\}e_n^{(m)} . \end{cases}$$

Let $\alpha_i \equiv \omega_i p_i - 1$, $i = 1, 2, \cdots, n$ and $\beta_i \equiv \omega_i l_i \omega_{i-1} u_{i-1}$, $i = 2, 3, \cdots, n$, $\beta_1 = \beta_{n+1} = 0$. Also let us consider $\tilde{e}_i^{(m)}$, $i = 1, 2, \cdots, n$ such that $\tilde{e}_1^{(m)} = e_1^{(m)}$, $e_i^{(m)} = \omega_i l_i \omega_{i-1} l_{i-1} \cdots \omega_2 l_2 \tilde{e}_i^{(m)}$, $i = 2, 3, \cdots, n$.

Then we obtain

$$\left.\begin{aligned}
\tilde{e}_1^{(m+1)} &= (\beta_1 - \alpha_1)\tilde{e}_1^{(m)} + \beta_2 \tilde{e}_2^{(m)} \\
\tilde{e}_2^{(m+1)} &= (\beta_1 - \alpha_1)\tilde{e}_1^{(m)} + (\beta_2 - \alpha_2)\tilde{e}_2^{(m)} + \beta_3 \tilde{e}_3^{(m)} \\
&\vdots \\
\tilde{e}_i^{(m+1)} &= (\beta_1 - \alpha_1)\tilde{e}_1^{(m)} + (\beta_2 - \alpha_2)\tilde{e}_2^{(m)} + \cdots + (\beta_i - \alpha_i)\tilde{e}_i^{(m)} + \beta_{i+1}\tilde{e}_{i+1}^{(m)} \\
&\vdots \\
\tilde{e}_{n-1}^{(m+1)} &= (\beta_1 - \alpha_1)\tilde{e}_1^{(m)} + (\beta_2 - \alpha_2)\tilde{e}_2^{(m)} + \cdots + (\beta_{n-1} - \alpha_{n-1})\tilde{e}_{n-1}^{(m)} + \beta_n \tilde{e}_n^{(m)} \\
\tilde{e}_n^{(m+1)} &= (\beta_1 - \alpha_1)\tilde{e}_1^{(m)} + (\beta_2 - \alpha_2)\tilde{e}_2^{(m)} + \cdots \\
&\quad + (\beta_{n-1} - \alpha_{n-1})\tilde{e}_{n-1}^{(m)} + (\beta_n - \alpha_n)\tilde{e}_n^{(m)} .
\end{aligned}\right\} \quad (1)$$

Now, let us consider the other deriviations to the conditions of the multiple relaxation parameters which satisfy $\rho(\mathcal{L}_\Phi) = 0$ (cf. [5]).

LEMMA 2.1. *In* (1), *for* $n \geq 1$, *if* $\alpha_1 = \beta_1$ $(= 0)$, *then we obtain*

$$\left.\begin{aligned}
\tilde{e}_1^{(m+1)} &= \beta_2 \tilde{e}_2^{(m)} \\
\tilde{e}_2^{(m+1)} &= (\beta_2 - \alpha_2)\tilde{e}_2^{(m)} + \beta_3 \tilde{e}_3^{(m)} \\
&\vdots \\
\tilde{e}_i^{(m+1)} &= (\beta_2 - \alpha_2)\tilde{e}_2^{(m)} + \cdots + (\beta_i - \alpha_i)\tilde{e}_i^{(m)} + \beta_{i+1}\tilde{e}_{i+1}^{(m)} \\
&\vdots \\
\tilde{e}_{n-1}^{(m+1)} &= (\beta_2 - \alpha_2)\tilde{e}_2^{(m)} + \cdots + (\beta_{n-1} - \alpha_{n-1})\tilde{e}_{n-1}^{(m)} + \beta_n \tilde{e}_n^{(m)} \\
\tilde{e}_n^{(m+1)} &= (\beta_2 - \alpha_2)\tilde{e}_2^{(m)} + \cdots + (\beta_{n-1} - \alpha_{n-1})\tilde{e}_{n-1}^{(m)} + (\beta_n - \alpha_n)\tilde{e}_n^{(m)} .
\end{aligned}\right\} \quad (2)$$

LEMMA 2.2. *In* (1), *for* $n \geq 2$, *if* $\alpha_n = \beta_{n+1}$ $(= 0)$, *then we obtain*

$$\left.\begin{aligned}
\tilde{e}_1^{(m+1)} &= (\beta_1 - \alpha_1)\tilde{e}_1^{(m)} + \beta_2 \tilde{e}_2^{(m)} \\
\tilde{e}_2^{(m+1)} &= (\beta_1 - \alpha_1)\tilde{e}_1^{(m)} + (\beta_2 - \alpha_2)\tilde{e}_2^{(m)} + \beta_3 \tilde{e}_3^{(m)} \\
&\vdots \\
\tilde{e}_i^{(m+1)} &= (\beta_1 - \alpha_1)\tilde{e}_1^{(m)} + (\beta_2 - \alpha_2)\tilde{e}_2^{(m)} + \cdots + (\beta_i - \alpha_i)\tilde{e}_i^{(m)} + \beta_{i+1}\tilde{e}_{i+1}^{(m)} \\
&\vdots \\
\tilde{e}_{n-1}^{(m+1)} &= (\beta_1 - \alpha_1)\tilde{e}_1^{(m)} + (\beta_2 - \alpha_2)\tilde{e}_2^{(m)} + \cdots + \{\beta_{n-1} - (\alpha_{n-1} - \beta_n)\}\tilde{e}_{n-1}^{(m)} \\
\tilde{e}_n^{(m+1)} &= \tilde{e}_{n-1}^{(m+1)} .
\end{aligned}\right\} \quad (3)$$

Both proofs are apparent from (1).

We can see that both of (2) and (3) imply that the essential dimensions for error vectors is reduced from n to $n - 1$. In this paper, we assume that every denominator does not zero (see [5, Theorem 3.2]). Thus, using these techniques, we can reduce the essential dimensions for the error vectors from n to $n-1, n-2, \cdots, 1$, sequentially. We call these process as "essential dimensions-reductions for error vectors".

In particular, we refer that (2) is the essential dimensions-reduction of type I from $(\tilde{\boldsymbol{e}}_1^{(m)}, \tilde{\boldsymbol{e}}_2^{(m)}, \cdots, \tilde{\boldsymbol{e}}_n^{(m)})^T$ to $(\tilde{\boldsymbol{e}}_2^{(m)}, \tilde{\boldsymbol{e}}_3^{(m)}, \cdots, \tilde{\boldsymbol{e}}_n^{(m)})^T$ and (3) is the essential dimensions-reduction of type II from $(\tilde{\boldsymbol{e}}_1^{(m)}, \tilde{\boldsymbol{e}}_2^{(m)}, \cdots, \tilde{\boldsymbol{e}}_n^{(m)})^T$ to $(\tilde{\boldsymbol{e}}_1^{(m)}, \tilde{\boldsymbol{e}}_2^{(m)}, \cdots, \tilde{\boldsymbol{e}}_{n-1}^{(m)})^T$.

EXAMPLE 2.1. For $1 \le k \le n$, first, using the essential dimensions-reduction of type I, we reduce the error vectors from $(\tilde{e}_1^{(m)}, \tilde{e}_2^{(m)}, \cdots, \tilde{e}_n^{(m)})^T$ to $(\tilde{e}_k^{(m)}, \tilde{e}_{k+1}^{(m)}, \cdots, \tilde{e}_n^{(m)})^T$ and next, using the essential dimensions-reduction of type II, we reduce from $(\tilde{e}_k^{(m)}, \tilde{e}_{k+1}^{(m)}, \cdots, \tilde{e}_n^{(m)})^T$ to $\tilde{e}_k^{(m)}$. Finally, we apply the essential dimensions-reduction of type I to $\tilde{e}_k^{(m)}$ and we reach the "final essential dimensions-reduction" for the error vectors. Then we obtain n conditions for the multiple relaxation parameters which satisfy $\rho(\mathcal{L}_\Phi) = 0$:

$$\left.\begin{array}{l} \alpha_1 = \beta_1 \\ \alpha_2 = \beta_2 \\ \quad\vdots \\ \alpha_{k-1} = \beta_{k-1} \\ \alpha_n = \beta_{n+1} \\ \alpha_{n-1} = \beta_n \\ \quad\vdots \\ \alpha_{k+1} = \beta_{k+2} \\ \alpha_k = \beta_k + \beta_{k+1} \end{array}\right\} \iff \begin{cases} \omega_i = \dfrac{1}{p_i - l_i\omega_{i-1}u_{i-1}}, & i = 1, 2, \cdots, k-1 \\ \omega_i = \dfrac{1}{p_i - u_i\omega_{i+1}l_{i+1}}, & i = n, n-1, \cdots, k+1 \\ \omega_k = \dfrac{1}{p_k - l_k\omega_{k-1}u_{k-1} - u_k\omega_{k+1}l_{k+1}}, \\ \quad \text{where} \quad l_1\omega_0u_0 = u_n\omega_{n+1}l_{n+1} = 0. \end{cases}$$

Note that these n conditions considered in [5] of the multiple relaxation parameters which satisfy $\rho(\mathcal{L}_\Phi) = 0$, and correspond to the reciprocal numbers of the pivots of Gaussian elimination (see [5, Theorem 2.2 and 3.1]).

On the other hand, we offer another type of the essential dimensions-reduction for error vectors.

LEMMA 2.3. *In* (1), *for* $2 \le k \le n-1$, *if* $-\alpha_{j-1} + \beta_j \ne 0,\ j = 2, 3, \cdots, k$ *and* $\alpha_k = 0$, *then for*

$$\begin{cases} -\alpha_1\tilde{e}_1^{(m)} + \beta_2\tilde{e}_2^{(m)} = (\beta_2 - \alpha_1)\tilde{\tilde{e}}_1^{(m,k)} \\ -\alpha_2\tilde{e}_2^{(m)} + \beta_3\tilde{e}_3^{(m)} = (\beta_3 - \alpha_2)\tilde{\tilde{e}}_2^{(m,k)} \\ \quad\vdots \\ -\alpha_{k-1}\tilde{e}_{k-1}^{(m)} + \beta_k\tilde{e}_k^{(m)} = (\beta_k - \alpha_{k-1})\tilde{\tilde{e}}_{k-1}^{(m,k)}, \end{cases}$$

we obtain

$$\begin{cases} \tilde{e}_1^{(m+1)} = (\beta_2-\alpha_1)\tilde{\tilde{e}}_1^{(m,k)} \\ \tilde{e}_2^{(m+1)} = (\beta_2-\alpha_1)\tilde{\tilde{e}}_1^{(m,k)} + (\beta_3-\alpha_2)\tilde{\tilde{e}}_2^{(m,k)} \\ \quad \vdots \\ \tilde{e}_{k-1}^{(m+1)} = (\beta_2-\alpha_1)\tilde{\tilde{e}}_1^{(m,k)} + \cdots + (\beta_{k-1}-\alpha_{k-2})\tilde{\tilde{e}}_{k-2}^{(m,k)} + (\beta_k-\alpha_{k-1})\tilde{\tilde{e}}_{k-1}^{(m,k)} \\ \tilde{e}_k^{(m+1)} = (\beta_2-\alpha_1)\tilde{\tilde{e}}_1^{(m,k)} + \cdots + (\beta_{k-1}-\alpha_{k-2})\tilde{\tilde{e}}_{k-2}^{(m,k)} + (\beta_k-\alpha_{k-1})\tilde{\tilde{e}}_{k-1}^{(m,k)} + \beta_{k+1}\tilde{e}_{k+1}^{(m)} \end{cases}$$

and

$$\begin{cases} \tilde{\tilde{e}}_1^{(m+1,k)} = (\beta_2-\alpha_1)\tilde{\tilde{e}}_1^{(m,k)} + \left(\beta_3 - \dfrac{\beta_2\alpha_2-\beta_3\alpha_1}{\beta_2-\alpha_1}\right)\tilde{\tilde{e}}_2^{(m,k)}, \quad \text{if } k \geq 3 \\ \tilde{\tilde{e}}_2^{(m+1,k)} = (\beta_2-\alpha_1)\tilde{\tilde{e}}_1^{(m,k)} + (\beta_3-\alpha_2)\tilde{\tilde{e}}_2^{(m,k)} + \left(\beta_4 - \dfrac{\beta_3\alpha_3-\beta_4\alpha_2}{\beta_3-\alpha_2}\right)\tilde{\tilde{e}}_3^{(m,k)}, \\ \qquad\qquad \text{if } k \geq 4 \\ \quad \vdots \\ \tilde{\tilde{e}}_{k-1}^{(m+1,k)} = (\beta_2-\alpha_1)\tilde{\tilde{e}}_1^{(m,k)} + (\beta_3-\alpha_2)\tilde{\tilde{e}}_2^{(m,k)} + \cdots + (\beta_k-\alpha_{k-1})\tilde{\tilde{e}}_{k-1}^{(m,k)} \\ \qquad + \left(\beta_{k+1} - \dfrac{(-\alpha_{k-1})\beta_{k+1}}{\beta_k-\alpha_{k-1}}\right)\tilde{e}_{k+1}^{(m)} \\ \tilde{e}_{k+1}^{(m+1)} = (\beta_2-\alpha_1)\tilde{\tilde{e}}_1^{(m,k)} + (\beta_3-\alpha_2)\tilde{\tilde{e}}_2^{(m,k)} + \cdots + (\beta_k-\alpha_{k-1})\tilde{\tilde{e}}_{k-1}^{(m,k)} \\ \qquad + (\beta_{k+1}-\alpha_{k+1})\tilde{e}_{k+1}^{(m)} + \beta_{k+2}\tilde{e}_{k+2}^{(m)} \\ \quad \vdots \\ \tilde{e}_n^{(m+1)} = (\beta_2-\alpha_1)\tilde{\tilde{e}}_1^{(m,k)} + (\beta_3-\alpha_2)\tilde{\tilde{e}}_2^{(m,k)} + \cdots + (\beta_k-\alpha_{k-1})\tilde{\tilde{e}}_{k-1}^{(m,k)} \\ \qquad + (\beta_{k+1}-\alpha_{k+1})\tilde{e}_{k+1}^{(m)} + \cdots + (\beta_{n-1}-\alpha_{n-1})\tilde{e}_{n-1}^{(m)} + (\beta_n-\alpha_n)\tilde{e}_n^{(m)}. \end{cases} \tag{4}$$

The proof is obtained by direct computations.

So we can reduce the error vectors from $(\tilde{e}_1^{(m)}, \tilde{e}_2^{(m)}, \cdots, \tilde{e}_n^{(m)})^T$ to $(\tilde{\tilde{e}}_1^{(m,k)}, \tilde{\tilde{e}}_2^{(m,k)}, \cdots, \tilde{\tilde{e}}_{k-1}^{(m,k)}, \tilde{e}_{k+1}^{(m)}, \tilde{e}_{k+2}^{(m)}, \cdots, \tilde{e}_n^{(m)})^T$, and we refer this the essential dimensions-reduction of type III.

Because $\rho(\mathcal{L}_\Phi) = 0$ implies that one of α_k, $k = 1, 2, \cdots, n$ must be zero (see the proof of [5, Theorem 2.1]), we have no other essential dimensions-reductions for error vectors.

For the $(n-1)$ equations in (2), (3) and (4) which are obtained after the essential dimensions-reductions of type I, II and III, respectively, we find that they are the similar types to n equations of (1) before the essential dimensions-reductions, and hence we can continue those essential dimensions-reductions of types I, II, or III for error vectors until we reach the "final essential dimensions-reduction" for the error vectors.

For each essential dimensions-reduction of types I, II, or III, the conditions of the multiple relaxation parameters are determined continuously and at the final essential dimensions-reduction for error vectors, we can find $\Phi = \mathrm{diag}(\omega_1, \omega_2, \cdots, \omega_n)$ such that $\rho(\mathcal{L}_\Phi) = 0$ (see for example, Example 2.4) which implies that the improved SOR method with orderings must stop at most n iterations (cf. [6, Theorem 3.1]). If we stop the essential dimensions-reductions for error vectors just before the final, then we can obtain the convergence theorems to the "one parameter improved SOR method with orderings" (see, for example, Example 2.5 ii) and cf. [7]).

Note that after the essential dimensions-reduction of type III, we can't apply the essential dimensions-reduction of type I anymore by the reason of $(-\alpha_1) + \beta_2 \neq 0$, but we can apply the essential dimensions-reduction of type I at the final essential dimensions-reduction for error vectors (see Example 2.5 i)).

As a result of these essential dimensions-reductions for error vectors, we can easily count the total numbers of conditions for the relaxation matrix Φ which satisfies $\rho(\mathcal{L}_\Phi) = 0$. Then we obtain the following theorem:

THEOREM 2.1. *For an $n \times n$ tridiagonal matrix $A = [-l_i, p_i, -u_i]$, there are totally 2^{n-1} conditions for $\alpha_i = \omega_i p_i - 1$ and $\beta_i = \omega_i l_i \omega_{i-1} u_{i-1}$ of the relaxation matrix $\Phi = \mathrm{diag}(\omega_1, \omega_2, \cdots, \omega_n)$ which satisfies $\rho(\mathcal{L}_\Phi) = 0$, and they are obtained by the essential dimensions-reductions of types* I, II *and* III.

PROOF. By inductions, we count the total number of conditions at essential dimensions-reductions for errors, whether $\alpha_1 = 0$ is used or not. Then we obtain the results. □

On the other hand, by [5, Theorem 2.1], we have the following theorem:

THEOREM 2.2. *Let $f_n(\lambda) = det(\lambda I - \mathcal{L}_\Phi)$. Then*

$$f_n(\lambda) = \begin{vmatrix} \lambda + \alpha_1 & -\beta_2 & & & 0 \\ -\lambda & \lambda + \alpha_2 & -\beta_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -\lambda & \lambda + \alpha_{n-1} & -\beta_n \\ 0 & & & -\lambda & \lambda + \alpha_n \end{vmatrix}$$

and we have the following recursion formula:

$$\begin{cases} f_0(\lambda) = 1\,, \quad f_1(\lambda) = \lambda + \alpha_1\,, \\ f_i(\lambda) = (\lambda + \alpha_i) f_{i-1}(\lambda) - \beta \lambda f_{i-2}(\lambda)\,, \quad i = 2, 3, \cdots, n\,. \end{cases}$$

Now, let us consider the conditions of $\rho(\mathcal{L}_\Phi) = 0$, that is, $f_n(\lambda) = \lambda^n$.

EXAMPLE 2.2. i) We first treat the case of $n = 3$. By Theorem 2.1, there are $2^{3-1} = 4$ conditions for the relaxation matrix Φ of $\rho(\mathcal{L}_\Phi) = 0$. On the other hand, by [5, Theorem

2.2], we have

$$f_3(\lambda) = \begin{vmatrix} \lambda+\alpha_1 & -\beta_2 & 0 \\ -\lambda & \lambda+\alpha_2 & -\beta_3 \\ 0 & -\lambda & \lambda+\alpha_3 \end{vmatrix}$$
$$= (\lambda+\alpha_1)(\lambda+\alpha_2)(\lambda+\alpha_3) - \{\beta_2(\lambda+\alpha_3)+\beta_3(\lambda+\alpha_1)\}\lambda$$
$$= \lambda^3 + \{(\alpha_1+\alpha_2+\alpha_3)-(\beta_2+\beta_3)\}\lambda^2$$
$$+ \{(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)-(\beta_2\alpha_3+\beta_3\alpha_1)\}\lambda + \alpha_1\alpha_2\alpha_3\,.$$

Thus, $f_3(\lambda) = \lambda^3$ if, and only if,

$$\begin{cases} \alpha_1+\alpha_2+\alpha_3 = \beta_2+\beta_3 \\ \alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1 = \beta_2\alpha_3+\beta_3\alpha_1 \\ \alpha_1\alpha_2\alpha_3 = 0\,. \end{cases}$$

Then for $\beta_1 = \beta_4 = 0$, we have the following four cases:

A$_1$) $\alpha_1 = \beta_1+\beta_2, \alpha_2 = \beta_3$ and $\alpha_3 = \beta_4$.

A$_2$) $\alpha_1 = \beta_1, \alpha_2 = \beta_2+\beta_3$ and $\alpha_3 = \beta_4$.

A$_3$) $\alpha_1 = \beta_1, \alpha_2 = \beta_2$ and $\alpha_3 = \beta_3+\beta_4$.

B$_2$) $\alpha_2 = \beta_1, \alpha_1+\alpha_3 = \beta_2+\beta_3$ and $\alpha_1\alpha_3 = \beta_3\alpha_1+\beta_2\alpha_3$.

These case A$_1$)–A$_3$) come from the essential dimensions-reductions of type I and II, and are treated in Example 2.1 and correspond to the reciprocal numbers of the pivots of Gaussian elimination (see [5]). On the other hand, if we use the essential dimensions-reductions of type III, then we obtain another case B$_2$) (see Lemma 2.4).

In case B$_2$), if $\alpha_1, \alpha_3 \neq 0$ and $\alpha_1 \neq \alpha_3$, then $\beta_2 = \frac{\alpha_1^2}{\alpha_1-\alpha_3}$ and $\beta_3 = \frac{-\alpha_3^2}{\alpha_1-\alpha_3}$. Hence $\frac{\beta_2}{\beta_3} = -\left(\frac{\alpha_1}{\alpha_3}\right)^2$, that is, $\frac{\omega_1 l_2 u_1}{\omega_3 l_3 u_2} = -\left(\frac{\omega_1 p_1 - 1}{\omega_3 p_3 - 1}\right)^2$. For the real matrix A, if $l_3u_2l_2u_1 > 0$ and ω_1 and ω_3 are real, then the signs of ω_1 and ω_3 are different. That is, one of ω_1 and ω_3 must be negative.

ii) Next, we take up the case of $n = 4$.

By Theorem 2.1, there are $2^{4-1} = 8$ conditions for the relaxation matrix Φ of $\rho(\mathcal{L}_\Phi) = 0$. On the other hand, by Theorem 2.2 and $f_2(\lambda) = \lambda^2 - (\alpha_1+\alpha_2-\beta_2)\lambda + \alpha_1\alpha_2$, we have

$$f_4(\lambda) = (\lambda+\alpha_4)f_3(\lambda) - \beta_4\lambda f_2(\lambda)$$
$$= \lambda^4 + \{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-(\beta_2+\beta_3+\beta_4)\}\lambda^3 + \{\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1$$
$$+ (\alpha_1+\alpha_2+\alpha_3)\alpha_4 - (\beta_2\alpha_3+\beta_3\alpha_1) - (\beta_2+\beta_3)\alpha_4 - \beta_4(\alpha_1+\alpha_2-\beta_2)\}\lambda^2$$
$$+ [\alpha_1\alpha_2\alpha_3 + \{(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)-(\beta_2\alpha_3+\beta_3\alpha_1)\}\alpha_4 - \alpha_1\alpha_2\beta_4]\lambda + \alpha_1\alpha_2\alpha_3\alpha_4\,.$$

Thus $f_4(\lambda) = \lambda^4$ if, and only if,

$$\begin{cases} \alpha_1+\alpha_2+\alpha_3+\alpha_4 = \beta_2+\beta_3+\beta_4 \\ \alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1+(\alpha_1+\alpha_2+\alpha_3)\alpha_4 \\ \quad = \beta_2\alpha_3+\beta_3\alpha_1+(\beta_2+\beta_3)\alpha_4+\beta_4(\alpha_1+\alpha_2-\beta_2) \\ \alpha_1\alpha_2\alpha_3+\{(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)-(\beta_2\alpha_3+\beta_3\alpha_1)\}\alpha_4-\alpha_1\alpha_2\beta_4 = 0 \\ \alpha_1\alpha_2\alpha_3\alpha_4 = 0\,. \end{cases}$$

Then for $\beta_1 = \beta_5 = 0$, we have the following eight cases:
A$_1$) $\alpha_1 = \beta_1 + \beta_2, \alpha_2 = \beta_3, \alpha_3 = \beta_4$ and $\alpha_4 = \beta_5$.
A$_2$) $\alpha_1 = \beta_1, \alpha_2 = \beta_2 + \beta_3, \alpha_3 = \beta_4$ and $\alpha_4 = \beta_5$.
A$_3$) $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \alpha_3 = \beta_3 + \beta_4$ and $\alpha_4 = \beta_5$.
A$_4$) $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \alpha_3 = \beta_3$ and $\alpha_4 = \beta_4 + \beta_5$.
B$_2$)

$$\begin{cases} \alpha_2 = 0\,, \quad \alpha_4 = \beta_5\,, \\ (\alpha_1 - \beta_1) + (\alpha_3 - \beta_4) = \beta_2 + \beta_3 \quad \text{and} \\ (\alpha_1 - \beta_1)(\alpha_3 - \beta_4) = \beta_2(\alpha_3 - \beta_4) + \beta_3(\alpha_1 - \beta_1)\,. \end{cases}$$

B$_3$)

$$\begin{cases} \alpha_1 = \beta_1\,, \quad \alpha_3 = 0\,, \\ (\alpha_2 - \beta_2) + (\alpha_4 - \beta_5) = \beta_3 + \beta_4 \quad \text{and} \\ (\alpha_1 - \beta_2)(\alpha_4 - \beta_5) = \beta_3(\alpha_4 - \beta_5) + \beta_4(\alpha_2 - \beta_2)\,. \end{cases}$$

C$_2$)

$$\begin{cases} \alpha_2 = 0\,, \quad \alpha_4 \neq \beta_5\,, \\ (\alpha_1 - \beta_2) + \alpha_4 = (\beta_3 - \alpha_3) + \beta_4\,, \\ (\alpha_1 - \beta_2)\alpha_4 = \beta_4(\alpha_1 - \beta_2) + (\beta_3 - \alpha_3)\alpha_4 \quad \text{and} \\ \alpha_1\alpha_3 = \beta_3\alpha_1 + \beta_2\alpha_3\,. \end{cases}$$

C$_3$)

$$\begin{cases} \alpha_1 \neq \beta_1\,, \quad \alpha_3 = 0\,, \\ \alpha_1 + (\alpha_4 - \beta_4) = \beta_2 + (\beta_3 - \alpha_2)\,, \\ (\alpha_4 - \beta_4)\alpha_1 = \beta_2(\alpha_4 - \beta_4) + (\beta_3 - \alpha_2)\alpha_1 \quad \text{and} \\ \alpha_2\alpha_4 = \beta_4\alpha_2 + \beta_3\alpha_4\,. \end{cases}$$

The cases A$_1$)–A$_4$) come from the essential dimensions-reductions of type I and II, and treated in Example 2.1 and correspond to the reciprocal numbers of the pivots of Gaussian elimination (see [5]). On the other hand, if we use the essential dimensions-reductions of type III, then we obtain the other four cases B$_2$), B$_3$), C$_2$) and C$_3$) (cf. Example 2.5 i)).

Note that if the essential dimensions-reductions of type III is used, then the corresponding conditions for Φ are a little complicate, and the essential dimensions-reduction of type III contains the cases that at least one of the multiple relaxation parameters ω_i becomes a negative or complex number.

The following lemma implies these facts.

LEMMA 2.4. *Let $A = [-l_i, 1, -u_i]$ be a 3×3 tridiagonal matrix. If $l_2u_1 = l_3u_2 = 1$ or $l_2u_1 + l_3u_2 = 1$, then ω_1 and ω_3 do not exist such that $\omega_2 = 1$ and $\rho(\mathcal{L}_\Phi) = 0$. Assume that l_i and u_i are real numbers such that $(l_2u_1 - 1)^2 + (l_3u_2 - 1)^2 > 0$ and $l_2u_1 + l_3u_2 \neq 1$. Then there are ω_1 and ω_2 such that $\omega_2 = 1$ and $\rho(\mathcal{L}_\Phi) = 0$, that is,*

$$\begin{cases} \omega_1 = \dfrac{1 \pm \sqrt{l_2u_1l_3u_2/(l_2u_1+l_3u_2-1)}}{1-l_2u_1}\,, \\ \omega_3 = \dfrac{1 \mp \sqrt{l_2u_1l_3u_2/(l_2u_1+l_3u_2-1)}}{1-l_3u_2}\,, \quad (\textit{double sign in order}) \ \textit{and} \\ \omega_1\omega_3 = \dfrac{1}{1-l_2u_1-l_3u_2}\,. \end{cases}$$

Hence

i) ω_1 *and* ω_3 *are real for* $l_2u_1l_3u_2(l_2u_1+l_3u_2-1) > 0$, *and* $\omega_1\omega_3 < 0$ *for* $l_2u_1 + l_3u_2 > 1$ *and* $\omega_1\omega_3 > 0$ *for* $l_2u_1 + l_3u_2 < 1$.

ii) ω_1 *and* ω_3 *are both imaginary numbers for* $l_2u_1l_3u_2(l_2u_1+l_3u_2-1) < 0$.

iii) $\omega_1 = 1$ *and* $\omega_3 = \frac{1}{1-l_3u_2}$ *for* $l_2u_1 = 0$.

iv) $\omega_1 = \frac{1}{1-l_2u_1}$ *and* $\omega_3 = 1$ *for* $l_3u_2 = 0$.

PROOF. In case of $\omega_2 = 1$, the selection of ω_i is case B in Example 2.2. Then

$$\alpha_1 + \alpha_3 = \beta_2 + \beta_3\,, \quad \alpha_1\alpha_3 = \beta_3\alpha_1 + \beta_2\alpha_3\,.$$

From the definition of β_2, β_3,

$$(1-l_2u_1)\omega_1 + (1-l_3u_2)\omega_3 = 2 \quad \text{and} \quad (1-l_2u_1-l_3u_2)\omega_1\omega_3 = 1\,.$$

Hence, $(l_2u_1-1)^2 + (l_3u_2-1)^2 \neq 0$ and $l_2u_1 + l_3u_2 \neq 1$ and we have

$$\begin{cases} \omega_1 = \dfrac{1 \pm \sqrt{l_2u_1l_3u_2/(l_2u_1+l_3u_2-1)}}{1-l_2u_1} \\ \omega_3 = \dfrac{1 \mp \sqrt{l_2u_1l_3u_2/(l_2u_1+l_3u_2-1)}}{1-l_3u_2}\,, \quad \text{(double sign in order)}\,. \end{cases}$$

Thus we obtain the conclusion. □

For the constant coefficient case, we have the following corollary:

COROLLARY 2.1. *Let* $A = [-l, 1, -u]$ *be a* 3×3 *tridiagonal matrix. If* $lu = 1$ *or* $lu = 1/2$, *then* ω_1 *and* ω_3 *do not exist under the conditions that* $\omega_2 = 1$ *and* $\rho(\mathcal{L}_\Phi) = 0$.

If $lu \neq 1, 1/2$, *then there are* ω_1 *and* ω_3 *such that* $\omega_2 = 1$ *and* $\rho(\mathcal{L}_\Phi) = 0$, *and*

$$\begin{cases} \omega_1 = \dfrac{1 \pm |lu|/\sqrt{2lu-1}}{1-lu} \\ \omega_3 = \dfrac{1 \mp |lu|/\sqrt{2lu-1}}{1-lu}\,, \quad (\textit{double sign in order})\,. \end{cases}$$

Moreover, if $lu > 1/2$, *then* ω_1 *and* ω_3 *are real numbers and* $\omega_1\omega_2 = \frac{1}{1-2lu} < 0$.

If $lu = 0$, *then* $\omega_1 = \omega_3 = 1$. *If* $lu < 1/2$ *and* $lu \neq 0$, *then* ω_1 *and* ω_3 *are both imaginary numbers.*

EXAMPLE 2.3. For a 3×3 tridiagonal matrix $A = [-3/4, 1, -5/4]$, we have $1 > lu = 15/16 > 1/2$, and A is not an H-matrix but $\rho(\mathcal{L}_\Phi) = 0$ for $\Phi = \mathrm{diag}(\omega_1, \omega_2, \omega_3)$, where $\omega_1 = 4(4 - 15/\sqrt{14}) < 0$, $\omega_2 = 1$ and $\omega_3 = 4(4 + 5/\sqrt{14}) > 2$.

This coincides with the conclusion of the above corollary.

We give other examples of the conditions of $\rho(\mathcal{L}_\Phi) = 0$.

EXAMPLE 2.4. For an $n \times n$ tridiagonal matrix $A = [-l_i, p_i, -u_i]$ and $2 \le k \le n-1$, $n \ge 3$, by the essential dimensions-reductions of types I and II, we see that $\omega_1, \omega_2, \cdots, \omega_{k-2}, \omega_{k+2}, \omega_{k+3}, \cdots, \omega_n$ are determined by

$$\begin{cases} \omega_1 = \dfrac{1}{p_1}, \quad \omega_i = \dfrac{1}{p_i - l_i\omega_{i-1}u_{i-1}}, \quad i = 2, 3, \cdots, k-2, \quad \text{if } k \ge 3, \\ \omega_n = \dfrac{1}{p_n}, \quad \omega_i = \dfrac{1}{p_i - u_i\omega_{i+1}l_{i+1}}, \quad i = n-1, n-2, \cdots, k+2, \quad \text{if } k \le n-2, \end{cases}$$

and the reduced error vectors $(\tilde{e}_{k-1}^{(m)}, \tilde{e}_k^{(m)}, \tilde{e}_{k+1}^{(m)})^T$ satisfy the following form:

$$\begin{bmatrix} \tilde{e}_{k-1}^{(m+1)} \\ \tilde{e}_k^{(m+1)} \\ \tilde{e}_{k+1}^{(m+1)} \end{bmatrix} = \begin{bmatrix} \beta_{k-1} - \alpha_{k-1} & \beta_k & 0 \\ \beta_{k-1} - \alpha_{k-1} & \beta_k - \alpha_k & \beta_{k+1} \\ \beta_{k-1} - \alpha_{k-1} & \beta_k - \alpha_k & \beta_{k+1} - (\alpha_{k+1} - \beta_{k+2}) \end{bmatrix} \begin{bmatrix} \tilde{e}_{k-1}^{(m)} \\ \tilde{e}_k^{(m)} \\ \tilde{e}_{k+1}^{(m)} \end{bmatrix}. \tag{5}$$

Moreover, for (5),

i) If we use only the essential dimensions-reductions of type I or II, then we obtain, for example,

$$\omega_{k-1} = \frac{1}{p_{k-1} - l_{k-1}\omega_{k-2}u_{k-2}}, \quad \omega_{k+1} = \frac{1}{p_{k+1} - l_{k+1}\omega_{k+2}u_{k+2}},$$
$$\omega_k = \frac{1}{p_k - l_k\omega_{k-1}u_{k-1} - u_k\omega_{k+1}u_{k+1}}.$$

ii) If we use the essential dimensions-reduction of type III, then under the conditions that $\alpha_k = 0$ (that is, $\omega_k p_k = 1$) and $\rho(\mathcal{L}_\Phi) = 0$, ω_{k-1} and ω_{k+1} are determined by

$$\begin{cases} (\alpha_{k-1} - \beta_{k-1}) + (\alpha_{k+1} - \beta_{k+2}) = \beta_k + \beta_{k+1} \\ (\alpha_{k-1} - \beta_{k-1})(\alpha_{k+1} - \beta_{k+2}) = \beta_k(\alpha_{k+1} - \beta_{k+2}) + \beta_{k+1}(\alpha_{k-1} - \beta_{k-1}). \end{cases} \tag{6}$$

EXAMPLE 2.5. i) We reduce the error vectors from $(\tilde{e}_1^{(m)}, \tilde{e}_2^{(m)}, \cdots, \tilde{e}_n^{(m)})^T$ to $(\tilde{\tilde{e}}_1^{(m,2)}, \tilde{e}_3^{(m)}, \cdots, \tilde{e}_n^{(m)})^T$ and next, from $(\tilde{\tilde{e}}_1^{(m,2)}, \tilde{e}_3^{(m)}, \tilde{e}_4^{(m)}, \cdots, \tilde{e}_n^{(m)})^T$ to $(\tilde{\tilde{e}}_1^{(m,3)}, \tilde{e}_4^{(m)}, \tilde{e}_5^{(m)}, \cdots, \tilde{e}_n^{(m)})^T$ and so on. From $(\tilde{\tilde{e}}_1^{(m,n-1)}, \tilde{e}_n^{(m)})^T$ to $\tilde{\tilde{e}}_1^{(m,n)}$ and finally, we apply the essential dimensions-reduction of type I to $\tilde{\tilde{e}}_1^{(m,n)}$. Then we obtain one of the conditions for the multiple relaxation parameters which satisfy $\rho(\mathcal{L}_\Phi) = 0$ (cf. [5]):

$$\left.\begin{aligned}
&\alpha_2 = 0\\
&\alpha_3 = \frac{(\beta_1-\alpha_1)\beta_3}{(\beta_1-\alpha_1)+(\beta_2-\alpha_2)}\\
&\alpha_4 = \frac{\{(\beta_1-\alpha_1)+(\beta_2-\alpha_2)\}\beta_4}{(\beta_1-\alpha_1)+(\beta_2-\alpha_2)+(\beta_3-\alpha_3)}\\
&\ \vdots\\
&\alpha_n = \frac{\{(\beta_1-\alpha_1)+(\beta_2-\alpha_2)+\cdots+(\beta_{n-2}-\alpha_{n-2})\}\beta_n}{(\beta_1-\alpha_1)+(\beta_2-\alpha_2)+\cdots+(\beta_{n-1}-\alpha_{n-1})}\\
&(\beta_1-\alpha_1)+(\beta_2-\alpha_2)+\cdots+(\beta_n-\alpha_n) = 0\,.
\end{aligned}\right\} \qquad (7)$$

ii) If (7) is satisfied and

$$|(\beta_1-\alpha_1)+(\beta_2-\alpha_2)+\cdots+(\beta_n-\alpha_n)| < 1\,,$$

then $\rho(\mathcal{L}_\Phi) < 1$ (cf. [7]).

REMARK 2.1. The condition ii) in Example 2.5 gives another convergence condition for the one parameter improved SOR method with orderings which is different from the results in [7].

These facts imply the difference between the convergent range of the multiple relaxation parameters ω_i, $i = 1, 2, \cdots, n$ for the improved SOR method with orderings and that of the relaxation parameter $0 < \omega < 2$ for the SOR method.

3. Applications to Hessenberg matrices.

Now, let us consider $A\boldsymbol{x} = \boldsymbol{b}$, where $A = [a_{i,j}]$ is an $n \times n$ nonsingular Hessenberg matrix such that

$$\begin{aligned}
&a_{i,j} = 0\,, \quad j = 1, 2, \cdots, i-2\,, \quad a_{i,i-1} = -l_i\,, \quad a_{i,i} = p_i\,,\\
&a_{i,j} = -u_{i,j}\,, \quad j = i+1, i+2, \cdots, n\,, \quad i = 1, 2, \cdots, n
\end{aligned}$$

For a proper starting vector $\boldsymbol{x}^{(0)} = [x_i^{(0)}]$, the improved SOR method with natural ordering is expressed as follows:

$$\begin{cases}
x_i^{(m+1)} = \omega_i l_i x_{i-1}^{(m+1)} + (1-\omega_i p_i)x_i^{(m)} + \omega_i u_{i,i+1}x_{i+1}^{(m)} + \cdots + \omega_i u_{i,n}x_n^{(m)} + \omega_i b_i\,,\\
i = 1, 2, \cdots, n\,, \quad x_0^{(m+1)} = 0\,, \quad x_{n+1}^{(m)} = 0\,, \quad m = 0, 1, 2, \cdots.
\end{cases}$$

For the exact solution $\hat{\boldsymbol{x}} = [\hat{x}_i]$ of $A\boldsymbol{x} = \boldsymbol{b}$, let $\boldsymbol{e}^{(m)} = \boldsymbol{x}^{(m)} - \hat{\boldsymbol{x}} \equiv [e_i^{(m)}]$ be the error vectors of $\boldsymbol{x}^{(m)} = [x_i^{(m)}]$, $m = 0, 1, 2, \cdots$. Then $\boldsymbol{e}^{(m+1)} = \mathcal{L}_\Phi \boldsymbol{e}^{(m)}$, where $\mathcal{L}_\Phi$ is the improved SOR matrix for A, that is,

$$\begin{cases} e_i^{(m+1)} - \omega_i l_i e_{i-1}^{(m+1)} = (1-\omega_i p_i) e_i^{(m)} + \omega_i u_{i,i+1} e_{i+1}^{(m)} + \omega_i u_{i,i+2} e_{i+2}^{(m)} + \cdots + \omega_i u_{i,n} e_n^{(m)}, \\ i = 1, 2, \cdots, n, \quad e_0^{(m+1)} = 0, \quad e_{n+1}^{(m)} = 0, \quad m = 0, 1, 2, \cdots. \end{cases}$$

The error vector $\boldsymbol{e}^{(m+1)} = [e_i^{(m+1)}]$ at the $(m+1)$-th iteration is expressed as follows:

$$\begin{cases} e_1^{(m+1)} = (1-\omega_1 p_1) e_1^{(m)} + \omega_1 u_{1,2} e_2^{(m)} + \omega_1 u_{1,3} e_3^{(m)} + \cdots + \omega_1 u_{1,n} e_n^{(m)} \\ e_2^{(m+1)} = \omega_2 l_2 (1-\omega_1 p_1) e_1^{(m)} + \{\omega_2 l_2 \omega_1 u_{1,2} + (1-\omega_2 p_2)\} e_2^{(m)} \\ \qquad + (\omega_2 l_2 \omega_1 u_{1,3} + \omega_2 u_{2,3}) e_3^{(m)} + \cdots + (\omega_2 l_2 \omega_1 u_{1,n} + \omega_2 u_{2,n}) e_n^{(m)} \\ e_3^{(m+1)} = \omega_3 l_3 \omega_2 l_2 (1-\omega_1 p_1) e_1^{(m)} + \omega_3 l_3 \{\omega_2 l_2 \omega_1 u_{1,2} + (1-\omega_2 p_2)\} e_2^{(m)} \\ \qquad + \{\omega_3 l_3 (\omega_2 l_2 \omega_1 u_{1,3} + \omega_2 u_{2,3}) + (1-\omega_3 p_3)\} e_3^{(m)} \\ \qquad + \{\omega_3 l_3 (\omega_2 l_2 \omega_1 u_{1,4} + \omega_2 u_{2,4}) + \omega_3 u_{3,4}\} e_4^{(m)} + \cdots \\ \qquad + \{\omega_3 l_3 (\omega_2 l_2 \omega_1 u_{1,n} + \omega_2 u_{2,n}) + \omega_3 u_{3,n}\} e_n^{(m)} \\ \quad \vdots \\ e_n^{(m+1)} = \omega_n l_n \omega_{n-1} l_{n-1} \cdots \omega_2 l_2 (1-\omega_1 p_1) e_1^{(m)} \\ \qquad + \omega_n l_n \omega_{n-1} l_{n-1} \cdots \omega_3 l_3 \{\omega_2 l_2 \omega_1 u_{1,2} + (1-\omega_2 p_2)\} e_2^{(m)} \\ \qquad + \omega_n l_n \omega_{n-1} l_{n-1} \cdots \omega_4 l_4 \{\omega_3 l_3 (\omega_2 l_2 \omega_1 u_{1,3} + \omega_2 u_{2,3}) + (1-\omega_3 p_3)\} e_3^{(m)} \\ \qquad + \cdots + \omega_n l_n \{(\omega_{n-1} l_{n-1} \omega_{n-2} u_{n-2,n} + \omega_{n-1} u_{n-1,n}) + (1-\omega_n p_n)\} e_n^{(m)}. \end{cases}$$

Let us consider $\tilde{e}_i^{(m)}$, $i = 1, 2, \cdots, n$ such that $\tilde{e}_1^{(m)} = e_1^{(m)}$, $\tilde{e}_i^{(m)} = (\omega_i l_i \omega_{i-1} l_{i-1} \cdots \omega_2 l_2)^{-1} e_i^{(m)}$, $i = 2, 3, \cdots, n$, and set $\alpha_i = \omega_i p_i - 1$, $i = 1, 2, \cdots, n$, $\beta_{1,j} = \omega_j l_j \omega_{j-1} u_{j-1} \cdots \omega_2 l_2 \omega_1 u_{1,j}$, $j = 2, 3, \cdots, n$, $\beta_{i,j} = \beta_{i-1,j} + \omega_j l_j \omega_{j-1} l_{j-1} \cdots \omega_{i+1} l_{i+1} \omega_i u_{i,j}$, $j = i, i+1, \cdots, n$, $i = 2, 3, \cdots, n-1$, $\beta_1 = \beta_{n+1} = 0$, and $\beta_{i+1} = \beta_{i,i+1} = \omega_{i+1} l_{i+1} \omega_i u_{i,i+1}$, $i = 1, 2, \cdots, n-1$.

Then we obtain

$$\left. \begin{array}{l} \tilde{e}_1^{(m+1)} = (\beta_1 - \alpha_1) \tilde{e}_1^{(m)} + \beta_2 \tilde{e}_2^{(m)} + \beta_{1,3} \tilde{e}_3^{(m)} + \beta_{1,4} \tilde{e}_4^{(m)} + \cdots + \beta_{1,n} \tilde{e}_n^{(m)} \\ \tilde{e}_2^{(m+1)} = (\beta_1 - \alpha_1) \tilde{e}_1^{(m)} + (\beta_2 - \alpha_2) \tilde{e}_2^{(m)} + \beta_3 \tilde{e}_3^{(m)} + \beta_{2,4} \tilde{e}_4^{(m)} + \cdots + \beta_{2,n} \tilde{e}_n^{(m)} \\ \quad \vdots \\ \tilde{e}_i^{(m+1)} = (\beta_1 - \alpha_1) \tilde{e}_1^{(m)} + (\beta_2 - \alpha_2) \tilde{e}_2^{(m)} + \cdots + (\beta_{n-1} - \alpha_{n-1}) \tilde{e}_{n-1}^{(m)} + \beta_n \tilde{e}_n^{(m)} \\ \tilde{e}_n^{(m+1)} = (\beta_1 - \alpha_1) \tilde{e}_1^{(m)} + (\beta_2 - \alpha_2) \tilde{e}_2^{(m)} + \cdots \\ \qquad + (\beta_{n-1} - \alpha_{n-1}) \tilde{e}_{n-1}^{(m)} + (\beta_n - \alpha_n) \tilde{e}_n^{(m)}. \end{array} \right\} \tag{8}$$

If we put $\tilde{\boldsymbol{e}}^{(m)} = (\tilde{e}_1^{(m)}, \tilde{e}_2^{(m)}, \cdots, \tilde{e}_n^{(m)})^T$ and

$$\begin{bmatrix} \beta_1 - \alpha_1 & \beta_2 & \beta_{1,3} & \beta_{1,4} & \cdots & \cdots & \beta_{1,n} \\ \beta_1 - \alpha_1 & \beta_2 - \alpha_2 & \beta_3 & \beta_{2,4} & \cdots & \cdots & \beta_{2,n} \\ \vdots & & & & \ddots & & \\ \beta_1 - \alpha_1 & \beta_2 - \alpha_2 & \beta_3 - \alpha_3 & \beta_4 - \alpha_4 & \cdots & \beta_{n-1} - \alpha_{n-1} & \beta_n \\ \beta_1 - \alpha_1 & \beta_2 - \alpha_2 & \beta_3 - \alpha_3 & \beta_4 - \alpha_4 & \cdots & \beta_{n-1} - \alpha_{n-1} & \beta_n - \alpha_n \end{bmatrix}, \tag{9}$$

then (8) may be written in the form $\tilde{e}^{(m+1)} = M_n \tilde{e}^{(m)}$. Hence we have the following theorem (cf. [4, Theorem 2.1]):

THEOREM 3.1. *Let* $f_n(\lambda) = \det(\lambda I - \mathcal{L}_\Phi)$. *Then*

$$f_n(\lambda) = \det(\lambda I - M_n)$$

$$= \begin{vmatrix} \lambda+\alpha_1 & -\beta_2 & -\beta_{1,3} & -\beta_{1,4} & \cdots & -\beta_{1,n} \\ -\lambda & \lambda+\alpha_2 & \beta_{1,3}-\beta_3 & \beta_{1,4}-\beta_{2,4} & \cdots & \beta_{1,n}-\beta_{2,n} \\ & -\lambda & \lambda+\alpha_3 & \beta_{2,4}-\beta_4 & \cdots & \beta_{2,n}-\beta_{3,n} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & -\lambda & \lambda+\alpha_{n-1} & \beta_{n-2,n}-\beta_n \\ & & & & -\lambda & \lambda+\alpha_n \end{vmatrix}$$

Moreover, we have the following recursion formula:

$$\begin{cases} f_0(\lambda) \equiv 1\,, \quad f_1(\lambda) = \lambda + \alpha_1\,, \\ f_i(\lambda) = (\lambda+\alpha_i) f_{i-1}(\lambda) - \beta_i \lambda f_{i-2}(\lambda) - \beta_{i-2,i}\lambda^2 f_{i-3}(\lambda) - \cdots \\ \qquad\qquad -\beta_{1,i}\lambda^{i-1} f_0(\lambda)\,, \quad i = 2, 3, \cdots, n\,. \end{cases} \tag{10}$$

The formula is verified by induction.

Then we have the n conditions of $\rho(\mathcal{L}_\Phi) = 0$, that is, $f_n(\lambda) = \lambda^n$ (cf. [5, Theorem 2.2]).

THEOREM 3.2. *If for* $1 \le k \le n$,

$$\begin{cases} \alpha_i = \beta_i\,, \quad i = 1, 2, \cdots, k-1 \\ \alpha_n = \beta_{n+1}\,, \quad \alpha_{n-1} = \beta_n - \beta_{n-2,n} \\ \alpha_i = (\beta_{i+1} - \beta_{i-1,i+1}) + (\beta_{i,i+2} - \beta_{i-1,i+2}) + \cdots + (\beta_{i,n} - \beta_{i-1,n})\,, \\ \qquad\qquad i = n-2\,,\ n-3, \cdots, k+1 \\ \alpha_k = \beta_k + (\beta_{k+1} - \beta_{k-1,k+1}) + (\beta_{k,k+2} - \beta_{k-1,k+2}) + \cdots + (\beta_{k,n} - \beta_{k-1,n})\,, \end{cases}$$

then the spectral radius of $\mathcal{L}_\Phi$ *equals to zero.*

The proof can be obtained from Theorem 3.1 by induction and direct computation. For other technique to have the above conditions and the recursion formula of ω_i, $i = 1, 2, \cdots, n$, see [8].

On the other hand, we can directly extend the essential dimensions-reductions of type I, II and III in Lemma 2.1–2.3, to the equation (8), and obtain the results which are similar to Lemma 2.1–2.3 and Theorem 2.1.

For example, we have the following theorem:

THEOREM 3.3. *For an* $n \times n$ *Hessenberg matrix, there are totally* 2^{n-1} *conditions for the relaxation matrix* $\Phi = \mathrm{diag}(\omega_1, \omega_2, \cdots, \omega_n)$ *which satisfies* $\rho(\mathcal{L}_\Phi) = 0$, *and they are obtained by the essential dimensions-reductions of the similar types* I, II *or* III.

Note that for an $n \times n$ nonsingular matrix, there are at most 2^{n-1} conditions for the relaxation matrix $\Phi = \mathrm{diag}(\omega_1, \omega_2, \cdots, \omega_n)$ which satisfies $\rho(\mathcal{L}_\Phi) = 0$. We will elsewhere perform a detailed study of this fact.

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