

The Bound of the Difference between Parameter Ideals and their Tight Closures

Shiro GOTO and Yukio NAKAMURA

Meiji University

1. Introduction.

Let R be a Noetherian local ring with the maximal ideal \mathfrak{m} and assume that R possesses positive characteristic $p > 0$. For each \mathfrak{m} -primary ideal I in R let $e_I(R)$ and I^* denote the multiplicity of R with respect to I and the tight closure of I , respectively. Here let us briefly recall the definition of tight closures. For an ideal \mathfrak{a} in R let $\mathfrak{a}^{[q]} = (a^q \mid a \in \mathfrak{a})R$ where $q = p^e$ with $e \geq 0$. Let $R^0 = R \setminus \bigcup_{\mathfrak{p} \in \text{Min } R} \mathfrak{p}$. Let \mathfrak{a}^* denote the set of elements $x \in R$ for which there exists $c \in R^0$ such that $cx^q \in \mathfrak{a}^{[q]}$ for all $q \gg 0$. Then the set \mathfrak{a}^* forms an ideal in R containing \mathfrak{a} , which we call the tight closure of \mathfrak{a} . The ideal \mathfrak{a} is said to be tightly closed if $\mathfrak{a}^* = \mathfrak{a}$. A local ring R is called F -rational if every parameter ideal in R is tightly closed.

In this paper we investigate the behavior of $\sup_{\mathfrak{q}} \ell_R(\mathfrak{q}^*/\mathfrak{q})$ and $\sup_{\mathfrak{q}} \{e_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}^*)\}$, where \mathfrak{q} moves all parameter ideals in R . These two values are very closely related. In fact, they agree if R is Cohen-Macaulay. By definition of F -rationality, it immediately follows that $\sup_{\mathfrak{q}} \ell_R(\mathfrak{q}^*/\mathfrak{q}) = 0$ if and only if R is F -rational. The second author studied $\sup_{\mathfrak{q}} \ell_R(\mathfrak{q}^*/\mathfrak{q})$ in [N2] and he gave some necessary conditions for $\sup_{\mathfrak{q}} \ell_R(\mathfrak{q}^*/\mathfrak{q})$ to be finite. On the other hand, Watanabe and Yoshida posed in [WY] the conjecture that the difference $e_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}^*)$ is not negative for every parameter ideal \mathfrak{q} in R if R is unmixed. Moreover, they conjectured that $e_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}^*) = 0$ for some parameter ideal \mathfrak{q} , then R is Cohen-Macaulay and F -rational. This problem is affirmatively solved in [GN] under the condition that R is a homomorphic image of a Cohen-Macaulay ring of characteristic $p > 0$ and $\text{Ass } R = \text{Assh } R$. We further investigate the finiteness of the supremum of the difference $e_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}^*)$ in this paper. The first result gives a necessary and sufficient condition for $\sup_{\mathfrak{q}} \ell_R(\mathfrak{q}^*/\mathfrak{q})$ to be finite. Namely,

THEOREM 1.1. *Let R be excellent and equidimensional. Then the following three conditions are equivalent.*

$$(1) \quad \sup_{\mathfrak{q}} \ell_R(\mathfrak{q}^*/\mathfrak{q}) < \infty.$$

Received February 7, 2001

Both authors are supported by a Grant-in-Aid for Scientific Research Japan.

1991 *Mathematics Subject classification.* Primary 13A35; Secondary 13H10; 13H15.

Key words and phrases. tight closures; F -rational ring; multiplicity.

- (2) Local rings $R_{\mathfrak{p}}$ are F -rational for all $\mathfrak{p} \in \text{Spec } R \setminus \{\mathfrak{m}\}$.
(3) R is FLC and $(0)_{\mathbb{H}_{\mathfrak{m}}^d(R)}^*$ is a module of finite length.

When this is the case, we have that $\sup_{\mathfrak{q}} \{e_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}^*)\} < \infty$.

Here $(0)_{\mathbb{H}_{\mathfrak{m}}^d(R)}^*$ stands for the tight closure of (0) in the local cohomology module $\mathbb{H}_{\mathfrak{m}}^d(R)$ ($d = \dim R$) (cf. [HH1, Chap 8]). We also refer e.g. [GY] for the definition of FLC rings. The theorem concludes, for instance, that $\sup_{\mathfrak{q}} \ell_R(\mathfrak{q}^*/\mathfrak{q})$ is finite for excellent local domains R having isolated singularity. The finiteness of $\sup_{\mathfrak{q}} \{e_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}^*)\}$ yields the following.

THEOREM 1.2. *Let R be a homomorphic image of a Cohen-Macaulay local ring of characteristic $p > 0$ and assume that $\text{Ass } R \subseteq \text{Assh } R \cup \{\mathfrak{m}\}$. If $\sup_{\mathfrak{q}} \{e_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}^*)\} < \infty$, Then*

$$\{\mathfrak{p} \mid R_{\mathfrak{p}} \text{ is Cohen-Macaulay}\} \setminus \{\mathfrak{m}\} = \{\mathfrak{p} \mid R_{\mathfrak{p}} \text{ is } F\text{-rational}\} \setminus \{\mathfrak{m}\}.$$

We state how this paper is organized. Section 2 is devoted to preliminaries. We recall the concept of d -sequences and USD -sequences, which plays a key role throughout our argument. We also briefly explain the tight closure of modules and argue the relation between the tight closure of parameter ideals and the tight closure of (0) in the local cohomology module $\mathbb{H}_{\mathfrak{m}}^d(R)$ under the assumption that R is excellent and equidimensional. In Section 3 we shall give the proof of our theorems.

Throughout this paper let (R, \mathfrak{m}) denote a Noetherian local ring of dimension d . For a finitely generated R -module M we denote by $\ell_R(M)$ the length of M and by $e_I(M)$ the multiplicity of M with respect to an \mathfrak{m} -primary ideal I . Let $\mathbb{H}_{\mathfrak{m}}^i(M)$ ($i \in \mathbb{Z}$) stand for the i -th local cohomology module of M with respect to \mathfrak{m} .

ACKNOWLEDGMENT. The authors are grateful to the referee for many suggestions.

2. Preliminaries.

We first recall the definition of d -sequences and USD -sequences. A sequence a_1, a_2, \dots, a_s of elements in R is called a d -sequence on R (cf. [H1]) if the equality

$$(a_1, a_2, \dots, a_{i-1}) : a_i a_j = (a_1, a_2, \dots, a_{i-1}) : a_j$$

holds for all $1 \leq i \leq j \leq s$, and moreover, it is called an unconditioned strong d -sequence (a USD -sequence for short) on R if $a_1^{n_1}, a_2^{n_2}, \dots, a_s^{n_s}$ is a d -sequence in any order and for all integers $n_1, n_2, \dots, n_s > 0$.

For a system a_1, a_2, \dots, a_d of parameters of R , it is known that

$$\{R/(a_1^n, a_2^n, \dots, a_d^n) \xrightarrow{a_1 \cdots a_d} R/(a_1^{n+1}, a_2^{n+1}, \dots, a_d^{n+1})\}$$

forms a direct system whose direct limit is $\mathbb{H}_{\mathfrak{m}}^d(R)$. Let K_n be the kernel of the canonical map from R , through $R/(a_1^n, a_2^n, \dots, a_d^n)R$, to the direct limit $\mathbb{H}_{\mathfrak{m}}^d(R)$. Then the system $\{R/K_n \xrightarrow{a_1 \cdots a_d} R/K_{n+1}\}$ forms a direct system again whose maps are injective and whose direct limit is also $\mathbb{H}_{\mathfrak{m}}^d(R)$. Hence we can write that $\varinjlim R/K_n = \bigcup_{n>0} R/K_n = \mathbb{H}_{\mathfrak{m}}^d(R)$.

Furthermore, when a_1, a_2, \dots, a_d is a *USD*-sequence, by the monomial property of a *USD*-sequence (cf. Theorem 2.3 of [GY]), K_n coincides with

$$\Sigma(a_1^n, a_2^n, \dots, a_d^n) := \sum_{i=1}^d ((a_1^n, \dots, \widehat{a_i^n}, \dots, a_d^n) : a_i^n) + (a_1^n, a_2^n, \dots, a_d^n).$$

We denote $\Sigma(\mathfrak{q}) = (a_1, a_2, \dots, a_d)$ when $\mathfrak{q} = (a_1, a_2, \dots, a_d)$. Also notice that $\Sigma(\mathfrak{q}) \subseteq \mathfrak{q}^*$, by colon capturing property (cf. [HH1, Theorem 7.9]) when R is a homomorphic image of a Cohen-Macaulay local ring of characteristic $p > 0$ and equidimensional.

Next we briefly explain the tight closure of the zero-module of arbitrary R -modules. Suppose that R has characteristic $p > 0$. The Frobenius endomorphism $F^e : R \rightarrow R$ is given by $F^e(x) = x^{p^e}$. When we consider R as an R -algebra via F^e , we denote it by eR . Let M be an R -module and $x \in M$. We say that $x \in (0)_M^*$ if there exists $c \in R^0$ such that $c \otimes x = 0$ in ${}^eR \otimes_R M$ for all $e \gg 0$. The following properties readily comes from the definition.

- (1) $(0)_{R/I}^* = I^*/I$ for any ideal I of R .
- (2) Let $f : M \rightarrow N$ be a homomorphism of R -modules. Then f induces $(0)_M^* \rightarrow (0)_N^*$.

Kawasaki's result [K, Corollary 1.2] guarantees that an excellent local ring R is a homomorphic image of a Cohen-Macaulay ring A . Furthermore, according to his theory, the Cohen-Macaulay ring A is given as a Rees algebra over R . Thus, an excellent local ring R of characteristic p can be a homomorphic image of a Cohen-Macaulay local ring of characteristic p . We put $H = H_m^d(R)$ for short.

PROPOSITION 2.1. *Let R be an excellent and equidimensional local ring of characteristic p . Suppose that the system a_1, a_2, \dots, a_d of parameters of R forms a *USD*-sequence. Then*

- (1) $(0)_H^* = \varinjlim (a_1^n, a_2^n, \dots, a_d^n)^*/(a_1^n, a_2^n, \dots, a_d^n)$.
- (2) $(0)_H^* = \bigcup_{n>0} (a_1^n, a_2^n, \dots, a_d^n)^*/\Sigma(a_1^n, a_2^n, \dots, a_d^n)$. In particular, the length of $(0)_H^*$ is finite if and only if $\sup_n \{\ell_R((a_1^n, a_2^n, \dots, a_d^n)^*/\Sigma(a_1^n, a_2^n, \dots, a_d^n))\} < \infty$.

PROOF. (1): We put $(\underline{a}^n) = (a_1^n, a_2^n, \dots, a_d^n)$. The natural map $\alpha_n : R/(\underline{a}^n) \rightarrow H$ induces $\beta_n : (\underline{a}^n)^*/(\underline{a}^n) = (0)_{R/(\underline{a}^n)}^* \rightarrow (0)_H^*$. Thus, we have the following commutative diagram:

$$\begin{array}{ccc} \varinjlim (\underline{a}^n)^*/(\underline{a}^n) & \xrightarrow{\beta} & (0)_H^* \\ \downarrow & & \downarrow \\ \varinjlim R/(\underline{a}^n) & \xrightarrow{\alpha} & H, \end{array}$$

where vertical arrows are injective. Since α is isomorphic, the injectivity of β follows. Next we shall prove that β is surjective. Let $\xi \in (0)_H^*$. Then there exist $n > 0$ and $x \in R$ such that $\bar{x} \in R/(\underline{a}^n)$ and $\alpha_n(\bar{x}) = \xi$, while there exists $c \in R^0$ such that $c \otimes \xi = 0$ in ${}^eR \otimes_R H$ for all $e \gg 0$. Hence $c \otimes \bar{x}$ belongs to the kernel of ${}^eR \otimes \alpha_n : {}^eR \otimes_R R/(\underline{a}^n) \rightarrow {}^eR \otimes_R H$. On the other hand, passing to isomorphisms ${}^eR \otimes_R R/(\underline{a}^n) \cong R/(\underline{a}^{nq})$ and ${}^eR \otimes_R H \cong H$, where $q = p^e$, ${}^eR \otimes \alpha_n$ coincides with α_{nq} , whence we have $\overline{cx^q} \in \text{Ker } \alpha_{nq}$. Therefore,

$cx^q \in K_{nq} = \Sigma(a_1^{nq}, a_2^{nq}, \dots, a_d^{nq}) \subseteq (a_1^{nq}, a_2^{nq}, \dots, a_d^{nq})^*$ since a_1, a_2, \dots, a_d is a *USD*-sequence.

Let $R_{red} = R/\sqrt{(0)}$. For $y \in R$, let y' stand for the image of y in R_{red} . Now, we have $(cx^q)' \in ((a_1^{nq}, a_2^{nq}, \dots, a_d^{nq})R_{red})^*$. Besides, we can take $t \in R^0$ such that t' is a test element of R_{red} because R_{red} is an excellent reduced local ring (cf. [HH1, Theorem 6.1]). Hence, it follows that

$$(tc)'x'^q \in (a_1'^n, a_2'^n, \dots, a_d'^n)^{[q]}R_{red}$$

for all $q \gg 0$. Hence $x' \in ((a_1'^n, a_2'^n, \dots, a_d'^n)R_{red})^*$, therefore we get $x \in (\underline{a}^n)^*$ by [HH1, Theorem 4.1]. This implies that the surjectivity of β .

(2): $\{(\underline{a}^n)^*/K_n\}$ is a direct system with injective maps. Besides, its direct limit coincides with that of $\{(\underline{a}^n)^*/(\underline{a}^n)\}$. The assertion follows. \square

3. Proof of the theorems.

Throughout this section, we assume that R is a homomorphic image of a Cohen-Macaulay local ring of characteristic $p > 0$ and equidimensional. For an ideal I in R we put $U(I) = \bigcup_{n>0} I : m^n$ and call it the unmixed component of I . A sequence a_1, a_2, \dots, a_s of elements of R is called a filter regular sequence if $(a_1, a_2, \dots, a_i) : a_{i+1} \subseteq U((a_1, a_2, \dots, a_i))$ for all $0 \leq i < s$. Note that any system a_1, a_2, \dots, a_d of parameters in R forms a filter regular sequence if R is *FLC*. For a parameter ideal $\mathfrak{q} = (a_1, a_2, \dots, a_d)$, we put $\mathfrak{q}_i = (a_1, a_2, \dots, a_i)$. We begin with the following Lemma.

LEMMA 3.1. *Let $\mathfrak{q} = (a_1, a_2, \dots, a_d)$ be a parameter ideal of R . We put the function*

$$f(n) = \ell_R((\mathfrak{q}_{d-1} + (a_d^n))^*/(U(\mathfrak{q}_{d-1}) + (a_d^n)))$$

for all $n > 0$. Then $f(n)$ is monotonous increasing function.

PROOF. We first note that $U(\mathfrak{q}_{d-1}) + (a_d^n) \subseteq (\mathfrak{q}_{d-1} + (a_d^n))^*$ by the colon capturing property (cf. [HH1, Theorem 7.9]). To get the lemma, it is enough to show that the following map is injective.

$$\Phi : (\mathfrak{q}_{d-1} + (a_d^n))^*/(U(\mathfrak{q}_{d-1}) + (a_d^n)) \rightarrow (\mathfrak{q}_{d-1} + (a_d^{n+1}))^*/(U(\mathfrak{q}_{d-1}) + (a_d^{n+1})),$$

where $\Phi(\bar{x}) = \overline{a_d x}$. It readily follows that Φ is well-defined. Take $x \in R$ such that $a_d x \in U(\mathfrak{q}_{d-1}) + (a_d^{n+1})$. Then there exists $y \in R$ such that $a_d(x - a_d^n y) \in U(\mathfrak{q}_{d-1})$. Therefore $x - a_d^n y \in U(\mathfrak{q}_{d-1})$ since $R/U(\mathfrak{q}_{d-1})$ is a 1-dimensional Cohen-Macaulay ring. Thus $x \in U(\mathfrak{q}_{d-1}) + (a_d^n)$ and lemma follows. \square

COROLLARY 3.2. *Let R be an FLC ring and $\mathfrak{q} = (a_1, a_2, \dots, a_d)$ a parameter ideal in R . Then for any integer $n > 0$, we have*

$$e_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}) \leq e_{(a_1^n, a_2^n, \dots, a_d^n)}(R) - \ell_R(R/(a_1^n, a_2^n, \dots, a_d^n)).$$

PROOF. Note that $e_{\mathfrak{q}}(R) = \ell_R(R/\mathfrak{U}(\mathfrak{q}_{d-1}) + (a_d))$ because a_1, a_2, \dots, a_d forms a filter regular sequence (cf. [GN, Proof of Theorem 1.1]). We now have

$$\begin{aligned} e_{(\mathfrak{q}_{d-1} + (a_d^n))}(R) - \ell_R(R/(\mathfrak{q}_{d-1} + (a_d^n))^*) \\ = \ell_R(R/\mathfrak{U}(\mathfrak{q}_{d-1}) + (a_d^n)) - \ell_R(R/(\mathfrak{q}_{d-1} + (a_d^n))^*) \\ = \ell_R((\mathfrak{q}_{d-1} + (a_d^n))^*/(\mathfrak{U}(\mathfrak{q}_{d-1}) + (a_d^n))). \end{aligned}$$

Hence by Lemma 3.1, we get

$$e_{(\mathfrak{q}_{d-1} + (a_d^n))}(R) - \ell_R(R/(\mathfrak{q}_{d-1} + (a_d^n))) \leq e_{(\mathfrak{q}_{d-1} + (a_d^{n+1}))}(R) - \ell_R(R/(\mathfrak{q}_{d-1} + (a_d^{n+1}))).$$

Changing the order of the system of parameters and repeating the same procedure, the assertion follows. \square

The following is an improvement of [N2, Proposition 4.4].

LEMMA 3.3. *Let a_1, a_2, \dots, a_{d-1} be a subsystem of parameters of R of length $d - 1$. Then $(a_1, a_2, \dots, a_{d-1})^* R_{\mathfrak{p}} = ((a_1, a_2, \dots, a_{d-1})R_{\mathfrak{p}})^*$ for all $\mathfrak{p} \in \text{Spec } R$.*

PROOF. Let $I = (a_1, a_2, \dots, a_{d-1})$. Notice that $\text{ht}_R I = d - 1$. We may assume that $\mathfrak{p} \supseteq I$ and $\mathfrak{p} \neq \mathfrak{m}$. Take $a_d \in R$ so that a_1, \dots, a_{d-1}, a_d forms a system of parameters. Let $S = R[1/a_d]$. Then it follows that $(IS)^* S_{\mathfrak{p}S} = (IR_{\mathfrak{p}})^*$ by [N1, Lemma 2.2]. To get the lemma, it is enough to show that $I^* S = (IS)^*$. The inclusion $I^* S \subseteq (IS)^*$ follows from [HH1, Lemma 4.11]. Conversely, take $x \in R$ with $x/1 \in (IS)^*$. Then there exists $c \in R^0$ such that $c/1 \cdot (x/1)^q \in I^{[q]} S$ for all $q \gg 0$. One can take n so that

$$a_d^n c x^q \in (a_1^q, a_2^q, \dots, a_{d-1}^q) \quad \text{for all } q \gg 0. \quad (\text{a})$$

(Here the integer n may depend on q .)

We may put $R = A/J$, where A is a Cohen-Macaulay local ring of characteristic p and J is an ideal of A with $\text{ht}_A J = 0$. Let $\tilde{x}, \tilde{c}, \tilde{a}_1, \dots, \tilde{a}_d \in A$ be preimages of $x, c, a_1, \dots, a_d \in R$, respectively. Here we can choose $\tilde{a}_1, \dots, \tilde{a}_d \in A$ to be a system of parameters of A . (cf. e.g., [BH, Lemma 10.1.10]). Taking the preimage of the whole of (a) in A yields

$$\tilde{a}_d^n \tilde{c} \tilde{x}^q \in (\tilde{a}_1^q, \tilde{a}_2^q, \dots, \tilde{a}_{d-1}^q) A + J \quad \text{for all } q \gg 0.$$

We take $d \in A \setminus \bigcup_{\mathfrak{p} \in \text{Min}_A A/J} \mathfrak{p}$ and $q' = p^{e'}$ so that $d \cdot J^{[q']} = (0)$. Then we have $d \tilde{a}_d^{nq'} \tilde{c} \tilde{x}^{qq'} \in (\tilde{a}_1^{qq'}, \tilde{a}_2^{qq'}, \dots, \tilde{a}_{d-1}^{qq'}) A$, whence $d \tilde{c} \tilde{x}^{qq'} \in (\tilde{a}_1^{qq'}, \tilde{a}_2^{qq'}, \dots, \tilde{a}_{d-1}^{qq'}) A$ because $\tilde{a}_1, \dots, \tilde{a}_d$ is a regular sequence on A . This implies that $d' x^q \in (a_1^q, \dots, a_{d-1}^q) R$ for some $d' \in R^0$ and for all $q \gg 0$. Thus we get $x \in (a_1, a_2, \dots, a_{d-1})^*$, whence $x/1 \in I^* S$. Thus the lemma follows. \square

We come to the place to give the proof of Theorem 1.1.

PROOF. (1) \Rightarrow (2): We first note that R is *FLC*. In fact, for all parameter ideal \mathfrak{q} , we have

$$\ell_R(\mathfrak{q}^*/\mathfrak{q}) = (\ell_R(R/\mathfrak{q}) - e_{\mathfrak{q}}(R)) + (e_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}^*)), \quad (*)$$

while $e_q(R) - \ell_R(R/q^*)$ is non negative by [GN, Theorem 1.1]. Hence $\sup_q\{\ell_R(R/q) - e_q(R)\}$ is finite. This implies that R is *FLC* by [CTS].

Take $\mathfrak{p} \in \text{Spec } R$ with $\mathfrak{p} \neq \mathfrak{m}$. Then $R_{\mathfrak{p}}$ is Cohen-Macaulay. We shall show that $R_{\mathfrak{p}}$ is *F*-rational. We may assume that $\text{ht}_{A_{\mathfrak{p}}} = d-1$. Let us take a parameter ideal $\mathfrak{q} = (a_1, a_2, \dots, a_d)$ such that $\mathfrak{q}_{d-1} = (a_1, a_2, \dots, a_{d-1}) \subseteq \mathfrak{p}$. Let $N = \sup_q\{\ell_R(\mathfrak{q}^*/\mathfrak{q})\}$. Then

$$\mathfrak{m}^N \mathfrak{q}_{d-1}^* \subseteq \mathfrak{m}^N (\mathfrak{q}_{d-1} + (a_d^n))^* \subseteq \mathfrak{q}_{d-1} + (a_d^n)$$

for all $n > 0$. The intersection theorem yields $\mathfrak{m}^N \mathfrak{q}_{d-1}^* \subseteq \mathfrak{q}_{d-1}$. Then it follows that $\mathfrak{q}_{d-1}^* R_{\mathfrak{p}} = \mathfrak{q}_{d-1} R_{\mathfrak{p}}$, while $\mathfrak{q}_{d-1}^* R_{\mathfrak{p}} = (\mathfrak{q}_{d-1} R_{\mathfrak{p}})^*$ by Lemma 3.3. Hence $R_{\mathfrak{p}}$ is *F*-rational since $\mathfrak{q}_{d-1} R_{\mathfrak{p}}$ is a parameter ideal of $R_{\mathfrak{p}}$ (cf. [FW, Proposition 2.2]).

(2) \Rightarrow (3): We may assume that $\dim R \geq 1$. $R_{\mathfrak{p}}$ is Cohen-Macaulay for any $\mathfrak{p} \in \text{Spec } R \setminus \{\mathfrak{m}\}$, since $R_{\mathfrak{p}}$ is excellent *F*-rational ring (cf. [H2, Theorem 4.2]). Thus R is *FLC* by [CTS] again. We shall show that the length of $(0)_H^*$ is finite. To do it, we may assume that R is reduced. In fact, $(\sqrt{(0)})_{\mathfrak{p}} = (0)$ for all $\mathfrak{p} \in \text{Spec } R \setminus \{\mathfrak{m}\}$. So $R_{red} = R/\sqrt{(0)}$ satisfies the assertion (2). Besides, $\sqrt{(0)}$ is a module of finite length, whence we have that R_{red} is *FLC* and $H_{\mathfrak{m}}^d(R) \cong H_{\mathfrak{m}}^d(R_{red})$. (Notice also that $R_{red}^0 = \{\bar{c} \mid c \in R^0\}$.)

Now let f_1, f_2, \dots, f_d be a system of parameters in R . Then for each i , f_i has a power f_i^n which is a test element for parameter ideals in R since $R[1/f_i]$ is *F*-rational (cf. [V, Theorem 3.9]). Put $\mathfrak{a} = (f_1^n, f_2^n, \dots, f_d^n)R$. Then \mathfrak{a} is an \mathfrak{m} -primary ideal for which $\mathfrak{a}I^* \subseteq I$ holds for all parameter ideals I . Let $(\underline{a}) = a_1, a_2, \dots, a_d$ be a system of parameters which forms a *USD*-sequence. (This choice is possible because R is *FLC*.) Since $(0)_H^* = \varinjlim (\underline{a})^*/(\underline{a})$ by Proposition 2.1, we have $\mathfrak{a}(0)_H^* = (0)$. Thus we have an embedding $(0)_H^* \hookrightarrow \text{Hom}_R(R/\mathfrak{a}, H)$ whose length are finite.

(3) \Rightarrow (1): From the equation (*) it is enough to prove that $\sup_q\{e_q(R) - \ell_R(R/q^*)\}$ is finite. To see this, by Corollary 3.2, we may assume that the generating system a_1, a_2, \dots, a_d of \mathfrak{q} is a *USD*-sequence. (Taking a suitable power, $a_1^n, a_2^n, \dots, a_d^n$ is a *USD*-sequence since R is *FLC*.)

Now $\mathfrak{q} \subseteq U(\mathfrak{q}_{d-1}) + (a_d) \subseteq \Sigma(\mathfrak{q}) \subseteq \mathfrak{q}^*$ (here we apply the colon capturing property), thus we have

$$\begin{aligned} e_q(R) - \ell_R(R/q^*) &= \ell_R(\mathfrak{q}^*/U(\mathfrak{q}_{d-1}) + (a_d)) \\ &= \ell_R(\mathfrak{q}^*/\Sigma(\mathfrak{q})) + \ell_R(\Sigma(\mathfrak{q})/U(\mathfrak{q}_{d-1}) + (a_d)) \\ &\leq \ell_R(\mathfrak{q}^*/\Sigma(\mathfrak{q})) + \ell_R(\Sigma(\mathfrak{q})/\mathfrak{q}), \end{aligned}$$

where the first equality follows from the fact that a_1, a_2, \dots, a_d is a filter regular sequence. Because $\ell_R(\Sigma(\mathfrak{q})/\mathfrak{q}) = \sum_{i=0}^{d-1} \binom{d}{i} \ell_R(H_{\mathfrak{m}}^i(R))$ (cf. [G, Proposition 3.6]) and $\mathfrak{q}^*/\Sigma(\mathfrak{q}) \subseteq (0)_H^*$ by Proposition 2.1. Hence the finiteness of $\sup_q\{e_q(R) - \ell_R(R/q^*)\}$ follows. \square

Before the proof of Theorem 1.2, we recall the following.

LEMMA 3.4 ([GN, Lemma 3.2]). *Suppose that $\text{Ass } R \subseteq \text{Assh } R \cup \{\mathfrak{m}\}$. Then*

$$\mathcal{F} := \{\mathfrak{p} \in \text{Spec } R \mid \text{ht}_R \mathfrak{p} > 1 = \text{depth } R_{\mathfrak{p}}, \mathfrak{p} \neq \mathfrak{m}\}$$

is a finite set.

Let us recall the statement of Theorem 1.2.

THEOREM 1.2. *Suppose that $\text{Ass } R \subseteq \text{Assh } R \cup \{\mathfrak{m}\}$. If $\sup_{\mathfrak{q}} \{e_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}^*)\} < \infty$, then*

$$\{\mathfrak{p} \mid R_{\mathfrak{p}} \text{ is } F\text{-rational}\} \setminus \{\mathfrak{m}\} = \{\mathfrak{p} \mid R_{\mathfrak{p}} \text{ is Cohen-Macaulay}\} \setminus \{\mathfrak{m}\}.$$

PROOF. The inclusion \subseteq follows from [H2, Theorem 4.2]. Let us show the opposite inclusion. Let $\mathfrak{a} = \prod_{i < d} \text{Ann}_R H_{\mathfrak{m}}^i(R)$. By [K, Proposition 2.3], $V(\mathfrak{a})$ is the non Cohen-Macaulay locus of R . We can choose a generating system f_1, f_2, \dots, f_n of \mathfrak{a} such that $f_i \in R^0$ for all i because $\text{ht}_R \mathfrak{a} > 0$. Let $\mathfrak{p} \in \text{Spec } R$ with $\mathfrak{p} \neq \mathfrak{m}$ and assume that $R_{\mathfrak{p}}$ is Cohen-Macaulay. Then there exists f_i with $f_i \notin \mathfrak{p}$. Let $S = R[1/f_i]$ and taken $\mathfrak{n} \in \text{Max } S$ with $\mathfrak{n} \supseteq \mathfrak{p}S$. Let $P = \mathfrak{n} \cap R$. Notice that R_P is Cohen-Macaulay. Besides, $S/\mathfrak{n} = S/P_S = R/P[1/f_i]$, which is a field, whence $\dim R/P = 1$. Thus $\text{ht}_R P = d - 1$.

CLAIM 1. *There exists $a_1, a_2, \dots, a_{d-1} \in P$ which forms a filter regular sequence.*

PROOF. When $d = 1$, there is nothing to prove. Let $d \geq 2$. Let \mathcal{F} be as in Lemma 3.4. Then $P \not\subseteq Q$ for any $Q \in \mathcal{F}$. We take

$$a_1 \in P \setminus \left(\bigcup_{Q \in \mathcal{F}} Q \right) \cup \left(\bigcup_{Q \in \text{Assh } R} Q \right).$$

Then it follows that $\text{Ass } R/(a_1) \subseteq \text{Assh } R/(a_1) \cup \{\mathfrak{m}\}$. In fact, take $\mathfrak{p}_1 \in \text{Ass } R/(a_1)$ with $\mathfrak{p}_1 \neq \mathfrak{m}$. Then $\text{depth } R_{\mathfrak{p}_1} = 1$ and it implies that $\text{ht}_R \mathfrak{p}_1 = 1$. Thus, $\dim R/\mathfrak{p}_1 = \dim R/(a_1)$, whence $\mathfrak{p}_1 \in \text{Assh } R/(a_1)$. From this choice it follows that a_1 is a filter regular sequence of length one. Repeating this procedure, the claim follows. \square

Let $N = \sup_{\mathfrak{q}} \{e_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}^*)\}$ and $a_1, a_2, \dots, a_{d-1} \in P$ be as above. We take $a_d \in R$ so that a_1, \dots, a_{d-1}, a_d is a system of parameters in R . Notice that it forms a filter regular sequence. Let $I = (a_1, a_2, \dots, a_{d-1})$ and $J = I + (a_d^n)$. Then

$$\begin{aligned} \ell_R(J^*/U(I) + (a_d^n)) &= \ell_R(R/U(I) + (a_d^n)) - \ell_R(R/J^*) \\ &= e_J(R) - \ell_R(R/J^*) \\ &\leq N \end{aligned}$$

Hence, $\mathfrak{m}^N I^* \subseteq \mathfrak{m}^N J^* \subseteq U(I) + (a_d^n)$ for all $n > 0$. The intersection theorem yields $\mathfrak{m}^N I^* \subseteq U(I)$. Hence $I^* = U(I)$ by colon capturing property. Now $I^* R_P = (I R_P)^*$ by Lemma 3.3, while $U(I)_P = I R_P$. It implies that $I R_P$ is tightly closed, whence R_P is F -rational Cohen-Macaulay ring (cf. [FW, Proposition 2.2]). Therefore, $R_{\mathfrak{p}}$ is F -rational. \square

References

- [BH] W. BRUNS and J. HERZOG, *Cohen-Macaulay rings*, Cambridge studies in advanced mathematics **39** (1993), Cambridge University Press.
- [CTS] N. T. CUONG, N. V. TRUNG and P. SCHENZEL, Verallgemeinerte Cohen-Macaulay-Moduln, *Math. Nachr.* **85** (1978), 57–73.
- [FW] R. FEDDER and K. WATANABE, A characterization of F -regularity in terms of F -purity, *Commutative algebra* (eds. M. HOCHSTER, J. D. SALLY and C. HUNEKE), *Math. Sc. Res. Inst. Publ.* **15** (1989) Springer, 227–245.
- [G] S. GOTO, On the associated graded rings of parameter ideals in Buchsbaum rings, *J. Algebra* **85** (1983), 490–534.
- [GN] S. GOTO and Y. NAKAMURA, Multiplicity and tight closures of parameters, *J. Algebra* **244** (2001), 302–311.
- [GY] S. GOTO and K. YAMAGISHI, The theory of unconditioned strong d -sequences and modules of finite local cohomology, preprint (1980).
- [H1] C. HUNEKE, The theory of d -sequences and powers of ideals, *Ad. in Math.* **46** (1982), 249–279.
- [H2] C. HUNEKE, *Tight closure and its applications*, C.M.B.S. Regional Conf. Ser. in Math. **88** (1996), Amer. Math. Soc.
- [HH1] M. HOCHSTER and C. HUNEKE, Tight closure, invariant theory, and the Briançon-Skoda theorem, *J. Amer. Math. Soc.* **3** (1990), 31–116.
- [K] T. KAWASAKI, Arithmetic Macaulayfication of a local ring, preprint (1999).
- [N1] Y. NAKAMURA, Local rings Cohen-Macaulay F -rational rings are F -rational, *Tokyo J. Math.* **14** (1991), 41–44.
- [N2] Y. NAKAMURA, On numerical invariants of Noetherian local rings of characteristic p , *J. Math. Kyoto Univ.* **34** (1994), 1–13.
- [V] J. D. VÉLEZ, Openness of the F -rational locus and Smooth Base Change, *J. Algebra* **172** (1995), 425–453.
- [WY] K. WATANABE and K. YOSHIDA, Hilbert-Kunz multiplicity and an inequality between multiplicity and colength, *J. Algebra* **230** (2000), 295–317.

Present Address:

DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE AND TECHNOLOGY,
MEIJI UNIVERSITY, KAWASAKI, 214–8571 JAPAN.

e-mail: goto@math.meiji.ac.jp

ynakamu@math.meiji.ac.jp