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Realization of Vassiliev Invariants by Unknotting Number One Knots

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Abstract. We show that for any natural number n and any knot K, there are infinitely many unknotting number one knots, all of whose Vassiliev invariants of order less than or equal to n coincide with those of K.

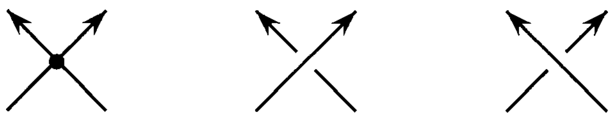
1. Introduction.

In 1990, V. A. Vassiliev [21] defined a sequence of knot invariants and J. S. Birman and X.-S. Lin [3] succeeded in giving an axiomatic description for Vassiliev invariants.

Our definition of Vassiliev invariants follows the Birman-Lin's axioms in [3] or D. Bar-Natan [1]. Whenever we have a knot invariant v which takes value in some abelian group, we can extend it to an invariant of singular knots by the Vassiliev skein relation:

$$v(K_D) = v(K_+) - v(V_-)\,.$$

Here *a singular knot* is an immersion of a circle in R^3 whose only singularities are transversal double points and K_D, K_+ and K_- denote the diagrams of singular knots which are identical except near one point as is shown in Fig. 1.1. An invariant v is called *a Vassiliev invariant of order n* and is denoted by v_n, if n is the smallest integer such that v vanishes on all singular knots with more than n double points.



K_D K_+ K_-

FIGURE 1.1.

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The set of all Vassiliev invariants is at least as powerful as all of quantum group invariants. However, for any knot K and for any positive integer n, some examples of knots have been constructed, all of whose Vassiliev invariants of order at most n coincide with those of K ([4][7][10][14]). Our purpose is to construct such examples of knots whose unknotting numbers are equal to one by using local moves called C_n-moves. Namely in this paper we show the following results.

THEOREM 1.1. *Let n be a natural number and K an oriented knot in S^3. Then there are infinitely many unknotting number one knots J_m $(m = 1, 2, \cdots)$ such that K and J_m are C_{n+1}-equivalent.*

LEMMA 1.2. *Let K and J be C_{n+1}-equivalent oriented knots. Then $v(K) = v(J)$ for any Vassiliev invariant v of order less than or equal to n,*

We will define C_n-moves and the C_n-equivalence in the next section. The following theorem is an immediate consequence of Theorem 1.1 and Lemma 1.2.

MAIN THEOREM. *Let n be a natural number and K an oriented knot in S^3. Then there are infinitely many unknotting number one knots J_m $(m = 1, 2, \cdots,)$ such that $v(J_m) = v(K)$ for any Vassiliev invariant v of order less than or equal to n.*

REMARK. A C_n-move is originally defined by K. Habiro in [5]. Habiro [6] showed that two oriented knots have the same Vassiliev invariants of order less than or equal to n if and only if they are C_{n+1}-equivalent by using the clasper theory. Lemma 1.2 is the 'if' part of Habiro's result and we give a simple proof of Lemma 1.2 in the next section. Our results are obtained not by using the clasper theory, only by using the argument of knot diagrams. We do not use the 'only if' part, the difficult half, of Habiro's result. Our proof of Theorem 1.1 is elementary and constructive. After finishing the first version of this paper the first author showed a simple proof of Main theorem in [11]. However the proof essentially uses the difficult half of Habiro's result. See also [22] and [13].

2. Band description of local moves.

We use a concept 'band description of knots' defined in [19] for the proof of Theorem 1.1. Note that the prototypes of band description appear in [17], [23] and [24]. In particular in [24] it is shown that any knot can be expressed as a band sum of a trivial knot and some Borromean rings. The concept of band description is a development of this fact.

A *tangle* T is a disjoint union of properly embedded arcs in the unit 3-ball B^3. A tangle T is *trivial* if there exists a properly embedded disk in B^3 contaning T. A *local move* is a pair of trivial tangles (T_1, T_2) with $\partial T_1 = \partial T_2$ such that for each component t of T_1 there exists a component u of T_2 with $\partial t = \partial u$.

Let (T_1, T_2) be a local move, t_1 a component of T_1 and t_2 a component of T_2 such that $\partial t_1 = \partial t_2$. Let N_1 and N_2 be regular neighbourhoods of t_1 and t_2 respectively such that $N_1 \cap \partial B^3 = N_2 \cap \partial B^3$. Let α be a disjoint union of properly embedded arcs in $B^2 \times [0, 1]$ as

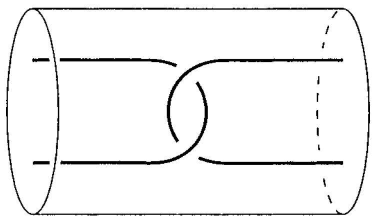

FIGURE 2.1.

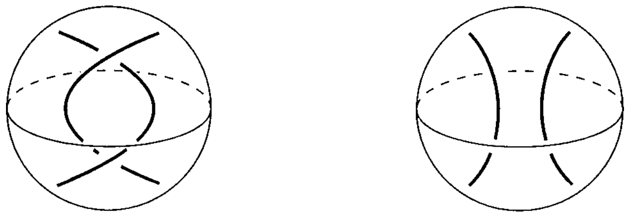

FIGURE 2.2.

illustrated in Fig. 2.1. Let $\psi_i : B^2 \times [0, 1] \to N_i$ be homeomorphisms with $\psi_i(B^2 \times \{0, 1\}) = N_i \cap \partial B^3$ for $i = 1, 2$. Suppose that $\psi_1(\partial\alpha) = \psi_2(\partial\alpha)$ and $\psi_1(\alpha)$ and $\psi_2(\alpha)$ are ambient isotopic in B^3 relative to ∂B^3. Then we say that a local move $((T_1 - t_1) \cup \psi_1(\alpha), (T_2 - t_2) \cup \psi_2(\alpha))$ is a *double of* (T_1, T_2) *with respect to the components* t_1 *and* t_2.

Two local moves (T_1, T_2) and (U_1, U_2) are *equivalent*, denoted by $(T_1, T_2) \cong (U_1, U_2)$, if there is an orientation preserving self-homeomorphism $\psi : B^3 \to B^3$ such that $\psi(T_i)$ and U_i are ambient isotopic in B^3 relative to ∂B^3 for $i = 1, 2$. Let K_1 and K_2 be oriented knots in the oriented three-sphere S^3. We say that K_1 and K_2 are *related by a local move* (T_1, T_2) if there is an orientation preserving embedding $h : B^3 \to S^3$ such that $K_i \cap h(B^3) = h(T_i)$ for $i = 1, 2$ and $K_1 - h(B^3) = K_2 - h(B^3)$ together with orientations. If K_1 and K_2 are related by a local move (T_1, T_2) and $(T_1, T_2) \cong (U_1, U_2)$, then K_1 and K_2 are related by (U_1, U_2).

A C_1*-move* is a local move as illustrated in Fig. 2.2. A double of a C_k-move is called a C_{k+1}*-move*. Note that any doubles of equivalent local moves with respect to the corresponding components are equivalent. Therefore we have that for each natural number n there are only finitely many C_n-moves up to equivalence. Two knots K_1 and K_2 are C_n*-equivalent* if K_1 and K_2 are related by a finite sequence of C_n-moves and ambient isotopies.

We note that our definition of C_k-move follows that in [5], and is different from the one in [6]. However by an easy induction on k it is shown that these two definitions are equivalent.

A local move (T_1, T_2) is *Brunnian* if for each pair of components t_1 and t_2 of T_1 and T_2 respectively with $\partial t_1 = \partial t_2$, $T_1 - t_1$ is ambient isotopic of $T_2 - t_2$ in B^3 relative to ∂B^3.

LEMMA 2.1. *A* C_n*-move is Brunnian.*

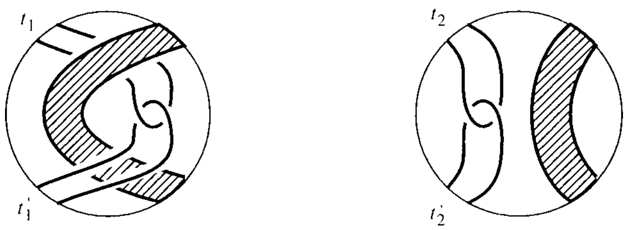

FIGURE 2.3.

PFOOF. It is easy to see that a double of a Brunnian local move is Brunnian. Since a C_1-move is Brunnian the result follows. □

Now we define the similarity of knots ([10][18]) to prove Lemma 1.2. We say that a knot K is *n-similar* to a knot L if the following occurs: There exists a diagram $D(K)$ of K and a collection $\mathcal{A} = \{A_1, A_2, \cdots, A_n\}$ of n pairwise disjoint, nonempty sets of crossings of $D(K)$ such that for any nonempty, not necessarily proper subcollection $\mathcal{A}'$ of $\mathcal{A}$, the diagram which is obtained from $D(K)$ by switching all the crossings in $\cup\mathcal{A}'$ is a diagram of L. The first author showed the following in [10].

LEMMA 2.2 ([10]). *If a knot K is $(n+1)$-similar to L, then the Vassiliev invariants of order less than or equal to n of K coincide with those of L.*

PROOF OF LEMMA 1.2. It is sufficient to show the case that K and J are related by a C_{n+1}-move (T_1, T_2). Let $h : B^3 \to S^3$ be the orientation preserving embedding such that $K \cap h(B^3) = h(T_1)$, $J \cap h(B^3) = h(T_2)$ and $K - h(B^3) = J - h(B^3)$. Let $t_1, t_2, \cdots, t_{n+2}$ be the components of T_1 and $u_1, u_2, \cdots, u_{n+2}$ the components of T_2 such that $\partial t_{n+2} = \partial u_{n+2}$. Let D be a properly embedded disk in B^3 containing T_1. By the Brunnian property of (T_1, T_2) we may suppose without loss of generality that $t_1 \cup \cdots \cup t_{n+1} = u_1 \cup \cdots \cup u_{n+1}$. Up to ambient isotopy in S^3 we may take regular projections of K and J respectively such that they differ only on the disk that is an injective image of $h(D)$. Let A_i be the set of crossing points of $h(u_i)$ and $h(u_{n+2})$ at which $h(u_i)$ goes over $h(u_{n+2})$ in the regular projection of J. Then the sets $A_1, \cdots, A_{n+1}$ show that J is $(n+1)$-similar to K. Then we have the result by Lemma 2.2. See also [16] for related results. □

We say that a C_n-move $(n \geq 2)$ as illustrated in Fig. 2.3 is *special* where each of the shaded regions represents $n-2$ times iteratedly doubled arcs. The arcs t_1, t_1', t_2 and t_2' are called *specified arcs* of this special C_n-move.

LEMMA 2.3. *Any C_n-move $(n \geq 2)$ is equivalent to a special C_n-move.*

PROOF. We will prove by an induction on n. Is is clear that the result holds for $n = 2$. Let (T_1, T_2) be a C_{n+1}-move. Then (T_1, T_2) is a double of a C_n-move (U_1, U_2). By the hypothesis of the induction, we may suppose that (U_1, U_2) is a special C_n-move. If (T_1, T_2) is

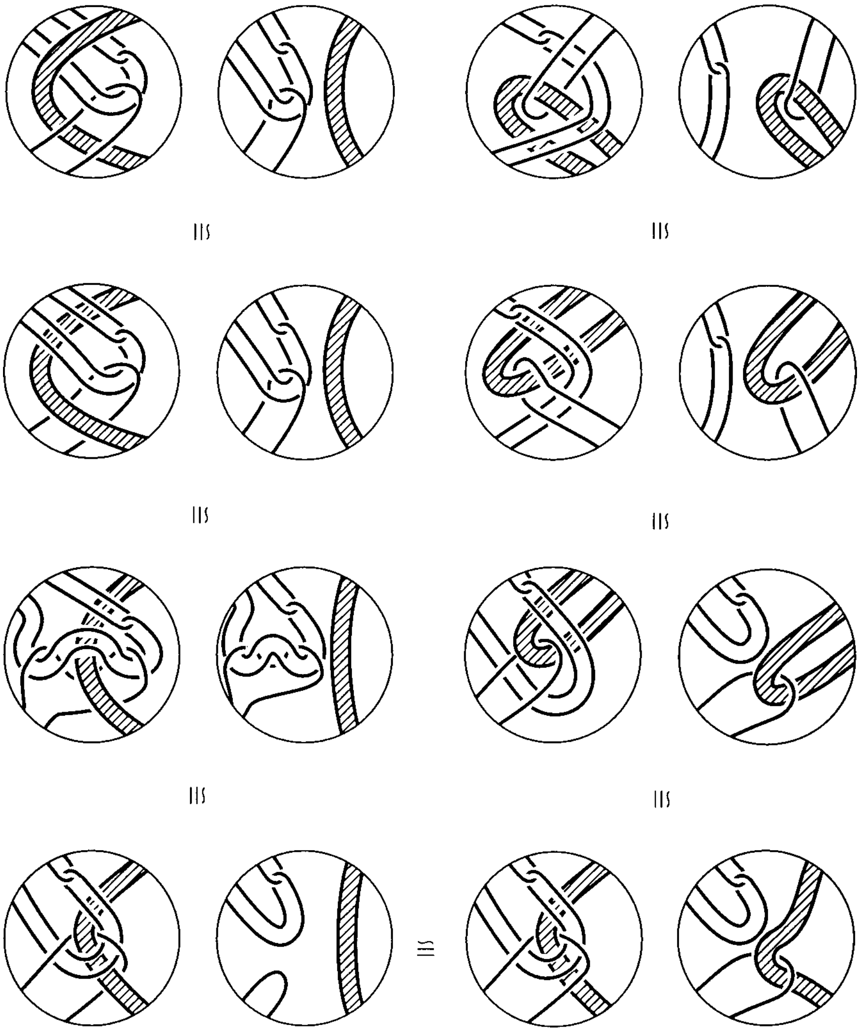

FIGURE 2.4.

a double of (U_1, U_2) with respect to the components that are not specified arcs, then (T_1, T_2) itself is a special C_{n+1}-move. Suppose that (T_1, T_2) is a double of (U_1, U_2) with respect to specified arcs. Then by the deformation illustrated in Fig. 2.4 we have that (T_1, T_2) is equivalent to a special C_{n+1}-move. □

COROLLARY 2.4. *C_{n+1}-equivalence implies C_n-equivalence.*

PROOF. It is easy to see that a special C_{n+1}-move is realized by twice applications of a C_n-move as is shown in Fig. 2.5. Thus we have the result. □

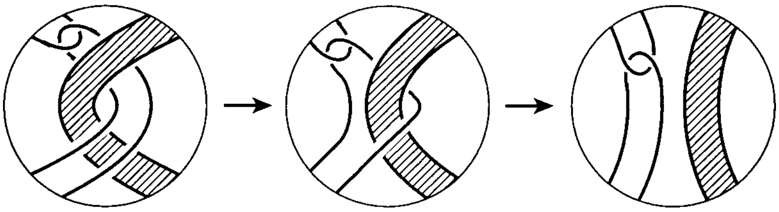

FIGURE 2.5.

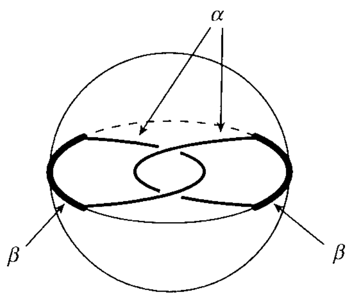

FIGURE 2.6.

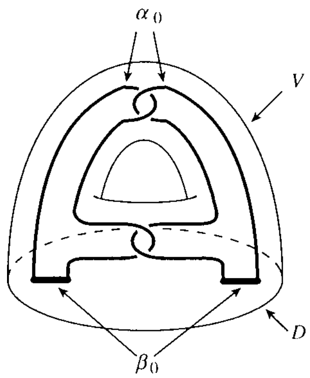

FIGURE 2.7.

A C_1-*link model* is a pair (α, β) where α is a disjoint union of $k + 1$ properly embedded arcs in B^3 and β is a disjoint union of arcs on ∂B^3 with $\partial\alpha = \partial\beta$ as illustrated in Fig. 2.6.

Suppose that a C_k-link model (α, β) has been defined where α is a disjoint union of $k+1$ properly embedded arcs in B^3 and β is a disjoint union of $k + 1$ arcs on ∂B^3 with $\partial\alpha = \partial\beta$ such that $\alpha \cup \beta$ is a disjoint union of $k + 1$ circles. Let γ be a component of $\alpha \cup \beta$ and N a regular neighbourhood of γ in B^3. Let V be an oriented solid torus, D a disk in ∂V, α_0 properly embedded arcs in V and β_0 arcs on D as illustrated in Fig. 2.7.

Let $\psi : V \to N$ be an orientation preserving homeomorphism such that $\psi(D) = N \cap \partial B^3$ and $\psi(\alpha_0 \cup \beta_0)$ bounds disjoint disks in B^3. We further assume for a technical reason that $\psi(\beta_0)$ does not contain $\gamma \cap \beta$. Then we call the pair $((\alpha - \gamma) \cup \psi(\alpha_0), (\beta - \gamma) \cup \psi(\beta_0))$ a C_{k+1}-*link model.* We also say that the pair $((\alpha - \gamma) \cup \psi(\alpha_0), (\beta - \gamma) \cup \psi(\beta_0))$ is a *double* of (α, β) with respect to the component γ. A *special* C_2-*link model* is illustrated in Fig. 2.8. The components γ_1 and γ_2 in Fig. 2.8 are called the *specified components* of this special C_2-link model. A double of a special C_n-link model with respect to a component γ that is not a specified component is called a *special* C_{n+1}-*link model.* And the specified components of this special C_{n+1}-link model are the same as those of the special C_n-link model. A *link model* is a C_n-link model for some n.

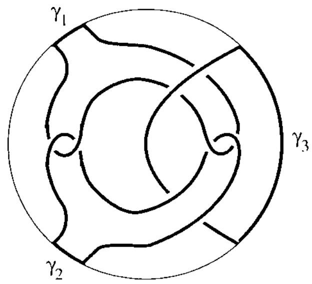

FIGURE 2.8.

Let $(\alpha_1, \beta_1), \cdots, (\alpha_\ell, \beta_\ell)$ be link models. Let K be an oriented knot. Let $\psi_i : B^3 \to S^3$ be an orientation preserving embedding for $i = 1, \cdots, \ell$ and $b_1, \cdots, b_m$ mutually disjoint disks embedded in S^3. Suppose that they satisfy the following conditions;

(1) $\psi_i(B^3) \cap \psi_j(B^3) = \emptyset$ if $i \neq j$,

(2) $\psi_i(B^3) \cap K = \emptyset$ for each i,

(3) $b_i \cap K = \partial b_i \cap K$ is an arc for each i,

(4) $b_i \cap \bigcup_{j=1}^{\ell} \psi_j(B^3) = \partial b_i \cap \bigcup_{j=1}^{\ell} \psi_j(B^3)$ is a component of $\psi_k(\beta_k)$ for some k for each i,

(5) $\bigcup_{i=1}^{m} b_i \cap \bigcup_{i=1}^{\ell} \psi_i(B^3) = \bigcup_{i=1}^{\ell} \psi_i(\beta_i)$.

Let J be an oriented knot defined by $J = K \cup (\bigcup_{i=1}^{m} \partial b_i) \cup (\bigcup_{i=1}^{\ell} \psi_i(\alpha_i)) - \bigcup_{i=1}^{m} \mathrm{int}(\partial b_i \cap K) - \bigcup_{i=1}^{\ell} \psi_i(\mathrm{int}\,\beta_i)$ where the orientation of J coincides with that of K on $K - \bigcup_{i=1}^{m} b_i$. We denote J by $J = \Omega(K; b_1, \cdots, b_m; (\alpha_1, \beta_1), \cdots, (\alpha_\ell, \beta_\ell); \psi_1, \cdots, \psi_\ell)$. Then we say that J is a *band sum* of K and link models $(\alpha_1, \beta_1), \cdots, (\alpha_\ell, \beta_\ell)$. We call each b_i a *band*. Each image $\psi_i(B^3)$ is called a *link ball*.

Let (T_1, T_2) be a local move. Then (T_2, T_1) is also a local move. We call (T_2, T_1) the *inverse* of (T_1, T_2). It is easy to see that the inverse of the C_1-move is equivalent to itself. Then it follows inductively that the inverse of a C_n-move is equivalent to a C_n-move (but possibly not equivalent to itself).

LEMMA 2.5. *Let n be a natural number greater than one. Let K and J be C_n-equivalent knots. Then J is a band sum of K and some special C_n-link models.*

PROOF. By Lemma 2.3, it is assumed that a C_n-move is special. We consider the sequence $K = K_0 \to K_1 \to \cdots \to K_\ell = J$, where K_i and K_{i+1} are related by a special C_n-move. We will prove by an induction on ℓ. Let (α, β) be a link model of a C_n-move. Let β' be a disjoint union of properly embedded arcs in B^3 that is a slight push off of β. Then we can show inductively on n that the local move (α, β') is equivalent to a C_n-move. Conversely we can show that a C_n-move is equivalent to (α, β') for some link model (α, β). In particular a special C_n-move corresponds to a special C_n-link model. See for example Fig. 2.9.

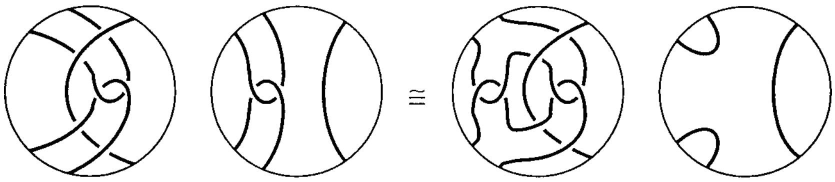

FIGURE 2.9.

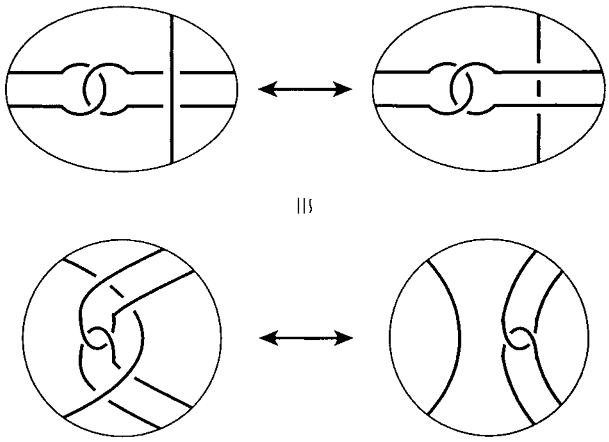

FIGURE 2.10.

Therefore we have the result in the case $\ell = 1$. By an inductive argument it is sufficient to consider the case that J is a band sum of a knot K_1 and some special C_n-link models where K and K_1 are related by a special C_n-move. Let (T_1, T_2) be the special C_n-move and $h : B^3 \to S^3$ the orientation preserving embedding such that $K \cap h(B^3) = h(T_1)$, $K_1 \cap h(B^3) = h(T_2)$ and $K - h(B^3) = K_1 - h(B^3)$. We can sweep the link balls and then slide the bands out of the ball $h(B^3)$ by an ambient isotopy of J which fixes K_1 setwisely. Note that this is possible by the triviality of the tangle T_2. Then we choose the link ball and bands in $h(B^3)$ so that K_1 is a band sum of K and a special C_n-link model. Note that the new link ball and the bands are disjoint from the previous ones. Therefore J is a band sum of K and the special C_n-link models. □

LEMMA 2.6. *Let (α, β) be a C_k-link model. Let $\beta_1, \beta_2, \cdots, \beta_{k+1}$ be the components of β. We give an arbitrary orientation to each β_i. Let K, J_1 and J_2 be oriented knots. Suppose that $J_1 = \Omega(K; b_1, \cdots, b_{k+1}; (\alpha, \beta); \varphi)$ and $J_2 = \Omega(K; c_1, \cdots, c_{k+1}; (\alpha, \beta); \psi)$ such that $\partial b_i \supseteq \varphi(\beta_i)$ and $\partial c_i \supseteq \psi(\beta_i)$ for each i. For each of $b_i \cap K$ and $c_i \cap K$ we give an orientation that is coherent to the orientation of $\varphi(\beta_i)$ and $\psi(\beta_i)$ in b_i and c_i respectively.*

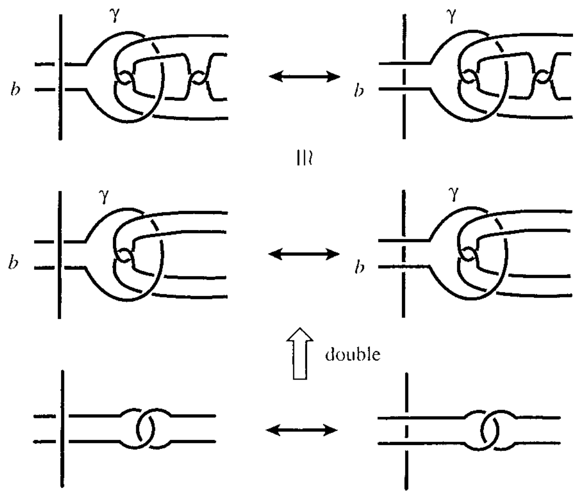

FIGURE 2.11.

Suppose that the ordered sets of oriented arcs $(b_1 \cap K, b_2 \cap K, \cdots, b_{k+1} \cap K)$ *and* $(c_1 \cap K, c_2 \cap K, \cdots, c_{k+1} \cap K)$ *are isotopic on the circle* K*. Then the knots* J_1 *and* J_2 *are* C_{k+1}*-equivalent.*

PROOF. First we claim that a crossing change between a band and a string is equivalent to a C_{k+1}-move. We will show this by an induction on k. When $k = 1$ the crossing change is nothing but a C_2-move. See Fig. 2.10.

Let b be the band and γ the component of $\alpha \cup \beta$ whose image intersects with b.

First suppose that (α, β) is a double of a C_{k-1}-link model (α', β') such that γ is still a component of $\alpha' \cup \beta'$. Then we have that the crossing change is a 'double' of a crossing change in the case $k - 1$. See for example Fig. 2.11. Then by the hypothesis of the induction we have the result. Therefore it is sufficient to show that up to equivalence (α, β) is a double of some C_{k-1}-link model (α'', β'') such that γ is still a component of $\alpha'' \cup \beta''$.

Consider the sequence of link models $(\alpha(1), \beta(1))$, $(\alpha(2), \beta(2)), \cdots, (\alpha(k), \beta(k)) = (\alpha, \beta)$ such that $(\alpha(j+1), \beta(j+1))$ is a double of $(\alpha(j), \beta(j))$ with respect to the component $\gamma(j)$ of $\alpha(j) \cup \beta(j)$ for each $1 \leq j \leq k-1$. Let $\gamma'(j+1)$ and $\gamma''(j+1)$ be the components of $\alpha(j+1) \cup \beta(j+1)$ that are not components of $\alpha(j) \cup \beta(j)$. If $\gamma'(k) \neq \gamma$ and $\gamma''(k) \neq \gamma$ then we set $(\alpha'', \beta'') = (\alpha(k-1), \beta(k-1))$ and have the conclusion. Therefore we may suppose without loss of generality that $\gamma'(k) = \gamma$. If $\{\gamma'(\ell), \gamma''(\ell)\} \cap \{\gamma(\ell), \gamma(\ell+1), \cdots, \gamma(k-1)\} = \emptyset$ for some $2 \leq \ell \leq k-1$, then by changing the order of doubling we have the conclusion. Suppose none of the cases above occur. Let $\gamma'(2), \gamma''(2)$ and $\gamma'''(2)$ be the components of $\alpha(2) \cup \beta(2)$. Then we easily have that $\gamma'(2)$ and $\gamma'''(2)$, or $\gamma''(2)$ and $\gamma'''(2)$ are still components of $\alpha(k) \cup \beta(k)$. Then by the deformation illustrated in Fig. 2.12 we have the conclusion. In Fig. 2.12 the shaded part represents iteratedly doubled arcs in the sense of link model. See for example Fig. 2.13. Thus we have shown the claim.

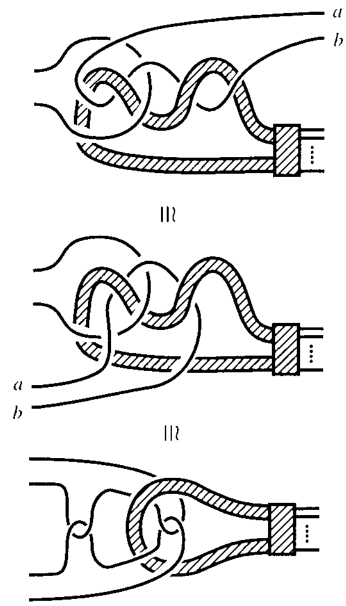

FIGURE 2.12.

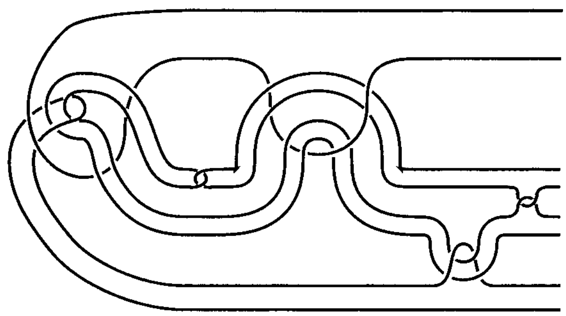

FIGURE 2.13.

Note that a full twist of a band is removable by the crossing change described above as illustrated in Fig. 2.14. Thus we have the result. □

LEMMA 2.7. *Let* (α, β) *be a special* C_k*-link model and* γ_1, γ_2 *the specified components of* $\alpha \cup \beta$. *Let* K *and* J *be oriented knots. Suppose that* $J = \Omega(K; b_1, \cdots, b_{k+1}; (\alpha, \beta); \varphi)$ *such that* $b_i \cap \varphi(\gamma_i) \neq \emptyset$ *for* $i = 1, 2$. *Then there is a special* C_k*-link model* (α', β') *with the same specified components* γ_1, γ_2 *and an oriented knot* $H = \Omega(K; c_1, \cdots, c_{k+1}; (\alpha', \beta'); \varphi')$ *that satisfies the following conditions*;

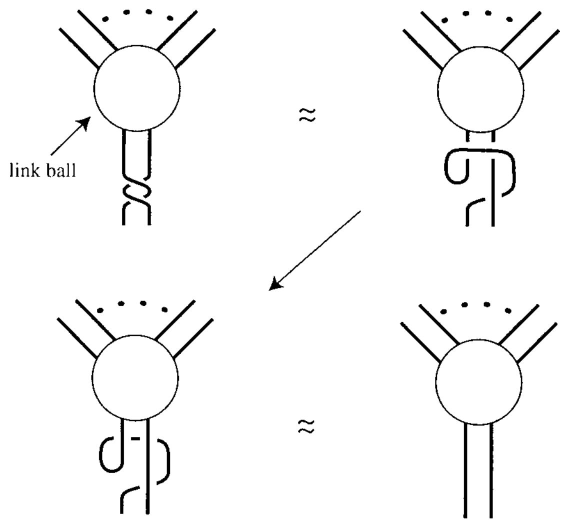

FIGURE 2.14.

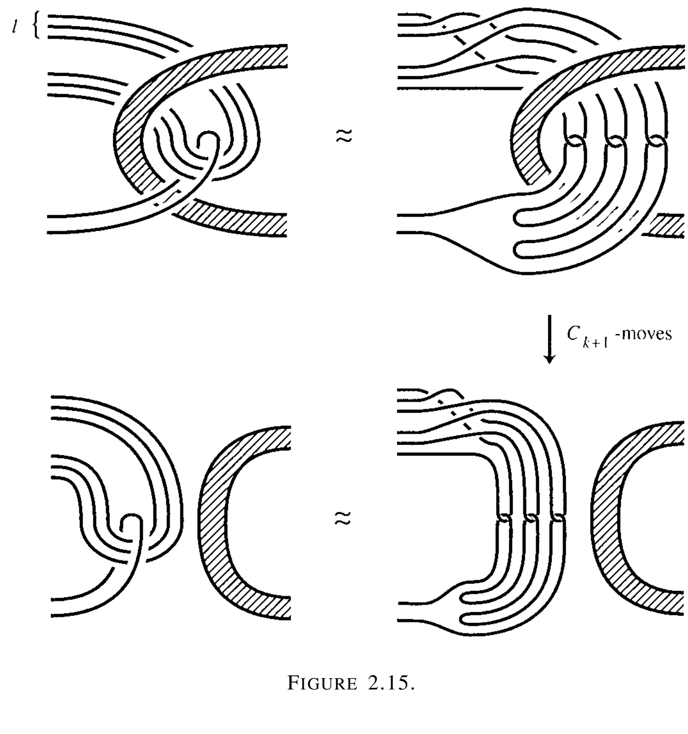

FIGURE 2.15.

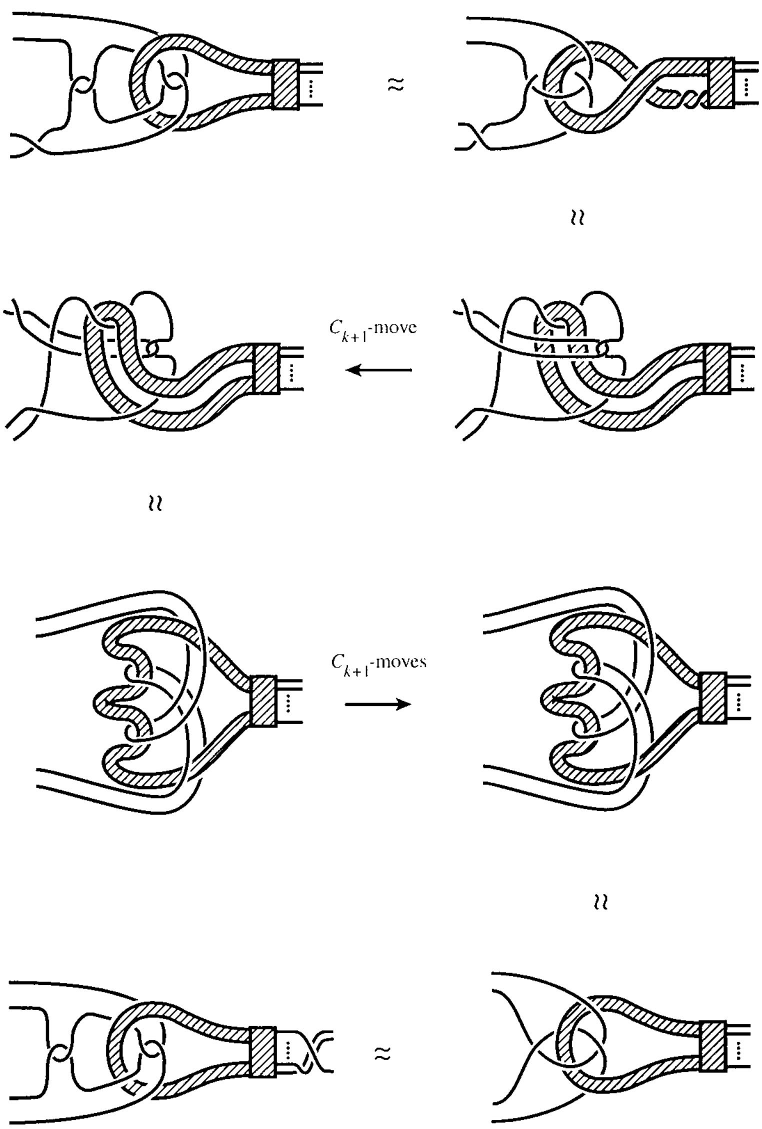

FIGURE 2.16.

(1) $K \cap b_i = K \cap c_i$ *for each* i,
(2) $\varphi'(B^3) = \varphi(B^3)$,
(3) $c_i \cap \varphi'(\gamma_i) = b_i \cap \varphi(\gamma_i)$ *for* $i = 1, 2$,
(4) $b_1 \cup c_1$ *is an annulus*,
(5) $b_2 \cup c_2$ *is a Möbius band*,

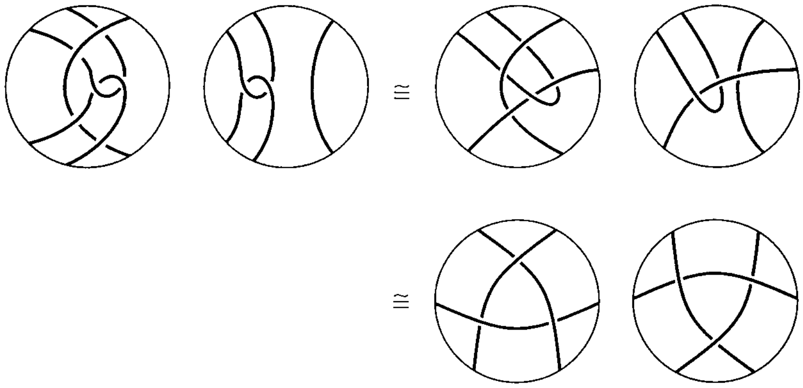

FIGURE 2.17.

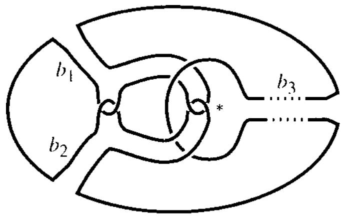

FIGURE 2.18.

(6) *J and H are C_{k+1}-equivalent.*

PROOF. First we note that the move illustrated in Fig. 2.15 is realized by ℓ-times applications of C_{k+1}-moves where the shaded region represents iteratedly doubled k arcs. Then by the deformation illustrated by Fig. 2.16 we have the result. □

PROOF OF THEOREM 1.1. First we note that a C_2-move is equivalent to a *delta move* defined in [9] as illustrated in Fig. 2.17. We note that the same move is defined in [8] independently. It is shown in [9] that knots are transformed into each other by delta moves. Then by Lemma 2.5 we have that K is a band sum of a trivial knot K_0 and some special C_2-link models. Let (α, β) be a special C_2-link model. Suppose that $K = \Omega(K_0; b_1, \cdots, b_{3\ell}; (\alpha, \beta), \cdots, (\alpha, \beta); \varphi_1, \cdots, \varphi_\ell)$ such that $b_i \cap \varphi_j(\beta) \neq \emptyset$ if and only if $3(j-1) < i \leq 3j$. We deform $K \cap (b_1 \cup b_2 \cup b_3 \cup \varphi_1(\alpha \cup \beta))$ up to C_3-equivalence using Lemmas 2.6 and 2.7, the result is still denoted by the same symbols, so that the knot $K' = \Omega(K_0; b_1, b_2, b_3; (\alpha, \beta); \varphi_1)$ is just as illustrated in Fig. 2.18. Note that by a crossing change at $*$ in Fig. 2.18 we have trivial knot.

Next we deform $K \cap (b_4 \cup b_5 \cup b_6 \cup \varphi_2(\alpha \cup \beta))$ up to C_3-equivalence using Lemmas 2.6 and 2.7 so that they are as illustrated in Fig. 2.19.

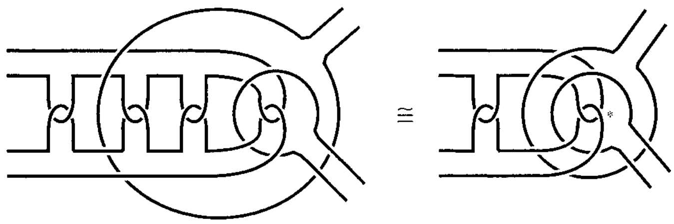

FIGURE 2.19.

We continue similar deformations and finally have a knot K_1 that is C_3-equivalent to K so that the crossing change at the crossing corresponding to $*$ in Fig. 2.19 deforms K_1 into a trivial knot.

Since K and K_1 are C_3-equivalent, we can express K as a band sum of K_1 and some special C_3-link models. Then by similar deformations of K up to C_4-equivalence we have a knot K_2 and a crossing $*$ whose change deforms K_2 into a trivial knot. Then we express K as a band sum of K_2 and some special C_4-link models. We continue the process above and finally have a knot K_{n-1} that is C_{n+1}-equivalent to K. Note that the unknotting number of K_{n-1} is 0 or 1. By the result in [12], we have the following: There exists a C_{n+1}-move such that by operating this C_{n+1}-move for K_{n-1} repeatedly, we have an infinite sequence of mutually C_{n+1}-equivalent knots $J_1'' = K_{n-1}, J_2'', J_3'', \cdots$, no two of whose order $n+1$ Vassiliev invariants coincide. Note that each J_m'' can be expressed as a band sum of K_{n-1} and some special C_{n+1}-link models. By a similar deformation up to C_{n+2}-equivalence we have a knot J_m' with unknotting number 0 or 1. Since C_{n+2}-equivalence does not change order $n+1$ Vassiliev invariants we have an infinite sequence of mutually C_{n+1}-equivalent knots $J_1' = K_{n-1}, J_2', J_3', \cdots$. At most one of them is a trivial knot. Therefore by removing it if it be we have the desired sequence $J_1, J_2, \cdots$. □

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