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Real Hypersurfaces of Complex Space Forms in Terms of Ricci $*$-Tensor

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Abstract. It is known that there are no Einstein real hypersurfaces in complex space forms equipped with the Kähler metric. In the present paper we classified the $*$-Einstein real hypersurfaces M in complex space forms $M_n(c)$ and such that the structure vector is a principal curvature vector.

1. Introduction.

The study of real hypersurfaces in complex space forms has been an active field over the past decade. Let M^{2n-1} be a connected real hypersurface of complex space forms $M_n(c)$, $n \geq 2$, $c \neq 0$ with the Kähler metric of constant holomorphic sectional curvature $4c$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the Kähler structure (J, G) of $M_n(c)$. T. E. Cecil and P. J. Ryan proved that there are no Einstein real hypersurfaces of $P_n(\mathbf{C})$[2]. And many differential geometers studied the real hypersurfaces of $M_n(c)$ in terms of Ricci tensor.

On the other hand, S. Tachibana introduced the notion of Ricci $*$-tensor field on almost Hermitian manifolds [11]. We apply this notion of Ricci $*$-tensor to almost contact manifolds. Moreover, we investigate the $*$-Einstein real hypersurfaces of complex space forms.

THEOREM 1.1. *Let M be a connected $*$-Einstein real hypersurface of $P_n(\mathbf{C})$ of constant holomorphic sectional curvature $4c > 0$, whose structure vector field ξ is a principal curvature vector. Then M is an open subset of one of the following*:

(i) *a geodesic hypersphere*,

(ii) *a tube over a totally geodesic complex projective space $P_k(\mathbf{C})$ of radius $\pi r/4$, where $0 < k < n-1$ and r is a positive number satisfying $4c = 4/r^2$*,

(iii) *a tube over a complex quadric Q_{n-1} and $P^n(\mathbf{R})$*.

THEOREM 1.2. *Let M be a connected $*$-Einstein real hypersurface of $H_n(\mathbf{C})$ of constant holomorphic sectional curvature $4c < 0$, whose structure vector field ξ is a principal curvature vector. Then M is an open subset of one of the following*:

(i) *a geodesic hypersphere*,

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(ii) *a tube over a totally geodesic complex hyperbolic hyperplane*,
(iii) *a tube over a totally real hyperbolic space* $H^n(\mathbf{R})$,
(iv) *a horosphere.*

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2. Preliminaries.

A complex n-dimensional Kähler manifold of constant holomorphic sectional curvature $4c$ is called a complex space form, which is denoted by $M_n(c)$. A complete and simply connected complex space form is a complex projective space $P_n(\mathbf{C})$, a complex Euclidean space $\mathbf{C}^n$ or a complex hyperbolic space $H_n(\mathbf{C})$, according as $c > 0, c = 0$ or $c < 0$. Let M be a real hypersurface of complex space forms $M_n(c)$, $c \neq 0$. In a neighborhood of each point, we take a unit normal vector field N in $M_n(c)$. The Riemannian connections $\widetilde{\nabla}$ in $M_n(c)$ and ∇ in M are related by the following formulas for arbitrary vector fields X and Y on M.

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N\,,$$

$$\widetilde{\nabla}_X N = -AX\,,$$

where g denotes the Riemannian metric of M induced from the Kähler metric G of $M_n(c)$ and A is the shape operator of M in $M_n(c)$. We denote by TM the tangent bundle of M. An eigenvector X of the shape operator A is called a *principal curvature vector*. Also an eigenvalue λ of A is called a *principal curvature*. We know that M has an almost contact metric structure induced from the Kähler structure (J, G) on $M_n(c)$: We define a $(1, 1)$-tensor field ϕ, a vector field ξ and a 1-form η on M by $g(\phi X, Y) = G(JX, Y)$ and $g(\xi, X) = \eta(X) = G(JX, N)$. Then we have

$$\phi^2 X = -X + \eta(X)\xi\,, \quad \eta(\xi) = 1\,, \quad \phi\xi = 0\,. \tag{1}$$

It follows from (1.1) that

$$\nabla_X \xi = \phi AX\,.$$

Let $\widetilde{R}$ and R be the curvature tensors of $M_n(c)$ and M, respectively. From the expression of the curvature tensor $\widetilde{R}$ of $M_n(c)$, we have the following equations of Gauss and Codazzi:

$$\begin{aligned} R(X, Y)Z = {} & c(g(Y, Z)X - g(X, Z)Y \\ & + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z) \\ & + g(AY, Z)AX - g(AX, Z)AY\,, \end{aligned}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = c(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi)\,.$$

By the Gauss equation, the Ricci tensor of M is defined by

$$S(X, Y) = \mathrm{trace}(Z \mapsto R(Z, X)Y)$$

as

$$S(X, Y) = c((2n + 1)g(X, Y) - 3\eta(X)\eta(Y)) + hg(AX, Y) - g(A^2X, Y),$$

where h denotes the trace of the shape operator A. Now we prepare without proof the following in order to prove our results.

LEMMA 2.1 ([7]). *If ξ is a principal curvature vector, then the corresponding principal curvature α is locally constant.*

LEMMA 2.2 ([7]). *Assume that ξ is a principal curvature vector and the corresponding principal curvature is α. If $AX = \lambda X$ for $X \perp \xi$, then we have $(2\lambda - \alpha)A\phi X = (\alpha\lambda + 2c)\phi X$.*

This result is due to M. Okumura for $P_n(\mathbf{C})$ and to S. Montiel and A. Romero for $H_n(\mathbf{C})$.

PROPOSITION 2.3 ([6], [9]). *Let M, where $n \geq 2$, be a real hypersurface in $M_n(c)$ of constant holomorphic sectional curvature $4c \neq 0$. Then $\phi A = A\phi$ if and only if M is an open subset of the following*:

(i) *a geodesic hypersphere*,

(ii) *a tube over totally geodesic complex space form $M_k(c)$, where $0 < k \leq n - 1$.*

These real hypersurfaces of $M_n(c)$ are homogeneous one. R. Takagi classified the homogeneous real hypersurfaces of $P_n(\mathbf{C})$.

PROPOSITION 2.4 ([12]). *Let M be a homogeneous real hypersurface of $P_n(\mathbf{C})$ of constant holomorphic sectional curvature $4c > 0$. Then M is locally congruent to the following*:

A_1: *Geodesic hyperspheres.*

A_2: *Tubes over totally geodesic complex projective spaces $P_k(\mathbf{C})$, where $0 < k < n - 1$.*

B: *Tubes over complex quadrics Q_{n-1} and $P^n(\mathbf{R})$.*

C: *Tubes over the Segre embedding of $P_1(C) \times P_{(n-1)/2}(\mathbf{C})$, and $n(\geq 5)$ and is odd.*

D: *Tubes over the Plücker embedding of the complex Grassmann manifold $G_{2,5}(\mathbf{C})$. Occur only for $n = 9$.*

E: *Tubes over the canonical embedding of the Hermitian symmetric space $SO(10)/U(5)$. Occur only for $n = 15$.*

We know that there are three types of homogeneous real hypersurfaces in $P_n(\mathbf{C})$ with at most three distinct principal curvatures. We will list the principal curvatures of these real hypersurfaces, see [10]. Here, r is a positive number and the holomorphic sectional curvature of $P_n(\mathbf{C})$ is $4c = 4/r^2$. The parameter u is chosen so that the tubes have radius ru.

PROPOSITION 2.5. *The geodesic hyperspheres (Type A_1) in $P_n(\mathbf{C})$ have two distinct principal curvatures*: $\lambda = (1/r)\cot u$ *of multiplicity* $2n-2$ *and* $\alpha = (2/r)\cot 2u$ *of multiplicity* 1, *where* $0 < u < \pi/2$.

PROPOSITION 2.6. *The Type A_2 real hypersurfaces in $P_n(\mathbf{C})$ have three distinct principal curvatures:* $\lambda_1 = -(1/r)\tan u$ *of multiplicity* $2k$, $\lambda_2 = (1/r)\cot u$ *of multiplicity* $2l$, *and* $\alpha = (2/r)\cot 2u$ *of multiplicity* 1, *where* $k > 0$, $l > 0$, *and* $k + l = n - 1$, $0 < u < \pi/2$.

PROPOSITION 2.7. *The Type B real hypersurfaces in $P_n(\mathbf{C})$ have three distinct principal curvatures:* $\lambda_1 = -(1/r)\cot u$ *of multiplicity* $n-1$, $\lambda_2 = (1/r)\tan u$ *of multiplicity* $n-1$, *and* $\alpha = (2/r)\tan 2u$ *of multiplicity* 1, *where* $0 < u < \pi/4$.

PROPOSITION 2.8. *The Type C, D and E real hypersurfaces in $P_n(\mathbf{C})$ have five distinct principal curvatures.*

In addition, many authors contributed to this result.

PROPOSITION 2.9 ([3]). *Let M be a real hypersurface of $P_n(\mathbf{C})$. Then M has constant principal curvatures and ξ is a principal curvature vector if and only if M is locally congruent to a homogeneous real hypersurface.*

In complex hyperbolic space $H_n(\mathbf{C})$, this classification was begun by S. Montiel and completed by J. Berndt.

PROPOSITION 2.10 ([1]). *Let M be a real hypersurface of $H_n(\mathbf{C})$ of constant holomorphic sectional curvature $4c < 0$. Then M has constant principal curvatures and ξ is a principal curvature vector if and only if M is locally congruent to the following*:

- A_1: *Geodesic hyperspheres (Type A_{11}) and tubes over totally geodesic complex hyperbolic hyperplanes (Type A_{12}).*
- A_2: *Tubes over totally geodesic $H_k(\mathbf{C})$, where* $0 < k < n-1$.
- B: *Tubes over totally real hyperbolic space $H^n(\mathbf{R})$.*
- N: *Horospheres.*

These real hypersurfaces have at most three constant principal curvatures. Here r is a positive number and the holomorphic sectional curvature of $H_n(\mathbf{C})$ is $4c = -4/r^2$. The parameter u is chosen so that the tubes have radius ru.

PROPOSITION 2.11. *The geodesic hyperspheres (Type A_{11}) in $H_n(\mathbf{C})$ have two distinct principal curvatures:* $\lambda = (1/r)\coth u$ *of multiplicity* $2n-2$ *and* $\alpha = (2/r)\coth 2u$ *of multiplicity* 1, *where* $u > 0$.

PROPOSITION 2.12. *The tubes around complex hyperbolic hyperplanes (Type A_{12}) in $H_n(\mathbf{C})$ have two distinct principal curvatures*: $\lambda = (1/r)\tanh u$ *of multiplicity* $2n-2$ *and* $\alpha = (2/r)\coth 2u$ *of multiplicity* 1, *where* $u > 0$.

PROPOSITION 2.13. *The Type A_2 real hypersurfaces in $H_n(\mathbf{C})$ have three distinct principal curvatures*: $\lambda_1 = (1/r)\tanh u$ *of multiplicity* $2k$, $\lambda_2 = (1/r)\coth u$ *of multiplicity* $2l$, *and* $\alpha = (2/r)\coth 2u$ *of multiplicity* 1, *where* $k > 0, l > 0, k + l = n - 1$ *and* $u > 0$.

PROPOSITION 2.14. *The Type B real hypersurfaces in $H_n(\mathbf{C})$ have three distinct principal curvatures*: $\lambda_1 = (1/r)\coth u$ *of multiplicity* $n - 1$, $\lambda_2 = (1/r)\tanh u$ *of multiplicity* $n - 1$, *and* $\alpha = (2/r)\tanh 2u$ *of multiplicity* 1. *These curvatures are distinct unless* $\coth u = \sqrt{3}$ *in which case* λ_1 *and* α *coincide to make a principal curvature of multiplicity* n, *where* $u > 0$.

PROPOSITION 2.15. *The horospheres* (*Type N*) *are real hypersurfaces of $H_n(\mathbf{C})$ that have two distinct principal curvatures*: $\lambda = 1/r$ *of multiplicity* $2n - 2$, *and* $\alpha = 2/r$ *of multiplicity* 1.

3. $*$-Einstein real hypersurfaces.

We denote by S^* the *Ricci $*$-tensor* of M defined by

$$S^*(X, Y) = \frac{1}{2}\mathrm{trace}(Z \mapsto R(X, \phi Y)\phi Z)\,,$$

as

$$S^*(X, Y) = 2cn(g(X, Y) - \eta(X)\eta(Y)) - g(\phi A\phi AX, Y)\,, \tag{2}$$

for any $X, Y \in TM$.

Further we denote by ρ^* the *$*$-scalar curvature* of M which is the trace of the linear endomorphism Q^* defined by $g(Q^*X, Y) = S^*(X, Y)$ for any $X, Y \in TM$.

We get immediately

$$S^*(X, \xi) = 0\,, \tag{3}$$

$$S^*(\xi, Y) = -\eta(A\phi A\phi Y)\,, \tag{4}$$

for any $X, Y \in TM$, and

$$S^*(\phi X, \phi Y) = S^*(Y, X) + \eta(A\phi A\phi X)\eta(Y)\,,$$

for any $X, Y \in TM$.

Let T^0M be a distribution defined by a subspace

$$T^0_xM = \{X \in T_xM : X \perp \xi_x\}$$

in the tangent space T_xM. From (1), this distribution T^0M is invariant with ϕ and called the *holomorphic distribution*. If Ricci $*$-tensor is a constant multiple of the Riemannian metric for the holomorphic distribution, i.e.

$$S^*(X, Y) = \frac{\rho^*}{2(n-1)}g(X, Y)$$

for $X, Y \in T^0M$ on M, then M is called a *$*$-Einstein* real hypersurface.

LEMMA 3.1. *If M is $*$-Einstein if and only if Ricci $*$-tensor S^* of M satisfies the following equation*:

$$S^*(X, Y) = \frac{\rho^*}{2(n-1)}(g(X, Y) - \eta(X)\eta(Y)) - \eta(X)\eta(A\phi A\phi Y)\,, \tag{5}$$

for any $X, Y \in TM$.

PROOF. By the assumption, we have

$$S^*(X - \eta(X)\xi, Y - \eta(Y)\xi) = \frac{\rho^*}{2(n-1)} g(X - \eta(X)\xi, Y - \eta(Y)\xi)\,, \tag{6}$$

for $X, Y \in TM$. On the other hand, we calculate

$$\begin{aligned} S^*(X - \eta(X)\xi, Y - \eta(Y)\xi) =& S^*(X, Y) - \eta(X)S^*(\xi, Y) - \eta(Y)S^*(X, \xi) \\ &+ \eta(X)\eta(Y)S^*(\xi, \xi)\,. \end{aligned}$$

From the equation (3) and (4), we get

$$S^*(X - \eta(X)\xi, Y - \eta(Y)\xi) = S^*(X, Y) + \eta(X)\eta(A\phi A\phi Y)\,. \tag{7}$$

Combine (6) and (7), we have the conclusion. □

We now discuss the standard examples of $*$-Einstein real hypersurfaces in $M_n(c)$.

Case 1. We will show that the real hypersurfaces of Type A_1 are $*$-Einstein. By Proposition 2.5, 2.11 and 2.12 $AX = \lambda X$ for any $X \in T^0M$. By Lemma 2.2, we have

$$\phi A\phi AX = -\lambda^2 X\,,$$

for $X \in T^0M$. Calculating the equation (2), we have

$$S^*(X, Y) = (2cn + \lambda^2)g(X, Y)\,,$$

for any $X, Y \in T^0M$.

From Proposition 2.5, we calculate Ricci $*$-tensor of geodesic hyperspheres (Type A_1) in $P_n(\mathbf{C})$ with radius ru,

$$S^*(X, Y) = c(2n + \cot^2 u)g(X, Y)\,,$$

for any $X, Y \in T^0M$.

On the other hand, by Proposition 2.11, we have Ricci $*$-tensor of geodesic hyperspheres (Type A_{11}) in $H_n(\mathbf{C})$ with radius ru,

$$S^*(X, Y) = c(2n - \coth^2 u)g(X, Y)\,,$$

for any $X, Y \in T^0M$. And using Proposition 2.12 we get Ricci $*$-tensor of tubes over totally geodesic complex hyperbolic hyperplanes (Type A_{12}) in $H_n(\mathbf{C})$ with radius ru,

$$S^*(X, Y) = c(2n - \tanh^2 u)g(X, Y)\,,$$

for any $X, Y \in T^0M$.

Case 2. The Type A_2 real hypersurfaces have three distinct principal curvatures. Now we decompose the holomorphic distribution that $T^0M = V_{\lambda_1} \oplus V_{\lambda_2}$. We set $\{e_i, e_j\}$ is an orthonormal basis for T^0M, such that e_i $(i = 1, \cdots, 2k)$ are principal curvature vectors with principal curvature λ_1 and e_j $(j = 2k+1, \cdots, 2n-2)$ are principal curvature vectors with principal curvature λ_2. By Proposition 2.3, we know that the principal curvature vectors are invariant with ϕ, so we get

$$A\phi e_i = \lambda_1 \phi e_i\,, \ (i = 1, \cdots, 2k)\,,$$

$$A\phi e_j = \lambda_2 \phi e_j\,, \ (j = 2k+1, \cdots, 2n-2)\,.$$

Any tangent vector field $X \in T^0M$ may be written using Einstein's convention as $X = a^i e_i + b^j e_j$, a^i and b^j are smooth functions on M. We calculate the Ricci $*$-tensor of Type A_2 real hypersurfaces by the equation (2), we have

$$S^*(X, Y) = 2cng(a^i e_i + b^j e_j, Y) + g(\lambda_1^2 a^i e_i + \lambda_2^2 b^j e_j, Y)\,.$$

If Type A_2 real hypersurfaces are $*$-Einstein, we need

$$\lambda_1^2 = \lambda_2^2\,. \tag{8}$$

By Proposition 2.6 we conclude that $\lambda_1 = -1/r$ and $\lambda_2 = 1/r$. The Type A_2 $*$-Einstein real hypersurface M in $P_n(\mathbf{C})$ is a tube of $P_k(\mathbf{C})$, $0 < k < n-1$ of radius $\pi r/4$. Then, we have Ricci $*$-tensor of M,

$$S^*(X, Y) = c(2n+1)g(X, Y)\,,$$

for any $X, Y \in T^0M$.

On the other hand, if there exists Type A_2 $*$-Einstein real hypersurfaces in $H_n(\mathbf{C})$, because of Proposition 2.13 and (8), we have

$$\tanh^4 u = 1\,.$$

It is a contradiction.

Case 3. The Type B real hypersurfaces have three distinct principal curvatures. Now we decompose the holomorphic distribution that $T^0M = V_{\lambda_1} \oplus V_{\lambda_2}$. We take an orthonormal basis $\{e_i, e_j\}$ for T^0M such that $Ae_i = \lambda_1 e_i$ $(i = 1, \cdots, n-1)$ and $Ae_j = \lambda_2 e_j$ $(j = n, \cdots, 2n-2)$. We may write $X = a^i e_i + b^j e_j$ for $X \in T^0M$. By Lemma 2.2 and Proposition 2.7 and 2.14, we have the following:

$$A\phi e_i = \lambda_2 \phi e_i\,, \ (i = 1, \cdots, n-1)\,,$$

$$A\phi e_j = \lambda_1 \phi e_j\,, \ (j = n, \cdots, 2n-2)\,.$$

We get

$$\phi A \phi A X = -\lambda_1 \lambda_2 X\,,$$

for any $X \in T^0M$. By Proposition 2.7, we have Ricci $*$-tensor of Type B real hypersurfaces in $P_n(\mathbf{C})$,

$$S^*(X, Y) = c(2n-1)g(X, Y)\,,$$

for any $X, Y \in T^0M$, where $c > 0$.

By Proposition 2.14, we conclude that Ricci $*$-tensor of Type B real hypersurfaces in $H_n(\mathbf{C})$,

$$S^*(X, Y) = c(2n-1)g(X, Y)\,,$$

for any $X, Y \in T^0M$, where $c < 0$.

Case 4. The Type N real hypersurfaces in $H_n(\mathbf{C})$ are horospheres. By Proposition 2.15, we calculate Ricci $*$-tensor of horospheres,

$$S^*(X, Y) = c(2n-1)g(X, Y)\,,$$

for any $X, Y \in T^0M$, where $c < 0$.

4. Proof of the Theorem.

Adding the result of Lemma 3.1 to the equation (2) we get

$$\begin{aligned}4cn(n-1)(g(X, Y) - \eta(X)\eta(Y)) - 2(n-1)g(\phi A\phi AX, Y)\\ = \rho^*(g(X, Y) - \eta(X)\eta(Y)) - 2(n-1)\eta(X)\eta(A\phi A\phi Y)\,.\end{aligned}$$

For any $X \in TM$ we have

$$(4cn(n-1) - \rho^*)(X - \eta(X)\xi) - 2(n-1)(\phi A\phi AX - \eta(X)\phi A\phi A\xi) = 0\,.$$

This, together with (1), shows that

$$\begin{aligned}&(4cn(n-1) - \rho^*)\phi X\\ &\quad + 2(n-1)(A\phi AX - \eta(A\phi AX)\xi - \eta(X)A\phi A\xi + \eta(X)\eta(A\phi A\xi)\xi) = 0\,.\end{aligned}$$

By the assumption, the structure vector field ξ is a principal curvature vector, we get

$$A\phi AX = \frac{\rho^* - 4cn(n-1)}{2(n-1)}\phi X\,.$$

Let X be a unit principal curvature vector in T^0M with principal curvature λ, we calculate the following:

$$A\phi AX = \frac{\lambda(\alpha\lambda + 2c)}{2\lambda - \alpha}\phi X\,.$$

Consequently we have

$$\frac{\lambda(\alpha\lambda + 2c)}{2\lambda - \alpha} = \frac{\rho^* - 4cn(n-1)}{2(n-1)}\,.$$

We get

$$2(n-1)\alpha\lambda^2 + 2(2c(n-1) + 4cn(n-1) - \rho^*)\lambda + \alpha(\rho^* - 4cn(n-1)) = 0\,.$$

From Lemma 2.1, α is constant and by the assumption, $*$-scalar curvature ρ^* is also constant. So, we conclude that M has at most three distinct constant principal curvatures. In the case of M in $P_n(\mathbf{C})$, by Proposition 2.9, M is homogeneous real hypersurface. Using the results of Proposition 2.4, M is locally congruent to one of homogeneous real hypersurfaces of Type

A_1, A_2 and B. And in the case of M in $H_n(\mathbf{C})$, by Proposition 2.10, M is locally congruent to one of real hypersurfaces of Type A_1, A_2, B and N. Because of the Section 3, we proved our theorems. □

5. Some remarks.

5.1. $*$-Einstein real hypersurfaces and pseudo-Einstein real hypersurfaces. A real hypersurface in a complex space form is said to be *pseudo-Einstein* if there are constants a and b such that

$$SX = aX + b\eta(X)\xi$$

for all tangent vectors X. The following results classify pseudo-Einstein real hypersurfaces in $P_n(\mathbf{C})$, the proof can be found in [2].

PROPOSITION 5.1. *Let M, where $n \geq 3$, be a real hypersurface in $P_n(\mathbf{C})$ of constant holomorphic sectional curvature $4c > 0$. Suppose that there are smooth functions a and b such that $SX = aX + b\eta(X)\xi$ for $X \in TM$. Then a and b must be constant and M is an open subset of one of*

(i) *a geodesic hypersphere*,

(ii) *a tube of radius ur over a complex projective space $P_k(\mathbf{C})$, with $0 < k < n-1$, $0 < u < \pi/2$, and $\cot^2 u = k/l$, or*

(iii) *a tube of radius ur over a complex quadric Q_{n-1} where $0 < u < \pi/4$ and $\cot^2 2u = n-2$.*

For complex hyperbolic space, the analogous result was proved by Montiel [8].

PROPOSITION 5.2. *Let M, where $n \geq 3$, be a real pseudo-Einstein hypersurface in $H_n(\mathbf{C})$ of constant holomorphic sectional curvature $4c < 0$. Then M is an open subset of one of*

(i) *a geodesic hypersphere*,

(ii) *a tube over a complex hyperbolic hyperplane*, *or*

(iii) *a horosphere.*

We investigate $*$-Einstein real hypersurfaces in complex space forms in this paper. From these results, we can consider that our results are independent from the classification of pseudo-Einstein real hypersurfaces.

5.2. Non-homogeneous $*$-Einstein real hypersurfaces. Take a regular curve γ in $M_n(c)$ with tangent vector field X. At each point of γ there is a unique complex projective or hyperbolic hyperplane cutting γ so as to be orthogonal not only to X but to JX. The union of these hyperplanes is called a *ruled real hypersurface*.

We remark that the following result:

PROPOSITION 5.3. *Ruled real hypersurfaces are $*$-Einstein real hypersurfaces.*

PROOF. We know that we may write the shape operator A of a ruled real hypersurface M in $M_n(c)$ [4]:

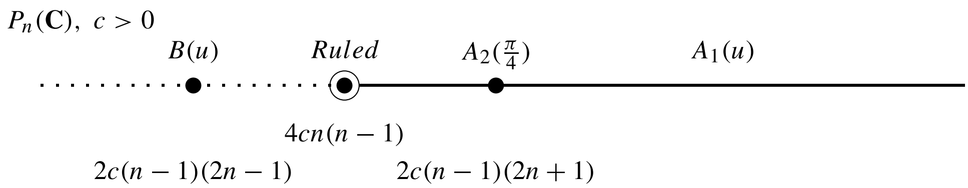
$$\begin{aligned} &A\xi = \mu\xi + \nu U \quad (\nu \neq 0)\,, \\ &AU = \nu\xi\,, \\ &AX = 0 \quad (\text{for any } X \perp \xi, U)\,, \end{aligned}$$

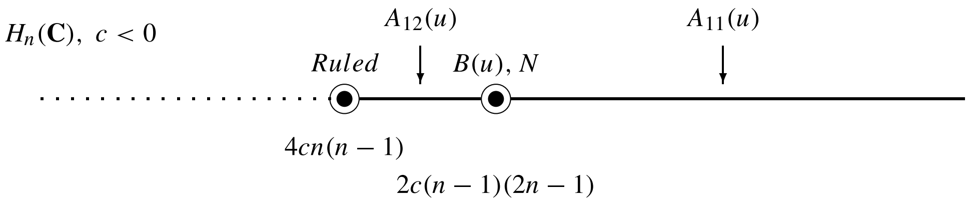
where U is a unit tangent vector field orthogonal to ξ, μ and ν are differential functions on M. The structure vector of ruled real hypersurface is not a principal curvature vector. By calculating the equation (2), we can show that

$$S^*(X, Y) = 2cng(X, Y)\,,$$

for any $X, Y \in T^0M$. □

5.3. $*$-scalar curvature of $*$-Einstein real hypersurfaces. We recall from section 3, we have constant $*$-scalar curvature ρ^* of $*$-Einstein real hypersurfaces. We can find the different situations of the real hypersurfaces in $P_n(\mathbf{C})$ and $H_n(\mathbf{C})$ by the following graphs of ρ^*.





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