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Classification of Sextics of Torus Type

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Dedicated to Professor Tatsuo Suwa on his 60th birthday

Abstract. In [7], the second author classified configurations of the singularities on tame sextics of torus type. In this paper, we give a complete classification of the singularities on irreducible sextic of torus type, without assuming the tameness of the sextics. We show that there exist 121 configurations and there are 5 pairs and a triple of configurations for which the corresponding moduli spaces coincide, ignoring the respective torus decomposition.

1. Introduction.

We consider an irreducible sextic of torus type C defined by

$$C : \{(X, Y, Z) \in \mathbf{P}^2;\ F_2(X, Y, Z)^3 + F_3(X, Y, Z)^2 = 0\} \tag{1}$$

where $F_i(X, Y, Z)$ is a homogeneous polynomial of degree i, for $i = 2, 3$. We consider the conic $C_2 = \{F_2(X, Y, Z) = 0\}$ and the cubic $C_3 = \{F_3(X, Y, Z) = 0\}$. Let $\Sigma(C)$ be the set of singular points of C. A singular point $P \in \Sigma(C)$ is called *inner* (respectively *outer*) with respect to the given torus decomposition (1) if $P \in C_2$ (resp. $P \notin C_2$). We say that C is *tame* if $\Sigma(C) \subset C_2 \cap C_3$. For tame sextics of torus type, there are 25 local singularity types among which 20 appear on irreducible sextics of torus type by [7]. As global singularities, there are 43 configurations of singularities on irreducible tame torus curves. The result in [7] is valid for non-tame sextics of torus type as the sub-configurations of the inner singularities on sextics of torus type. We call them *the inner configuration*. In this paper, we complete the classification of configurations of the singularities on irreducible sextics of torus type.

This paper is composed as follows. In §2, we give the list of topological types for outer singularities and explain basic degenerations among singularities. In §3, we study possible outer configurations of singularities. We start from a given inner configuration, and we determine the possible singularities which can be inserted outside of the conic C_2. We prove that there exist 121 configurations of singularities of non-tame sextics among which there exist 21 maximal configurations (Theorem 4, Corollary 6). In §4, we introduce the notions of *a distinguished configuration moduli space* and *a reduced configuration moduli space* and *a minimal moduli slice*. Minimal moduli slices are very convenient for the topological study of

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plane curves. We prove that the dimension of a minimal slice is equal to the expected minimal dimension (Theorem 11). In §5, we give normal forms for the maximal configurations. In this process, we show that the moduli spaces of certain configurations are not irreducible but their minimal slices have dimension zero and they have normal forms which are mutually interchangeable by a Galois action. However it is not clear if they are isomorphic in the classical topology. See Proposition 17 and Proposition 19. We also prove that there exist 5 pairs and a triple of configurations for which the moduli spaces are identical if we ignore the distinction of inner and outer singularities (Theorem 20).

For reduced non-irreducible sextics of torus type, we will study their configurations in [4].

2. Inner and outer singularities.

2.1. Inner and outer singularities. Let C be an irreducible sextic defined by $f(x, y) = 0$ where $f = f_2^3 + f_3^2$ and $f_i(x, y)$ is a polynomial of degree i for $i = 2, 3$. Here (x, y) is the affine coordinates $x = X/Z$, $y = Y/Z$. Let C_2, C_3 be the conic and the cubic defined by $f_2 = 0$ and $f_3 = 0$ respectively. We assume that the line at infinity is not a component of any of C_2, C_3 and C. Let P be a singular point of C. A singular point P of C is called an *inner* singularity (respectively an *outer* singularity) if P is on the intersection $C_2 \cap C_3$ (resp. $P \notin C_2 \cap C_3$) with respect to the torus decomposition (1). We will see later that the notion of *inner* or *outer* singularity depends on the choice of a torus expression. In [7], second author classified inner singularities. Simple singularities which appear as singularities on sextics of torus type are $A_2, A_5, A_8, A_{11}, A_{14}, A_{17}, E_6$ as inner singularities and $A_1, \cdots, A_5, D_4, D_5, E_6$ as outer singularities (see Proposition 1). We use the following normal forms.

$$A_n : y^2 + x^{n+1} = 0, \quad E_6 : y^3 + x^4 = 0, \quad D_k : y^2 x + x^{k-1}.$$

Non-simple inner singularities on irreducible sextics of torus type are the following ([6]): $B_{3,2j}$, $j = 3, 4, 5$, $B_{4,6}$, $C_{3,k}$, $k = 7, 8, 9, 12, 15$, $C_{6,6}$, $C_{6,9}$, $C_{9,9}$ and Sp_1 where

$$\begin{cases} B_{p,q} : \ y^p + x^q = 0 \text{ (Brieskorn-Pham type)} \\ C_{p,q} : \ y^p + x^q + x^2y^2 = 0, \quad \frac{2}{p} + \frac{2}{q} < 1 \\ Sp_1 : \ (y^2 - x^3)^2 + (xy)^3 = 0\,. \end{cases}$$

Note that $B_{3,3} = D_4$. For outer singularities, a direct computation gives the following.

PROPOSITION 1. *Assume that C is an irreducible sextic of torus type and $P \in C$ is an outer singularity with multiplicity m. Then $m \le 3$ and the local topological type (C, P) is a simple singularity and it takes one of the following.*

1. *If $m = 2$, (C, P) is equivalent to one of $A_1, A_2, \cdots, A_5$.*
2. *If $m = 3$, (C, P) is equivalent to one of D_4, D_5, E_6.*

REMARK 2. The assertion is true for reduced sextics of torus type without the assumption of irreducibility.

PROOF. First observe that the sum of Milnor numbers of inner singularities is bounded by 12 from below, as the generic sextic of torus type has 6 A_2-singularities. By this observation and by the genus formula (see §3), the sum of Milnor numbers of outer singularities is less than or equal to $20-12=8$. By the lower semi-continuity of Milnor number, the Milnor number of (C, P) is greater than or equal to $(m-1)^2$, where m is the multiplicity of C at P. Thus we get $m \le 3$. The rest of the assertion is proved by an easy computation. We may assume that $P = O$, where O is the origin. The generic form of f_2, f_3 are given as

$$(2)\qquad \begin{cases} f_2(x,y) := a_{02}y^2 + (a_{11}x + a_{01})y + a_{20}x^2 + a_{10}x + a_{00} \\ f_3(x,y) := b_{03}y^3 + (b_{12}x + b_{02})y^2 + (b_{21}x^2 + b_{11}x + b_{01})y \\ \qquad\qquad +b_{30}x^3 + b_{20}x^2 + b_{10}x + b_{00}\,. \end{cases}$$

The condition $P \in C$ and $P \notin C_2$ says that $a_{00} = -t^2$, $b_{00} = -t^3$ for some $t \in \mathbf{C}^*$. Using the condition $f_x(O) = f_y(O) = 0$ where f_x, f_y are partial derivatives in x and y respectively, we eliminate coefficients b_{01} and b_{10} as

$$b_{01} := \frac{3}{2}t_0 a_{01}\,, \quad b_{10} := \frac{3}{2}t_0 a_{10}\,.$$

We denote the Newton principal part of f by $NPP(f, x, y)$. Assume that $m = 2$. Then $(C, O) = A_1$ generically. By the action of GL(3, $\mathbf{C}$), we can assume that the tangent direction of (C, O) is given by $y = 0$. The degeneration $A_1 \to A_2$ is given by putting $f_{xx}(O) = f_{xy}(O) = 0$. A direct computation shows that the equivalent class (C, O) can be A_k for $k \le 5$. For example, to make the degeneration $A_2 \to A_3$, we put the coefficient of x^3 in $NPP(f, x, y)$ to be zero. Then $NPP(f, x, y)$ takes the form $c_2y^2 + c_1yx^2 + c_0x^4$ with $c_2 \ne 0$ as $m = 2$. The degeneration $A_3 \to A_4$ takes place when the discriminant of the above polynomial vanishes. Then we take a new coordinate system (x, y_1) so that $c_2y^2 + c_1yx^2 + c_0x^4 = c_2y_1^2$. Then we repeat a similar argument. We can see that $A_5 \to A_6$ makes f to be divisible by y^2 by an easy computation.

Assume that $m = 3$. Generically this gives $(C, O) \cong D_4$. Assume that the 3-jet is degenerated. We may assume (by a linear change of coordinates) that the tangent cone is defined by y^2x or y^3 corresponding either the number of the components in the tangent cone is 2 or 1. Assume that it is given by $y^2x = 0$. Thus the Newton principal part of f is given by

$$-\frac{3}{64}\frac{(a_{01}^2 + 4a_{02}t_0^2)^2y^4}{t_0^2} - \frac{1}{8}(16t_0^3b_{12} + 12t_0^2a_{11}a_{01} + 12t_0^2a_{02}a_{10} + 3a_{01}^2a_{10})xy^2 \\ - \frac{3}{64}\frac{(4a_{20}t_0^2 + a_{10}^2)^2x^4}{t_0^2}\,.$$

Thus $(C,O) \cong D_5$. Further we observe by a direct computation that $a_{10}^2 + 4t_0^2 a_{20} = 0$ makes f reducible. Thus no D_k ($k \geq 6$) appears. If the tangent cone is given by $y^3 = 0$, a similar argument shows that the only possible singularity (C,O) is E_6. □

2.2. Degenerations on sextics of torus type. We consider the basic degenerations among singularities. First, the possibility of the degeneration of outer singularities under fixing the inner singularities is as usual: $A_1 \to A_2 \to A_3 \to A_4 \to A_5 \to E_6$ and $D_4 \to D_5 \to E_6$. Of course, some of the above singularities does not exist when the inner configuration is very restrictive (i.e., far from the generic one $= [6A_2]$).

The degenerations of inner singularities are studied in [7]: $A_2 \to A_5 \to A_8 \to A_{11} \to A_{14} \to A_{17}$ and $A_5 \to E_6$. The degeneration of an outer singularity into an inner singularity is described by the following.

PROPOSITION 3. 1. *An outer A_1 and two inner A_2's degenerate into an E_6.*
2. *An outer A_2 and three inner A_2's degenerate into a $B_{3,6}$.*
3. *An outer A_3 and three inner A_2's degenerate into a $C_{3,7}$.*
4. *An outer A_4 and three inner A_2's degenerate into a $C_{3,8}$.*
5. *An outer A_5 and three inner A_2's degenerate into a $C_{3,9}$.*
There are no other degenerations.

PROOF. The proof is computational. We show the first two degenerations in detail and leave the other cases to the reader. We start from the normal form $f = f_2^3 + f_3^2$ where f_2, f_3 are given as in (2). We assume that C has a node at O which is not on the conic C_2. Putting $f_2(0,0) = -t_0^2$ and $f_3(0,0) = -t_0^3$ for some $t_0 \in \mathbf{C}^*$, we get the normal form:

$$\begin{cases} f_2(x,y) = a_{02}y^2 + (a_{11}x + a_{01})y + a_{20}x^2 + a_{10}x - t_0^2 \\ f_3(x,y) = b_{03}y^3 + (b_{12}x + b_{02})y^2 + (b_{21}x^2 + b_{11}x + 3/2\, t_0 a_{01})y \\ \qquad\qquad +b_{30}x^3 + b_{20}x^2 + 3/2 t_0 a_{10} x - t_0^3\,. \end{cases}$$

We can put $t_0 \to 0$ in this form to see that two inner A_2 singularities are used to create a E_6 singularity: $2A_2 + A_1 \to E_6$. Note that as $f(O) = -t_0^2$, $t_0 \to 0$ implies the conic C_2 approaches to O so that O becomes an inner singularity for $t_0 = 0$. To check the degeneration of inner A_2 singularities, we can look at the resultant $R(f_2, f_3, y)$ of f_2 and f_3 and find that $x = 0$ has a multiplicity two in $R = 0$.

Next we consider that the case $(C,O) = A_2$. We may assume that the tangent cone at O is given by $y = 0$. The corresponding normal form is given by

$$\begin{aligned} f_2(x,y) =& a_{02}y^2 + (a_{11}x + a_{01})y + a_{20}x^2 + A_{10}t_0 x - t_0^2 \\ f_3(x,y) =& b_{03}y^3 + (b_{12}x + b_{02})y^2 + (b_{21}x^2 + (-3/4\, a_{01}A_{10} + 3/2\, a_{11}t_0)x)y \\ & + 3/2\, t_0 a_{01} y + b_{30}x^3 - 3/8\, t_0(A_{10}^2 - 4a_{20})x^2 + 3/2\, t_0^2 A_{10}\, x - t_0^3\,. \end{aligned}$$

Here we have substituted $a_{10} = A_{10}t_0$ so that we can easily see the limit $\lim_{t_0 \to 0} f_i(x,y)$. We can see easily $(C,O) \to B_{3,6}$. We observe also that the cubic C_3 has a node at O as the limit

$t_0 = 0$ and the intersection multiplicity of C_2 and C_3 at O is 3. See [7] for the degeneration $B_{3,6} \to C_{3,7} \to C_{3,8} \to C_{3,9}$.

For $(C, O) = A_3$, the normal form is given as follows and the assertion is easily checked by putting $t_0 = 0$.

$$\begin{aligned} f_2(x, y) =& a_{02}y^2 + (a_{11}x + a_{01})y + a_{20}x^2 + A_{10}t_0x - t_0^2 \\ f_3(x, y) =& b_{03}y^3 + (b_{12}x + b_{02})y^2 + (b_{21}x^2 + (-3/4\, a_{01}A_{10} + 3/2\, a_{11}t_0)x)y \\ & + 3/2\, t_0a_{01}y - 1/16\, A_{10}(A_{10}^2 + 12a_{20})x^3 - 3/8\, t_0(A_{10}^2 - 4a_{20})x^2 \\ & + 3/2\, t_0^2 A_{10}x - t_0^3\, . \end{aligned}$$

The other cases is similar.

2.3. List of configurations of inner singularities. For the classification of non-tame configurations, we start from the classification of the configurations of singularities on tame sextics of torus type [7]. The list of configurations in [7] is valid as the sub-configuration defined by the inner singularities for a sextic which may have outer singularities. Let $C_2 \cap C_3 = \{P_1, \cdots, P_k\}$. The i-vector is by definition the k-tuple of integers given by the intersection numbers $I(C_2, C_3; P_i)$, $i = 1, \cdots, k$. There exist 43 possible configurations as follows, assuming C is irreducible. Put $\mathbf{v}$:= i-vector(C)

1. $\mathbf{v} = [1, 1, 1, 1, 1, 1]$: $t1 = [6A_2]$.
2. $\mathbf{v} = [1, 1, 1, 1, 2]$: $t2 = [4\,A_2,\ A_5]$, $t3 = [4\,A_2,\ E_6]$.
3. $\mathbf{v} = [1, 1, 2, 2]$: $t4 = [2\,A_2,\ 2\,A_5]$, $t5 = [2\,A_2,\ A_5,\ E_6]$, $t6 = [2\,A_2,\ 2\,E_6]$.
4. $\mathbf{v} = [1, 1, 1, 3]$: $t7 = [3\,A_2,\ A_8]$, $t8 = [3\,A_2,\ B_{3,6}]$, $t9 = [3\,A_2,\ C_{3,7}]$, $t10 = [3\,A_2,\ C_{3,8}]$, $t11 = [3\,A_2,\ C_{3,9}]$
5. $\mathbf{v} = [2, 2, 2]$: $t12 = [3\,A_5]$, $t13 = [2\,A_5,\ E_6]$, $t14 = [A_5, 2E_6]$, $t15 = [3\,E_6]$.
6. $\mathbf{v} = [1, 2, 3]$: $t16 = [A_2,\ A_5,\ A_8]$, $t17 = [A_2,\ A_5,\ B_{3,6}]$, $t18 = [A_2,\ A_5,\ C_{3,7}]$, $t19 = [A_2,\ E_6,\ A_8]$, $t20 = [A_2,\ E_6,\ B_{3,6}]$, $t21 = [A_2,\ E_6,\ C_{3,7}]$,
7. $\mathbf{v} = [1, 1, 4]$: $t22 = [2\,A_2,\ A_{11}]$, $t23 = [2\,A_2,\ C_{3,9}^{\natural}]$, $t24 = [2\,A_2,\ B_{3,8}]$, $t25 = [2A_2,\ C_{6,6}]$, $t26 = [2A_2, B_{4,6}]$.
8. $\mathbf{v} = [3, 3]$: $t27 = [2\,A_8]$, $t28 = [A_8,\ B_{3,6}]$, $t29 = [A_8,\ C_{3,7}]$,
9. $\mathbf{v} = [2, 4]$: $t30 = [A_5,\ A_{11}]$, $t31 = [A_5,\ C_{3,9}^{\natural}]$, $t32 = [A_5,\ B_{3,8}]$, $t33 = [E_6,\ A_{11}]$, $t34 = [E_6,\ C_{3,9}^{\natural}]$, $t35 = [E_6,\ B_{3,8}]$
10. $\mathbf{v} = [1, 5]$: $t36 = [A_2,\ A_{14}]$, $t37 = [A_2,\ C_{3,12}]$, $t38 = [A_2,\ B_{3,10}]$, $t39 = [A_2,\ C_{6,9}]$, $t40 = [A_2,\ Sp_1]$.
11. $\mathbf{v} = [6]$: $t41 = [A_{17}]$, $t42 = [C_{3,15}]$, $t43 = [C_{9,9}]$.

Here $C_{3,9}^{\natural}$ is the notation in Pho [7]. The singularities $C_{3,9}$ and $C_{3,9}^{\natural}$ are topologically isomorphic but they are distinguished by $\iota = 3$ and 4 respectively where ι is the local intersection number of the conic C_2 and the cubic C_3.

3. Configurations of non-tame sextics.

3.1. Genus admissible configurations. For the classification, we consider two inequalities by the positivity of the genus formula:

$$g(C) = \frac{(d-1)(d-2)}{2} - \sum_{P\in\Sigma(C)} \delta(C,P) \geq 0 \tag{3}$$

and by the positivity of the class number $n^*(C)$:

$$n^*(C) = d(d-1) - \sum_{P\in\Sigma(C)} (\mu(C,P) + m(C,P) - 1) \geq 0\,. \tag{4}$$

Here $d = \mathrm{degree}(C)$, $\Sigma(C)$ is the set of singular points of C and $\delta(C,P)$ is the δ-genus of C at P which is equal to $\frac{1}{2}(\mu(C,P) + r(C,P) - 1)$ with $r(C,P)$ being the number of local irreducible components at P (see Milnor [2]). The class number $n^*(C)$ of C is defined by the degree of the dual curve C^* where $m(C,P)$ is the multiplicity of C at P. See [3,5] for the class number formula (4).

A configuration Σ is called a *genus-admissible* if the genus and the class number given by the above formulae (3), (4) are non-negative.

There exist 145 configurations which satisfy those inequalities. See Tables 1–5 in the end of this paper. In the list, the first bracket shows the configuration of the inner singularities and the second is that of the outer singularities. For example, $[[6A_2],[3A_1]]$ shows that C has 6 A_2's as inner singularities and 3 A_1's as outer singularities. The vector $(g(C), \mu^*(C), n^*(C), i(C))$ denotes the invariants of C, where $g(C)$ is the genus of the normalization, $\mu^*(C)$ is the sum of Milnor numbers at singular points, $n^*(C)$ is the class number and $i(C)$ is defined by $3d(d-2) - \sum_P \delta(P)$ which is the number of flex points on C. For the calculation of $\delta(P)$, we refer Oka [5]. (In Corollary 12 of [5], there is a trivial mistake. The correct formula is $\bar{\delta}(A_{2p-1}) = 6p$ for any p which follows from Theorem 10 of [5].)

3.2. Existing configurations. The main problem is how to know those configurations which do exist and which do not exist in the list of Tables 1–5 in subsection 6.1.

THEOREM 4. *The possible configurations of singularities of irreducible sextics of torus type with at least one outer singularity are given by Table* 1*–Table* 5 *in the last subsection* 6.1. *There are* 24 *configurations in the table which do not exist* (*they are marked 'No'*) *and the other* 121 *configurations exist.*

Combining the list of the configuration of tame sextics of torus type, there exist 164 configurations on irreducible sextics of torus type.

The column of the table "Existence ?" provides the informations about existence and non-existence and typical degenerations. "No" implies the corresponding configuration does not exist. "Max" implies that the configuration is maximal among irreducible sextics of torus type. The arrow shows a possible degeneration. The last column gives the expected minimal moduli slice dimension, which is defined in §4.

COROLLARY 5. *The fundamental group* $\pi_1(\mathbf{P}^2 - C)$ *of the complement of a sextic* C *with a configuration corresponding to one of the following is isomorphic to* $\mathbf{Z}_2 * \mathbf{Z}_3$ *by* [6].

$$nt\,j,\ j = 1, 2, 3, 4, 5, 19, 25, 26, 27, 33, 43, 44, 45, 54, 61, 68, 72, 73, 74, 90, 92\,.$$

COROLLARY 6. *There exist* 21 *maximal configurations on non-tame sextics of torus type*:

$$nt23 = [[6\,A_2], [3A_2]],\ nt32 = [[4\,A_2, A_5], [E_6]],\ nt47 = [[4\,A_2, E_6], [A_5]]$$
$$nt64 = [[2\,A_2, A_5, E_6], [A_4]], nt67 = [[2\,A_2, A_5, E_6], [2\,A_2]],$$
$$nt70 = [[2\,A_2, 2\,E_6], [A_3]]$$
$$nt78 = [[3\,A_2, A_8], [D_5]], nt83 = [[3\,A_2, A_8], [A_1, A_4]], nt91 = [[3\,A_2, B_{3,6}], [A_2]]$$
$$nt99 = [[A_5, 2E_6], [A_2]], nt100 = [[3E_6], [A_1]], nt104 = [[A_2, A_5, A_8], [A_4]]$$
$$nt110 = [[A_2, E_6, A_8], [A_3]], nt113 = [[A_2, E_6, A_8], [A_1, A_2]],$$
$$nt118 = [[2\,A_2, A_{11}], [A_4]]$$
$$nt123 = [[2\,A_2, C_{3,9}^{\natural}], [A_2]], nt128 = [[2A_8], [A_3]], nt136 = [[E_6, A_{11}], [A_2]]$$
$$nt139 = [[A_2, A_{14}], [A_3]], nt142 = [[A_2, A_{14}], [A_1, A_2]], nt145 = [[A_{17}], [A_2]]\,.$$

In the table, $C_{3,9}$ and $C_{3,9}^{\natural}$ are topologically isomorphic but they are distinguished by $\iota = 3$ and 4 respectively ([7]).

We prove the existence of the maximal configurations by giving explicit minimal slices later in §5. The proof of the existence of other configurations follows from Lemma 12. Note also that the existence of the other configurations is implicitly confirmed when we construct the minimal slices for maximal configurations.

3.3. Proof of the non-existence of 24 configurations. In this subsection, we prove the non-existence of the configurations nt j, j =14,16, 17, 18, 38, 39, 48, 76, 79, 84, 85, 86, 89, 93, 107, 111, 114, 121, 125, 129, 131, 132, 140, 143 in Tables 1–5. It is well-known that the total sum of the Milnor numbers μ^* on sextics is bounded by 19 if the singularities are all simple ([1], [8]). Thus the configurations nt79, nt86, nt111, nt114, nt129, nt132, nt140, nt143 do not exist.

Another powerful tool is to consider the dual curves. We know that the dual singularities of $A_k, k \geq 3$, $C_{3,p}$, $p \geq 7$ and $B_{3,q}$, $q \geq 6$ are generically isomorphic to themselves [5]. If the singularity is not generic, the dual singularity has a bigger Milnor number. The singularity $B_{3,3}$ corresponds to a tri-tangent line in the dual curve C^*. By Bezout theorem, a tri-tangent line does not exist for curves of degree ≤ 5. Thus the existence of $B_{3,3}$ implies $n^*(C) \geq 6$.

The non-existence of the configurations nt14, nt16, nt17, nt18, nt38, nt39, nt48, nt89, nt93, and nt125 can be proved by taking the dual curve information into consideration. For example, consider the configuration nt14 $= [[6A_2], [A_1, B_{3,3}]]$. If such a curve C exists, the dual curve C^* has degree 4, which is impossible. Next we show that the configurations nt16–nt18 do not exist. Assume a curve C with the configuration nt16$= [[6A_2], [A_2, A_3]]$ for example. Then the dual curve C^* has degree 5 and C^* has one A_3 and $4A_2$'s as singularities. By the class formula, the dual curve $C^{**} = (C^*)^*$ have degree 4 which is absurd. The other two can be eliminated in the same discussion.

For nt93, we use the fact that the dual singularity of $C_{3,7}$ is again $C_{3,7}$ ([6]). Assume that there exists a sextic C with configuration nt93. Then the dual curve C^* is a quintic with $C_{3,7}$ and A_2. Then by the Plücker formula, this is ridiculous as $\delta(C_{3,7}) = 6$. Suppose that a sextic with the configuration nt125 exists. Then the dual curve have degree 5 and $B_{3,8}$ as a singularity. However the total sum of the Milnor numbers on an irreducible quintic is bounded by 12, a contradiction. The other configurations are treated in a similar way.

The configurations nt76, nt85, nt107, nt121 and nt131 do not exist as they are not in the list of Yang table [10]. The non-existence of these configurations can be also checked by a direct maple computation. The non-existence of nt84 has to be checked by a direct computation. □

REMARK 7. We remark here that a configuration in the list of Yang does not necessarily exist as a configuration of a sextic of torus type. There are also a certain configurations with only simple singularities which is not a sublattice of a lattice of maximal rank in Yang's list.

4. Moduli spaces.

4.1. Distinguished configuration moduli and reduced configuration moduli. Let Σ_1, Σ_2 be configurations of singularities. In this paper, a configuration is a finite set of topological equivalent classes of germs of isolated curve singularities. We say that $\Sigma := [\Sigma_1, \Sigma_2]$ be a *distinguished configuration* on a sextic of torus type if Σ_1 is the configuration of inner singularities and Σ_2 is the configuration of outer singularities. We put $\Sigma_{red} := \Sigma_1 \cup \Sigma_2$ and we call Σ_{red} a *reduced configuration*. We now introduce several moduli spaces which we consider in this paper. First, recall that spaces of conics and cubics are 6 and 10 dimensional respectively. Let $\tilde{\mathcal{T}}$ be the vector space of dimension 16 which is defined by

$$\tilde{\mathcal{T}} := \{\mathbf{f} = (f_2, f_3);\, \text{degree } f_2 = 2,\ \text{degree } f_3 = 3\}\,.$$

There is a canonical GL(3, **C**)-action on $\tilde{\mathcal{T}}$. The center of GL(3, **C**) is identified with $\mathbf{C}^*$. It defines a canonical weighted homogeneous action on $\tilde{\mathcal{T}}$ and we introduce an equivalence relation $\sim$ by $(f_2, f_3) \sim (f_2', f_3') \Leftrightarrow f_2' = f_2 t^2,\ f_3' = f_3 t^3$ for some $t \in \mathbf{C}^*$. In particular, $(f_2, f_3) \sim (f_2\,\omega^j, \pm f_3)$ for $j = 1, 2$ where $\omega = (\sqrt{3}I - 1)/2$. (We use the notation $I = \sqrt{-1}$.) Let $\mathcal{T}$ be the weighted projective space by the $\mathbf{C}^*$-action and let $\pi : \tilde{\mathcal{T}} \to \mathcal{T}$ be the quotient map. Then PGL(3, **C**) = GL(3, **C**)/$\mathbf{C}^*$ acts on $\mathcal{T}$. Each equivalence class (**f**) defines a sextic of torus type $C(\mathbf{f})$ defined by $f_2(x, y)^3 + f_3(x, y)^2 = 0$. We put

$$\Sigma(\mathbf{f})_{in} := \{(C(\mathbf{f}), P_i);\, f_2(P_i) = 0\}, \quad \Sigma(\mathbf{f})_{out} := \{(C(\mathbf{f}), P_i);\, f_2(P_i) \neq 0\}$$

where $\{P_1, \cdots, P_k\}$ are the singular points of $C(\mathbf{f})$ and $(C(\mathbf{f}), P_i)$ is the topological equivalent class of the germ at P_i. Let $\Sigma = [\Sigma_1, \Sigma_2]$ be a distinguished configuration. The *distinguished* *configuration moduli* $\mathcal{M}(\Sigma) \subset \mathcal{T}$ is defined by the quotient $\tilde{\mathcal{M}}(\Sigma)/\mathbf{C}^*$

$$\tilde{\mathcal{M}}(\Sigma) := \{\mathbf{f} = (f_2, f_3) \in \tilde{\mathcal{T}};\ \Sigma(\mathbf{f})_{in} = \Sigma_1,\ \Sigma(\mathbf{f})_{out} = \Sigma_2\}.$$

The space of sextics, denoted by $\tilde{\mathcal{S}}$, is a vector space of dimension 28 and its quotient by the homogeneous $\mathbf{C}^*$-action is denoted by $\mathcal{S}$. There exist a canonical GL(3, **C**)-equivariant mapping $\tilde{\psi}_{red} : \tilde{\mathcal{T}} \to \tilde{\mathcal{S}}$ which is defined by $\tilde{\psi}_{red}(f_2, f_3) = f_2^3 + f_3^2$ and it induces a canonical PGL(3, **C**)-equivariant mapping $\psi_{red} : \mathcal{T} \to \mathcal{S}$. Let Σ_0 be a reduced configuration. The *reduced configuration moduli* $\mathcal{M}_{red}(\Sigma_0)$ is defined by $\tilde{\mathcal{M}}_{red}(\Sigma_0)/\mathbf{C}^*$ where $\mathbf{C}^*$-action is the scalar multiplication and

$$\tilde{\mathcal{M}}_{red}(\Sigma_0) = \{f \in \tilde{\mathcal{S}}; \exists \Sigma = [\Sigma_1, \Sigma_2],\ \Sigma_0 = \Sigma_1 \cup \Sigma_2,\ \exists \mathbf{f} \in \mathcal{M}(\Sigma),\ \tilde{\psi}_{red}(\mathbf{f}) = f\}.$$

The map $\psi_{red} : \mathcal{M}(\Sigma) \to \mathcal{M}_{red}(\Sigma_{red})$ is not necessarily injective (see Observation 14).

REMARK 8. Let $\mathbf{f} = (f_2, f_3) \in \tilde{\mathcal{M}}([\Sigma_1, \Sigma_2])$ and assume that $f_2^3 + f_3^2 = 0$ is an irreducible sextic and assume that Σ_2 is not empty. Consider the family of sextics C_t : $tf_2^3(x, y) + f_3(x, y)^2 = 0$. By the Bertini theorem, for a generic $t \neq 0$, C_t has only inner singularities and $\Sigma(C_t) = \Sigma_1'$, where simple singularities in Σ_1 are unchanged in Σ_1' and non-simple singularities are replaced by the first generic singularities fixing the singularities of the conic $f_2 = 0$ and the cubic $f_3 = 0$ and their local intersection numbers in Table A' of [7]. For example, inner singularities with a nodal cubic and a smooth conic, with the intersection number 3, any singularity in the series $B_{3,6} \to C_{3,7} \to C_{3,8} \to C_{3,9}$ is replaced by $B_{3,6}$. This is the reason why we need the information of defining polynomials f_2, f_3, not only the geometry of C_2 and C_3.

4.2. Moduli slice and irreducibility. A subspace $A \subset \tilde{\mathcal{M}}(\Sigma)$ is called a *moduli slice* of $\mathcal{M}(\Sigma)$ if its GL(3, **C**)-orbit covers the whole moduli space $\tilde{\mathcal{M}}(\Sigma)$ and A is an algebraic set. A moduli slice is called *minimal* if the dimension is minimum. As we are mainly interested in the topology of the pair $(\mathbf{P}^2, C)$ where C is a sextic defined by $f_2^3 + f_3^2 = 0$, the important point is the connectedness of the moduli. Thus we are interested, not in the algebraic structure of the moduli spaces but in the explicit form of a minimal moduli slice, which we call *a normal form*. Note that the moduli space $\mathcal{M}(\Sigma)$ might be irreducible even if a minimal slice $\mathcal{A}$ is not irreducible. (In such a case, we can replace $\mathcal{A}$ by its irreducible components.)

Points $P_1, P_2, \cdots, P_k$ in $\mathbf{P}^2$ are called *generic* if any three of them are not on a line. Let P_1, P_2, P_3 are generic points and let L_i be lines through P_i, for $i = 1, 2$. We say L_i is a *generic* line through P_i with respect to $\{P_1, P_2, P_3\}$ if L_i does not pass through any of other two points $\{P_j; j \neq i\}$. Observe that two set of generic four points, or of generic three points and two generic lines through two of them are transformed each other by PGL(3, **C**)-action. Note that the dimension of the isotropy group of a point (respectively a point and a line through it) is codimension 2 (resp. 3). As $\dim \mathrm{PGL}(3, \mathbf{C}) = 8$, we can fix, using the above principle either

(a) location of four singularities at generic positions or

(b) three singularities at generic positions and two generic tangent cones.

This technique is quite useful to compute various normal forms.

4.3. Virtual dimension and transversality. In general, the dimension of the moduli space of a given configuration of singularities is difficult to be computed. However in the space of sextics of torus type, the situation is quite simple. Suppose that we are given a sextic defined by (2). Take a point $P = (\alpha, \beta) \in \mathbf{C}^2$ and consider the condition for P to be a singular point of C. For simplicity we assume that $P = (0, 0)$.

(I) First assume that P to be an inner singularity. Let σ be the topological equivalence class of (C, P). We define the integer i-codim(σ) by (the number of independent conditions on the coefficients) -2. Here 2 is the freedom to choose P. For example, the condition for P to be an inner A_2 singularity is simply $f_2(P) = f_3(P) = 0$. So i-codim(A_2) $=$ 0. Assume that $(C, P) \cong A_5$. Then the corresponding condition is $f_2(P) = f_3(P) = 0$ and the intersection multiplicity of C_2 and C_3 at P is 2. This condition is equivalent to $(f_{2x} f_{3y} - f_{2y} f_{3x})(P) = 0$. Thus i-codim($A_5$) $= 1$. Similarly the condition $(C, P) \cong E_6$ is given by $f_2(P) = f_3(P) = 0$ and the partial derivatives f_{3x} and f_{3y} vanishes at P. See Pho [7] for the characterization of inner singularities. Thus we have i-codim(E_6) $= 2$.

Let $\iota = I(C_2, C_3; P)$ be the intersection number of C_2 and C_3 at P. Similar discussion proves that

PROPOSITION 9. *For the inner singularities on sextics of torus type, i-codim is given as follows.*

icodim	1	2	3	4
singularity	A_5	E_6, A_8	$A_{11}, B_{3,6}$	$A_{14}, C^{\natural}_{3,9}$ $C_{3,7}, C_{6,6}$
icodim	5	6	7	8
singularity	$A_{17}, C_{3,12}$ $B_{3,8}, C_{3,8}$ $C_{6,9}, B_{4,6}$	$C_{3,15}, B_{3,10}$ $C_{3,9}, Sp_1$ $C_{9,9}, C_{6,12}$ $D_{4,7}$	$B_{3,12}, Sp_2$	$B_{6,6}$

The proof is immediate from the above consideration and the existence of the degeneration series where each step is codimension one ([7]). The vertical degenerations keep the intersection number ι and it is observed to have codimension one for each arrow in [7]. The first and the second horizontal sequence are induced by increasing ι by one for each arrow. Thus each arrow has codimension one. Recall that P is $C_{6,6}$ singularity if both of C_2 and C_3 has a node at P. Thus we can easily see that i-codim($C_{6,6}$) $= 4$. The degenerations $C_{6,6} \to C_{6,9} \to C_{9,9}$ or $C_{6,9} \to C_{6,12}$ has also codimension one for each arrow ([7]). So the rest of assertion follows immediately from the above consideration.

$$
\begin{array}{ccccccccc}
A_5 & \rightarrow & A_8 & \rightarrow & A_{11} & \rightarrow & A_{14} & \rightarrow & A_{17} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
E_6 & \rightarrow & B_{3,6} & \rightarrow & C^{\natural}_{3,9} & \rightarrow & C_{3,12} & \rightarrow & C_{3,15} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & C_{3,7} & \rightarrow & B_{3,8} & \rightarrow & B_{3,10} & \rightarrow & B_{3,12} \\
 & & \downarrow & & & & & & \\
 & & C_{3,8} & & C_{6,6} & \rightarrow & C_{6,9} & \rightrightarrows & \begin{array}{c} C_{9,9} \\ C_{6,12} \end{array} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & C_{3,9} & & B_{4,6} & \rightarrow & Sp_1 & \rightarrow & Sp_2 \\
 & & & & \downarrow & & & & \\
 & & & & D_{4,7} & & & &
\end{array}
$$

(II) Now we assume that P is an outer singularity. This means $f_2(P) \neq 0$. Let σ be the topological equivalent class of (C, P). We define the integer o-codim(P) by the number of conditions on the space of coefficients of f minus 2. By the argument in the proof of Proposition 1, we can easily see that

PROPOSITION 10. *For an outer singularity on sextics of torus type, we have o-codim* $(A_i) = i,\ i = 1, \cdots, 5$ *and o-codim*$(D_i) = i,\ i = 4, 5$ *and o-codim*$(E_6) = 6$. *Thus in all cases, o-codim*(σ) *is equal to the Milnor number.*

PROOF. For A_1, we need three condition $f(P) = f_x(P) = f_y(P) = 0$. Here f_x, f_y are partial derivatives. Thus o-codim$(A_1) = 3 - 2 = 1$. The other assertion follows from the basic degeneration series of codimension one:

$$A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow A_4 \rightarrow A_5, \quad B_{3,3} = D_4 \rightarrow D_5 \rightarrow E_6 .$$

Note that $B_{3,3}$ singularity is defined by 6 equations, $f(P) = f_x(P) = f_y(P) = f_{x,x}(P) = f_{xy}(P) = f_{yy}(P) = 0$. Thus we have o-codim$(B_{3,3}) = 4$ and other assertion follows from the above degeneration sequence. □

For a given configuration $\Sigma = [\Sigma_1, \Sigma_2]$ on sextics of torus type, we define the *expected minimal moduli slice dimension*, denoted by ems-dim(Σ) by the integer

$$\text{ems-dim}(\Sigma) := 16 - \sum_{\sigma \in \Sigma_1} \text{i-codim}(\sigma) - \sum_{\sigma \in \Sigma_2} \text{o-codim}(\sigma) - 9 .$$

Here 16 is the dimension of sextics of torus type and 9 is the dimension of GL$(3, \mathbf{C})$. On the other hand, we denote the dimension of minimal moduli slice of the distinguished moduli $\mathcal{M}(\Sigma)$ by ms-dim(Σ). When $\mathcal{M}(\Sigma)$ has several irreducible components $\mathcal{M}_1, \cdots, \mathcal{M}_k$ with possibly different dimensions, we define the dimension of minimal moduli slice of each component $\mathcal{M}_i$ similarly and we denote it by ms-dim$(\Sigma, \mathcal{M}_i)$. By the above definition, it is obvious that

$$\text{ms-dim}(\Sigma, \mathcal{M}_i) \geq \text{ems-dim}(\Sigma)\,, \quad \text{ms-dim}(\Sigma) \geq \text{ems-dim}(\Sigma)\,.$$

We say that a component $\mathcal{M}_i$ of $\mathcal{M}(\Sigma)$ has a *transverse moduli slice* or the moduli component $\mathcal{M}_i$ is *transversal* if ms-dim$(\Sigma, \mathcal{M}_i) =$ ems-dim(Σ). If every component $\mathcal{M}_i$ is transverse, we say that $\mathcal{M}(\Sigma)$ has a *transverse moduli slice*.

THEOREM 11. *For any configuration Σ of sextics of torus type, there exists a component $\mathcal{M}_0$ of $\mathcal{M}(\Sigma)$ which is transversal.*

It is probably true that $\mathcal{M}(\Sigma)$ is transverse for any Σ but we do not want to check this assertion for 121 cases. The proof of the above weaker assertion is reduced to the assertion on maximal configurations (see the next section) and to the following proposition.

LEMMA 12. *Assume that Σ degenerates into a maximal configuration Σ' which has a transverse moduli slice. Then the moduli $\mathcal{M}(\Sigma)$ has also a component which is transversal.*

PROOF. By the definition, a minimal moduli slice for $\mathcal{M}(\Sigma')$ can be obtained by adding ν equations on the space of coefficients where $\nu :=$ ems-dim$(\Sigma) -$ ems-dim(Σ'). Thus we have

$$\begin{aligned}\text{ms-dim}(\Sigma') &\geq \text{ms-dim}(\Sigma) - (\text{ems-dim}(\Sigma) - \text{ems-dim}(\Sigma'))\\ &\geq \text{ms-dim}(\Sigma') + (\text{ms-dim}(\Sigma) - \text{ems-dim}(\Sigma)) \geq \text{ms-dim}(\Sigma')\end{aligned}$$

which implies the assertion. □

5. Minimal moduli slices for maximal configurations.

In this section, we give normal forms of minimal moduli slices for the maximal configurations. Using the degeneration argument and Lemma 12, this guarantees the existence of any other non-maximal configurations in Tables 1–5 in the subsection 6.1. We also show that they have transverse minimal moduli slices.

nt23. We consider the minimal moduli slice of $\mathcal{M}(\Sigma_{23})$ with $\Sigma_{23} = [[6A_2], [3A_2]]$ by the following minimal slice condition:

($\star$) three outer A_2's are at $P_0 := (0, 0)$ and $P_1 := (1, 1)$ and $P_3 := (1, -1)$. The (reduced) tangent cones of C at $(1, \pm 1)$ are given by $y = \pm 1$ respectively.

The calculation is easy. We start from the normal form $f = f_2^3 + f_3^2$ where f_2, f_3 are given as in (2). Necessary conditions are

$$f_2(P_i) = -t_i^2, \quad f_3(P_i) = -t_i^3, \quad f_x(P_i) = f_y(P_i) = 0, \ i = 0, 1, 2\,.$$

The assumption on the tangential cones gives $f_{xy}(P_i) = f_{xx}(P_i) = 0, \quad i = 1, 2$. Solving these equations, we get the following normal form with one free parameter $t := t_0$. As ems-dim(Σ_{23}) = 1, it has a transverse minimal moduli slice.

$$
\begin{aligned}
f_2(x,y) =& y^2 - 9x^2t^2 - 3x^2 + 6t^2x + 2x - t^2\\
f_3(x,y) =& \frac{1}{2}(-9y^2t^2x - 3y^2x + 3y^2t^2 + 3y^2 + 3x^3 + 27x^3t^2 + 54x^3t^4 - 3x^2\\
&- 27x^2t^2 - 54t^4x^2 + 18xt^4 + 6t^2x - 2t^4)/t\,.
\end{aligned}
$$

As is well-known, the corresponding sextics are the dual of smooth cubics.

nt32. We consider the moduli space $\mathcal{M}(\Sigma_{32})$ with $\Sigma_{32} = [[4A_2, A_5], [E_6]]$. The irreducibility is easily observed using the slice condition:
($\star$) an inner A_5 is at $(0,0)$ and an outer E_6 is at $(0,1)$ with respective tangent cones defined by $y = 0$ and $y = 1$.

We usually use the $\mathbf{C}^*$-action to normalize the coefficient of y^2 in f_2 to be 1. The normal forms are given by

$$
\begin{aligned}
f_2(x,y) =& y^2 + (-1 - t_1^2)y + a_{02}x^2\\
f_3(x,y) =& \frac{1}{8t_1}(t_1^4 + 6t_1^2 - 3)y^3 + \frac{1}{8t_1}(6 - 6t_1^4)y^2 + \frac{1}{8t_1}(-6a_{02}x^2 - 6t_1^2\\
&+ 6t_1^2a_{02}x^2 - 3t_1^4 - 3)y + \frac{1}{8t_1}(6t_1^2a_{02}x^2 + 6a_{02}x^2)\,.
\end{aligned}
$$

Observe that $\mathcal{M}(\Sigma_{32})$ is irreducible by this expression. We have used 6 dimension of PGL (3, $\mathbf{C}$) for the above slice. To get a minimal slice, we have two more dimension to use, so we can fix a location of an inner A_2. Here, we have two choices: either (a) to choose a location which is on $\mathbf{Q}^2$ or (b) to choose a simple normal form. The case (a) gives as a little complicated normal form. So we choose (b). We choose $t_1 = a_{02} = 1$. This can be done by taking an inner A_2-singularity at (α, β) where

$$
\alpha = -\frac{1}{2}\sqrt{6 - 2I\sqrt{3}}, \quad \beta = (3 + I\sqrt{3})/2\,.
$$

Note that α is not well-defined but α^2 is well-defined. This is enough as $f_2(x,y)$, $f_3(x,y)$ are even in x in the above normal form and the condition implies also $(-\alpha, \beta)$ is another inner A_2. The corresponding minimal slice has dimension 0, and consists of two points and as the moduli is irreducible, we can take the normal form

$$
\begin{aligned}
&f_2(x,y) = y^2 - 2y + x^2, \quad f_3(x,y) = (y^3 - 3y + 3x^2)/2\\
&f(x,y) = (y^2 - 2y + x^2)^3 + (y^3 - 3y + 3x^2)^2/4\,.
\end{aligned}
\tag{5}
$$

Let $f_{(32)}(x,y)$ be the corresponding sextic.

nt47. The moduli space $\mathcal{M}(\Sigma_{47})$ is irreducible and ems-dim(Σ_{47}) = 0 where $\Sigma_{47} := [[4A_2, E_6], [A_5]]$. This can be checked easily using the slice as in nt32:

($\star$) an outer A_5 is at $(0,0)$ and an inner E_6 is at $(0,1)$ with respective tangent cones given by $y = 0$ and $y = 1$.

The corresponding normal form is given as

$$f_2(x, y) = y^2 + (-1 + t_0^2)y - t_0^2 + a_{02}x^2$$
$$f_3(x, y) = \left(-\frac{3}{2}t_0 - \frac{1}{2}t_0^3\right) y^3 + 3y^2 t_0 + y \left(\frac{3}{2}t_0(-1 + t_0^2) + \frac{3}{2t_0}a_{02}x^2\right) - t_0^3 + \frac{3}{2}t_0 a_{02}x^2 .$$

Thus the irreducibility follows from this expression. Observe also that f_2, f_3 are even in x. Now we compute the minimal moduli slice with an additional condition, an inner A_2 at (α, β) where α, β are as in nt32. As a minimal slice, we can take

$$f_2(x, y) = y^2 - \frac{5}{2}y + \frac{3}{2} + \frac{1}{2}x^2$$
$$f_3(x, y) = \sqrt{6}I \left(-\frac{3}{8}y^3 + \frac{3}{2}y^2 + \frac{1}{6}\left(-\frac{45}{4} - \frac{3}{2}x^2\right) y + \frac{1}{6}\left(\frac{9}{2} + \frac{9}{4}x^2\right)\right) .$$

Let $f_{(47)}(x, y)$ be the corresponding sextic. The above normal form proves ms-dim(Σ_{47}) = ems-dim$(\Sigma_{47}) = 0$. It is easy to observe that $8 f_{(32)}(x, y) = f_{(47)}(x, y)$ by a direct computation.

nt67. The moduli space $\mathcal{M}(\Sigma_{67})$ with $\Sigma_{67} = [[2A_2, A_5, E_6], [2A_2]]$ is not irreducible. First we observe that ems-dim$(\Sigma_{67}) = 0$ as before. We consider the minimal moduli slice with the slice condition:

($\star$) two outer A_2's are at $(0, \pm 1)$, an inner E_6 is at $(1, 0)$ and an inner A_5 at $(-1, 0)$.

The corresponding slice reduces to two points defined by $\mathbf{f}_a = (f_{2a}, f_{3a})$ and $\mathbf{f}_b = (f_{2b}, f_{3b})$ where

$$\mathbf{f}_a : \begin{cases} f_{2a} = y^2 + \frac{1}{2} - \frac{1}{2}x^2 + \frac{1}{2}Ix^2\sqrt{3} - \frac{1}{2}I\sqrt{3} \\ f_{3a} = \frac{1}{4}\sqrt{18 - 6I\sqrt{3}}(1 - x + I\sqrt{3}y^2 - x^2 + x^3 + xy^2) \end{cases}$$

$$\mathbf{f}_b : \begin{cases} f_{2b} = y^2 + \frac{1}{2} - \frac{1}{2}x^2 - \frac{1}{2}Ix^2\sqrt{3} + \frac{1}{2}I\sqrt{3} \\ f_{3b} = \frac{1}{4}\sqrt{18 + 6I\sqrt{3}}(1 - x - I\sqrt{3}y^2 - x^2 + x^3 + xy^2) . \end{cases}$$

OBSERVATION 13. They are not in the same orbit of PGL(3, **C**) in $\mathcal{M}(\Sigma_{67})$.

PROOF. For a matrix $A \in$ GL(3, **C**), we define as usual $\phi_A : \mathbf{P}^2 \to \mathbf{P}^2$ by the multiplication from the left. Assume that there is a matrix $A \in$ GL(3, **C**) such that $\mathbf{f}_a^A := \phi_A^*(\mathbf{f}_a) = (\mathbf{f}_b)$, it must keep the singular points $(-1, 0)$, $(1, 0)$. Moreover we observe that $f_{2a}, f_{3a}, f_{2b}, f_{3b}$ are even in y variable. Thus the involution $(x, y) \to (x, -y)$ keep the above polynomials. As the image of outer singularities must be outer singularities, we may assume that $(0, 1)$, $(0, -1)$ are also invariant by ϕ_A. This implies that $A =$ Id in PGL(3, **C**). This is ridiculous. □

OBSERVATION 14. Each of $\psi_{red}(\mathbf{f}_a)$, $\psi_{red}(\mathbf{f}_b)$ has two different torus decompositions in $\mathcal{M}(\Sigma_{67})$.

PROOF. We will show the assertion for $\mathbf{f}_a$. First, two inner A_2's are located at

$$P_1 := \left(\frac{-1}{3}I\sqrt{3}, \frac{1}{3}\sqrt{3}+I\right), \quad P_2 := \left(\frac{-1}{3}I\sqrt{3}, -\frac{1}{3}\sqrt{3}-I\right).$$

We choose a conic $h_2(x, y) = 0$ which passes through four A_2 singularities $(-1, 0)$, $(1, 0)$, P_1, P_2, and cut x-axis vertically at $(1, 0)$. Then another decomposition is given by $\psi_{red}(\mathbf{f}_a) = h_2^3 + h_3^2$ where $h_2(x, y) := y^2 - 1 + x^2$ and

$$h_3(x, y) := \frac{1}{4}(x^3 - x^2 - Iy^2x\sqrt{3} - x + 1 - y^2)\sqrt{18 - 6I\sqrt{3}}.$$

OBSERVATION 15. $\mathbf{h} = (h_2, h_3)$ and $\mathbf{f}_b = (f_{2b}, f_{3b})$ are in the same GL(3, $\mathbf{C}$)-orbit in $\tilde{\mathcal{M}}(\Sigma_{67})$. In particular, $\psi_{red}(\mathbf{f}_a)$ and $\psi_{red}(\mathbf{f}_b)$ are PGL(3, $\mathbf{C}$)-equivalent.

In fact, a direct computation shows that $\phi_B^*(h_2, h_3) = (f_{2b}, f_{3b})$ where

$$B = \begin{bmatrix} \frac{3}{4} & 0 & \frac{-1}{4}I\sqrt{3} \\ 0 & \frac{1}{4}\sqrt{3}+\frac{3}{4}I & 0 \\ \frac{-1}{4}I\sqrt{3} & 0 & \frac{3}{4} \end{bmatrix}.$$

PROPOSITION 16. *The images of the moduli spaces* $\mathcal{M}([[4A_2, A_5], [E_6]])$, $\mathcal{M}([[4A_2, E_6], [A_5]])$ *and* $\mathcal{M}([[2A_2, A_5, E_6], [2A_2]])$ *by the morphism* ψ_{red} *into* $\mathcal{M}([4A_2, A_5, E_6])$ *are the same.*

The first equality $\psi_{red}(\mathcal{M}([[4A_2, A_5], [E_6]])) = \psi_{red}(\mathcal{M}([[4A_2, E_6], [A_5]]))$ is already observed by the above normal forms. Observation 15 proves that $\mathcal{M}([[2A_2, A_5, E_6], [2A_2]])$ is irreducible. Thus it is enough to show that $\psi_{red}(\mathbf{f}_a) \in \psi_{red}(\mathcal{M}([[4A_2, A_5], [E_6]])$. In fact, we have $\psi_{red}(\mathbf{f}_a) = \psi_{red}(\mathbf{g})$ where $\mathbf{g} = (g_2, g_3) \in \psi_{red}(\mathcal{M}([[4A_2, A_5], [E_6]])$ and g_2, g_3 are given by

$$\begin{aligned} g_2(x, y) =& y^2 - 1 + (1 + 2I\sqrt{3})x^2 + 2I\sqrt{3}x \\ g_3(x, y) =& \frac{1}{28}(7xy^2 + 2y^2 + I\sqrt{3}y^2 + 4x^3 + 9Ix^3\sqrt{3} - 2x^2 + 13Ix^2\sqrt{3} - 8x \\ & + 3Ix\sqrt{3} - 2 - I\sqrt{3})\sqrt{-54 - 78I\sqrt{3}}. \end{aligned}$$

nt64. We consider the distinguished configuration moduli $\mathcal{M}(\Sigma_{64})$ with $\Sigma_{64} = [[2A_2, A_5, E_6], [A_4]]$. We have ems-dim$(\Sigma_{64}) = 0$. We consider the minimal slice with respect to:

($\star$) an inner A_5 is at (0,1), an inner E_6 is at $(1, -1)$ with tangent cone $x = 1$ and an outer A_4 is at $(0, 0)$ with tangent cone $y = 0$.

The minimal slice consists of two points $\mathbf{f}_a = (f_{2a}, f_{3a})$ and $\mathbf{f}_b = (f_{2b}, f_{3b})$:

$$\begin{cases} f_{2a}(x,y) = \frac{1}{5}(5y^2\sqrt{5} - 10 + 5yx + 4x\sqrt{5} + 16x - y\sqrt{5} + 5y + 5x^2\sqrt{5} - x^2 \\ \quad -4\sqrt{5} + 11yx\sqrt{5} + 5y^2)/(1+\sqrt{5}) \\ f_{3a} = \frac{1}{125}\sqrt{50+30\sqrt{5}}(250 + 110x^3\sqrt{5} + 88x^3 - 420yx + 110\sqrt{5} \\ \quad -192y^2\sqrt{5} - 210x\sqrt{5} - 15y\sqrt{5} - 48x^2\sqrt{5} + 155y^3 + 366yx^2 \\ \quad +348y^2x\sqrt{5} + 336yx^2\sqrt{5} + 97y^3\sqrt{5} - 510x - 75y + 498y^2x \\ \quad -300yx\sqrt{5} + 30x^2 - 330y^2)/(1+\sqrt{5})^3 . \end{cases}$$

$$\begin{cases} f_{2b}(x,y) = \frac{1}{5}(11yx\sqrt{5} - 5yx - y\sqrt{5} + 10 + 5y^2\sqrt{5} + x^2 + 5x^2\sqrt{5} - 5y - 16x \\ \quad +4x\sqrt{5} - 4\sqrt{5} - 5y^2)/(-1+\sqrt{5}) \\ f_{3b}(x,y) = \frac{1}{125}I\sqrt{-50+30\sqrt{5}}(-250 + 110x^3\sqrt{5} - 88x^3 + 420yx + 110\sqrt{5} \\ \quad -192y^2\sqrt{5} - 210x\sqrt{5} - 15y\sqrt{5} - 48x^2\sqrt{5} - 155y^3 - 366yx^2 \\ \quad +348y^2x\sqrt{5} + 336yx^2\sqrt{5} + 97y^3\sqrt{5} + 510x + 75y - 498y^2x \\ \quad -300yx\sqrt{5} - 30x^2 + 330y^2)/(-1+\sqrt{5})^3 . \end{cases}$$

Note that the stabilizer in PGL(3, $\mathbf{C}$) of three points $(0, 0)$, $(1, -1)$, $(0, 1)$ and two lines $x = 1$ and $y = 0$ is trivial. Thus $\mathbf{f}_a$ and $\mathbf{f}_b$ are not in the same orbit even in the reduced moduli space $\mathcal{M}_{red}([2A_2, A_5, E_6, A_4])$. Thus the reduced moduli has two irreducible components.

PROPOSITION 17. *Two sextics $f_a := f_{2a}^3 + f_{3a}^2$ and $f_b := f_{2b}^3 + f_{3b}^2$ are defined over $\mathbf{Q}(\sqrt{5})$. Let $\iota : \mathbf{Q}(\sqrt{5}) \to \mathbf{Q}(\sqrt{5})$ be the involution induced by the Galois automorphism defined by $\iota(\sqrt{5}) = -\sqrt{5}$. Then $\iota(f_a) = f_b$.*

We do not know if there exists an explicit homeomorphism of the complements of the sextics $f_a = 0$ and $f_b = 0$ in $\mathbf{P}^2$.

nt70. The moduli space $\mathcal{M}(\Sigma_{70})$ with $\Sigma_{70} = [[2A_2, 2E_6], [A_3]]$. The distinguished configuration moduli is irreducible and transversal and ems-dim(Σ_{70}) = ms-dim$(\Sigma_{70}) = 0$. For the computation of a minimal slice, we use the slice condition:

($\star$) an outer A_3 is at the origin with tangent cone $x = 0$ and two inner E_6's are at $(1, \pm 1)$. The tangent cone at $(1, 1)$ is given by $y = 1$.

The normal form is given by

$$\begin{cases} f_2(x,y) = \frac{1}{3}(3y^2 + (6x-6)y - 2x^2 - 2x + 1) \\ f_3(x,y) = \frac{I\sqrt{3}}{9}(x-1)(18y^2 + (9x-9)y - 17x^2 - 2x + 1) . \end{cases}$$

nt78. We consider the moduli slice of $\mathcal{M}(\Sigma_{78})$ where $\Sigma_{78} = [[3A_2, A_8], [D_5]]$. We have ems-dim($\Sigma_{78}$) $= 0$. However the computation of minimal slice turns out to be messy. So we consider the slice $\mathcal{A}$ under the condition:

($\star$) an outer D_5 is at $O = (0, 0)$ with $y = 0$ as the tangent cone of multiplicity 1, and an inner A_8 is at $(1, 1)$ with $y = 1$ as the tangent cone.

The normal form $\mathbf{f} = (f_2, f_3)$ is given by

$$\begin{aligned} f_2(x, y) =& \frac{1}{8t_1^2}(8t_1^4 y - 8t_1^4 + 8y^2t_1^2 + 8a_{10}xt_1^2 - 8ya_{10}xt_1^2 - 8yt_1^2 + 2ya_{10}^2x \\ & - ya_{10}^2 - a_{10}^2x^2) \end{aligned}$$

$$\begin{aligned} f_3(x, y) =& \frac{1}{512}(-24y^3a_{10}^2t_1^4 + 48a_{10}^2t_1^2y^2 + 192y^2t_1^4 + 288t_1^4x^2a_{10}^2 + 512t_1^8 \\ & - 48y^2a_{10}^2xt_1^2 - 16a_{10}^3x^3t_1^2 + 24ya_{10}^3x^2t_1^2 + 1152ya_{10}xt_1^6 \\ & - 192ya_{10}^2xt_1^4 - 48ya_{10}^3xt_1^2 + 3ya_{10}^4x^2 - 768yt_1^8 + 24ya_{10}^2x^2t_1^2 \\ & - 264yt_1^4x^2a_{10}^2 + 384y^2a_{10}xt_1^4 + 48y^2{a_{10}}^3xt_1^2 - 384y^2a_{10}xt_1^6 \\ & + 144y^2a_{10}^2xt_1^4 - 48a_{10}^2y^2t_1^4 - 192y^3{t_1}^4 + 384t_1^6y^3 - 768a_{10}xt_1^6 \\ & + 64y^3t_1^8 - 1152t_1^6y^2 + 768yt_1^6 + 3a_{10}^4y^2 - 8y^3t_1^2a_{10}^3 - 24y^3{t_1}^2a_{10}^2 \\ & - 384ya_{10}xt_1^4 + 192t_1^8y^2 - 6y^2a_{10}^4x + 96yt_1^4{a_{10}}^2)/t_1^5\,. \end{aligned}$$

From this normal form, we see that $\mathcal{A}$ is irreducible and we can fix one special point $\mathbf{f}_a = (f_{2a}, f_{3a})$, substituting $t_1 = 1, a_{10} = -1$, where

$$\begin{aligned} f_{2a}(x, y) =& -\frac{1}{8}y - 1 + y^2 - x + \frac{5}{4}xy - \frac{1}{8}x^2 \\ f_{3a} :=& 1 - \frac{57}{32}xy - \frac{261}{512}x^2y + \frac{21}{256}xy^2 + \frac{1}{32}x^3 - \frac{765}{512}y^2 + \frac{27}{64}y^3 + \frac{3}{16}y + \frac{3}{2}x \\ & + \frac{9}{16}x^2\,. \end{aligned}$$

The isotropy subgroup fixing $(0, 0)$, $(1, 1)$ and two lines $y = 0$ and $y = 1$ is generated by

$$A = \begin{bmatrix} a_1 & a_2 & 0 \\ 0 & a_1 + a_2 & 0 \\ 0 & a_1 + a_2 - a_9 & a_9 \end{bmatrix}.$$

We can easily see that the orbit of $\mathbf{f}_a$ by this isotropy group is the whole slice $\mathcal{A}$. Thus $\mathcal{M}(\Sigma_{78})$ has also a transversal minimal moduli slice which is given by one point $\mathbf{f}_a$. In fact, we can see that $\mathbf{f}_a^A = \mathbf{f}$ where A is defined by

$$a_1 = -\frac{a_{10}}{t_1}\,, \quad a_2 = \frac{1}{25}\frac{8t_1^4 + a_{10}^2 + 17a_{10}t_1^2 + 8t_1^2}{t_1^3}\,, \quad a_9 = t_1\,.$$

nt83. The moduli space $\mathcal{M}(\Sigma_{83})$ with $\Sigma_{83} = [[3A_2, A_8], [A_1, A_4]]$ is irreducible. Here we compute the minimal slice $\mathcal{S}$ with the following slice condition:

($\star$) an outer A_4 is at the origin, an outer A_1 is at $(1, -1)$ and an inner A_8 is at $(1, 1)$. The tangent cones at the origin and at $(1, 1)$ are given by $x = 0$ and $y = 1$ respectively.

Then ems-dim$(\Sigma_{83}) = 0$ and it has a transverse minimal slice which consists of a single point $\{(f_2, f_3)\}$ where

$$\begin{cases} f_2(x, y) = \dfrac{1}{565}(565y^2 + 126yx - 176y + 405x^2 - 936x + 16) \\ f_3(x, y) = \dfrac{1}{319225} I\sqrt{565}(13321y^3 + 28215y^2x - 6294y^2 + 16767yx^2 - 31644yx \\ \qquad +1056y + 18225x^3 - 45198x^2 + 5616x - 64)\,. \end{cases}$$

nt91. We consider the moduli space $\mathcal{M}(\Sigma_{91})$ with $\Sigma_{91} = [[3A_2, B_{3,6}], [A_2]]$. The distinguished configuration moduli is irreducible and transversal and ems-dim$(\Sigma_{83}) =$ ms-dim(Σ_{83}) $= 2$. For the computation of a minimal slice, we use the slice condition:

($\star$) an outer A_2 is at $O = (0, 0)$ with the tangent cone $x = 0$, an inner $B_{3,6}$ is at $(1, 1)$ with the tangent cone $y = 1$ and an inner A_2 is at $(1, -1)$.

The normal form are given by the following polynomials with two-parameters t_1, t_2 ($t_1 \neq 0, t_2 \neq 0$):

$$\begin{cases} f_2(x, y) = y^2 - (t_2x - t_2)y + (1 + t_2 - t_1^2)x^2 + (2t_1^2 - t_2 - 2)x - t_1^2 \\ f_3(x, y) = \dfrac{1}{8t_1}(6t_2y^3 + ((-6t_2 + 12t_1^2 - 3t_2^2)x - 12t_1^2 + 3t_2^2)y^2 + ((6t_2 + 6t_2^2 - 12t_2t_1^2)x^2 \\ \qquad +(-6t_2^2 + 24t_2t_1^2 - 12t_2)x - 12t_2t_1^2)y + (-8t_1^4 + 12t_1^2 - 6t_2 - 3t_2^2 + 12t_2t_1^2)x^3 \\ \qquad +(3t_2^2 + 24t_1^4 - 24t_2t_1^2 + 12t_2 - 36t_1^2)x^2 + (24t_1^2 - 24t_1^4 + 12t_2t_1^2)x + 8t_1^4)\,. \end{cases}$$

nt99. The moduli space $\mathcal{M}(\Sigma_{99})$ with $\Sigma_{99} = [[A_5, 2E_6], [A_2]]$ is not irreducible. First we observe that ems-dim$(\Sigma_{99}) = 0$ as before. We consider the minimal moduli slice with the slice condition:

($\star$) an outer A_2 is at $(-1, 0)$, two inner E_6's are at $(0, \pm 1)$ and an inner A_5 is at $(1, 0)$.

The corresponding slice reduces to two points $\mathbf{f}_a = (f_{2a}, f_{3a})$ and $\mathbf{f}_b = (f_{2b}, f_{3b})$ where

$$\begin{cases} f_{2a}(x, y) = -\dfrac{1}{23}(5 + 4\sqrt{3})(5y^2 - 5 + 23x^2 - 18x - 4x\sqrt{3} - 4y^2\sqrt{3} + 4\sqrt{3}) \\ f_{3a}(x, y) = 2\sqrt{3 + 2\sqrt{3}}(1 + \sqrt{3})(\sqrt{3} + 3x^2 - x\sqrt{3} - 3x - y^2\sqrt{3})x\,. \end{cases}$$

$$\begin{cases} f_{2b}(x, y) = \dfrac{1}{23}(-5 + 4\sqrt{3})(5y^2 - 5 + 23x^2 - 18x + 4x\sqrt{3} + 4y^2\sqrt{3} - 4\sqrt{3}) \\ f_{3b}(x, y) = -2\sqrt{3 - 2\sqrt{3}}(-1 + \sqrt{3})(-\sqrt{3} + 3x^2 + x\sqrt{3} - 3x + y^2\sqrt{3})x\,. \end{cases}$$

The isotropy subgroup fixing the configuration of singularity, except possibly exchanging two E_6 is generated by the involution $\iota(x, y) \to (x, -y)$. However the defining conics and cubics are even in y. Thus $\mathbf{f}_a, \mathbf{f}_b$ are invariant under this involution. Thus the moduli spaces $\mathcal{M}(\Sigma_{99})$ and $\mathcal{M}_{red}((\Sigma_{99})_{red})$ has two irreducible components, like the case nt64. Also we have a similar assertion:

PROPOSITION 18. *Both sextics* $\psi_{red}(\mathbf{f}_a) = f_{2a}^3 + f_{3a}^2 = 0$ *and* $\psi_{red}(\mathbf{f}_b) = f_{2b}^3 + f_{3b}^2 = 0$ *are defined over* $\mathbf{Q}(\sqrt{3})$. *Let* $\iota : \mathbf{Q}(\sqrt{3}) \to \mathbf{Q}(\sqrt{3})$ *be the involution induced by the Galois automorphism defined by* $\iota(\sqrt{3}) = -\sqrt{3}$. *Then* $\iota(\psi_{red}(\mathbf{f}_a)) = \psi_{red}(\mathbf{f}_b)$.

nt100. Let us consider the moduli space $\mathcal{M}(\Sigma_{100})$ with $\Sigma_{100} = [[3E_6], [A_1]]$. The distinguished configuration moduli is irreducible and transversal and ems-dim(Σ_{100}) = ms-dim $(\Sigma_{100}) = 0$. For the computation of a minimal slice, we use the slice condition:

($\star$) an outer A_1 is at $(-1, 0)$ and three inner E_6's are at $(0, \pm 1)$, and $(1, 0)$.

The normal forms are given by

$$f_2(x, y) = y^2 - 5x^2 + 6x - 1\,, \quad f_3(x, y) = 6\sqrt{3}x(x + y - 1)(x - y - 1)\,.$$

This curve has been studied in our previous paper [6].

nt104. The moduli space $\mathcal{M}(\Sigma_{104})$ with $\Sigma_{104} = [[A_2, A_5, A_8], [A_4]]$ is not irreducible. First we observe that ems-dim$(\Sigma_{104}) = 0$ as before. We consider the minimal moduli slice with the slice condition:

($\star$) an outer A_4 is at $(0, 0)$ with the tangent cone $x = 0$, an inner A_8 is at $(1, 1)$ with the tangent cone $y = 1$ and an inner A_5 is at $(1, -1)$.

The corresponding slice reduces to two points $\mathbf{f}_a = (f_{2a}, f_{3a})$ and $\mathbf{f}_b = (f_{2b}, f_{3b})$ where

$$\begin{cases} f_{2a}(x, y) = y^2 + \dfrac{11}{5}yx - \dfrac{11}{5}y - \dfrac{1}{6}x^2 - \dfrac{28}{15}x + \dfrac{31}{30} \\ \qquad\qquad +I\left(-\dfrac{2}{5}yx + \dfrac{2}{5}y - \dfrac{1}{6}x^2 + \dfrac{11}{15}x - \dfrac{17}{30}\right) \\ f_{3a}(x, y) = \dfrac{1}{443682000}\sqrt{-537594690 - 620415330I}(-14148y^3 + 7532 - 25008x \\ \qquad\qquad +3925x^3 - 41895y^2x - 21546yx^2 + 72522yx - 36828y + 12849x^2 \\ \qquad\qquad +42597y^2 + I(-24093x^2 - 18324 + 25497yx^2 - 29529y^2 - 74754yx \\ \qquad\qquad +42696y + 43956x + 6561y^3 + 27990y^2x))\,. \end{cases}$$

$$\begin{cases} f_{2b}(x,y) = y^2 + \frac{11}{5}yx - \frac{11}{5}y - \frac{1}{6}x^2 - \frac{28}{15}x + \frac{31}{30} \\ \qquad -I\left(-\frac{2}{5}yx + \frac{2}{5}y - \frac{1}{6}x^2 + \frac{11}{15}x - \frac{17}{30}\right) \\ f_{3b}(x,y) = \frac{1}{443682000}\sqrt{-537594690 + 620415330I}(-14148y^3 + 7532 - 25008x \\ \qquad +3925x^3 - 41895y^2x - 21546yx^2 + 72522yx - 36828y + 12849x^2 \\ \qquad +42597y^2 - I(-24093x^2 - 18324 + 25497yx^2 - 29529y^2 - 74754yx \\ \qquad +42696y + 43956x + 6561y^3 + 27990y^2x)). \end{cases}$$

PROPOSITION 19. *Let $f_a := f_{2a}^3 + f_{3a}^2$ and $f_b := f_{2b}^3 + f_{3b}^2$ and we consider the sextics $C_a := \{f_a = 0\}$ and $C_b := \{f_b = 0\}$. Let $\varphi : \mathbf{C}[x,y] \to \mathbf{C}[x,y]$ be the Galois involution defined by the complex conjugation on the coefficients. We first observe that $f_b = \varphi(f_a)$. Let $\xi : \mathbf{P}^2 \to \mathbf{P}^2$ be the homeomorphism defined by the complex conjugation $\xi((X,Y,Z) = (\bar{X}, \bar{Y}, \bar{Z})$, or, $\xi(x,y) = (\bar{x}, \bar{y})$ in the affine coordinate. The above observation gives the homeomorphism of the pairs of spaces $\xi : (\mathbf{P}^2, C_a) \to (\mathbf{P}^2, C_b)$. In particular, their complements $\mathbf{P}^2 - C_a$ and $\mathbf{P}^2 - C_b$ are homeomorphic.*

nt110. We consider the moduli space $\mathcal{M}(\Sigma_{110})$ with $\Sigma_{110} = [[A_2, E_6, A_8], [A_3]]$. The distinguished configuration moduli is irreducible and transversal and ems-dim$(\Sigma_{110}) =$ ms-dim $(\Sigma_{110}) = 0$. For the computation of a minimal slice, we use the slice condition:

($\star$) an outer A_3 is at $(0,0)$ with the tangent cone $x = 0$, an inner A_8 is at $(1,1)$ with the tangent cone $y = 1$ and an inner E_6 is at $(1,-1)$.

The defining polynomials are given by

$$\begin{aligned} f_2(x,y) =& \frac{1}{15}(15y^2 + 12yx - 12y + 5x^2 - 22x + 2) \\ f_3(x,y) =& \frac{I\sqrt{30}}{450}(81y^3 + 180y^2x - 99y^2 + 117yx^2 - 234yx + 36y + 40x^3 \\ & - 183x^2 + 66x - 4). \end{aligned}$$

nt113. We consider the moduli space $\mathcal{M}(\Sigma_{113})$ with $\Sigma_{113} = [[A_2, E_6, A_8], [A_1, A_2]]$. First we observe that ems-dim$(\Sigma_{113}) = 0$. We first compute the minimal slice of $\mathcal{M}([[2A_2, E_6, A_5], [A_1, A_1]])$ with respect to:

($\star$) an outer A_1 is at $P := (0,-1)$, an outer A_1 is at $O := (0,0)$, an inner E_6 is at $Q := (-1,-1)$ and an inner A_5 is at $R := (-1,1)$.

The corresponding normal form is given by

$$\begin{aligned} f_2(x,y) =& -(-y^2ta_{01} + t^3y^2 + y^2 + y^2a_{01} - y^2t - t^2y^2 - xta_{01}y + yxa_{01} \\ & - a_{01}yt + a_{01}y + x^2t^3 + 5x^2t^2 - 2x^2ta_{01} - 4x^2a_{01} - 6x^2 - 3xta_{01} \\ & - 3a_{01}x - 6x + 4xt^2 + 2xt^3 + t - 1)/(t-1) \end{aligned}$$

$$
\begin{aligned}
f_3(x,y) = &-\frac{1}{2}(-2+2t+12yxt^2+6y^2+4y^3-20x^2-3a_{01}yt-9xta_{01}\\
&-15x^2ta_{01}-9y^2ta_{01}+9x^2ta_{01}y-8yxt^3+16yxt-9x^2t^2a_{01}\\
&-4yxt^4-6yxa_{01}+6t^2y^2+2t^3y^3-16yx+4x^2t^4+20x^2t^3+6x^2t^2\\
&-20yx^2+2t^4x^3+16t^3x^3-2y^2xt^3-2y^2xt+2y^2xt^4+26yx^2t\\
&-2yx^2t^4-10yx^2t^3+6yx^2t^2-18tx^3+2y^2x-10x^2t-2t^4y^3\\
&+6t^2y^3-12ta_{01}x^3-12x^2a_{01}y+3a_{01}y-9a_{01}x-6y^3ta_{01}\\
&+3y^3t^2a_{01}+3y^2t^2a_{01}+3y^2a_{01}x-12y^2t+6y^2a_{01}+6xt^3+12xt^2\\
&-12x^2a_{01}+3y^3a_{01}-10y^3t+3yx^2t^2a_{01}-18x-3y^2ta_{01}x\\
&-6x^3a_{01}t^2+6yxt^2a_{01})/(t-1)\,.
\end{aligned}
$$

Now the conditions for R (respectively O) to be A_8 (resp. A_2) singularities are given by

$$
A_8:\begin{cases}
g_1 = 2a_{01}-2a_{01}s-a_{01}s^2+4-4s-5s^2+3s^3=0 \quad \text{or}\\
g_2 = 16-7a_{01}{}^2+12a_{01}+59s^5-49s^4-23s^6+3s^7+85a_{01}s^2+2s^3\\
\qquad -40s^2+8a_{01}s+56s-118a_{01}s^3+44a_{01}s^4-2a_{01}s^5-a_{01}s^6\\
\qquad +23a_{01}{}^2s-3a_{01}{}^2s^2-7a_{01}{}^2s^3+2a_{01}{}^2s^4=0
\end{cases}
$$

$$
\begin{aligned}
A_2: H_1 = &-20s^6+120s^5+12a_{01}s^5-12a_{01}s^4-144s^4-24a_{01}{}^2s^3-448s^3\\
&-240a_{01}s^3+768a_{01}s^2+1344s^2+108a_{01}{}^2s^2-768a_{01}s-1152s\\
&-144a_{01}{}^2s+192a_{01}+48a_{01}{}^2+256=0
\end{aligned}
$$

where we put $t = s-1$. It turns out that $g_1 = H_1 = 0$ gives three points, defined by

$$
\begin{aligned}
f_2(x,y) = &\Big((x^2-1-5yx+x-5y+y^2)s^2+\Big(-2-7x-4yx-\frac{17}{2}y^2-4y\\
&+\frac{7}{2}x^2\Big)s+2+\frac{11}{2}y^2-2x^2+\frac{11}{2}y+\frac{11}{2}yx+\frac{11}{2}x\Big)/(-2+2s+s^2)
\end{aligned}
$$

$$
\begin{aligned}
f_3(x,y) = &\Big((6yx^2-15y+19x^3-2-36y^2x-33y^2+3y^3-6yx+3x+21x^2)s^2\\
&+\Big(-21x+9y^2x-\frac{33}{2}y^2-\frac{51}{2}y^3-12y-\frac{33}{2}x^2-4-25x^3-21yx\\
&+\frac{33}{2}yx^2\Big)s+4+\frac{33}{2}y^3+30y^2+3x^2-12yx^2+\frac{33}{2}x+7x^3+\frac{27}{2}y^2x\\
&+21yx+\frac{33}{2}y\Big)/(-4+4s+2s^2)
\end{aligned}
$$

where $-1+2s^3=0$. The other pair $g_2 = H_1 = 0$ is equivalent to $g_2 = 0$ and $2s2^4 - 13s2^3 + 27s2^2 - 19s2 + 5 = 0$. As g_2 has degree 2 in a_{01}, this gives 8 points. Anyway we have that ms-dim$(\Sigma_{113}) = 0$.

nt118. We consider the moduli slice of $\mathcal{M}(\Sigma_{118})$ where $\Sigma_{118} = [[2A_2, A_{11}], [A_4]]$. We have ems-dim$(\Sigma_{118}) = 0$. However the computation of minimal slice turns out to be complicated. So we consider the slice $\mathcal{A}$ under the condition:

($\star$) an inner A_{11} is at $O = (0, 0)$ with the tangent cone $x = 0$ and an outer A_4 is at $(1, 0)$ with $x = 1$ as the tangent cone.

The normal form is given by

$$\begin{cases} f_2(x,y) = -\dfrac{864}{125}\dfrac{a_{11}^2 y^2}{a_{10}^4} + a_{11}xy + \left(-a_{10} - \dfrac{25}{576}a_{10}^4\right)x^2 + a_{10}x \\ f_3(x,y) = \dfrac{1}{8640000}(-155271168y^3a_{11}^3 + 59719680y^2a_{10}a_{11}^2x \\ \qquad +34214400y^2a_{10}^4xa_{11}^2 - 59719680y^2a_{10}a_{11}^2 - 31104000a_{10}^5x^2ya_{11} \\ \qquad -2700000ya_{10}^8x^2a_{11} + 31104000a_{10}^5xya_{11} + 2700000a_{10}^9x^3 \\ \qquad +8640000a_{10}^6x^3 + 78125a_{10}^{12}x^3 - 2700000a_{10}^9x^2 - 17280000x^2a_{10}^6 \\ \qquad +8640000a_{10}^6x)/a_{10}^6 . \end{cases}$$

We can easily see that $\mathcal{A}$ is irreducible and we can fix one special point $\mathbf{f}_a = (f_{2a}, f_{3a})$, substituting $a_{11} = a_{10} = 1$, where

$$\begin{cases} f_{2a}(x,y) = -\dfrac{864}{125}y^2 + yx - \dfrac{601}{576}x^2 + x \\ f_{3a}(x,y) = -\dfrac{11232}{625}y^3 + \dfrac{1359}{125}y^2x - \dfrac{864}{125}y^2 - \dfrac{313}{80}yx^2 + \dfrac{18}{5}yx + \dfrac{18269}{13824}x^3 \\ \qquad -\dfrac{37}{16}x^2 + x . \end{cases}$$

The isotropy subgroup G_0 fixing $(0, 0)$, $(1, 0)$ and two lines $x = 0$ and $x = 1$ is given by

$$G_0 = \left\{ \begin{bmatrix} u & 0 & 0 \\ 0 & w & 0 \\ u-v & 0 & v \end{bmatrix} \in \mathrm{PGL}(3, \mathbf{C})\,;\; u, v, w \in \mathbf{C}^* \right\} .$$

We can also show that the orbit of $\mathbf{f}_a$ by this isotropy group is the whole slice $\mathcal{A}$. Thus $\mathcal{M}(\Sigma_{118})$ has also a transversal minimal moduli slice which is given by one point $\mathbf{f}_a$.

nt123. For the normal forms of $\mathcal{M}(\Sigma_{123})$ with $\Sigma_{123} = [[2A_2, C_{3,9}^{\natural}], [A_2]]$, see the next section.

nt128. Now we consider the moduli space $\mathcal{M}(\Sigma_{128})$ with $\Sigma_{128} = [[2A_8], [A_3]]$. The distinguished configuration moduli is irreducible and transversal and ems-dim$(\Sigma_{128}) = 0$. For the computation of a minimal slice, we use the slice condition:

($\star$) an outer A_3 is at $(-1,0)$ with the tangent cone $x=-1$, an inner A_8 is at $(0,1)$ with the tangent cone $y=1$ and another inner A_8 is at $(0,-1)$.

The defining polynomials are given by

$$\begin{cases} f_2(x,y) = -3y^2 - 6xy - x^2 + 6x + 3 \\ f_3(x,y) = \dfrac{1}{16}(81y^3 + 252y^2x + 207x^2y - 162xy - 81y + 38x^3 - 180x^2 - 90x)\,. \end{cases}$$

nt136. We consider the moduli space $\mathcal{M}(\Sigma_{136})$ with $\Sigma_{136} = [[E_6, A_{11}], [A_2]]$. The distinguished configuration moduli is irreducible and transversal and ems-dim(Σ_{136}) = ms-dim $(\Sigma_{136}) = 0$. For the computation of a minimal slice, we use the slice condition:

($\star$) an inner A_{11} at $(0,0)$ with the tangent cone $x=0$, an outer A_2 at $(1,1)$ with the tangent cone $y+x=2$ and an inner E_6 at $(1,-1)$.

The normal form is given by

$$f_2(x,y) = y^2 + \frac{4}{3}xy - \frac{11}{3}x^2 + 4x$$
$$f_3(x,y) = \frac{1}{36}I(14y^3 + 18y^2x + 12y^2 - 54x^2y + 72xy - 10x^3 - 36x^2 + 48x)\sqrt{6}\,.$$

nt139. We consider the moduli slice of $\mathcal{M}(\Sigma_{139})$ where $\Sigma_{139} = [[A_2, A_{14}], [A_3]]$. We have ems-dim$(\Sigma_{139}) = 0$. We consider the slice $\mathcal{A}$ under the condition:

($\star$) an inner A_{14} is at $O=(0,0)$ with the tangent cone $x=0$ and an outer A_3 at $(1,0)$ with $x=1$ as the tangent cone, and an inner A_2 at $(-1,-1)$.

The corresponding slice is reduced to a single point and we can take the normal form as follows.

$$f_2(x,y) = y^2 - \frac{10}{3}xy + \frac{41}{18}x^2 - \frac{1}{18}x$$
$$\begin{aligned} f_3(x,y) = &-\frac{7}{16}I\sqrt{5}y^3 + \frac{433}{192}Iy^2\sqrt{5}x - \frac{1}{192}Iy^2\sqrt{5} - \frac{27}{8}Iy\sqrt{5}x^2 + \frac{1}{24}Iy\sqrt{5}x \\ &+ \frac{1771}{1152}I\sqrt{5}x^3 - \frac{97}{1728}I\sqrt{5}x^2 + \frac{1}{3456}I\sqrt{5}x\,. \end{aligned}$$

nt142. We consider the moduli slice of $\mathcal{M}(\Sigma_{142})$ where $\Sigma_{142} = [[A_2, A_{14}], [A_1, A_2]]$. We have ems-dim$(\Sigma_{142}) = 0$. The minimal slice under the condition:

($\star$) an inner A_{14} is at $O=(0,0)$ with the tangent cone $x=0$, an outer A_2 is at $(1,0)$ with the tangent cone $x=1$ and an outer A_1 at $(-1,1)$.

The normal form is given by one point described by

$$f_2(x,y) = y^2 + \frac{16}{3}xy + \frac{106}{45}x^2 - 2x$$
$$\begin{aligned} f_3 = &\frac{41}{27}Iy^3\sqrt{5} + \frac{403}{45}Iy^2x\sqrt{5} - \frac{5}{9}Iy^2\sqrt{5} + \frac{122}{15}Iyx^2\sqrt{5} - 6Iyx\sqrt{5} \\ &+ \frac{1354}{675}Ix^3\sqrt{5} - \frac{136}{45}Ix^2\sqrt{5} + \frac{10}{9}I\sqrt{5}x\,. \end{aligned}$$

nt145. We consider the moduli slice of $\mathcal{M}(\Sigma_{145})$ where $\Sigma_{145} = [[A_{17}], [A_2]]$. We have ems-dim$(\Sigma_{145}) = 0$. However the computation of minimal slice turns out to be complicated. So we consider the slice $\mathcal{A}$ under the condition:

($\star$) an A_{17} is at $O = (0, 0)$ with the tangent cone $x = 0$ and an A_2 is at $(1, 0)$.

We note that the tangent cone at A_2 can not be generic. In fact, we see, by computation, that the tangent cone at A_2 must pass through A_{17}. The normal form is given by three dimensional family:

$$\begin{cases} f_2(x, y) = a_{10}b_{02}y^2 + \frac{1}{2}a_{10}b_{11}xy + \left(-a_{10} - \frac{9}{64}a_{10}^4\right)x^2 + a_{10}x \\ f_3(x, y) = \frac{1}{2}b_{02}b_{11}y^3 - \frac{27}{64}y^2xa_{10}^3b_{02} - b_{02}y^2x + \frac{1}{4}y^2xb_{11}^2 + b_{02}y^2 \\ \qquad -\frac{9}{32}b_{11}x^2ya_{10}^3 - b_{11}x^2y + b_{11}xy + x^3 + \frac{9}{16}x^3a_{10}^3 + \frac{27}{512}x^3a_{10}^6 \\ \qquad -\frac{9}{16}x^2a_{10}^3 - 2x^2 + x\,. \end{cases}$$

We can easily see that $\mathcal{A}$ is irreducible and we can fix one special point $\mathbf{f}_a = (f_{2a}, f_{3a})$, substituting $b_{11} = 0$ and $a_{10} = b_{02} = 1$, where

$$f_{2a}(x, y) = y^2 - \frac{73}{64}x^2 + x\,, \quad f_{3a}(x, y) = -\frac{91}{64}xy^2 + y^2 + \frac{827}{512}x^3 - \frac{41}{16}x^2 + x\,.$$

The isotropy subgroup J fixing $(0, 0)$, $(1, 0)$ and one lines $x = 0$ is 3-dimensional and it is given by

$$J = \left\{ M = \begin{bmatrix} v+s & 0 & 0 \\ 0 & u & 0 \\ v & w & s \end{bmatrix} \in \mathrm{PGL}(3, \mathbf{C});\, u, s \neq 0, v \neq -1 \right\}.$$

We can also show that the orbit of $\mathbf{f}_a$ by this isotropy group is the whole slice $\mathcal{A}$. Thus $\mathcal{M}(\Sigma_{145})$ has also a transversal minimal moduli slice which is given by one point $\mathbf{f}_a$.

6. Coincidence of some moduli spaces.

We have seen that there exist $121 (= 145 - 24)$ different distinguished configurations. On the other hand, we assert

THEOREM 20. *For the following six reduced configurations, the corresponding distinguished configurations are not unique*: $[6A_2, A_5]$, $[6A_2, E_6]$, $[6A_2, A_1, A_5]$, $[4A_2, 2A_5]$, $[4A_2, A_5, E_6]$, $[3A_2, C_{3,9}]$. *More precisely, we have*

1. $\psi_{red}([[6A_2], [A_5]]) = \psi_{red}([[4A_2, A_5], [2A_2]])$ (nt5 *and* nt37).
2. $\psi_{red}([[6A_2], [E_6]]) = \psi_{red}([[4A_2, E_6], [2A_2]])$ (nt8, nt52).
3. $\psi_{red}([[6A_2], A_1, A_5]) = \psi_{red}([[4A_2, A_5], A_1, 2A_2])$ (nt 13, nt42).
4. $\psi_{red}([[4A_2, A_5], [A_5]]) = \psi_{red}([[2A_2, 2A_5], 2A_2])$ (nt29, nt60).

5. $\psi_{red}([[4A_2, A_5], E_5]) = \psi_{red}([[4A_2, E_6], [A_5]]) = \psi_{red}([[2A_2, A_5, E_6], [2A_2]])$ (nt32, nt47, nt67).
6. $\psi_{red}([[2A_2, C_{3,9}^{\natural}], [A_2]]) = \psi_{red}([3A_2, C_{3,9}])$ (nt123 *and* t11).

PROOF. We prove the assertion by giving explicit torus decompositions for a given $f \in \mathcal{M}_{red}(\Sigma)$ using minimal moduli slices.

I. We will show that the respective images of $\mathcal{M}(\Sigma_5)$ and $\mathcal{M}(\Sigma_{37})$ into the reduced moduli space $\mathcal{M}_{red}([6A_2, A_5])$ coincide, where $\Sigma_5 = [[6A_2], [A_5]])$ and $\Sigma_{37} = [[4A_2, A_5], [2A_2]])$. As their minimal slice dimensions are both equal to two, this case requires a heavy computation. So we need a special device for the computation. We first compute the normal form of the minimal moduli slice of $\mathcal{M}(\Sigma_5)$, with the slice conditions:

($\star_1$): an outer A_5 is at $O := (0, 0)$ with the tangent cone $x = 0$.

($\star_2$) Two inner A_2's are at $P := (1, 1)$ and $Q := (1, -1)$. The tangent cone at P is given by $y = 1$.

First, we can easily observe that $\mathcal{M}(\Sigma_5)$ is irreducible, by looking at the slice with respect to ($\star_1$). Then we compute the minimal slice with respect to ($\star_1 + \star_2$). There are several components but we can use the following component $\mathcal{A}$ by the irreducibility of $\mathcal{M}(\Sigma_5)$.

$$(6) \qquad \begin{cases} f_2(x, y) = y^2 + (-1 - a_{10} + t_0^2)x^2 + a_{10}x - t_0^2 \\ f_3(x, y) = -\dfrac{1}{2}(-3y^2xt_0^2 + 3y^2a_{10}x + 6y^2x - 3t_0^2y^2 + 4x^3t_0^4 - 9x^3t_0^2 \\ \qquad -3x^3a_{10}t_0^2 + 3x^3a_{10} + 6x^3 - 6x^2t_0{}^4 + 15x^2t_0^2 + 6x^2a_{10}t_0^2 \\ \qquad -6x^2a_{10} - 12x^2 - 3a_{10}t_0^2x + 2t_0^4)/t_0 \,. \end{cases}$$

Note that $t_0 = f_3(0, 0)/f_2(0, 0)$. We observe that $f_2(x, y)$, $f_3(x, y)$ are even in y-variable and t_0 is even in $f_2(x, y)$ and in $t_0 f_3(x, y)$. Thus the sextic $f_2^3 + f_3^2 = 0$ is symmetric with respect to x-axis and the change $t_0 \to -t_0$ does not change the class of (f_2, f_3) in $\mathcal{M}(\Sigma_5)$. In fact, this is the reason we consider the above slice condition. For the computation of the minimal slice $\mathcal{M}(\Sigma_{37})$, we consider the slice $\mathcal{B}$ with the condition:

($\star_3$) Two outer A_2's at P, Q and an inner A_5 at O. The tangent cone at O and P are given by $x = 0$ and $y = 1$.

The normal form is given by $g(x, y) = g_2(x, y)^3 + g_3(x, y)^2$ where

$$(7) \qquad \begin{cases} g_2(x, y) = y^2 + a_{20}x^2 + (-1 - a_{20} - t_1^2)x \\ g_3(x, y) = -\dfrac{1}{8}(-6xt_1^2 + 6a_{20}x - 6a_{20} + 6 - 6x - 6t_1^2)y^2/t_1 \\ \qquad -\dfrac{1}{8}(6x^2t_1^2 - 6a_{20}x^2 + 3xt_1^4 + 6xt_1^2 + 6a_{20}x - 9x - 3x^3 \\ \qquad +3x^3a_{20}^2 - 6x^3a_{20}t_1^2 - x^3t_1^4 + 6x^2t_1^4 + 12x^2 - 6x^2a_{20}^2 \\ \qquad +3xa_{20}^2 + 6xa_{20}t_1^2)/t_1 \,. \end{cases}$$

Here $t_1 = f_3(P)/f_2(P)$. We observe that g_2, g_3 are also even in y-variable, while t_1 is even in f_2 and in $t_1 f_3$. The assertion follows from

PROPOSITION 21. *There are canonical bijective morphisms $\xi_1 : \mathcal{A} \to \mathcal{B}$ and $\xi_2 : \mathcal{B} \to \mathcal{A}$ so that $\xi_1 \circ \xi_2$ and $\xi_2 \circ \xi_1$ induce the identity maps on the images $\pi(\mathcal{A})$ and $\pi(\mathcal{B})$.*

PROOF. First we construct ξ_1. Take a $\mathbf{f}_a = (f_2, f_3)$ in $\mathcal{A}$ written as (6). First we show the existence of a conic $g_2(x, y) = 0$ which contains four A_2 singularities of $f_2^3 + f_3^2 = 0$ other than P, Q and A_5 with the tangent line $y = 0$ at O. Four A_2's are symmetric with respect to x-axis and their x-coordinates are the solutions of

$$R_1 = 3x^2t_0^2 + 6b_{12}x^2t_0 + 4x^2b_{12}^2 + 3xt_0^2 + 6b_{12}xt_0 + 3t_0^2 = 0\,.$$

We do not need to solve these solutions explicitly. We start from the form $h_2(x, y) = y^2 + ax^2 + bx + c$. First we put the condition $h_2(0, 0) = 0$. Then we compute the resultant $S(x)$ of h_2 and f_3 in y. Then by the above symmetry condition, S can be written as $S(x) = S_1(x)^2$ where S_1 is a polynomial of degree 3. Then S_1 must be divisible by R_1. This condition is enough to solve the coefficient of h_2 up to a multiplication of a constant, and we have

$$\begin{aligned} h_2(x, y) =&(4t_0^4x + 8x^2t_0^4 - 19x^2t_0^2 - 2a_{10}t_0^2x + t_0^2y^2 - 10x^2a_{10}t_0^2 - 6t_0^2x \\ &+ 12x^2a_{10} + 12x^2 + 3a_{10}^2x^2)/t_0^2\,. \end{aligned}$$

Now we have to find the partner cubic polynomial $g_3(x, y)$ such that $f(x, y) = h_2(x, y)^3t + h_3(x, y)^2$ for some polynomial $h_3(x, y)$. The argument by Tokunaga [9] can not be used as we have an A_5. Instead of using that, we introduce a systematic computational method. For that purpose, we consider the family of polynomial $f_t(x, y) := f(x, y) - h_2(x, y)^3t$. Assuming the existence of such h_3, this family of sextics $f_t = 0$ has four A_2's at the same location as $f = 0$ and an A_5 at the origin. (Note that the tangent line of the conic $h_2 = 0$ at O is the same with that of $f_3 = 0$.) If there is a τ_0 such that f_{τ_0} is a square of a cubic polynomial, $f_{\tau_0}(x, y) = 0$ has an non-isolated singularity at O. So we look for a special value for which the singularity at O is bigger than A_5. In fact such a τ_0 is given by $\tau_0 = 1$ and then we see that f_{τ_0} is a square of a polynomial of degree 3. This technique is quite useful to find the partner cubic for other cases and hereafter we refer this technique as *degeneration method*. The corresponding cubic form is given by

$$\begin{aligned} h_3(x, y) =&\frac{-1}{2} I(t_0^4y^2 - 4t_0^4x + 2t_0^6x - 6y^2xt_0^2 + 5y^2xt_0^4 + 36x^2t_0^2 - 53x^2t_0^4 \\ &+ 20x^2t_0^6 - 48x^3 + 114x^3t_0^2 - 93x^3t_0^4 + 26x^3t_0^6 - t_0^4xa_{10} \\ &- 3y^2a_{10}t_0^2x + 30x^2a_{10}t_0^2 - 22x^2t_0^4a_{10} - 72x^3a_{10} + 117x^3a_{10}t_0^2 \\ &- 49x^3t_0^4a_{10} + 6x^2t_0^2a_{10}^2 - 36x^3a_{10}^2 + 30x^3a_{10}^2t_0^2 - 6a_{10}^3x^3)\sqrt{3}/t_0^3\,. \end{aligned}$$

Thus we define $\xi_1(f_2, f_3) = (h_2, h_3)$. In terms of the parameters, ξ_1 is defined by $\xi_1(a_{10}, t_0) = (a_{20}, t_1)$ where

$$a_{20} = \frac{3a_{10}^2 - 10a_{10}t_0^2 + 12a_{10} - 19t_0^2 + 8t_0^4 + 12}{t_0^2}, \quad t_1 = \frac{I\sqrt{3}(a_{10} - 2t_0^2 + 2)}{t_0}.$$

Now the construction of the morphism $\xi_2 : \mathcal{B} \to \mathcal{A}$ is done in the exact same way. Take $\mathbf{g} = (g_2, g_3) \in \mathcal{B}$ as in (7). First find a conic which pass through six A_2's of $g(x, y) = 0$, and then find the partner cubic by degeneration method. In term of parameters, we define $\xi_2(a_{20}, t_1) = (a_{10}, t_0)$ where

$$a_{10} = -\frac{1}{2}\frac{3a_{20}^2 + 5a_{20}t_1^2 - 6a_{20} - t_1^2 + 3 + 2t_1^4}{t_1^2}, \quad t_0 = \frac{-1}{2}\frac{I\sqrt{3}(-1 + t_1^2 + a_{20})}{t_1}.$$

We can easily check that $\xi_1 \circ \xi_2(g_2, g_3) = (g_2, -g_3)$ and $\xi_2 \circ \xi_1(f_2, f_3) = (f_2, -f_3)$ which implies the assertion. (Recall that $(f_2, f_3) \sim (f_2, -f_3)$.) □

REMARK 22. We remark that the generic element of $\mathcal{A}$ is contained in the moduli space $\mathcal{M}(\Sigma_5)$. However for non-generic element $(f_2, f_3) \in \mathcal{A}$, the slice condition $(\star_1 + \star_2)$ guarantee only that $f_2^3 + f_3^2 = 0$ has an outer A_5 at O and two inner A_2's at P, Q. As Σ_{13} or Σ_{29} has an outer A_5 and four inner A_2's, their slices with respect to the slice condition $(\star_1 + \star_2)$ are subvarieties of $\mathcal{A}$. Here Σ_j is the configuration corresponding to nt-j in the table at the end. Similarly the slices of $\mathcal{M}(\Sigma_{42})$, $\mathcal{M}(\Sigma_{60})$ with respect to the slice condition $(\star_3)$ are subvarieties of $\mathcal{B}$. This observation will be used in the next two pairs.

II. The equalities $\psi_{red}(\mathcal{M}(\Sigma_{13})) = \psi_{red}(\mathcal{M}(\Sigma_{42}))$ and $\psi_{red}(\mathcal{M}(\Sigma_{29})) = \psi_{red}(\mathcal{M}(\Sigma_{60}))$ follow from the above argument (Proposition 21), where $\Sigma_{13} = [[6A_2], [A_1, A_5]]$, $\Sigma_{42} = [[4A_2, A_5], [A_1, 2A_2]]$, $\Sigma_{29} = [[4A_2, A_5], [A_5]]$ and $\Sigma_{60} = [[2A_2, 2A_5], [2A_2]]$. First we consider the equality $\psi_{red}(\mathcal{M}(\Sigma_{13})) = \psi_{red}(\mathcal{M}(\Sigma_{42}))$. In fact, we may consider the slice $\mathcal{A}', \mathcal{B}'$ of $\mathcal{M}(\Sigma_{13})$ or $\mathcal{M}(\Sigma_{42})$ subject to the slice condition $(\star_1 + \star_2)$ or $(\star_3)$. Then we have the canonical inclusions $\mathcal{A}' \subset \mathcal{A}$, $\mathcal{B}' \subset \mathcal{B}$. For example, $\mathcal{A}'$ consist of $(f_2, f_3) \in \mathcal{A}$ such that $f_2^3 + f_3^2 = 0$ has also an outer A_1. As f_2, f_3 are symmetric with respect to x-axis, A_1 must be on $y = 0$. Thus the condition for (f_2, f_3) to be in $\mathcal{A}'$ is given by the vanishing of the discriminant polynomial of $f(x, 0)/x^2$ in x, which is

$$\begin{aligned}(-8 &+ 12t_0^2 + 3a_{10}t_0^2 - 6t_0^4)(3a_{10}^4 - 24a_{10}^3t_0^2 + 30a_{10}^3 \\ &+ 72t_0^4a_{10}^2 - 180t_0^2a_{10}^2 + 120a_{10}^2 - 96t_0^6a_{10} + 360t_0^4a_{10} - 474a_{10}t_0^2 \\ &+ 216a_{10} + 48t_0^8 - 240t_0^6 + 469t_0^4 - 420t_0^2 + 144) = 0.\end{aligned}$$

Similarly $\mathcal{B}'$ is described in $\mathcal{B}$ by the equation

$$\begin{aligned}(27 &- 45a_{20} + 9a_{20}^2 + 9a_{20}^3 + 19t_1^2 + 18t_1^2a_{20} + 27t_1^2a_{20}^2 + 9t_1^4 + 27t_1^4a_{20} \\ &+ 9t_1^6)(t_1^4{a_{20}}^2 + 3a_{20}^2 - 6a_{20} - 6t_1^2a_{20} + 2t_1^6a_{20} + 2t_1^4a_{20} + 6t_1^2 + t_1^8 \\ &+ 4t_1^4 + 3 + 2t_1^6) = 0.\end{aligned}$$

One can check that the generic sextic in $\mathcal{A}'$ is contained in $\mathcal{M}(\Sigma_{13})$, putting explicit values to parameters. It is obvious that $\xi_1(\mathcal{A}') \subset \mathcal{B}'$ and $\xi_2(\mathcal{B}') \subset \mathcal{A}'$. Thus the assertion follows.

Next we consider the equality $\psi_{red}(\mathcal{M}(\Sigma_{29})) = \psi_{red}(\mathcal{M}(\Sigma_{60}))$ with $\Sigma_{29} = [[4A_2, A_5], [A_5]]$ and $\Sigma_{60} = [[2A_2, 2A_5], [2A_2]]$. Consider the slice $\mathcal{A}'', \mathcal{B}''$ of $\mathcal{M}(\Sigma_{29})$ and $\mathcal{M}(\Sigma_{60})$ subject to the slice condition $(\star_1 + \star_2)$ or $(\star_3)$. Then we have the canonical inclusions $\mathcal{A}'' \subset \mathcal{A}, \quad \mathcal{B}'' \subset \mathcal{B}$. The slices $\mathcal{A}'', \mathcal{B}''$ are at the "boundary" of $\mathcal{A}, \mathcal{B}$ respectively. For example, consider $(f_2, f_3) \in \mathcal{A}$. Then if the sextic $f_2^3 + f_3^2 = 0$ has one inner A_5, it must be on x-axis. Thus this is the case if and only if the resultant $S(y)$ of $f_2(x, y)$ and $f_3(x, y)$ in x, which is an even polynomial in y, has $y = 0$ as a solution. This condition is described as

$$\begin{aligned}&-96t_0^6a_{10} - 180a_{10}^2t_0^2 - 24a_{10}^3t_0^2 + 72t_0^4a_{10}^2 - 240t_0^6 + 469t_0^4 + 360a_{10}t_0^4\\ &\qquad + 3a_{10}^4 - 420t_0^2 + 216a_{10} + 48t_0^8 + 144 + 30a_{10}^3 + 120a_{10}^2 - 474a_{10}t_0^2 = 0\,.\end{aligned}$$

Now we consider $\mathcal{B}''$. Take $(g_2, g_3) \in \mathcal{B}$ and let $S(y)$ be the resultant of g_2 and g_3 in x-variable. As it has an inner A_5 at O, $S(y)$ is divisible by y^2. The condition that the sextic $g_2^3 + g_3^2 = 0$ has two inner A_5 is equivalent to $S(y)$ is divisible by y^4. Thus the slice $\mathcal{B}''$ consist of $(g_2, g_3) \in \mathcal{B}$ which satisfy

$$\begin{aligned}&t_1^8 + 2a_{20}t_1^6 + 2t_1{}^6 + a_{20}^2t_1^4 + 4t_1^4 + 2a_{20}t_1^4 - 6t_1^2a_{20} + 6t_1^2 + 3a_{20}^2 - 6a_{20}\\ &\quad + 3 = 0\,.\end{aligned}$$

Then after checking that a generic sextic of $\mathcal{A}'', \mathcal{B}''$ have the prescribed singularities, the assertion follows from $\xi_1(\mathcal{A}'') \subset \mathcal{B}'', \ \xi_2(\mathcal{B}'') \subset \mathcal{A}''$.

III. We show that the coincidence of moduli spaces $\psi_{red}(\mathcal{M}(\Sigma_8)) = \psi_{red}(\mathcal{M}(\Sigma_{52}))$ where $\Sigma_8 = [[6A_2], [E_6]]$ and $\Sigma_{52} = [[4A_2, E_6], [2A_2]]$. First we observe that ems-dim(Σ_6) = ems-dim$(\Sigma_{52}) = 1$. In fact, it is easy to see that both moduli spaces are irreducible and have transverse slice. We consider the minimal slices $\mathcal{S}_{52}$ of $\mathcal{M}(\Sigma_{52})$ (respectively $\mathcal{S}_8$ of $\mathcal{M}(\Sigma_8)$) with respect to the slice condition:

$(\star)$: two outer (resp. inner) A_2's are at $P = (1, 1)$ and $Q = (1, -1)$ and an inner (resp. outer) E_6 is at $O = (0, 0)$. The tangent cones at P and O are given by $y = 1$ and $x = 0$ respectively.

The normal forms of the slice $\mathcal{S}_{52}$ and $\mathcal{S}_8$ can be given as follows.

$$\mathcal{S}_{52} : \begin{cases} f_2(x, y) = y^2 + (-3 - t_1^2)x^2 + 2x \\ f_3(x, y) = -\dfrac{1}{2}\dfrac{-6y^2x - 3y^2xt_1^2 + 6y^2 + 6x^3 + 9x^3t_1^2 + 2x^3t_1^4 - 6x^2t_1^2 - 6x^2}{t_1} \end{cases}$$

$$\mathcal{S}_8 : \begin{cases} g_2 := y^2 + (3 - t_0^2)x^2 + (-4 + 2t_0^2)x - t_0^2 \\ g_3 := \dfrac{1}{2}(6y^2x - 3y^2xt_0^2 + 3y^2t_0^2 + 6x^3 - 9x^3t_0^2 + 2x^3t_0^4 - 12x^2 \\ \qquad +21x^2t_0^2 - 6x^2t_0^4 - 12xt_0^2 + 6xt_0^4 - 2t_0^4)/t_0\,. \end{cases}$$

We can see that $f_2^3 + f_3^2 = g_2^3 + g_3^2$ under the correspondence $t_0 = 2I\sqrt{3}/t_1$.

IV. We have already seen the coincidence of the images of three moduli spaces $\mathcal{M}([[4A_2, A_5], [E_6]])$, $\mathcal{M}([[4A_2, E_6], [A_5]])$ and $\mathcal{M}([[2A_2, A_5, E_6], [2A_2]])$ in the previous section (Proposition 16).

V. We show that $\psi_{red}(\mathcal{M}(\Sigma_{123})) = \psi_{red}(\mathcal{M}(\Sigma_{t11}))$ where $\Sigma_{123} = [[2A_2, C^{\natural}_{3,9}], [A_2]]$ and $\Sigma_{t11} = [3A_2, C_{3,9}]$. By Maple computation, we can show that both moduli spaces are irreducible and the dimensions of minimal moduli slices are 1. First, we consider the minimal moduli slices $\mathcal{A}$ of $\mathcal{M}(\Sigma_{123})$ and $\mathcal{B}$ of $\mathcal{M}(\Sigma_{t11})$ with three singularities are specialized as follows.

($\star_1$) for $\mathcal{A}$: $C_{3,9}$-singularity is at $(0, 0)$ with $y = 0$ as the tangent line, $P := (0, 1)$ is an outer A_2-singularity with $x = 0$ as the tangent line and $Q := (1, -1)$ is an inner A_2-singularity.

($\star_2$) for $\mathcal{B}$: $C_{3,9}$-singularity is at $(0, 0)$ with $y = 0$ as the tangent line, $P := (0, 1)$ is an inner A_2-singularity with $x = 0$ as the tangent line and $Q := (1, -1)$ is an inner A_2-singularity.

The normal form of $\mathcal{A}$ is given by

$$f_2(x, y) = y^2 - y - yt_0^2 - yxt_0^2 + \frac{1}{3}Iyxt_0^2\sqrt{3} - 2x^2t_0^2 - 2x^2 + \frac{1}{3}Ix^2t_0^2\sqrt{3}$$

$$\begin{aligned} f_3(x, y) = &\frac{1}{4}(2y^3t_0^2 + 3Ix^3t_0^2\sqrt{3} - 13yx^2t_0^2 - 6y^2t_0^2 - 5x^3t_0^2 + Iyxt_0^2\sqrt{3} \\ &+ Iy^2xt_0^2\sqrt{3} + 3Iyx^2t_0^2\sqrt{3} - 3yxt_0^2 - 3y^2xt_0^2 + 6y^3 - 6x^3 - 6y^2 \\ &- Iy^2x\sqrt{3} - 3yx + Iyx\sqrt{3} + 3y^2x + 2Ix^3\sqrt{3} - 12yx^2)t_0 . \end{aligned}$$

We observe that $f_3(x, y) = 0$ has a node at O and the intersection number $I(C_2, C_3; f g O) = 4$. See also [7]. Now we look for another torus decomposition $f(x, y) = g_2(x, y)^3 + g_3(x, y)^2$ so that $I(g_2, g_3; O) = 3$ and thus the conic $g_2(x, y) = 0$ passes through three A_2 singularities and $C_{3,9}$-singularity. By an easy computation, we find g_2 is given by

$$\begin{aligned} g_2(x, y) = &-\frac{1}{6}(-6y^2 - 6y^2t_0^2 + 6y + 6yt_0^2 - 3yxt_0^2 + Iyxt_0^2\sqrt{3} \\ &+ Ix^2t_0^2\sqrt{3} + 9x^2t_0^2 + 12x^2)/(1 + t_0^2) . \end{aligned}$$

To look for a partner cubic form g_3, we apply the degeneration method to the family $g_t := f - tg_2^3$. We can take $t = (1 + t_0^2)^3$ and the partner cubic form is given by

$$\begin{aligned} g_3(x, y) = &\frac{1}{8}(10x^3t_0^2 + 4yx^2t_0^2 + 2Iyx^2t_0^2\sqrt{3} - 6y^2xt_0^2 + 6yxt_0^2 - 3y^3t_0^2 \\ &- I\sqrt{3}y^3t_0^2 + 3y^2t_0^2 + Iy^2t_0^2\sqrt{3} + 12x^3 + 6yx^2 + 2Iyx^2\sqrt{3} - 6y^2x \\ &+ 6yx - 3y^3 - I\sqrt{3}y^3 + 3y^2 + Iy^2\sqrt{3})\sqrt{-2 + 2I\sqrt{3}}\,t_0 . \end{aligned}$$

where $t_0 = f_3(0, 1)/f_2(0, 1)$. This gives an isomorphism $\phi : \mathcal{A} \to \mathcal{B}$ which completes the proof.

6.1. Configuration table.

TABLE 1.

i-vector	No	Σ	$[g, \mu^*, n^*, i(C)]$	Existence?	ems-dim
[1,1,1,1,1,1]	nt1	$[[6A_2], [A_1]]$	[3,13,10,18]	$\to$ t3, nt2	6
	nt2	$[[6A_2], [A_2]]$	[3,14,9,16]	$\to$ t8, nt3	5
	nt3	$[[6A_2], [A_3]]$	[2,15,8,12]	$\to$ t9, nt4	4
	nt4	$[[6A_2], [A_4]]$	[2,16,7,9]	$\to$ t10, nt5	3
	nt5	$[[6A_2], [A_5]]$	[1,17,6,6]	$\to$ t11, nt29	2
	nt6	$[[6A_2], [B_{3,3}]]$	[1,16,6,6]	$\to$ nt30, nt7	3
	nt7	$[[6A_2], [D_5]]$	[1,17,5,4]	$\to$ nt31, nt8	2
	nt8	$[[6A_2], [E_6]]$	[1,18,4,2]	$\to$ nt32	1
	nt9	$[[6A_2], [2A_1]]$	[2,14,8,12]	$\to$ nt43, nt10	5
	nt10	$[[6A_2], [A_1, A_2]]$	[2,15,7,10]	$\to$ nt44, nt11	4
	nt11	$[[6A_2], [A_1, A_3]]$	[1,16,6,6]	$\to$nt45, nt12	3
	nt12	$[[6A_2], [A_1, A_4]]$	[1,17,5,3]	$\to$ nt46, nt13	2
	nt13	$[[6A_2], [A_1, A_5]]$	[0,18,4,0]	$\to$ nt47	1
	nt14	$[[6A_2], [A_1, B_{3,3}]]$	[0,17,4,0]	No	2
	nt15	$[[6A_2], [2A_2]]$	[2,16,6,8]	$\to$ nt23	3
	nt16	$[[6A_2], [A_2, A_3]]$	[1,17,5,4]	No	2
	nt17	$[[6A_2], [A_2, A_4]]$	[1,18,4,1]	No	1
	nt18	$[[6A_2], [2A_3]]$	[0,18,4,0]	No	1
	nt19	$[[6A_2], [3A_1]]$	[1,15,6,6]	$\to$ t15, nt20	4
	nt20	$[[6A_2], [2A_1, A_2]]$	[1,16,5,4]	$\to$ nt69, nt21	3
	nt21	$[[6A_2], [2A_1, A_3]]$	[0,17,4,0]	$\to$ nt70	2
	nt22	$[[6A_2], [A_1, 2A_2]]$	[1,17,4,2]	$\to$ nt52, nt23	2
	nt23	$[[6A_2], [3A_2]]$	[1,18,3,0]	Max	1
	nt24	$[[6A_2], [4A_1]]$	[0,16,4,0]	$\to$ nt100	3

TABLE 2.

i-vector	No	Σ	$[g, \mu^*, n^*, i(C)]$	Existence?	ems-dim
[1,1,1,1,2]	nt25	$[[4A_2, A_5], [A_1]]$	[2,14,10,16]	→ t5, nt26	5
	nt26	$[[4A_2, A_5], [A_2]]$	[2,15,9,14]	→t17, nt27	4
	nt27	$[[4A_2, A_5], [A_3]]$	[1,16,8,10]	→ t18, nt28	3
	nt28	$[[4A_2, A_5], [A_4]]$	[1,17,7,7]	→ nt29	2
	nt29	$[[4A_2, A_5], [A_5]]$	[0,18,6,4]	→ nt32, nt47	1
	nt30	$[[4A_2, A_5], [B_{3,3}]]$	[0,17,6,4]	→ nt31	2
	nt31	$[[4A_2, A_5], [D_5]]$	[0,18,5,2]	→ nt32	1
	nt32	$[[4A_2, A_5], [E_6]]$	[0,19,4,0]	Max	0
	nt33	$[[4A_2, A_5], [2A_1]]$	[1,15,8,10]	t14, nt34	4
	nt34	$[[4A_2, A_5], [A_1, A_2]]$	[1,16,7,8]	→ nt62, nt35	3
	nt35	$[[4A_2, A_5], [A_1, A_3]]$	[0,17,6,4]	→ nt63, nt36	2
	nt36	$[[4A_2, A_5], [A_1, A_4]]$	[0,18,5,1]	→ nt64	1
	nt37	$[[4A_2, A_5], [2A_2]]$	[1,17,6,6]	→ nt60	2
	nt38	$[[4A_2, A_5], [A_2, A_3]]$	[0,18,5,2]	No	1
	nt39	$[[4A_2, A_5], [A_2, A_4]]$	[0,19,4,-1]	No	0
	nt40	$[[4A_2, A_5], [3A_1]]$	[0,16,6,4]	→ nt65, nt41	3
	nt41	$[[4A_2, A_5], [2A_1, A_2]]$	[0,17,5,2]	→nt66, nt42	2
	nt42	$[[4A_2, A_5], [A_1, 2A_2]]$	[0,18,4,0]	→ nt67	1
	nt43	$[[4A_2, E_6], [A_1]]$	[2,15,8,12]	→t6, nt44	4
	nt44	$[[4A_2, E_6], [A_2]]$	[2,16,7,10]	→ t20, nt45	3
	nt45	$[[4A_2, E_6], [A_3]]$	[1,17,6,6]	→ t21, nt46	2
	nt46	$[[4A_2, E_6], [A_4]]$	[1,18,5,3]	→ nt47	1
	nt47	$[[4A_2, E_6], [A_5]]$	[0,19,4,0]	Max	0
	nt48	$[[4A_2, E_6], [B_{3,3}]]$	[0,18,4,0]	No	1
	nt49	$[[4A_2, E_6], [2A_1]]$	[1,16,6,6]	→nt68, nt50	3
	nt50	$[[4A_2, E_6], [A_1, A_2]]$	[1,17,5,4]	→ nt69, nt51	2
	nt51	$[[4A_2, E_6], [A_1, A_3]]$	[0,18,4,0]	→ nt70	1
	nt52	$[[4A_2, E_6], [2A_2]]$	[1,18,4,2]	→ nt67	1
	nt53	$[[4A_2, E_6], [3A_1]]$	[0,17,4,0]	→ nt71	2

TABLE 3.

i-vector	No	Σ	$[g, \mu^*, n^*, i(C)]$	Existence?	ems-dim
[1,1,2,2]	nt54	$[[2A_2, 2A_5], [A_1]]$	[1,15,10,14]	$\to$ t13	4
	nt55	$[[2A_2, 2A_5], [A_2]]$	[1,16,9,12]	$\to$ nt56	3
	nt56	$[[2A_2, 2A_5], [A_3]]$	[0,17,8,8]	$\to$ nt57	2
	nt57	$[[2A_2, 2A_5], [A_4]]$	[0,18,7,5]	$\to$ nt64	1
	nt58	$[[2A_2, 2A_5], [2A_1]]$	[0,16,8,8]	$\to$ nt96, nt59	3
	nt59	$[[2A_2, 2A_5], [A_1, A_2]]$	[0,17,7,6]	$\to$ nt97, nt60	2
	nt60	$[[2A_2, 2A_5], [2A_2]]$	[0,18,6,4]	$\to$ nt67	1
	nt61	$[[2A_2, A_5, E_6], [A_1]]$	[1,16,8,10]	$\to$ t14, nt62	3
	nt62	$[[2A_2, A_5, E_6], [A_2]]$	[1,17,7,8]	$\to$ nt63	2
	nt63	$[[2A_2, A_5, E_6], [A_3]]$	[0,18,6,4]	$\to$ nt64	1
	nt64	$[[2A_2, A_5, E_6], [A_4]]$	[0,19,5,1]	Max	0
	nt65	$[[2A_2, A_5, E_6], [2A_1]]$	[0,17,6,4]	$\to$nt98, nt66	2
	nt66	$[[2A_2, A_5, E_6], [A_1, A_2]]$	[0,18,5,2]	$\to$ nt67	1
	nt67	$[[2A_2, A_5, E_6], [2A_2]]$	[0,19,4,0]	Max	0
	nt68	$[[2A_2, 2E_6], [A_1]]$	[1,17,6,6]	$\to$t15, nt69	2
	nt69	$[[2A_2, 2E_6], [A_2]]$	[1,18,5,4]	$\to$ nt70	1
	nt70	$[[2A_2, 2E_6], [A_3]]$	[0,19,4,0]	Max	0
	nt71	$[[2A_2, 2E_6], [2A_1]]$	[0,18,4,0]	$\to$ nt100	1
[1,1,1,3]	nt72	$[[3A_2, A_8], [A_1]]$	[2,15,10,15]	$\to$ t19, nt73	4
	nt73	$[[3A_2, A_8], [A_2]]$	[2,16,9,13]	$\to$ t28, nt74	3
	nt74	$[[3A_2, A_8], [A_3]]$	[1,17,8,9]	$\to$ t29, nt75	2
	nt75	$[[3A_2, A_8], [A_4]]$	[1,18,7,6]	$\to$ nt78	1
	nt76	$[[3A_2, A_8], [A_5]]$	[0,19,6,3]	No	0
	nt77	$[[3A_2, A_8], [B_{3,3}]]$	[0,18,6,3]	$\to$ nt78	1
	nt78	$[[3A_2, A_8], [D_5]]$	[0,19,5,1]	Max	0
	nt79	$[[3A_2, A_8], [E_6]]$	[0,20,4,-1]	No	-1
	nt80	$[[3A_2, A_8], [2A_1]]$	[1,16,8,9]	$\to$ nt108, nt81	3
	nt81	$[[3A_2, A_8], [A_1, A_2]]$	[1,17,7,7]	$\to$ nt82	2
	nt82	$[[3A_2, A_8], [A_1, A_3]]$	[0,18,6,3]	$\to$ nt83	1
	nt83	$[[3A_2, A_8], [A_1, A_4]]$	[0,19,5,0]	Max	0
	nt84	$[[3A_2, A_8], [2A_2]]$	[1,18,6,5]	No	1
	nt85	$[[3A_2, A_8], [A_2, A_3]]$	[0,19,5,1]	No	0
	nt86	$[[3A_2, A_8], [A_2, A_4]]$	[0,20,4,-2]	No	-1
	nt87	$[[3A_2, A_8], [3A_1]]$	[0,17,6,3]	$\to$ nt112, nt88	2
	nt88	$[[3A_2, A_8], [2A_1, A_2]]$	[0,18,5,1]	$\to$ nt113	1
	nt89	$[[3A_2, A_8], [A_1, 2A_2]]$	[0,19,4,-1]	No	0
	nt90	$[[3A_2, B_{3,6}], [A_1]]$	[0,17,7,6]	$\to$ t20, nt91	3
	nt91	$[[3A_2, B_{3,6}], [A_2]]$	[0,18,6,4]	Max	2
	nt92	$[[3A_2, C_{3,7}], [A_1]]$	[0,18,6,3]	$\to$ t21	2
	nt93	$[[3A_2, C_{3,7}], [A_2]]$	[0,19,5,1]	No	1

TABLE 4.

i-vector	No	Σ	$[g, \mu^*, n^*, i(C)]$	Existence?	ems-dim
[2,2,2]	nt94	$[[3A_5], [A_1]]$	[0,16,10,12]	$\to$ nt95, nt100	3
	nt95	$[[3A_5], [A_2]]$	[0,17,9,10]	$\to$ nt97	2
	nt96	$[[2A_5, E_6], [A_1]]$	[0,17,8,8]	$\to$ nt97, nt98	2
	nt97	$[[2A_5, E_6], [A_2]]$	[0,18,7,6]	$\to$ nt99	1
	nt98	$[[A_5, 2E_6], [A_1]]$	[0,18,6,4]	$\to$ nt99, nt100	1
	nt99	$[[A_5, 2E_6], [A_2]]$	[0,19,5,2]	Max	0
	nt100	$[[3E_6], [A_1]]$	[0,19,4,0]	Max	0
[1,2,3]	nt101	$[[A_2, A_5, A_8], [A_1]]$	[1,16,10,13]	$\to$ nt102	3
	nt102	$[[A_2, A_5, A_8], [A_2]]$	[1,17,9,11]	$\to$ nt103	2
	nt103	$[[A_2, A_5, A_8], [A_3]]$	[0,18,8,7]	$\to$ nt104	1
	nt104	$[[A_2, A_5, A_8], [A_4]]$	[0,19,7,4]	Max	0
	nt105	$[[A_2, A_5, A_8], [2A_1]]$	[0,17,8,7]	$\to$ nt106	2
	nt106	$[[A_2, A_5, A_8], [A_1, A_2]]$	[0,18,7,5]	$\to$ nt113	1
	nt107	$[[A_2, A_5, A_8], [2A_2]]$	[0,19,6,3]	No	0
	nt108	$[[A_2, E_6, A_8], [A_1]]$	[1,17,8,9]	$\to$ nt109	2
	nt109	$[[A_2, E_6, A_8], [A_2]]$	[1,18,7,7]	$\to$ nt110	1
	nt110	$[[A_2, E_6, A_8], [A_3]]$	[0,19,6,3]	Max	0
	nt111	$[[A_2, E_6, A_8], [A_4]]$	[0,20,5,0]	No	-1
	nt112	$[[A_2, E_6, A_8], [2A_1]]$	[0,18,6,3]	$\to$ nt113	1
	nt113	$[[A_2, E_6, A_8], [A_1, A_2]]$	[0,19,5,1]	Max	0
	nt114	$[[A_2, E_6, A_8], [2A_2]]$	[0,20,4,-1]	No	-1
[1,1,4]	nt115	$[[2A_2, A_{11}], [A_1]]$	[1,16,10,14]	$\to$ t33, nt116	3
	nt116	$[[2A_2, A_{11}], [A_2]]$	[1,17,9,12]	$\to$ nt117	2
	nt117	$[[2A_2, A_{11}], [A_3]]$	[0,18,8,8]	$\to$ nt118	1
	nt118	$[[2A_2, A_{11}], [A_4]]$	[0,19,7,5]	Max	0
	nt119	$[[2A_2, A_{11}], [2A_1]]$	[0,17,8,8]	$\to$ nt135, nt120	2
	nt120	$[[2A_2, A_{11}], [A_1, A_2]]$	[0,18,7,6]	$\to$ nt136	1
	nt121	$[[2A_2, A_{11}], [2A_2]]$	[0,19,6,4]	No	0
	nt122	$[[2A_2, C_{3,9}^\natural], [A_1]]$	[0,18,7,5]	$\to$ t34, nt123	2
	nt123	$[[2A_2, C_{3,9}^\natural], [A_2]]$	[0,19,6,3]	Max	1
	nt124	$[[2A_2, B_{3,8}], [A_1]]$	[0,19,6,2]	$\to$ t35	1
	nt125	$[[2A_2, B_{3,8}], [A_2]]$	[0,20,5,0]	No	0

TABLE 5.

i-vector	No	Σ	$[g, \mu^*, n^*, i(C)]$	Existence?	ems-dim
[3,3]	nt126	$[[2A_8], [A_1]]$	[1,17,10,12]	→ nt127	2
	nt127	$[[2A_8], [A_2]]$	[1,18,9,10]	→ nt128	1
	nt128	$[[2A_8], [A_3]]$	[0,19,8,6]	Max	0
	nt129	$[[2A_8], [A_4]]$	[0,20,7,3]	No	-1
	nt130	$[[2A_8], [2A_1]]$	[0,18,8,6]	→ nt128	1
	nt131	$[[2A_8], [A_1, A_2]]$	[0,19,7,4]	No	0
	nt132	$[[2A_8], [2A_2]]$	[0,20,6,2]	No	-1
[2,4]	nt133	$[[A_5, A_{11}], [A_1]]$	[0,17,10,12]	→ nt134	2
	nt134	$[[A_5, A_{11}], [A_2]]$	[0,18,9,10]	→ nt136	1
	nt135	$[[E_6, A_{11}], [A_1]]$	[0,18,8,8]	→ nt136	1
	nt136	$[[E_6, A_{11}], [A_2]]$	[0,19,7,6]	Max	0
[1,5]	nt137	$[[A_2, A_{14}], [A_1]]$	[1,17,10,13]	→ nt138	2
	nt138	$[[A_2, A_{14}], [A_2]]$	[1,18,9,11]	→ nt139	1
	nt139	$[[A_2, A_{14}], [A_3]]$	[0,19,8,7]	Max	0
	nt140	$[[A_2, A_{14}], [A_4]]$	[0,20,7,4]	No	-1
	nt141	$[[A_2, A_{14}], [2A_1]]$	[0,18,8,7]	→ nt142	1
	nt142	$[[A_2, A_{14}], [A_1, A_2]]$	[0,19,7,5]	Max	0
	nt143	$[[A_2, A_{14}], [2A_2]]$	[0,20,6,3]	No	-1
[6]	nt144	$[[A_{17}], [A_1]]$	[0,18,10,12]	→ nt145	1
	nt145	$[[A_{17}], [A_2]]$	[0,19,9,10]	Max	0

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