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Square Integrable Solutions of $\Delta u + \lambda u = 0$ on Noncompact Manifolds

Reiji KONNO

Meiji University

Abstract. The Schrödinger-type equation $-\Delta u + Vu = \lambda u$ on a noncompact Riemannian manifold $\mathcal{M}$ has no nontrivial square integrable solution u for any positive constant λ, if the metric and the function V satisfy certain conditions near the infinity. A set of conditions of that kind was given by the author in the case that the metric is rotationally symmetric. It contained a condition which required smallness of the curvatures of $\mathcal{M}$ in the distance. But we have had a question whether the set could remain sufficient even if we remove that condition. The present paper answers it negatively by constructing a square integrable solution for a metric which satisfies all the conditions except the one in question.

Let $\mathcal{M}$ be a d-dimensional Riemannian manifold ($d \geq 2$) which admits a global system of coordinates $(r, \omega) \in (r_0, \infty) \times S^{d-1}$ therewith the metric is represented as

$$ds^2 = dr^2 + \rho(r)^2 d\omega^2 \tag{1}$$

where $d\omega$ is the standard metric of S^{d-1} and ρ is a positive function. Let Δ denote the Laplacian (Laplace-Beltrami operator) of $\mathcal{M}$ and $V(r, \omega)$ be a function defined on $\mathcal{M}$; then consider the equation

$$-\Delta u + Vu = \lambda u \tag{2}$$

where λ is an arbitrary positive constant. What we are concerned in the present paper is the behavior of the solution u near $r = \infty$, especially the existence or nonexistence of square integrable solutions.

Many authors dealt with this or a similar type of problem when $\mathcal{M}$ is a complete noncompact manifold having negative or positive definite curvatures (e.g. [1]–[3] and [6]–[8]). But in this article we do not require the completeness of the manifolds. We only assume an asymptotic behavior of $\rho(r)$ for large r. In that sense, we are treating a local problem at the infinity. As to the curvatures, the definiteness of the sign of the curvatures is not asked. Only their absolute values are of interest.

There is a theorem due to the author which offers a set of conditions assuring the nonexistence of square integrable solutions. Our purpose is to examine the efficiency of those conditions. Let us quote it here.

THEOREM A ([5, Theorem 1 combined with its corollary]). *Let $\psi(r)$ be a positive function of r ($r_0 \leq r < \infty$) which is locally absolutely continuous and satisfies the following*

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conditions:

$$\int_{r_0}^{\infty} \psi(r)dr = \infty\,,$$

$$\psi(r)^{-1}\dot{\psi}(r) + \psi(r) \geq -\alpha \quad (\textit{for a.e. large } r)$$

(*the dot representing d/dr*) *with some positive constant* α, *and*

$$\int_{r_0}^{\infty} \exp\left(-\int_{r_0}^{r} \psi(s)ds\right) dr = \infty\,.$$

Suppose that the function $\rho(r)$ *satisfies the following conditions* (i)–(v):

(i) $\rho \in C^1(r_0, \infty)$, $\rho(r) > 0$, $\dot{\rho}(r) \geq 0$ *in* $r_0 < r < \infty$, $\rho(r) \to \infty$ $(r \to \infty)$ *and* $\dot{\rho}$ *is locally absolutely continuous in* $r_0 < r < \infty$,

(ii) $\rho(r)^{-1}\dot{\rho}(r) = o(1)\ \ (r \to \infty)$,

(iii) $2\rho(r)^{-1}\dot{\rho}(r) \geq \psi(r)\ \ (\textit{for large } r)$,

(iv) $\rho(r)^{-3}\dot{\rho}(r)^3 = o(\psi(r))\ \ (r \to \infty)$,

(v) $\rho(r)^{-1}\ddot{\rho}(r) = o(\psi(r))\ \ (r \to \infty, \textit{a.e. } r)$.

Furthermore, *let*

$$V(r, \omega) = V_1(r, \omega) + V_2(r, \omega)$$

where $V_1(r, \omega)$ *and* $V_2(r, \omega)$ *are functions which satisfy the following conditions*:

(vi) $V_1(r, \omega)$ *is real-valued*, *continuous and locally absolutely continuous in r for almost every fixed* $\omega \in S^{n-1}$, *and*

$$V_1(r, \omega) = o(1)\,, \quad \dot{V}_1(r, \omega) = o(\psi(r)) \quad (r \to \infty, \textit{uniformly in } \omega)\,,$$

(vii) $V_2(r, \omega)$ *is complex-valued, bounded and measurable*, *and satisfies*

$$V_2(r, \omega) = o(\psi(r)) \quad (r \to \infty, uniformly\ in\ \omega)\,.$$

Let λ *be an arbitrary positive constant. Then no solution u of* (2) *is square integrable except* $u \equiv 0$.

The condition (v) implies that the absolute values of the curvatures of $\mathcal{M}$ should decrease in a sufficiently rapid manner. But it has been an open problem whether that condition was indispensable. In this article we will illustrate an example of ρ which satisfies (i)–(iv) with a certain ψ for which (2) has a square integrable solution. The existence of such an example indicates that (i)–(iv) alone are insufficient and hence (v) or some other condition is needed in order to guarantee the nonexistence of nontrivial L^2-solutions.

Let us consider the simple case where $V \equiv 0$, because that seems to tell best the essential point. Our construction goes in the reverse direction. That means, we at first pick up an L^2-function u, and then study the property of ρ for which

$$\Delta u + \lambda u \equiv \frac{1}{\rho^{d-1}}\frac{\partial}{\partial r}\left(\rho^{d-1}\frac{\partial u}{\partial r}\right) + \frac{1}{\rho^2}\Lambda u + \lambda u = 0 \tag{3}$$

holds (Λ is the Laplacian of S^{d-1}). Before beginning the construction, we change the function u to

$$v(r, \omega) = \rho(r)^{\frac{d-1}{2}} u(r, \omega)\,. \tag{4}$$

Then v satisfies

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{\rho^2}\Lambda v-\left[\frac{(d-1)(d-3)}{4}\frac{\dot\rho^2}{\rho^2}+\frac{d-1}{2}\frac{\ddot\rho}{\rho}\right]v+\lambda v=0\,, \tag{5}$$

where a dot represents d/dr. Note that $u\in L^2(\mathcal{M})$ if and only if $v\in L^2((r_0,\infty)\times S^{d-1};drdS)$, where dS is the measure of S^{d-1}.

Now, let $r_0\geq 1$, write $n=[r]$, the largest integer not exceeding r, and set

$$\begin{cases} r=n+s,\\ \lambda=\pi^2,\\ v(r,\omega)=v(r)=\dfrac{\sin\pi r}{n(n+1)}\left(n+1-s+\dfrac{1}{2\pi}\sin 2\pi r\right)\\ \qquad\qquad\qquad\qquad (n\leq r<n+1;\quad n=1,2,\cdots)\,. \end{cases} \tag{6}$$

The function v depends only on r, hence $\Lambda u=0$ and, as is easily seen, $v\in C^2(r_0,\infty)\cap L^2(r_0,\infty)$ as a function of the single variable r. (The number $\lambda=\pi^2$ is not essential. We can get a similar example for any $\lambda>0$ by changing the scale of the variable r.) What we intend to show is the following statement:

THEOREM B. *We can find a positive C^1-function $\rho(r)$ and positive constants k_1, k_2 and r_* such that ρ satisfies the relation* (5) *with the v given by* (6) *and yields the estimate*

$$\frac{k_1}{r}\leq\frac{\dot\rho(r)}{\rho(r)}\leq\frac{k_2}{r} \tag{7}$$

for $r\geq r_*$.

If we prove this theorem, the conditions (i)–(iv) of Theorem A are fulfilled with $\psi(r)=k_1/2r$, although (3) has a square integrable solution u. Therefore, we shall be able to conclude that the condition (v) is significant.

Let us consider the following function

$$x(r)=\frac{(d-1)r}{2}\frac{\dot\rho(r)}{\rho(r)}\,. \tag{8}$$

Then a straightforward calculation and the equation (5) show that

$$\frac{d}{dr}x(r)=\frac{1}{r}x(r)(1-x(r))+h(r) \tag{9}$$

where $h(r)$ is the continuous function given by

$$\begin{aligned} h(r)&=r\left(\pi^2+\frac{\ddot v(r)}{v(r)}\right)\\ &=-\frac{4\pi r\sin 2\pi r}{n+1-s+\dfrac{1}{2\pi}\sin 2\pi r}\\ &=-\frac{4\pi(n+s)\sin 2\pi s}{n+1-s+\dfrac{1}{2\pi}\sin 2\pi s}\,, \end{aligned} \tag{10}$$

for $n \le r = n + s < n + 1$, $(n = 1, 2, \cdots)$. Note that the inequality (7) is equivalent to

$$c_1 \le x(r) \le c_2 \quad (r \ge r_*)$$

where c_1 and c_2 are some positive constants. Therefore, our purpose will be achieved by proving the following proposition.

PROPOSITION 0. *We can find a positive integer n_0 and positive constants c_1 and c_2 such that the initial value problem*

$$\begin{cases} \dfrac{d}{dr}x(r) = \dfrac{1}{r}x(r)(1 - x(r)) + h(r) \quad (n_0 < r)\,, \\ x(n_0) = \alpha_0 \end{cases} \tag{11}$$

has a unique solution $x(r)$ throughout $n_0 \le r < \infty$ which satisfies

$$c_1 \le x(r) \le c_2 \quad (n_0 \le r < \infty)$$

provided the initial value α_0 lies in the interval

$$4.2 \le \alpha_0 \le 4.4\,. \tag{12}$$

We will prove this proposition in stages. First we show

PROPOSITION 1. *For $n \le r \le n + 1$ $(n = 1, 2, \cdots)$ we have*

$$h(r) = -4\pi \sin 2\pi s + \frac{1}{n}(4\pi \sin 2\pi s - 8\pi s \sin 2\pi s + 2\sin^2 2\pi s) + R(r), \quad |R(r)| \le \frac{4\pi}{n^2}\,.$$

PROOF. One has

$$\begin{aligned} h(r) &= -\frac{4\pi\left(1 + \dfrac{s}{n}\right)\sin 2\pi s}{1 + \dfrac{1}{n}\left(1 - s + \dfrac{1}{2\pi}\sin 2\pi s\right)} \\ &= -4\pi \sin 2\pi s + \frac{4\pi}{n}\sin 2\pi s\left(1 - 2s + \frac{1}{2\pi}\sin 2\pi s\right) + R(r)\,, \\ R(r) &= -\frac{1}{n^2}\cdot\frac{4\pi \sin 2\pi s\left(1 - s + \dfrac{1}{2\pi}\sin 2\pi s\right)\left(1 - 2s + \dfrac{1}{2\pi}\sin 2\pi s\right)}{1 + \dfrac{1}{n}\left(1 - s + \dfrac{1}{2\pi}\sin 2\pi s\right)}\,, \end{aligned}$$

for $n \le r < n + 1$, which shows $|R(r)| \le 4\pi/n^2$ because

$$0 \le 1 - s + \frac{1}{2\pi}\sin 2\pi s \le 1\,, \quad -1 \le 1 - 2s + \frac{1}{2\pi}\sin 2\pi s \le 1\,.$$

Next we consider an initial value problem in the interval $n \le r$ for each integer n separately in order to observe the influence of the magnitude of $x(n)$ on that of $x(n + 1)$.

PROPOSITION 2. *We can find a positive integer n_1 and a positive constant β_1 such that for each integer $n \geq n_1$ and any real number α, $4.2 \leq \alpha \leq 4.4$, the solution of*

$$\begin{cases} \dfrac{d}{dr}x(r) = \dfrac{1}{r}x(r)(1-x(r)) + h(r) \quad (n < r) \\ x(n) = \alpha \end{cases} \tag{13}$$

satisfies $x(r) \leq \beta_1$ in the interval of the form $n \leq r \leq r_1$, r_1 being any number, as long as $x(r)$ exists there.

PROOF. The proposition is clear from

$$x(n+s) = \alpha + \int_n^{n+s} h(r)dr + \int_n^{n+s} \frac{1}{r}x(r)(1-x(r))dr\,,$$

because $|h(r)| \leq 8\pi$ $(r \geq 1)$ and $x(1-x)/r \leq 1/4r$.

PROPOSITION 3. *There exist an integer n_2 and a number β_2 such that if $n \geq n_2$ and $4.2 \leq \alpha \leq 4.4$ then the solution of* (13) *exists everywhere in the interval $n \leq r \leq n + \frac{3}{2}$ and fulfills*

$$x(r) \geq \beta_2 \tag{14}$$

for $n \leq r \leq n + \frac{3}{2}$.

REMARK. The number 3/2 does not have a special meaning. One has only to show that $x(r)$ exists for $n \leq r \leq n+1$ and satisfies the differential equation (for the left derivative) even at $r = n$.

PROOF. (I) Consider the differential equation for a function $\varphi_n(r)$:

$$\begin{cases} \dfrac{d}{dr}\varphi_n(r) = \dfrac{1}{r}\varphi_n(r)(1-\varphi_n(r)) - \dfrac{5\pi n + \frac{1}{4}}{r} \quad (n < r)\,, \\ \varphi_n(n) = \alpha - \varepsilon\,, \end{cases} \tag{15}$$

where ε is an arbitrary constant, $0 < \varepsilon < 1/2$. Put $k_n = \sqrt{5\pi n}$, then we have

$$\tan^{-1}\frac{\varphi_n - \frac{1}{2}}{k_n} = \tan^{-1}\frac{\alpha - \varepsilon - \frac{1}{2}}{k_n} - k_n \log\left(1 + \frac{s}{n}\right)$$

($\tan^{-1}$ is the principal value) at least in some interval. But since $k_n \log\left(1 + \frac{s}{n}\right) = O(1/\sqrt{n})$ for large n, the solution φ_n exists in $n \leq r \leq n + \frac{3}{2}$ if n is sufficiently large. We therefore get

$$\varphi_n = \frac{1}{2} + \frac{\alpha - \varepsilon - \frac{1}{2} - k_n \tan\left\{k_n \log\left(1 + \frac{s}{n}\right)\right\}}{1 + \dfrac{\alpha - \varepsilon - \frac{1}{2}}{k_n}\tan\left\{k_n \log\left(1 + \frac{s}{n}\right)\right\}}\,.$$

It is clear that if n is not less than some number, say, n_2, then φ_n fulfills

$$\beta_2 \leq \varphi_n(r) \leq \beta_2' \quad \left(n \leq r \leq n + \frac{3}{2}\right)$$

for some real numbers β_2, β_2' which are independent of n.

(II) We will show that $x(r)$ exists in $n \leq r \leq n + \frac{3}{2}$. Set $y(r) := x(r) - \varphi_n(r)$. Suppose contrary to the conclusion that $x(r)$ ceases to exist somewhere before $n + \frac{3}{2}$. Due to Proposition 2 and the existence theorem for ordinary differential equations, such a case occurs only when $x(r)$ diverges to $-\infty$ at that point. Hence we can find a number $\gamma (n < \gamma < n + \frac{3}{2})$ such that $\lim_{r\to\gamma-0} y(r) = 0$ and $y(r) > 0$ in the interval $n \leq r < \gamma$.

Now, take n_2 so large that for any $n \geq n_2$, $h(r)$ admits the following estimate from below:

$$h(r) \geq -4\pi\left(1 + \frac{s}{n}\right) \geq -\frac{5\pi n + \frac{1}{4}}{r} \quad \left(n \leq r \leq n + \frac{3}{2}\right).$$

Then from Proposition 2 and from the first part of this proof, one sees

$$\begin{aligned}\frac{dy}{dr} &= \frac{1}{r}x(1-x) + h(r) - \frac{1}{r}\varphi_n(1-\varphi_n) + \frac{1}{r}\left(5\pi n + \frac{1}{4}\right)\\ &\geq \frac{1}{r}(x-\varphi_n)(1-x-\varphi_n)\\ &\geq -\frac{\beta_1+\beta_2'}{n}y \quad (n < r < \gamma)\end{aligned}$$

if $n \geq n_2$ and r stays in the interval $n < r < \gamma$. Therefore, putting $M = (\beta_1 + \beta_2')/n$, we have

$$y(r) \geq y(n)e^{-M(r-n)} \geq \varepsilon e^{-3M/2} > 0 \quad (n \leq r < \gamma).$$

But this is incompatible with $y(r) \to 0$ $(r \to \gamma - 0)$. Hence $x(r)$ exists throughout the interval $n \leq r \leq n + \frac{3}{2}$ and is not less than $\varphi_n(r)$ there. This establishes the proposition.

PROPOSITION 4. *Suppose $n \geq n_2$. Let x be the solution of* (13) *and write $\alpha = a + 2$. If* $4.2 \leq \alpha \leq 4.4$, *then $x(r)$ admits the expression*

$$\begin{cases} x(r) = a + 2\cos 2\pi s + \dfrac{1}{n}[(-a^2 + a - 1 + 4\cos 2\pi s)\cdot s + 2(1-\cos 2\pi s) \\ \qquad -\dfrac{2a+1}{\pi}\sin 2\pi s - \dfrac{3}{4\pi}\sin 4\pi s] + R^*(r) \quad (n \leq r \leq n+1), \\ |R^*(r)| \leq \dfrac{\beta_3}{n^2}, \end{cases} \tag{16}$$

where β_3 is some positive number independent of n and r.

PROOF. For saving the description, we write $\eta \pm \delta$ to denote an entity which lies between $\eta - \delta$ and $\eta + \delta$ so that the expression

$$\xi = \eta \pm \delta$$

stands for

$$|\xi - \eta| \le \delta \,.$$

Moreover, by the calculation

$$\xi = \eta \pm \text{smaller} = \eta \pm \text{bigger}$$

we state

$$|\xi - \eta| \le \text{smaller} \quad \textit{therefore} \quad |\xi - \eta| \le \text{bigger}\,.$$

Now, let $n \ge n_2$ and $n \le r \le n+1$. From $\beta_2 \le x(r) \le \beta_1$ we get $|x(r)(1-x(r))| \le \beta_4$ where β_4 dose not depend on n nor on r. Hence, from (13) it follows that

$$x(r) = a + 2 + \int_n^{n+s} h(r)dr \pm \beta_4 \log\left(1 + \frac{s}{n}\right). \tag{17}$$

On the other hand, Proposition 1 shows that

$$\begin{aligned}\int_n^{n+s} h(r)dr = &- 2 + 2\cos 2\pi s + \frac{1}{n}\biggl(s + 2 - 2\cos 2\pi s + 4s\cos 2\pi s \\ &- \frac{2}{\pi}\sin 2\pi s - \frac{1}{4\pi}\sin 4\pi s\biggr) + \int_n^{n+s} R(r)dr\,.\end{aligned} \tag{18}$$

Therefore, since $|R(r)| \le 4\pi/n^2$, one sees

$$x(r) = a + 2\cos 2\pi s \pm \frac{\beta_5}{n}$$

for some β_5 and hence one can choose a number β_6 to compute

$$\begin{aligned}\frac{1}{r}x(1-x) &= \frac{1}{n\left(1+\frac{s}{n}\right)}\left(a + 2\cos 2\pi s \pm \frac{\beta_5}{n}\right)\left(1 - a - 2\cos 2\pi s \pm \frac{\beta_5}{n}\right) \\ &= \frac{1}{n}(a + 2\cos 2\pi s)(1 - a - 2\cos 2\pi s) \pm \frac{\beta_6}{n^2}\,.\end{aligned}$$

Substituting this estimate together with (18) to the equation (13) and integrating both sides from n to $n+s$ again, we obtain, by setting $\beta_3 = \beta_6 + 4\pi$, that

$$\begin{aligned}x(r) &= a + 2 + \int_n^{n+s} h(r)dr + \frac{1}{n}\int_0^s (a + 2\cos 2\pi t)(1 - a - 2\cos 2\pi t)dt \pm \frac{\beta_3}{n^2} \\ &= a + 2\cos 2\pi s + \frac{1}{n}\biggl[(-a^2 + a - 1 + 4\cos 2\pi s)\cdot s + 2(1 - \cos 2\pi s) \\ &\quad - \frac{1+2a}{\pi}\sin 2\pi s - \frac{3}{4\pi}\sin 4\pi s\biggr] \pm \frac{\beta_3}{n^2}\,.\end{aligned}$$

PROPOSITION 5. *We can find an integer n_3 such that if $n \ge n_3$ and $4.2 \le \alpha \le 4.4$ then x of (13) satisfies $4.2 \le x(n+1) \le 4.4$.*

PROOF. Set $\alpha = a + 2$. By Proposition 4, the solution of (13) fulfills

$$x(n+1) = a + 2 - \frac{1}{n}(a^2 - a - 3) \pm \frac{\beta_3}{n^2}$$

provided $n \geq n_2$ and $4.2 \leq \alpha \leq 4.4$. But since $a + 2 - (a^2 - a - 3)/n$ is an increasing function of a in the interval $2.2 \leq a \leq 2.4$ for fixed $n \geq 4$, it follows that

$$4.2 + \frac{0.36}{n} - \frac{\beta_3}{n^2} \leq x(n+1) \leq 4.4 - \frac{0.36}{n} + \frac{\beta_3}{n^2}\,.$$

Hence if n_3 is an integer $\geq \max(n_2, 4, \beta_3/0.36)$, we have $4.2 \leq x(n+1) \leq 4.4$, provided $n \geq n_3$.

PROOF OF PROPOSITION 0. At first we note that if $x_1(r)$ and $x_2(r)$ are the solutions of

$$\begin{cases} \dfrac{d}{dr}x_1(r) = \dfrac{1}{r}x_1(r)(1 - x_1(r)) + h(r) & \left(n < r < n + \dfrac{3}{2}\right) \\ x_1(n) = \alpha \end{cases}$$

and

$$\begin{cases} \dfrac{d}{dr}x_2(r) = \dfrac{1}{r}x_2(r)(1 - x_2(r)) + h(r) & (n + 1 < r < n + 2) \\ x_2(n+1) = x_1(n+1) \end{cases} \tag{19}$$

for large n respectively, then the connected function

$$x(r) = \begin{cases} x_1(r) & (n \leq r \leq n + 1)\,, \\ x_2(r) & (n + 1 \leq r \leq n + 2) \end{cases}$$

is the solution in the interval $n \leq r \leq n + 2$, because the solution of the initial value problem (19) is unique. Hence Proposition 5 tells that if $n_0 \geq n_3$ and $4.2 \leq \alpha_0 \leq 4.4$, then the solution of

$$\begin{cases} \dot{x}(r) = \dfrac{1}{r}x(r)(1 - x(r)) + h(r) & (n_0 < r)\,, \\ x(n_0) = \alpha_0 \end{cases} \tag{20}$$

exists throughout $n_0 \leq r < \infty$ and fulfills $4.2 \leq x(n) \leq 4.4$ for any integer $n \geq n_0$. Take n_0 so large that

$$\frac{1}{n_0}|(-a^2 + a - 1 + 4\cos 2\pi s)\cdot s + 2(1 - \cos 2\pi s) - \frac{1+2a}{\pi}\sin 2\pi s - \frac{3}{4\pi}\sin 4\pi s| + \frac{\beta_3}{n_0^2} \leq 0.1$$

holds for $0 \leq s \leq 1$. Then from (16) we ontain

$$\begin{aligned} 0.1 &\leq x([r]) - 4 - 0.1 \\ &\leq x(r) \\ &\leq x([r]) + 0.1 \\ &\leq 4.5 \end{aligned}$$

and Theorem B as well as Proposition 0 is established.

REMARK 1. Since $\dfrac{\ddot{\rho}}{\dot{\rho}} = \dfrac{h}{x} - \dfrac{d-3}{2}\dfrac{\dot{\rho}}{\rho}$, we have $\dfrac{\ddot{\rho}}{\dot{\rho}} \not\to 0$. Hence the condition (v) of Theorem A can not be fulfilled by any ψ satisfying (iii).

REMARK 2. The Schrödinger equation $-\Delta u + q(x)u = \lambda u$, $\lambda > 0$ in a Euclidean space E^d can possess a nontrivial square integrable solution u if we simply assume $q(x) = o(1)$ as $|x| \to \infty$. Such $q(x)$ and u were shown first by von Neumann and Wigner [9] and then generalized by Kato [4]. The solution $u(x)$ of [9] has, in effect, the form

$$\begin{cases} u(x) = u(r) = r^{-\frac{d-1}{2}} v_1(r), \quad r = |x|, \\ v_1(r) = \dfrac{\sin\sqrt{\lambda}r}{1 + (2\sqrt{\lambda}r - \sin 2\sqrt{\lambda}r)^2} \end{cases} \tag{6'}$$

which corresponds to the potential $q(x) = q(r)$ through

$$q(r) = \frac{\ddot{v}_1(r)}{v_1(r)} + \lambda - \frac{(d-1)(d-3)}{4r^2}.$$

It is natural to consider the equation

$$\frac{d}{dr}x(r) = \frac{1}{r}x(r)(1 - x(r)) + r\left(\frac{\ddot{v}_1(r)}{v_1(r)} + \lambda\right) \tag{9'}$$

instead of (6) and (9) of the present paper. But the solution of (9′) is not positive definite. It means, the function $\rho(r)$ which satisfies (8) is not monotone increasing, hence the condition (i) of Theorem A is partly violated besides (v). Another choice, for example,

$$v_2(r) = \frac{\sin\sqrt{\lambda}r}{2 + 2\sqrt{\lambda}r - \sin 2\sqrt{\lambda}r}$$

also gives an oscillating $\rho(r)$, and so forth. The choice of $v(r)$ is thus delicate.

References

[1] H. DONNELLY, Spectral geometry for certain noncompact Riemannian manifolds, Math. Z. **169** (1979), 63–76.

[2] H. DONNELLY, Eigenvalues embedded in the continuum for negatively curved manifolds, Michigan Math. J. **28** (1981), 53–62.

[3] L. KARP, Noncompact Riemannian manifolds with purely continuous spectrum, Michigan Math. J. **31** (1984), 339–347.

[4] T. KATO, Growth property of solutions of the reduced wave equation with a variable coefficient, Comm. Pure Appl. Math. **12** (1959), 403–425.

[5] R. KONNO, A remark on the Schrödinger-type equation on manifolds, I, Memoir of the Inst. Sci. and Tech. Meiji Univ. **38** No. 3 (1999), 27–34.

[6] H. P. MCKEAN, An upper bound to the spectrum of Δ on a manifold of negative curvature, J. Differential Geom. **4** (1970), 359–366.

[7] M. A. PINSKY, The spectrum of the Laplacian on a manifold of negative curvature. I, J. Differential Geom. **13** (1978), 87–91.

[8] M. A. PINSKY, The spectrum of the Laplacian on a manifold of negative curvature. II, J. Differential Geom. **14** (1979), 609–620.

[9] J. VON NEUMANN and E. P. WIGNER, Über merkwürdige diskrete Eigenwerte, Phisik. Z. **50** (1929), 465–467.

Present Address:
Department of Mathematics, School of Science and Technology,
Meiji University,
Higashimita, Tama-ku, Kawasaki, 214–8571 Japan.