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A New Proof for Some Relations among Axial Distances and Hook-Lengths

Masashi KOSUDA

University of the Ryukyus

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Introduction

In this paper, we give a new proof of several relations among axial distances and products of hook-lengths using a generalized version of Lagrange's interpolation polynomial (Theorem 0.1). Applying this method to another case, a new relation is obtained (Theorem 0.2).

To state the main theorems, we introduce some terminology and notation following the books [4, 5]. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ be a partition of n. The *Young diagram* or *shape* of λ is an array of n boxes having l left-justified rows with row i containing λ_i boxes for $1 \le i \le l$. The box in row i and column j has coordinates (i, j), as in a matrix. The *conjugate* of a partition λ is the partition λ^* whose diagram is the transpose of the diagram λ. More precisely, for λ, $\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_{l'}^*)$ is defined by

$$\lambda_j^* = |\{k; \lambda_k \ge j\}| \quad (j = 1, 2, \dots, l' = \lambda_1)\,.$$

The *hook-length* of λ at $(i, j) \in \lambda$ is denoted by $h_\lambda(i, j)$ and defined by

$$h_\lambda(i, j) = \lambda_i + \lambda_j^* - i - j + 1\,.$$

For a partition λ, let $h(\lambda)$ denote the product of all the hook-lengths in λ, namely

$$h(\lambda) = \prod_{(i,j)\in\lambda} h_\lambda(i, j)\,.$$

For an indeterminate q, we define the *q-integer* $[i]$ by

$$[0] = 0 \quad \text{and} \quad [i] = \frac{q^i - q^{-i}}{q - q^{-1}} = q^{i-1} + q^{i-3} + \cdots + q^{-i+1} \ (i \ge 1)\,.$$

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Replacing each factor of $h(\lambda)$ with the corresponding q-integer, we define $h[\lambda]$ as follows:

$$h[\lambda] = \prod_{(i,j)\in\lambda} [h_\lambda(i,j)] .$$

For each $x = (i, j) \in \lambda$, the *content* of x is defined by $c(x) = j - i$. For an ordered pair (x, y) of coordinates in a Young diagram, the *axial distance* $d(x, y)$ is defined by

$$d(x, y) = c(y) - c(x) .$$

Let λ, μ be partitions of n and $n-1$ respectively. If μ is obtained from λ by removing one box, then we write $\mu \lhd \lambda$. If there exists a 3-tuple of partitions (ν, μ, λ) such that $\nu \lhd \mu \lhd \lambda$, then the axial distance $d(\nu, \mu, \lambda)$ is defined to be $d(\nu, \mu, \lambda) = d(\mu \setminus \nu, \lambda \setminus \mu) = c(\lambda \setminus \mu) - c(\mu \setminus \nu)$. For example, if $\lambda \setminus \mu = \{(r_1, c_1)\}$ and $\mu \setminus \nu = \{(r_0, c_0)\}$, then

$$d(\nu, \mu, \lambda) = c_1 - r_1 - (c_0 - r_0) = \begin{cases} h_\mu(r_1, c_0) & \text{if } r_0 \geq r_1 , \\ -h_\mu(r_0, c_1) & \text{if } r_0 < r_1 . \end{cases}$$

Throughout this paper, we use "Young diagram λ" instead of "the Young diagram of a partition λ". Under these preparations, we state the following theorems.

THEOREM 0.1. *Suppose that two Young diagrams μ and λ satisfy $\mu \lhd \lambda$. Let $\{\lambda^+_{(r)}\}_{r=1,2,\ldots,b(\lambda)}$ be the set of all the Young diagrams which satisfy $\lambda^+_{(r)} \rhd \lambda$ and $\{\mu^-_{(s)}\}_{s=1,2,\ldots,b'(\mu)}$ the set of all the Young diagrams which satisfy $\mu^-_{(s)} \lhd \mu$. Then the following relations hold*:

$$(1) \qquad \sum_{r=1}^{b(\lambda)} \frac{h[\lambda]}{h[\lambda^+_{(r)}]} \cdot q^{\pm c(\lambda^+_{(r)} \setminus \lambda)} = 1 ,$$

$$(2) \qquad \sum_{r=1}^{b(\lambda)} \frac{h[\lambda]}{h[\lambda^+_{(r)}]} \cdot \frac{1}{[d(\mu, \lambda, \lambda^+_{(r)})]} = 0 ,$$

$$(3) \qquad \sum_{r=1}^{b(\lambda)} \frac{h[\lambda]}{h[\lambda^+_{(r)}]} \cdot \frac{q^{\pm d(\mu,\lambda,\lambda^+_{(r)})}}{[d(\mu, \lambda, \lambda^+_{(r)})]^2} = \frac{h[\mu]}{h[\lambda]} ,$$

$$(4) \qquad \sum_{s=1}^{b'(\mu)} \frac{h[\mu]}{h[\mu^-_{(s)}]} \cdot q^{\pm c(\mu \setminus \mu^-_{(s)})} = \sum_{x \in \mu} q^{\pm 2c(x)} ,$$

$$(5) \qquad \sum_{s=1}^{b'(\mu)} \frac{h[\mu]}{h[\mu^-_{(s)}]} \cdot \frac{1}{[d(\mu^-_{(s)}, \mu, \lambda)]} = [c(\lambda \setminus \mu)] ,$$

$$(6) \qquad \sum_{s=1}^{b'(\mu)} \frac{h[\mu]}{h[\mu^-_{(s)}]} \cdot \frac{q^{\pm d(\mu^-_{(s)},\mu,\lambda)}}{[d(\mu^-_{(s)}, \mu, \lambda)]^2} = \frac{h[\lambda]}{h[\mu]} - q^{\mp c(\lambda \setminus \mu)} .$$

Further, let μ' be a Young diagram which differs from μ and satisfies $\mu' \lhd \lambda$, and λ' a Young diagram which differs from λ and satisfies $\lambda' \rhd \mu$. Then the following relations hold:

$$(7)\qquad \sum_{r=1}^{b(\lambda)} \frac{h[\lambda]}{h[\lambda^+_{(r)}]} \cdot \frac{q^{\pm c(\lambda^+_{(r)}\setminus\lambda)}}{[d(\mu,\lambda,\lambda^+_{(r)})][d(\mu',\lambda,\lambda^+_{(r)})]} = 0\,,$$

$$(8)\qquad \sum_{s=1}^{b'(\mu)} \frac{h[\mu]}{h[\mu^-_{(s)}]} \cdot \frac{q^{\pm c(\mu\setminus\mu^-_{(s)})}}{[d(\mu^-_{(s)},\mu,\lambda)][d(\mu^-_{(s)},\mu,\lambda')]} = -q^{\pm(c(\lambda\setminus\mu)+c(\lambda'\setminus\mu))}.$$

THEOREM 0.2. *Let λ, μ, $\{\lambda^+_{(r)}\}_{r=1,2,\dots,b(\lambda)}$ and $\{\mu^-_{(s)}\}_{s=1,2,\dots,b'(\mu)}$ be as in the previous theorem. Then the following relation holds*:

$$h[\mu]^3 \sum_{s=1}^{b'(\mu)} \frac{1}{h[\mu^-_{(s)}]} \cdot \frac{1}{[d(\mu^-_{(s)},\mu,\lambda)]^3} = h[\lambda]^3 \sum_{r=1}^{b(\lambda)} \frac{1}{h[\lambda^+_{(r)}]} \cdot \frac{1}{[d(\mu,\lambda,\lambda^+_{(r)})]^3}\,.$$

Theorem 0.1(1) describes the branching rule from the irreducible representations of the symmetric groups $\mathfrak{S}_{n-1}$ to those of $\mathfrak{S}_n$, if we specialize the parameter q to 1. As for this special case, we already have a proof by induction on the depth of λ which utilize the equation in [4],I,§1,Ex.1. As we mention later, as for the q-analogue version of this relation, we also have another proof as well as the other relations in Theorem 0.1. In this paper, however, we would like to propose the following new approach which utilize a generalized version of Lagrange's interpolation polynomial.

By direct calculations or using the equation in [4],I,§1,Ex.1 we find that the ratio $h[\mu]$ to $h[\lambda]$ (resp. $h[\lambda]$ to $h[\mu]$) has the following presentation by axial distances $\{d_s = d(\mu^-_{(s)},\mu,\lambda)\}(=D)$ and $\{e_r = d(\mu,\lambda,\lambda^+_{(r)})\}(=E)$:

$$(1.2)\qquad \frac{h[\mu]}{h[\lambda]} = (-1)^{|D|} \cdot \frac{\prod_{d_s\in D}[d_s]}{\prod_{e_r\in\hat{E}}[e_r]}\,,$$

$$(1.5)\qquad \frac{h[\lambda]}{h[\mu]} = (-1)^{|E|-1} \cdot \frac{\prod_{e_r\in E}[e_r]}{\prod_{d_s\in\hat{D}}[d_s]}\,,$$

where $\hat{S} = S \setminus \{1,-1\}$. The important trick for the new proof is the fact that using the same axial distances in D and E, we obtain the following presentations of $h[\lambda]/h[\lambda^+_{(r)}]$ and $h[\mu]/h[\mu^-_{(s)}]$:

$$(1.6)\qquad \frac{h[\lambda]}{h[\lambda^+_{(r)}]} = \frac{\prod_{d_s\in\hat{D}_0}[e_r+d_s]}{\prod_{e_i\in E\setminus\{e_r\}}[e_r-e_i]}\,,$$

$$(1.8)\qquad \frac{h[\mu]}{h[\mu^-_{(s)}]} = -\frac{\prod_{e_r\in\hat{E}_0}[d_s+e_r]}{\prod_{d_i\in D\setminus\{d_s\}}[d_s-d_i]}\,,$$

where $\hat{S}_0 = S \cup \{0\} \setminus \{1, -1\}$. Summing up both sides of the equations (1.6),(1.8), we find these presentations are versions of Lagrange's interpolation polynomial (The former is just the classical one). Showing a generalized version of Lagrange's interpolation polynomial, we prove Theorem 0.1(1)–(6). Theorem 0.1(7) and (8) will follow them. Moreover, we obtain a new relation in Theorem 0.2.

In the paper [6], Wenzl proved a more general version of Theorem 0.1(1) to show the existence of a Markov trace for a series of the Iwahori-Hecke algebras $\{H_n(q)\}_n$ of type A, modifying λ and $\lambda^+_{(r)}$ ingeniously so that the equation in [4],I,§3,Ex.3 can be applied. His proof is, in other words, based on the Schur-Weyl reciprocity. By mimicking Wenzl's method, we can also prove the other relations in Theorem 0.1. In fact, the author used some of the relations in Theorem 0.1 to construct irreducible representations of the generalized Hecke algebra and those of the Hecke category [1, 3]. The relation in Theorem 0.2, on the other hand, does not seem to be obtained from the Schur-Weyl reciprocity, even in the case $q = 1$. Our new method is, however, valid for the proof of Theorem 0.2 as well as Theorem 0.1(1)–(6).

When the author was writing a paper concerning the announcement [2], he found a special version ($q = 1$ version) of the relation in Theorem 0.2. In his study, in the case $q = 1$, the main theorems above were used to show the well-definedness of the irreducible representations of a new algebra called *the party algebra*. The party algebra is obtained from the group algebra of the symmetric group $\mathfrak{S}_k$ by adding an element which satisfies $f^2 = \beta f$ and some other relations. If we specialize the parameter β to a positive integer k, then we find that the party algebra is isomorphic to the centralizer algebra of the unitary reflection group of type $G(r, 1, k)$ in the endomorphism ring $\mathrm{End}(V^{\otimes n})$ of tensor space under the condition that $\dim\ V = k \geq n, r > n$ and $G(r, 1, k)$ acts diagonally on $V^{\otimes n}$. The existence of a q-analogue version of these relations indicates the existence of a q-analogue version of the party algebra.

This paper is organized as follows. In Section 1, we deduce the equation (1.6),(1.8). Some other relations which will be used to prove the main theorems are also proved. In Section 2, we prove the generalized version of Lagrange's interpolation polynomial using the Vandermonde determinant or the Schur function. Note that we do not do any calculation concerning the Schur-Weyl reciprocity. Finally, in Section 3 and 4, we prove Theorem 0.1 and 0.2 respectively.

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1. Preliminaries

First we describe $h[\mu]/h[\lambda]$ and $h[\lambda]/h[\mu]$ in terms of axial distances $d_s = d(\mu^-_{(s)}, \mu, \lambda)$, $(s = 1, 2, \ldots, b'(\mu))$ and $e_r = d(\mu, \lambda, \lambda^+_{(r)})$, $(r = 1, 2, \ldots, b(\lambda))$.

Let $\{\lambda^+_{(r)}\}_{r=1,2,\ldots,b(\lambda)}$ and $\{\mu^-_{(s)}\}_{s=1,2,\ldots,b'(\mu)}$ be the sets of all the Young diagrams which satisfy $\lambda^+_{(r)} \rhd \lambda$ and $\mu^-_{(s)} \lhd \mu$ respectively. We assume that they also satisfy $c(\lambda^+_{(1)} \setminus \lambda) <$

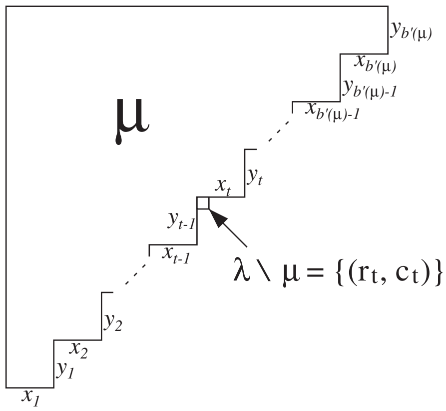

FIGURE 1. $\lambda \setminus \mu$ as an outer corner of μ.

$\cdots < c(\lambda^+_{(b(\lambda))} \setminus \lambda)$ and $c(\mu \setminus \mu^-_{(1)}) < \cdots < c(\mu \setminus \mu^-_{(b'(\mu))})$. In other words, the outer corners of λ are indexed by $\lambda^+_{(1)} \setminus \lambda, \ldots, \lambda^+_{(b(\lambda))} \setminus \lambda$ from left to right and so are the inner corners of μ by $\mu \setminus \mu^-_{(1)}, \ldots, \mu \setminus \mu^-_{(b'(\mu))}$ from left to right.

Similarly, let $\{\mu^+_{(r)}\}_{r=1,2,\ldots,b(\mu)}$ and $\{\lambda^-_{(s)}\}_{s=1,2,\ldots,b'(\lambda)}$ be the sets of all the Young diagrams which satisfy $\mu^+_{(r)} \rhd \mu$ and $\lambda^-_{(s)} \lhd \lambda$ respectively and assume that they also satisfy $c(\lambda \setminus \lambda^-_{(1)}) < \cdots < c(\lambda \setminus \lambda^-_{(b'(\lambda))})$ and $c(\mu^+_{(1)} \setminus \mu) < \cdots < c(\mu^+_{(b(\mu))} \setminus \mu)$. We note that there exist indices t and u such that $\mu^+_{(t)} = \lambda$ and $\lambda^-_{(u)} = \mu$.

Using these notation we calculate $h[\mu]/h[\lambda]$. First we regard $\lambda\setminus\mu$ as an outer corner of μ and put $\lambda\setminus\mu = \mu^+_{(t)}\setminus\mu = \{(r_t, c_t)\}$. Let $x_1, \ldots, x_{b'(\mu)}$ be the lengths of horizontal edges of μ and $y_1, \ldots, y_{b'(\mu)}$ the lengths of vertical edges of μ as illustrated in Figure 1 . More precisely, x_k and y_k are defined by $x_k = c(\mu \setminus \mu^-_{(k)}) - c(\mu^+_{(k)} \setminus \mu)$ and $y_k = c(\mu^+_{(k+1)} \setminus \mu) - c(\mu \setminus \mu^-_{(k)})$ respectively. Then we have the following:

$$\begin{aligned}
\frac{h[\mu]}{h[\lambda]} &= \frac{\prod_{(i,j)\in\mu}[h_\mu(i,j)]}{\prod_{(i,j)\in\lambda}[h_\lambda(i,j)]} \\
&= \left(\prod_{j=1}^{c_t-1} \frac{[h_\mu(r_t,j)]}{[h_\mu(r_t,j)+1]}\right) \times \left(\prod_{i=1}^{r_t-1} \frac{[h_\mu(i,c_t)]}{[h_\mu(i,c_t)+1]}\right) \\
&= \Bigg(\prod_{p=1}^{x_1} \frac{[x_1+\cdots+x_{t-1}+y_1+\cdots+y_{t-1}-p]}{[x_1+\cdots+x_{t-1}+y_1+\cdots+y_{t-1}-p+1]} \\
&\qquad \cdot \prod_{p=1}^{x_2} \frac{[x_2+\cdots+x_{t-1}+y_2+\cdots+y_{t-1}-p]}{[x_2+\cdots+x_{t-1}+y_2+\cdots+y_{t-1}-p+1]}
\end{aligned}$$

$$\begin{aligned}
&\qquad\vdots\\
&\cdot\prod_{p=1}^{x_{t-1}}\frac{[x_{t-1}+y_{t-1}-p]}{[x_{t-1}+y_{t-1}-p+1]}\Bigg)\\
&\times\Bigg(\prod_{p=1}^{y_{b'(\mu)}}\frac{[x_t+\cdots+x_{b'(\mu)}+y_t+\cdots+y_{b'(\mu)}-p]}{[x_t+\cdots+x_{b'(\mu)}+y_t+\cdots+y_{b'(\mu)}-p+1]}\\
&\cdot\prod_{p=1}^{y_{b'(\mu)-1}}\frac{[x_t+\cdots+x_{b'(\mu)-1}+y_t+\cdots+y_{b'(\mu)-1}-p]}{[x_t+\cdots+x_{b'(\mu)-1}+y_t+\cdots+y_{b'(\mu)-1}-p+1]}\\
&\qquad\vdots\\
&\cdot\prod_{p=1}^{y_t}\frac{[x_t+y_t-p]}{[x_t+y_t-p+1]}\Bigg)\\
=&\Bigg(\frac{[x_2+\cdots+x_{t-1}+y_1+\cdots+y_{t-1}]}{[x_1+\cdots+x_{t-1}+y_1+\cdots+y_{t-1}]}\\
&\cdot\frac{[x_3+\cdots+x_{t-1}+y_2+\cdots+y_{t-1}]}{[x_2+\cdots+x_{t-1}+y_2+\cdots+y_{t-1}]}\cdot\cdots\cdot\frac{[y_{t-1}]}{[x_{t-1}+y_{t-1}]}\Bigg)\\
&\times\Bigg(\frac{[x_t+\cdots+x_{b'(\mu)}+y_t+\cdots+y_{b'(\mu)-1}]}{[x_t+\cdots+x_{b'(\mu)}+y_t+\cdots+y_{b'(\mu)}]}\\
&\cdot\frac{[x_t+\cdots+x_{b'(\mu)-1}+y_t+\cdots+y_{b'(\mu)-2}]}{[x_t+\cdots+x_{b'(\mu)-1}+y_t+\cdots+y_{b'(\mu)-1}]}\cdot\cdots\cdot\frac{[x_t]}{[x_t+y_t]}\Bigg).
\end{aligned}$$

Here we find that if $1\le s<t$ then

$$x_{s+1}+\cdots+x_{t-1}+y_s+\cdots+y_{t-1}=c(\lambda\setminus\mu)-c(\mu\setminus\mu^-_{(s)})$$

and if $t\le s\le b'(\mu)$ then

$$x_t+\cdots+x_s+y_t+\cdots+y_{s-1}=c(\mu\setminus\mu^-_{(s)})-c(\lambda\setminus\mu).$$

Similarly, we find that if $1\le r<t$

$$x_r+\cdots+x_{t-1}+y_r+\cdots+y_{t-1}=c(\lambda\setminus\mu)-c(\mu^+_{(r)}\setminus\mu)$$

and if $t\le r\le b'(\mu)$ then

$$x_t+\cdots+x_r+y_t+\cdots+y_r=c(\mu^+_{(r+1)}\setminus\mu)-c(\lambda\setminus\mu).$$

Hence we have

$$\frac{h[\mu]}{h[\lambda]}=\Bigg(\frac{[c(\lambda\setminus\mu)-c(\mu\setminus\mu^-_{(1)})]}{[c(\lambda\setminus\mu)-c(\mu^+_{(1)}\setminus\mu)]}$$

$$
\begin{aligned}
&\cdot\frac{[c(\lambda\setminus\mu)-c(\mu\setminus\mu^-_{(2)})]}{[c(\lambda\setminus\mu)-c(\mu^+_{(2)}\setminus\mu)]}\cdot\ \cdots\ \cdot\frac{[c(\lambda\setminus\mu)-c(\mu\setminus\mu^-_{(t-1)})]}{[c(\lambda\setminus\mu)-c(\mu^+_{(t-1)}\setminus\mu)]}\Bigg)\\
&\times\Bigg(\frac{[c(\mu\setminus\mu^-_{(b'(\mu))})-c(\lambda\setminus\mu)]}{[c(\mu^+_{(b(\mu))})-c(\lambda\setminus\mu)]}\\
&\cdot\frac{[c(\mu\setminus\mu^-_{(b'(\mu)-1)})-c(\lambda\setminus\mu)]}{[c(\mu^+_{(b(\mu)-1)})-c(\lambda\setminus\mu)]}\cdot\ \cdots\ \cdot\frac{[c(\mu\setminus\mu^-_{(t)})-c(\lambda\setminus\mu)]}{[c(\mu^+_{(t+1)})-c(\lambda\setminus\mu)]}\Bigg)\\
&=\frac{(-1)^{b'(\mu)}\prod_{s=1}^{b'(\mu)}[c(\lambda\setminus\mu)-c(\mu\setminus\mu^-_{(s)})]}{\prod_{1\le r\le t-1,t+1\le r\le b(\mu)}[c(\mu^+_{(r)}\setminus\mu)-c(\lambda\setminus\mu)]}\,.
\end{aligned}
$$

Since

(1.1)
$$
\begin{aligned}
&\{c(\mu^+_{(r)}\setminus\mu)-c(\lambda\setminus\mu)|1\le r\le t-1,t+1\le r\le b(\mu)\}\\
&=\{c(\lambda^+_{(r)}\setminus\lambda)-c(\lambda\setminus\mu)|1\le r\le b(\lambda)\}\setminus\{1,-1\}\\
&=\{d(\mu,\lambda,\lambda^+_{(r)})|r=1,\dots,b(\lambda)\}\setminus\{1,-1\}\,,
\end{aligned}
$$

if we put

$$E=\{d(\mu,\lambda,\lambda^+_{(r)})\mid r=1,\dots,b(\lambda)\}\,,\quad \hat{E}=E\setminus\{1,-1\}$$

and

$$D=\{d(\mu^-_{(s)},\mu,\lambda)\mid s=1,\dots,b'(\mu)\}\,,$$

then we find

(1.2)
$$\frac{h[\mu]}{h[\lambda]}=(-1)^{|D|}\cdot\frac{\prod_{d\in D}[d]}{\prod_{e\in\hat{E}}[e]}\,.$$

Thus we also have

(1.3)
$$\frac{h[\mu]^2}{h[\lambda]^2}=\frac{\prod_{d\in D}[d]^2}{\prod_{e\in E}[e]^2}\,.$$

REMARK 1.1. Comparing the cardinality of the sets in the equation (1.1), we have

$$|\hat{E}|=b(\mu)-1=b'(\mu)=|D|\,.$$

Similarly, we can calculate $h[\lambda]/h[\mu]$. This time we regard $\lambda\setminus\mu$ as an inner corner of λ and put $\lambda\setminus\mu=\lambda\setminus\lambda^-_{(u)}=\{(r'_u,c'_u)\}$. Let $z_1,\dots,z_{b'(\lambda)}$ be the lengths of horizontal edges of λ and $w_1,\dots,w_{b'(\lambda)}$ the lengths of vertical edges of λ as illustrated in Figure 2. More precisely, we define z_k and w_k by $z_k=c(\lambda\setminus\lambda^-_{(k)})-c(\lambda^+_{(k)}\setminus\lambda)$ and $w_k=c(\lambda^+_{(k+1)}\setminus\lambda)-c(\lambda\setminus\lambda^-_{(k)})$

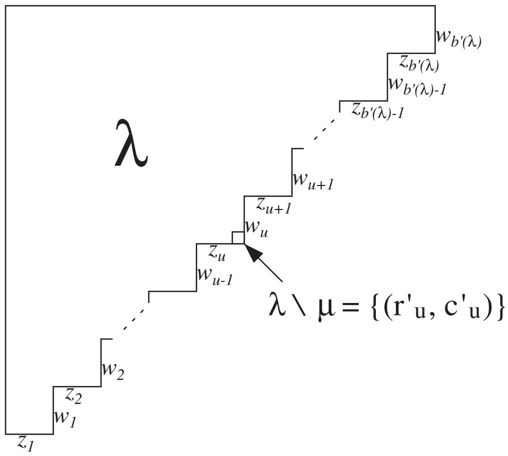

FIGURE 2. $\lambda \setminus \mu$ as an inner corner of λ.

respectively. Then we have the following:

$$\frac{h[\lambda]}{h[\mu]} = \frac{\prod_{(i,j)\in\lambda}[h_\lambda(i,j)]}{\prod_{(i,j)\in\mu}[h_\mu(i,j)]}$$

$$= \left(\prod_{j=1}^{c'_u-1} \frac{[h_\lambda(r'_u, j)]}{[h_\lambda(r'_u, j)-1]}\right) \times \left(\prod_{i=1}^{r'_u-1} \frac{[h_\lambda(i, c'_u)]}{[h_\lambda(i, c'_u)-1]}\right)$$

$$= \left(\prod_{p=1}^{z_1} \frac{[z_1+\cdots+z_u+w_1+\cdots+w_{u-1}-p+1]}{[z_1+\cdots+z_u+w_1+\cdots+w_{u-1}-p]}\right.$$

$$\cdot \prod_{p=1}^{z_2} \frac{[z_2+\cdots+z_u+w_2+\cdots+w_{u-1}-p+1]}{[z_2+\cdots+z_u+w_2+\cdots+w_{u-1}-p]}$$

$$\left.\cdots\cdot \prod_{p=1}^{z_{u-1}} \frac{[z_{u-1}+z_u+w_{u-1}-p+1]}{[z_{u-1}+z_u+w_{u-1}-p]} \cdot \prod_{p=1}^{z_u-1} \frac{[z_u-p+1]}{[z_u-p]}\right)$$

$$\times \left(\prod_{p=1}^{w_{b'(\lambda)}} \frac{[z_{u+1}+\cdots+z_{b'(\lambda)}+w_u+\cdots+w_{b'(\lambda)}-p+1]}{[z_{u+1}+\cdots+z_{b'(\lambda)}+w_u+\cdots+w_{b'(\lambda)}-p]}\right.$$

$$\cdot \prod_{p=1}^{w_{b'(\lambda)-1}} \frac{[z_{u+1}+\cdots+z_{b'(\lambda)-1}+w_u+\cdots+w_{b'(\lambda)-1}-p+1]}{[z_{u+1}+\cdots+z_{b'(\lambda)-1}+w_u+\cdots+w_{b'(\lambda)-1}-p]}$$

$$\left.\cdots\cdot \prod_{p=1}^{w_{u+1}} \frac{[z_{u+1}+w_u+w_{u+1}-p+1]}{[z_{u+1}+w_u+w_{u+1}-p]} \cdot \prod_{p=1}^{w_u-1} \frac{[w_u-p+1]}{[w_u-p]}\right)$$

$$= \Big(\frac{[z_1 + \cdots + z_u + w_1 + \cdots + w_{u-1}]}{[z_2 + \cdots + z_u + w_1 + \cdots + w_{u-1}]} \cdot \frac{[z_2 + \cdots + z_u + w_2 + \cdots + w_{u-1}]}{[z_3 + \cdots + z_u + w_2 + \cdots + w_{u-1}]} \cdot \cdots \cdot \frac{[z_{u-1} + z_u + w_{u-1}]}{[z_u + w_{u-1}]} \cdot \frac{[z_u]}{[1]} \Big)$$

$$\times \Big(\frac{[z_{u+1} + \cdots + z_{b'(\lambda)} + w_u + \cdots + w_{b'(\lambda)}]}{[z_{u+1} + \cdots + z_{b'(\lambda)} + w_u + \cdots + w_{b'(\lambda)-1}]} \cdot \frac{[z_{u+1} + \cdots + z_{b'(\lambda)-1} + w_u + \cdots + w_{b'(\lambda)-1}]}{[z_{u+1} + \cdots + z_{b'(\lambda)-1} + w_u + \cdots + w_{b'(\lambda)-2}]} \cdot \cdots \cdot \frac{[z_{u+1} + w_u + w_{u+1}]}{[z_{u+1} + w_u]} \cdot \frac{[w_u]}{[1]} \Big)$$

$$= \Big(\frac{[c(\lambda \setminus \mu) - c(\lambda^+_{(1)} \setminus \lambda)]}{[c(\lambda \setminus \mu) - c(\lambda \setminus \lambda^-_{(1)})]} \cdot \frac{[c(\lambda \setminus \mu) - c(\lambda^+_{(2)} \setminus \lambda)]}{[c(\lambda \setminus \mu) - c(\lambda \setminus \lambda^-_{(2)})]} \cdot \cdots \cdot \frac{[c(\lambda \setminus \mu) - c(\lambda^+_{(u-1)} \setminus \lambda)]}{[c(\lambda \setminus \mu) - c(\lambda \setminus \lambda^-_{(u-1)})]} \cdot [c(\lambda \setminus \mu) - c(\lambda^+_{(u)} \setminus \lambda)] \Big)$$

$$\times \Big(\frac{[c(\lambda^+_{(b(\lambda))} \setminus \lambda) - c(\lambda \setminus \mu)]}{[c(\lambda \setminus \lambda^-_{(b'(\lambda))}) - c(\lambda \setminus \mu)]} \cdot \frac{[c(\lambda^+_{(b(\lambda)-1)} \setminus \lambda) - c(\lambda \setminus \mu)]}{[c(\lambda \setminus \lambda^-_{(b'(\lambda)-1)}) - c(\lambda \setminus \mu)]} \cdot \cdots \cdot \frac{[c(\lambda^+_{(u+2)} \setminus \lambda) - c(\lambda \setminus \mu)]}{[c(\lambda \setminus \lambda^-_{(u+1)}) - c(\lambda \setminus \mu)]} \cdot [c(\lambda^+_{(u+1)} \setminus \lambda) - c(\lambda \setminus \mu)] \Big)$$

$$= \frac{(-1)^{b'(\lambda)} \prod_{r=1}^{b(\lambda)} [c(\lambda^+_{(r)} \setminus \lambda) - c(\lambda \setminus \mu)]}{\prod_{1 \le s \le u-1, u+1 \le s \le b'(\lambda)} [c(\lambda \setminus \mu) - c(\lambda \setminus \lambda^-_{(s)})]} .$$

Since

$$\begin{aligned} &\{c(\lambda \setminus \mu) - c(\lambda \setminus \lambda^-_{(s)}) | 1 \le s \le u-1, u+1 \le s \le b'(\lambda)\} \\ &= \{c(\lambda \setminus \mu) - c(\mu \setminus \mu^-_{(s)}) | s = 1, \ldots, b'(\mu)\} \setminus \{1, -1\} \\ &= \{d(\mu^-_{(s)}, \mu, \lambda) | s = 1, \ldots, b'(\mu)\} \setminus \{1, -1\} \ (= D \setminus \{1, -1\}) , \end{aligned} \tag{1.4}$$

if we put $\hat{D} = D \setminus \{1, -1\}$, then we obtain

$$\frac{h[\lambda]}{h[\mu]} = (-1)^{|E|-1} \cdot \frac{\prod_{e \in E} [e]}{\prod_{d \in \hat{D}} [d]} . \tag{1.5}$$

Further, we find that

$$\frac{h[\lambda]^2}{h[\mu]^2} = \frac{\prod_{e \in E} [e]^2}{\prod_{d \in D} [d]^2} . \tag{1.3$'$}$$

This is consistent with the equation (1.3).

REMARK 1.2. Comparing the cardinality of the sets in the equation (1.4), we have

$$|\hat{D}| = b'(\lambda) - 1 = b(\lambda) - 2 = |E| - 2\,.$$

Now we prove a formula among the axial distances in the sets D and E.

LEMMA 1.3. *Let D, $\hat{D}$, E and $\hat{E}$ be sets of axial distances previously defined*:

$$D = \{d_s = d(\mu^-_{(s)}, \mu, \lambda) \mid s = 1, \dots, b'(\mu)\}\,, \quad \hat{D} = D \setminus \{1, -1\}\,,$$
$$E = \{e_r = d(\mu, \lambda, \lambda^+_{(r)}) \mid r = 1, \dots, b(\lambda)\}\,, \quad \hat{E} = E \setminus \{1, -1\}\,.$$

Then the following identity holds:

$$\sum_{d_s \in \hat{D}} d_s + \sum_{e_r \in E} e_r = \sum_{d_s \in D} d_s + \sum_{e_r \in \hat{E}} e_r = -c(\lambda \setminus \mu)\,.$$

PROOF. Put $\hat{D}_0 = \hat{D} \cup \{0\}$. By the equation (1.4), we have

$$\hat{D}_0 = \{c(\lambda \setminus \mu) - c(\lambda \setminus \lambda^-_{(s)}) \mid s = 1, \dots, b'(\lambda)\} \quad \text{(recall that } \lambda^-_{(u)} = \mu)$$

and

$$E = \{c(\lambda^+_{(r)} \setminus \lambda) - c(\lambda \setminus \mu) \mid r = 1, \dots, b(\lambda)\}\,.$$

Hence we obtain

$$\begin{aligned}\sum_{d_s \in \hat{D}} d_s + \sum_{e_r \in E} e_r &= \sum_{d_s \in \hat{D}_0} d_s + \sum_{e_r \in E} e_r \\ &= c(\lambda \setminus \mu) \cdot b'(\lambda) - \sum_{s=1}^{b'(\lambda)} c(\lambda \setminus \lambda^-_{(s)}) + \sum_{r=1}^{b(\lambda)} c(\lambda^+_{(r)} \setminus \lambda) - c(\lambda \setminus \mu) \cdot b(\lambda) \\ &= -c(\lambda \setminus \mu) + \sum_{r=1}^{b(\lambda)} c(\lambda^+_{(r)} \setminus \lambda) - \sum_{s=1}^{b'(\lambda)} c(\lambda \setminus \lambda^-_{(s)})\,.\end{aligned}$$

Let $\lambda^+_{(r)} \setminus \lambda = \{(r_r, c_r)\}$, $(r = 1, \dots, b(\lambda))$ and let $\lambda \setminus \lambda^-_{(s)} = \{(r'_s, c'_s)\}$, $(s = 1, \dots, b'(\lambda))$. Then we find that $(r'_s, c'_s) = (r_s - 1, c_{s+1} - 1)$ for $s = 1, \dots, b'(\mu)$. Hence we find

$$\begin{aligned}\sum_{r=1}^{b(\lambda)} c(\lambda^+_{(r)} \setminus \lambda) - \sum_{s=1}^{b'(\lambda)} c(\lambda \setminus \lambda^-_{(s)}) &= \sum_{r=1}^{b(\lambda)} (c_r - r_r) - \sum_{s=1}^{b'(\lambda)} (c'_s - r'_s) \\ &= \sum_{r=1}^{b(\lambda)} (c_r - r_r) - \sum_{s=1}^{b'(\lambda)} (c_{s+1} - r_s) \\ &= c_1 - r_{b(\lambda)}\end{aligned}$$

$$= 1 - 1$$
$$= 0\,.$$

Thus we obtain

$$\sum_{d_s \in \hat{D}} d_s + \sum_{e_r \in E} e_r = -c(\lambda \setminus \mu)\,.$$

Similarly, using the equation (1.1) we can check that

$$\sum_{d_s \in D} d_s + \sum_{e_r \in \hat{E}} e_r = -c(\lambda \setminus \mu)\,.$$

□

Next we calculate $h[\lambda]/h[\lambda^+_{(r)}]$ and $h[\mu]/h[\mu^-_{(s)}]$.

Replacing μ and λ with λ and $\lambda^+_{(r)}$ respectively in the calculation of $h[\mu]/h[\lambda]$, we have the following:

$$\begin{aligned}
\frac{h[\lambda]}{h[\lambda^+_{(r)}]} &= \frac{(-1)^{b'(\lambda)} \prod_{s=1}^{b'(\lambda)}[c(\lambda^+_{(r)} \setminus \lambda) - c(\lambda \setminus \lambda^-_{(s)})]}{\prod_{1 \le i \le r-1, r+1 \le i \le b(\lambda)}[c(\lambda^+_{(i)} \setminus \lambda) - c(\lambda^+_{(r)} \setminus \lambda)]} \\
&= \frac{(-1)^{b'(\lambda)} \prod_{s=1}^{b'(\lambda)}[c(\lambda^+_{(r)} \setminus \lambda) - c(\lambda \setminus \mu) + c(\lambda \setminus \mu) - c(\lambda \setminus \lambda^-_{(s)})]}{(-1)^{b(\lambda)-1} \prod_{1 \le i \le r-1, r+1 \le i \le b(\lambda)}[c(\lambda^+_{(r)} \setminus \lambda) - c(\lambda^+_{(i)} \setminus \lambda)]} \\
&= \frac{\prod_{s=1}^{b'(\lambda)}[d(\mu, \lambda, \lambda^+_{(r)}) + c(\lambda \setminus \mu) - c(\lambda \setminus \lambda^-_{(s)})]}{\prod_{1 \le i \le r-1, r+1 \le i \le b(\lambda)}[d(\mu, \lambda, \lambda^+_{(r)}) - d(\mu, \lambda, \lambda^+_{(i)})]}\,.
\end{aligned}$$

Since $\lambda \setminus \lambda^-_{(u)} = \lambda \setminus \mu$, using the equation (1.4) we have

$$\begin{aligned}
&\{c(\lambda \setminus \mu) - c(\lambda \setminus \lambda^-_{(s)}) | s = 1, \dots, b'(\lambda)\} \\
&\quad = \{d(\mu^-_{(s)}, \mu, \lambda) | s = 1, \dots, b'(\mu)\} \cup \{0\} \setminus \{1, -1\}\,.
\end{aligned}$$

Hence putting

$$\hat{D}_0 = \{d_s = d(\mu^-_{(s)}, \mu, \lambda) | s = 1, \dots, b'(\mu)\} \cup \{0\} \setminus \{1, -1\}$$

and

$$E = \{e_i = d(\mu, \lambda, \lambda^+_{(i)}) | i = 1, \dots, b(\lambda)\}$$

we have

$$\frac{h[\lambda]}{h[\lambda^+_{(r)}]} = \frac{\prod_{d_s \in \hat{D}_0}[e_r + d_s]}{\prod_{e_i \in E \setminus \{e_r\}}[e_r - e_i]}\,. \tag{1.6}$$

By Remark 1.2, we find that $|\hat{D}_0| = |E| - 1$. Hence we have

$$\begin{aligned}
\frac{\prod_{d_s \in \hat{D}_0}[e_r + d_s]}{\prod_{e_i \in E\setminus\{e_r\}}[e_r - e_i]} &= \frac{\prod_{d_s \in \hat{D}_0}(q^{e_r+d_s} - q^{-e_r-d_s})}{\prod_{e_i \in E\setminus\{e_r\}}(q^{e_r-e_i} - q^{e_i-e_r})} \\
&= \frac{\prod_{d_s \in \hat{D}_0}(q^{-e_r+d_s})(q^{2e_r} - q^{-2d_s})}{\prod_{e_i \in E\setminus\{e_r\}}(q^{-e_r-e_i})(q^{2e_r} - q^{2e_i})} \\
&= \frac{q^{-|\hat{D}_0|e_r} \cdot q^{\sum_{d_s \in \hat{D}_0} d_s}}{q^{-(|E|-1)e_r} \cdot q^{e_r - \sum_{e_i \in E} e_i}} \cdot \frac{\prod_{d_s \in \hat{D}_0}(q^{2e_r} - q^{-2d_s})}{\prod_{e_i \in E\setminus\{e_r\}}(q^{2e_r} - q^{2e_i})} \\
&= q^{-e_r} \cdot q^{\sum_{d_s \in \hat{D}_0} d_s + \sum_{e_i \in E} e_i} \cdot \frac{\prod_{d_s \in \hat{D}_0}(q^{2e_r} - q^{-2d_s})}{\prod_{e_i \in E\setminus\{e_r\}}(q^{2e_r} - q^{2e_i})} \\
&\qquad \left(\textstyle\sum_{d_s \in \hat{D}_0} d_s + \sum_{e_i \in E} e_i = -c(\lambda \setminus \mu) \text{ by Lemma 1.3}\right) \\
&= q^{-c(\lambda^+_{(r)} \setminus \lambda)} \cdot \frac{\prod_{d_s \in \hat{D}_0}(q^{2e_r} - q^{-2d_s})}{\prod_{e_i \in E\setminus\{e_r\}}(q^{2e_r} - q^{2e_i})}\,.
\end{aligned}$$

Thus we have obtained

$$\frac{h[\lambda]}{h[\lambda^+_{(r)}]} = q^{-c(\lambda^+_{(r)} \setminus \lambda)} \cdot \frac{\prod_{d_s \in \hat{D}_0}(q^{2e_r} - q^{-2d_s})}{\prod_{e_i \in E\setminus\{e_r\}}(q^{2e_r} - q^{2e_i})}\,. \tag{1.7}$$

Similarly, replacing λ and μ with μ and $\mu^-_{(s)}$ respectively in the calculation of $h[\lambda]/h[\mu]$, we have the following:

$$\begin{aligned}
\frac{h[\mu]}{h[\mu^-_{(s)}]} &= \frac{(-1)^{b'(\mu)} \prod_{r=1}^{b(\mu)}[c(\mu^+_{(r)} \setminus \mu) - c(\mu \setminus \mu^-_{(s)})]}{\prod_{1 \le i \le s-1, s+1 \le i \le b'(\mu)}[c(\mu \setminus \mu^-_{(s)}) - c(\mu \setminus \mu^-_{(i)})]} \\
&= \frac{\prod_{r=1}^{b(\mu)}[c(\mu^+_{(r)} \setminus \mu) - c(\lambda \setminus \mu) + d(\mu^-_{(s)}, \mu, \lambda)]}{(-1)^{b'(\mu)} \prod_{1 \le i \le s-1, s+1 \le i \le b'(\mu)}[-d(\mu^-_{(s)}, \mu, \lambda) + d(\mu^-_{(i)}, \mu, \lambda)]} \\
&= -\frac{\prod_{r=1}^{b(\mu)}[c(\mu^+_{(r)} \setminus \mu) - c(\lambda \setminus \mu) + d(\mu^-_{(s)}, \mu, \lambda)]}{\prod_{1 \le i \le s-1, s+1 \le i \le b'(\mu)}[d(\mu^-_{(s)}, \mu, \lambda) - d(\mu^-_{(i)}, \mu, \lambda)]}\,.
\end{aligned}$$

Since $\mu^+_{(t)} \setminus \mu = \lambda \setminus \mu$, using the equation (1.1) we have

$$\begin{aligned}
&\{c(\mu^+_{(r)} \setminus \mu) - c(\lambda \setminus \mu) | r = 1, \dots, b(\mu)\} \\
&\qquad = \{d(\mu, \lambda, \lambda^+_{(r)}) | r = 1, \dots, b(\lambda)\} \cup \{0\} \setminus \{1, -1\}\,.
\end{aligned}$$

Hence putting

$$\hat{E}_0 = \{e_r = d(\mu, \lambda, \lambda^+_{(r)}) | r = 1, \dots, b(\lambda)\} \cup \{0\} \setminus \{1, -1\}$$

and

$$D = \{d_i = d(\mu^-_{(i)}, \mu, \lambda) | i = 1, \dots, b'(\mu)\}$$

we have

$$\frac{h[\mu]}{h[\mu^-_{(s)}]} = -\frac{\prod_{e_r \in \hat{E}_0}[d_s + e_r]}{\prod_{d_i \in D \setminus \{d_s\}}[d_s - d_i]}. \tag{1.8}$$

By Remark 1.1, we find that $|\hat{E}_0| = |D| + 1$. Hence we have

$$\begin{aligned}
\frac{\prod_{e_r \in \hat{E}_0}[d_s + e_r]}{\prod_{d_i \in D \setminus \{d_s\}}[d_s - d_i]} &= \frac{1}{(q - q^{-1})^2} \cdot \frac{\prod_{e_r \in \hat{E}_0}(q^{d_s + e_r} - q^{-d_s - e_r})}{\prod_{d_i \in D \setminus \{d_s\}}(q^{d_s - d_i} - q^{d_i - d_s})} \\
&= \frac{1}{(q - q^{-1})^2} \cdot \frac{\prod_{e_r \in \hat{E}_0}(q^{e_r - d_s})(q^{2d_s} - q^{-2e_r})}{\prod_{d_i \in D \setminus \{d_s\}}(q^{-d_i - d_s})(q^{2d_s} - q^{2d_i})} \\
&= \frac{1}{(q - q^{-1})^2} \cdot \frac{\prod_{e_r \in \hat{E}_0} q^{e_r}}{\prod_{d_i \in D \setminus \{d_s\}} q^{-d_i}} \cdot \frac{q^{-|\hat{E}_0| d_s}}{q^{-(|D|-1)d_s}} \\
&\quad \times \frac{\prod_{e_r \in \hat{E}_0}(q^{2d_s} - q^{-2e_r})}{\prod_{d_i \in D \setminus \{d_s\}}(q^{2d_s} - q^{2d_i})} \\
&= \frac{q^{\sum_{e_r \in \hat{E}_0} e_r + \sum_{d_i \in D} d_i - d_s} \cdot q^{(-|\hat{E}_0| + |D| - 1) \cdot d_s}}{(q - q^{-1})^2} \\
&\quad \times \frac{\prod_{e_r \in \hat{E}_0}(q^{2d_s} - q^{-2e_r})}{\prod_{d_i \in D \setminus \{d_s\}}(q^{2d_s} - q^{2d_i})} \\
&\quad \text{(Lemma 1.3)} \\
&= \frac{q^{-c(\lambda \setminus \mu) - d_s} \cdot q^{-2d_s}}{(q - q^{-1})^2} \cdot \frac{\prod_{e_r \in \hat{E}_0}(q^{2d_s} - q^{-2e_r})}{\prod_{d_i \in D \setminus \{d_s\}}(q^{2d_s} - q^{2d_i})} \\
&= \frac{q^{-2c(\lambda \setminus \mu) + c(\mu \setminus \mu^-_{(s)})}}{(q - q^{-1})^2} \cdot \frac{q^{-2d_s} \prod_{e_r \in \hat{E}_0}(q^{2d_s} - q^{-2e_r})}{\prod_{d_i \in D \setminus \{d_s\}}(q^{2d_s} - q^{2d_i})}.
\end{aligned}$$

Hence we have obtained the following:

$$\frac{h[\mu]}{h[\mu^-_{(s)}]} = -\frac{q^{-2c(\lambda \setminus \mu) + c(\mu \setminus \mu^-_{(s)})}}{(q - q^{-1})^2} \cdot \frac{q^{-2d_s} \prod_{e_r \in \hat{E}_0}(q^{2d_s} - q^{-2e_r})}{\prod_{d_i \in D \setminus \{d_s\}}(q^{2d_s} - q^{2d_i})}. \tag{1.9}$$

Before concluding this section, we show an identity among the contents of the outer corners and the inner corners of a Young diagram μ.

PROPOSITION 1.4. *Let $\{\mu^+_{(r)} \setminus \mu\}_{r=1,\dots,b(\mu)}$ be the set of all the outer corners of a Young diagram μ and $\{\mu \setminus \mu^-_{(s)}\}_{s=1,\dots,b'(\mu)}$ the set of all the inner corners of the Young diagram*

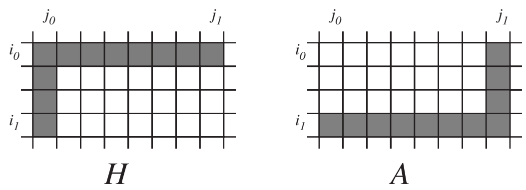

FIGURE 3. {Contents in H} = {Contents in A}.

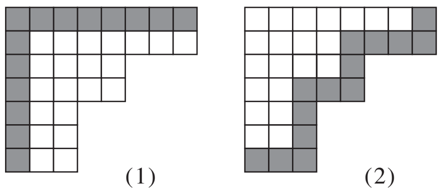

FIGURE 4. {Contents in (1,1) hook of λ} = {Contents in the rim of λ}.

μ. Then the following identity holds.

$$(q+q^{-1}-2)\sum_{x\in\mu} q^{c(x)} = \sum_{r=1}^{b(\mu)} q^{c(\mu^+_{(r)}\backslash\mu)} - \sum_{s=1}^{b'(\mu)} q^{c(\mu\backslash\mu^-_{(s)})} - 1\,.$$

To prove this proposition, we use the following lemma.

LEMMA 1.5. *Let $\lambda = (\alpha_1,\dots,\alpha_r|\beta_1,\dots,\beta_r)$ be a Young diagram by the notation due to Frobenius. Namely, let $\alpha_i = \lambda_i - i$ [resp. $\beta_i = \lambda^*_i - i$] be the number of nodes in the i-th row [resp. column] of λ to the right [resp. below] of (i,i), for $1 \le i \le r$. Here, we suppose that the main diagonal of the diagram λ consists of r nodes (i,i) $(1 \le i \le r)$. Then contents in $(1,1)$ hook of λ coincide with contents in the rim of λ, i.e.*

$$\begin{aligned}&\{c(x)|\ x\in(\alpha_1|\beta_1)\}\\ &\quad=\{c(x)|\ x\in(\alpha_1,\dots,\alpha_r|\beta_1,\dots,\beta_r)\setminus(\alpha_2,\dots,\alpha_r|\beta_2,\dots,\beta_r)\}\,.\end{aligned}$$

PROOF. It is easy to check that two sets of contents in a hook and in the corresponding arm coincide (See Figure 3).

Iterative use of this fact shows that the contents in (1,1) hook of λ and the contents in the rim of λ coincide (See Figure 4(1)(2)). □

PROOF OF PROPOSITION 1.4. Let $S = \sum_{x\in\mu} q^{c(x)}$. Then

$$(q-1)S = \sum_{x\in\mu} q^{c(x)+1} - \sum_{x\in\mu} q^{c(x)} = \sum_{i=1}^{l(\mu)} q^{c((i,\mu_i+1))} - \sum_{i=1}^{l(\mu)} q^{c((i,1))}\,, \qquad (1.10)$$

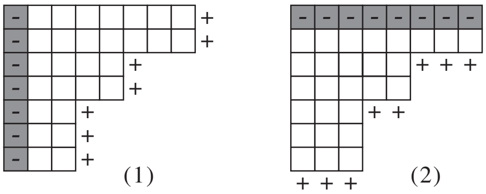

FIGURE 5. Interpretation of the equation (1.10) and (1.11).

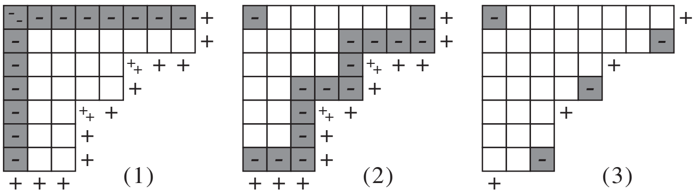

FIGURE 6. Interpretation of Proposition 1.4.

where $\{(i, \mu_i + 1) | i = 1, \ldots, l(\mu)\}$ is the union of outer vertical rims of μ and $\{(i, 1) | i = 1, \ldots, l(\mu)\}$ is the first column of μ (See Figure 5(1)). Similarly,

$$(1.11) \qquad (q^{-1} - 1)S = \sum_{x \in \mu} q^{c(x)-1} - \sum_{x \in \mu} q^{c(x)} = \sum_{j=1}^{\mu_1} q^{c((\mu'_j+1, j))} - \sum_{i=1}^{\mu_1} q^{c((1, j))},$$

where $\{(\mu'_j + 1, j) | j = 1, \ldots, \mu_1\}$ is the union of outer horizontal rims of μ and $\{(1, j) | j = 1, \ldots, \mu_1\}$ is the first row of μ (See Figure 5(2)).

Hence we obtain

$$\begin{aligned}(q + q^{-1} - 2) \sum_{x \in \mu} q^{c(x)} &= \sum_{i=1}^{l(\mu)} q^{c((i, \mu_i+1))} + \sum_{j=1}^{\mu_1} q^{c((\mu'_j+1, j))} - \sum_{i=1}^{l(\mu)} q^{c((i,1))} - \sum_{i=1}^{\mu_1} q^{c((1,j))} \\ &= \sum_{i=1}^{l(\mu)} q^{c((i, \mu_i+1))} + \sum_{j=1}^{\mu_1} q^{c((\mu'_j+1, j))} \\ &\quad - \sum_{x \in \{\text{the rim of } \mu\}} q^{c(x)} - q^{c((1,1))}.\end{aligned}$$

By Lemma 1.5 and Figure 6(1)–(3), we will find that the last term is equal to

$$\sum_{r=1}^{b(\mu)} q^{c(\mu^+_{(r)} \setminus \mu)} - \sum_{s=1}^{b'(\mu)} q^{c(\mu \setminus \mu^-_{(s)})} - 1.$$

This completes the proof. □

2. Lagrange's interpolation polynomial

In this section we show a version of Lagrange's interpolation polynomial.

For an arbitrary integer sequence $\gamma = (\gamma_1, \dots, \gamma_n)$ of size n on which the symmetric group $\mathfrak{S}_n$ acts by transposition of the components, we define $a_\gamma(x_1, \dots, x_n)$ by

$$a_\gamma(x_1, \dots, x_n) = \det[x_i^{\gamma_j}] = \sum_{w \in S_n} (\operatorname{sgn} w) x^{w\gamma} .$$

The Schur function s_γ is defined by

$$s_\gamma(x_1, \dots, x_n) = \frac{a_{\gamma+\delta}(x_1, \dots, x_n)}{a_\delta(x_1, \dots, x_n)} ,$$

where δ is the sequence $(n-1, n-2, \dots, 1, 0)$.

LEMMA 2.1. *Let $X = \{x_1, \dots, x_n\}$ be a set of indeterminates whose cardinality is equal to n. Then the following identity holds*:

$$\sum_{x_r \in X} \frac{x_r^j}{\prod_{x_i \in X \setminus \{x_r\}} (x_r - x_i)} = s_{(j-n+1,0,\dots,0)}(x_1, \dots, x_n) .$$

PROOF. Let $\Delta_n(z_1, \dots, z_n)$ be the simplest alternating function in the variables $z_1, \dots, z_n$:

$$\Delta_n(z_1, \dots, z_n) = \prod_{i<j} (z_i - z_j) .$$

As is well known, this is equal to $a_\delta(z_1, \dots, z_n)$. Using this notation, the lemma is verified by the following direct calculation:

$$\begin{aligned}
&s_{(j-n+1,0,\dots,0)}(x_1, x_2, x_3, \dots, x_n) \\
&= \begin{vmatrix} x_1^j & x_1^{n-2} & x_1^{n-3} & \cdots & x_1 & 1 \\ x_2^j & x_2^{n-2} & x_2^{n-3} & \cdots & x_2 & 1 \\ x_3^j & x_3^{n-2} & x_3^{n-3} & \cdots & x_3 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n^j & x_n^{n-2} & x_n^{n-3} & \cdots & x_n & 1 \end{vmatrix} \cdot \{\Delta_n(x_1, x_2, x_3, \cdots, x_n)\}^{-1} \\
&= \left(x_1^j \cdot \begin{vmatrix} x_2^{n-2} & x_2^{n-3} & \cdots & x_2 & 1 \\ x_3^{n-2} & x_3^{n-3} & \cdots & x_3 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n^{n-2} & x_n^{n-3} & \cdots & x_n & 1 \end{vmatrix} - x_2^j \cdot \begin{vmatrix} x_1^{n-2} & x_1^{n-3} & \cdots & x_1 & 1 \\ x_3^{n-2} & x_3^{n-3} & \cdots & x_3 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n^{n-2} & x_n^{n-3} & \cdots & x_n & 1 \end{vmatrix} \right.
\end{aligned}$$

$$+\cdots+(-1)^{n-1}\cdot x_n^j\cdot\begin{vmatrix} x_1^{n-2} & x_1^{n-3} & \cdots & x_1 & 1\\ x_2^{n-2} & x_2^{n-3} & \cdots & x_2 & 1\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ x_{n-1}^{n-2} & x_{n-1}^{n-3} & \cdots & x_{n-1} & 1\end{vmatrix}\Bigg)$$

$$\times\{\Delta_n(x_1,x_2,x_3,\cdots,x_n)\}^{-1}$$

$$=x_1^j\cdot\frac{\Delta_{n-1}(x_2,x_3,x_4,\ldots,x_n)}{\Delta_n(x_1,x_2,x_3,\ldots,x_n)}-x_2^j\cdot\frac{\Delta_{n-1}(x_1,x_3,x_4,\ldots,x_n)}{\Delta_n(x_1,x_2,x_3,\ldots,x_n)}$$

$$+\cdots+(-1)^{n-1}\cdot x_n^j\cdot\frac{\Delta_{n-1}(x_1,x_2,x_3,\ldots,x_{n-1})}{\Delta_n(x_1,x_2,x_3,\ldots,x_n)}$$

$$=x_1^j\cdot\frac{\Delta_{n-1}(x_2,x_3,x_4,\ldots,x_n)}{\Delta_n(x_1,x_2,x_3,\ldots,x_n)}+x_2^j\cdot\frac{\Delta_{n-1}(x_1,x_3,x_4,\ldots,x_n)}{\Delta_n(x_2,x_1,x_3,\ldots,x_n)}$$

$$+\cdots+x_n^j\cdot\frac{\Delta_{n-1}(x_1,x_2,x_3\ldots,x_{n-1})}{\Delta_n(x_n,x_1,x_2,\ldots,x_{n-1})}$$

$$=\sum_{x_r\in X}\frac{x_i^j}{\prod_{x_i\in X\setminus\{x_r\}}(x_r-x_i)}\,.$$

□

EXAMPLE 2.2. If $0\le j<n-1=|X|-1$, then we have

$$\sum_{x_r\in X}\frac{x_r^j}{\prod_{x_i\in X\setminus\{x_r\}}(x_r-x_i)}=0\,. \tag{2.1}$$

If $|X|-1=n-1\le j$, then we have

$$\sum_{x_r\in X}\frac{x_r^j}{\prod_{x_i\in X\setminus\{x_r\}}(x_r-x_i)}=s_{(j-n+1,0,\ldots,0)}(x_1\ldots x_n)=h_{j-n+1}(x_1,\ldots,x_n)\,.$$

Here $h_r(x_1,\ldots,x_n)$ is the r-th complete symmetric function. In particular, if $j=n=|X|$ then we have

$$\sum_{x_r\in X}\frac{x_r^{|X|}}{\prod_{x_i\in X\setminus\{x_r\}}(x_r-x_i)}=h_1(x_1,\ldots,x_n)=\sum_{x_i\in X}x_i\,, \tag{2.2}$$

and if $j=n-1=|X|-1$ then we have

$$\sum_{x_r\in X}\frac{x_r^{|X|-1}}{\prod_{x_i\in X\setminus\{x_r\}}(x_r-x_i)}=h_0(x_1,\ldots,x_n)=1\,. \tag{2.3}$$

On the other hand, if j is negative, then putting $j=-k$ we have

$$\sum_{x_r\in X}\frac{x_r^j}{\prod_{x_i\in X\setminus\{x_r\}}(x_r-x_i)}$$

$$= s_{(-k-n+1,0,\dots,0,0)}(x_1,\dots,x_n)$$
$$= (x_1\cdots x_n)^{-k}\cdot s_{(1-n,k,\dots,k,k)}(x_1,\dots,x_n)$$
$$= (-1)^{n-1}\cdot(x_1\cdots x_n)^{-k}\cdot s_{(k-1,k-1,\dots,k-1,0)}(x_1,\dots,x_n)\,.$$

In particular, if $j=-1$ then we have

$$\sum_{x_r\in X}\frac{x_r^{-1}}{\prod_{x_i\in X\setminus\{x_r\}}(x_r-x_i)}=\frac{(-1)^{n-1}s_{(0,\dots,0)}(x_1,\dots,x_n)}{x_1\cdots x_n}=\frac{(-1)^{|X|-1}}{x_1\cdots x_n}\,, \tag{2.4}$$

and if $j=-2$ then we have

$$\begin{aligned}\sum_{x_r\in X}\frac{x_r^{-2}}{\prod_{x_i\in X\setminus\{x_r\}}(x_r-x_i)}&=\frac{(-1)^{n-1}s_{(1,1,\dots,1,0)}(x_1,\dots,x_n)}{(x_1\cdots x_n)^2}\\&=\frac{(-1)^{|X|-1}\sum_{x_r\in X}(\prod_{x_i\in X\setminus\{x_r\}}x_i)}{\prod_{x_i\in X}x_i^2}\,.\end{aligned}\tag{2.5}$$

3. Proof of Theorem 0.1

In this section, we prove Theorem 0.1 one by one using Example 2.2. Throughout this section we follow the notation in Lemma 1.3. We note that although the equations (1), (3), (4), (6), (7) and (8) contain both plus and minus signs in the exponents of the parameter q, if we prove the equations hold for either sign then replacing q with q^{-1} we find that they also hold for the other sign.

First we prove (1)–(3).

PROOF OF (1). Using the equation (1.7), we find

$$\sum_{r=1}^{b(\lambda)}q^{c(\lambda^+_{(r)}\setminus\lambda)}\cdot\frac{h[\lambda]}{h[\lambda^+_{(r)}]}=\sum_{e_r\in E}\frac{\prod_{d_s\in\hat{D}_0}(q^{2e_r}-q^{-2d_s})}{\prod_{e_i\in E\setminus\{e_r\}}(q^{2e_r}-q^{2e_i})}\,.$$

By Remark 1.2 we find $|\hat{D}_0|=|\hat{D}|+1=|E|-1$. Hence using the equations (2.1),(2.3) we obtain the following:

$$\sum_{e_r\in E}\frac{\prod_{d_s\in\hat{D}_0}(q^{2e_r}-q^{-2d_s})}{\prod_{e_i\in E\setminus\{e_r\}}(q^{2e_r}-q^{2e_i})}=\sum_{e_r\in E}\frac{(q^{2e_r})^{|E|-1}}{\prod_{e_i\in E\setminus\{e_r\}}(q^{2e_r}-q^{2e_i})}=1\,.$$

Thus (1) holds. □

PROOF OF (2). Similarly, using the equation (1.7) we have

$$\sum_{r=1}^{b(\lambda)}\frac{1}{[e_r]}\cdot\frac{h[\lambda]}{h[\lambda^+_{(r)}]}=\sum_{r=1}^{b(\lambda)}\frac{q-q^{-1}}{q^{e_r}-q^{-e_r}}\cdot\frac{q^{-c(\lambda^+_{(r)}\setminus\lambda)}\prod_{d_s\in\hat{D}_0}(q^{2e_r}-q^{-2d_s})}{\prod_{e_i\in E\setminus\{e_r\}}(q^{2e_r}-q^{2e_i})}$$

$$
\begin{aligned}
&= \sum_{r=1}^{b(\lambda)} \frac{q^{-c(\lambda^+_{(r)}\setminus\lambda)+e_r}(q-q^{-1})}{q^{2e_r}-q^0} \cdot \frac{\prod_{d_s\in\hat{D}_0}(q^{2e_r}-q^{-2d_s})}{\prod_{e_i\in E\setminus\{e_r\}}(q^{2e_r}-q^{2e_i})} \\
&= q^{-c(\lambda\setminus\mu)}(q-q^{-1}) \sum_{e_r\in E} \frac{\prod_{d_s\in\hat{D}_0}(q^{2e_r}-q^{-2d_s})}{(q^{2e_r}-q^{2\cdot 0})\prod_{e_i\in E\setminus\{e_r\}}(q^{2e_r}-q^{2e_i})} \\
&\qquad (\ \hat{D}_0 = \hat{D}\cup\{0\}\ ) \\
&= q^{-c(\lambda\setminus\mu)}(q-q^{-1}) \sum_{e_r\in E} \frac{\prod_{d_s\in\hat{D}}(q^{2e_r}-q^{-2d_s})}{\prod_{e_i\in E\setminus\{e_r\}}(q^{2e_r}-q^{2e_i})} \\
&= 0\,.
\end{aligned}
$$

Here we applied the equation (2.1) using the fact that $|\hat{D}| = |E| - 2$. □

PROOF OF (3). Similarly as in the proof of (1) and (2), using the equation (1.7) we have

$$
\begin{aligned}
\sum_{r=1}^{b(\lambda)} \frac{q^{-e_r}}{[e_r]^2}\cdot\frac{h[\lambda]}{h[\lambda^+_{(r)}]} &= \sum_{r=1}^{b(\lambda)} \frac{(q-q^{-1})^2 q^{-e_r}}{(q^{e_r}-q^{-e_r})^2}\cdot\frac{q^{-c(\lambda^+_{(r)}\setminus\lambda)}\prod_{d_s\in\hat{D}_0}(q^{2e_r}-q^{-2d_s})}{\prod_{e_i\in E\setminus\{e_r\}}(q^{2e_r}-q^{2e_i})} \\
&= \sum_{r=1}^{b(\lambda)} \frac{(q-q^{-1})^2 q^{e_r}}{(q^{2e_r}-q^0)^2}\cdot\frac{q^{-c(\lambda^+_{(r)}\setminus\lambda)}\prod_{d_s\in\hat{D}_0}(q^{2e_r}-q^{-2d_s})}{\prod_{e_i\in E\setminus\{e_r\}}(q^{2e_r}-q^{2e_i})} \\
&= \sum_{r=1}^{b(\lambda)} \frac{(q-q^{-1})^2 q^{-c(\lambda\setminus\mu)}}{(q^{2e_r}-q^{2\cdot 0})^2}\cdot\frac{\prod_{d_s\in\hat{D}_0}(q^{2e_r}-q^{-2d_s})}{\prod_{e_i\in E\setminus\{e_r\}}(q^{2e_r}-q^{2e_i})} \\
&\qquad (\ \hat{D}_0 = \hat{D}\cup\{0\}\ ) \\
&= (q-q^{-1})^2 q^{-c(\lambda\setminus\mu)} \sum_{e_r\in E} \frac{\prod_{d_s\in\hat{D}}(q^{2e_r}-q^{-2d_s})}{(q^{2e_r}-q^{2\cdot 0})\prod_{e_i\in E\setminus\{e_r\}}(q^{2e_r}-q^{2e_i})} \\
&= (q-q^{-1})^2 q^{-c(\lambda\setminus\mu)} \sum_{e_r\in E} \frac{\prod_{d_s\in\hat{D}}((q^{2e_r}-1)-(q^{-2d_s}-1))}{(q^{2e_r}-1)\prod_{e_i\in E\setminus\{e_r\}}((q^{2e_r}-1)-(q^{2e_i}-1))} \\
&\qquad (\ |\hat{D}| = |E|-2 \text{ by Remark 1.2 and apply the equation (2.1),(2.4)}\ ) \\
&= (q-q^{-1})^2 q^{-c(\lambda\setminus\mu)}\cdot\left((-1)^{|\hat{D}|}\prod_{d_s\in\hat{D}}(q^{-2d_s}-1)\right)\cdot\frac{(-1)^{|\hat{E}|-1}}{\prod_{e_r\in E}(q^{2e_r}-1)} \\
&= -(q-q^{-1})^2 q^{-c(\lambda\setminus\mu)}\cdot\frac{\prod_{d_s\in\hat{D}}(1-q^{-2d_s})}{\prod_{e_r\in E}(1-q^{2e_r})}\,.
\end{aligned}
$$

On the other hand, by the equation (1.5) we have

$$\begin{aligned}
\frac{h[\mu]}{h[\lambda]} &= \frac{\prod_{d_s\in\hat{D}}[d_s]}{(-1)^{|E|-1}\prod_{e_r\in E}[e_r]} \\
&\qquad (\ |\hat{D}| = |E| - 2 \text{ by Remark 1.2 }) \\
&= -(q-q^{-1})^2\cdot\frac{\prod_{d_s\in\hat{D}}(q^{d_s}-q^{-d_s})}{(-1)^{|E|}\prod_{e_r\in E}(q^{e_r}-q^{-e_r})} \\
&= -(q-q^{-1})^2\cdot\frac{\prod_{d_s\in\hat{D}}(q^{d_s})(1-q^{-2d_s})}{(-1)^{|E|}\prod_{e_r\in E}(q^{-e_r})(q^{2e_r}-1)} \\
&= -(q-q^{-1})^2\cdot q^{\sum_{d_s\in\hat{D}}d_s+\sum_{e_r\in E}e_r}\cdot\frac{\prod_{d_s\in\hat{D}}(1-q^{-2d_s})}{\prod_{e_r\in E}(1-q^{2e_r})} \\
&\qquad (\text{ Lemma 1.3 }) \\
&= -(q-q^{-1})^2\cdot q^{-c(\lambda\backslash\mu)}\cdot\frac{\prod_{d_s\in\hat{D}}(1-q^{-2d_s})}{\prod_{e_r\in E}(1-q^{2e_r})}\,.
\end{aligned}$$

Hence (3) holds. □

Next we prove (4)–(6).

PROOF OF (4). Using the equation (1.9) we have

$$\sum_{s=1}^{b'(\mu)} q^{-c(\mu\backslash\mu^-_{(s)})}\cdot\frac{h[\mu]}{h[\mu^-_{(s)}]} = -\frac{q^{-2c(\lambda\backslash\mu)}}{(q-q^{-1})^2}\sum_{d_s\in D}\frac{q^{-2d_s}\prod_{e_r\in\hat{E}_0}(q^{2d_s}-q^{-2e_r})}{\prod_{d_i\in D\backslash\{d_s\}}(q^{2d_s}-q^{2d_i})}\,.$$

Since $\hat{E}_0 = \hat{E}\cup\{0\}$, we have $|\hat{E}_0| = |D| + 1$ by Remark 1.1. Thus applying the equations (2.1),(2.2),(2.3),(2.4) we find

$$\begin{aligned}
&\sum_{d_s\in D}\frac{q^{-2d_s}\prod_{e_r\in\hat{E}_0}(q^{2d_s}-q^{-2e_r})}{\prod_{d_i\in D\backslash\{d_s\}}(q^{2d_s}-q^{2d_i})} \\
&= \sum_{d_s\in D}\frac{(q^{2d_s})^{|D|}}{\prod_{d_i\in D\backslash\{d_s\}}(q^{2d_s}-q^{2d_i})} + \sum_{d_s\in D}\frac{(q^{2d_s})^{|D|-1}\sum_{e_r\in\hat{E}_0}(-q^{-2e_r})}{\prod_{d_i\in D\backslash\{d_s\}}(q^{2d_s}-q^{2d_i})} \\
&\qquad + \sum_{d_s\in D}\frac{(q^{2d_s})^{-1}\prod_{e_r\in\hat{E}_0}(-q^{-2e_r})}{\prod_{d_i\in D\backslash\{d_s\}}(q^{2d_s}-q^{2d_i})} \\
&= \sum_{d_s\in D}q^{2d_s} - \sum_{e_r\in\hat{E}_0}q^{-2e_r} + (-1)^{|D|-1}\prod_{d_s\in D}q^{-2d_s}\prod_{e_r\in\hat{E}_0}(-q^{-2e_r}) \\
&\qquad (\ |\hat{E}_0| = |D| + 1 \text{ by Remark 1.1 })
\end{aligned}$$

$$= \sum_{d_s \in D} q^{2d_s} - \sum_{e_r \in \hat{E}_0} q^{-2e_r} + q^{-2(\sum_{d_s\in D} d_s + \sum_{e_r \in \hat{E}_0} e_r)}$$

(Lemma 1.3)

$$= \sum_{d_s \in D} q^{2d_s} - \sum_{e_r \in \hat{E}_0} q^{-2e_r} + q^{2c(\lambda\backslash\mu)} .$$

Thus we obtain

$$\sum_{s=1}^{b'(\mu)} q^{-c(\mu\backslash\mu^-_{(s)})} \cdot \frac{h[\mu]}{h[\mu^-_{(s)}]} = - \frac{q^{-2c(\lambda\backslash\mu)}}{(q-q^{-1})^2}\left(\sum_{d_s \in D} q^{2d_s} - \sum_{e_r \in \hat{E}_0} q^{-2e_r} + q^{2c(\lambda\backslash\mu)} \right)$$

$$= - \frac{1}{(q-q^{-1})^2}\left(\sum_{s=1}^{b'(\mu)} q^{-2c(\mu\backslash\mu^-_{(s)})} - \sum_{r=1}^{b(\mu)} q^{-2c(\mu^+_{(r)}\backslash\mu)} + 1 \right).$$

On the other hand by Proposition 1.4 we have

$$\sum_{x\in\mu} q^{-2c(x)} = \frac{1}{(q-q^{-1})^2}\left(\sum_{r=1}^{b(\mu)} q^{-2c(\mu^+_{(r)}\backslash\mu)} - \sum_{s=1}^{b'(\mu)} q^{-2c(\mu\backslash\mu^-_{(s)})} - 1 \right).$$

Hence we obtain (4). □

PROOF OF (5). Similarly using the equation (1.9), we have

$$\sum_{s=1}^{b'(\mu)} \frac{1}{[d_s]} \cdot \frac{h[\mu]}{h[\mu^-_{(s)}]} = - \sum_{s=1}^{b'(\mu)} \frac{q-q^{-1}}{q^{d_s}-q^{-d_s}} \cdot \frac{q^{-2c(\lambda\backslash\mu)+c(\mu\backslash\mu^-_{(s)})}}{(q-q^{-1})^2} \cdot \frac{q^{-2d_s}\prod_{e_r\in\hat{E}_0}(q^{2d_s}-q^{-2e_r})}{\prod_{d_i\in D\backslash\{d_s\}}(q^{2d_s}-q^{2d_i})}$$

$$= - \frac{1}{q-q^{-1}} \sum_{d_s\in D} \frac{q^{-c(\lambda\backslash\mu)-d_s}}{q^{-d_s}(q^{2d_s}-q^{2\cdot 0})} \cdot \frac{q^{-2d_s}\prod_{e_r\in\hat{E}_0}(q^{2d_s}-q^{-2e_r})}{\prod_{d_i\in D\backslash\{d_s\}}(q^{2d_s}-q^{2d_i})}$$

($\hat{E}_0 = \hat{E} \cup \{0\}$)

$$= - \frac{q^{-c(\lambda\backslash\mu)}}{q-q^{-1}} \sum_{d_s\in D} \frac{q^{-2d_s}\prod_{e_r\in\hat{E}}(q^{2d_s}-q^{-2e_r})}{\prod_{d_i\in D\backslash\{d_s\}}(q^{2d_s}-q^{2d_i})}$$

($|\hat{E}| = |D|$ by Remark 1.1 and apply the equation (2.1))

$$= - \frac{q^{-c(\lambda\backslash\mu)}}{q-q^{-1}} \sum_{d_s\in D} \frac{(q^{2d_s})^{|\hat{E}|-1} + (-1)^{|\hat{E}|}\cdot(q^{2d_s})^{-1}\prod_{e_r\in\hat{E}} q^{-2e_r}}{\prod_{d_i\in D\backslash\{d_s\}}(q^{2d_s}-q^{2d_i})}$$

(apply the equations (2.3),(2.4))

$$= - \frac{q^{-c(\lambda\backslash\mu)}}{q-q^{-1}} (1 + (-1)^{|\hat{E}|+|D|-1} \cdot q^{-\sum_{d_s\in D} 2d_s} q^{-\sum_{e_r\in\hat{E}} 2e_r})$$

($|\hat{E}| = |D|$ by Remark 1.1 and use Lemma 1.3)

$$
\begin{aligned}
&= -\frac{q^{-c(\lambda\backslash\mu)}}{q-q^{-1}}(1-q^{2c(\lambda\backslash\mu)}) \\
&= [c(\lambda \setminus \mu)]\,.
\end{aligned}
$$

□

PROOF OF (6). Similarly as in the proof of (4) and (5), using the equation (1.9) we have the following:

$$
\begin{aligned}
\frac{q^{d_s}}{[d_s]^2}\cdot\frac{h[\mu]}{h[\mu^-_{(s)}]} &= -\frac{q^{d_s}(q-q^{-1})^2}{(q^{d_s}-q^{-d_s})^2}\cdot\frac{q^{-2c(\lambda\backslash\mu)+c(\mu\backslash\mu^-_{(s)})}}{(q-q^{-1})^2}\cdot\frac{q^{-2d_s}\prod_{e_r\in\hat{E}_0}(q^{2d_s}-q^{-2e_r})}{\prod_{d_i\in D\backslash\{d_s\}}(q^{2d_s}-q^{2d_i})} \\
&= -\frac{q^{d_s-2c(\lambda\backslash\mu)+c(\mu\backslash\mu^-_{(s)})}}{q^{-2d_s}(q^{2d_s}-q^{2\cdot 0})^2}\cdot\frac{q^{-2d_s}\prod_{e_r\in\hat{E}_0}(q^{2d_s}-q^{-2e_r})}{\prod_{d_i\in D\backslash\{d_s\}}(q^{2d_s}-q^{2d_i})} \\
&\qquad (\ \hat{E}_0=\hat{E}\cup\{0\}\ ) \\
&= -q^{-c(\lambda\backslash\mu)}\cdot\frac{1}{(q^{2d_s}-q^{2\cdot 0})}\cdot\frac{\prod_{e_r\in\hat{E}}(q^{2d_s}-q^{-2e_r})}{\prod_{d_i\in D\backslash\{d_s\}}(q^{2d_s}-q^{2d_i})}\,.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\sum_{s=1}^{b'(\mu)}\frac{q^{d_s}}{[d_s]^2}\cdot\frac{h[\mu]}{h[\mu^-_{(s)}]} &= -q^{-c(\lambda\backslash\mu)}\sum_{d_s\in D}\frac{1}{(q^{2d_s}-1)}\frac{\prod_{e_r\in\hat{E}}((q^{2d_s}-1)-(q^{-2e_r}-1))}{\prod_{d_i\in D\backslash\{d_s\}}((q^{2d_s}-1)-(q^{2d_i}-1))} \\
&\qquad (\ |\hat{E}|=|D| \text{ by Remark 1.1 and apply the equation (2.3), (2.1),(2.4) }) \\
&= -q^{-c(\lambda\backslash\mu)}\left(1+((-1)^{|\hat{E}|}\prod_{e_r\in\hat{E}}(q^{-2e_r}-1))\cdot\frac{(-1)^{|\hat{E}|-1}}{\prod_{d_s\in D}(q^{2d_s}-1)}\right) \\
&= -q^{-c(\lambda\backslash\mu)}\left(1-\frac{\prod_{e_r\in\hat{E}}(1-q^{-2e_r})}{\prod_{d_i\in D}(1-q^{2d_i})}\right) \\
&= -q^{-c(\lambda\backslash\mu)}\left(1-\frac{\prod_{e_r\in\hat{E}}q^{-e_r}(q^{e_r}-q^{-e_r})}{\prod_{d_i\in D}q^{d_i}(q^{-d_i}-q^{d_i})}\right) \\
&= -q^{-c(\lambda\backslash\mu)}\left(1-q^{-\sum_{e_r\in\hat{E}}e_r-\sum_{d_s\in D}d_s}\cdot\frac{\prod_{e_r\in\hat{E}}(q^{e_r}-q^{-e_r})}{(-1)^{|D|}\prod_{d_s\in D}(q^{d_i}-q^{-d_i})}\right) \\
&\qquad (\text{ Lemma 1.3 and Remark 1.1 }) \\
&= -q^{-c(\lambda\backslash\mu)}+(-1)^{|D|}\cdot\frac{\prod_{e_r\in\hat{E}}[e_r]}{\prod_{d_i\in D}[d_i]} \\
&\qquad (\text{ the equation (1.2) }) \\
&= -q^{c(\lambda\backslash\mu)}+\frac{h[\lambda]}{h[\mu]}\,.
\end{aligned}
$$

This completes the proof of (6). □

PROOF OF (7). First we note that it is easily checked that the following identity holds for an arbitrary pair (d, e) of non-zero distinct integers:

$$\frac{1}{[d][e]} = \frac{1}{[e-d]}\left(\frac{q^d}{[d]} - \frac{q^e}{[e]}\right).$$

Putting $d = d(\mu, \lambda, \lambda^+_{(r)})$ and $e = d(\mu', \lambda, \lambda^+_{(r)})$, we have the following:

$$\begin{aligned}&\frac{q^{-c(\lambda^+_{(r)}\setminus\lambda)}}{[d(\mu,\lambda,\lambda^+_{(r)})][d(\mu',\lambda,\lambda^+_{(r)})]}\\ &\quad= \frac{1}{[c(\lambda\setminus\mu) - c(\lambda\setminus\mu')]}\left(\frac{q^{-c(\lambda\setminus\mu)}}{[d(\mu,\lambda,\lambda^+_{(r)})]} - \frac{q^{-c(\lambda\setminus\mu')}}{[d(\mu',\lambda,\lambda^+_{(r)})]}\right).\end{aligned}$$

Hence using the result of (2), we have the following:

$$\begin{aligned}&\sum_{r=1}^{b(\lambda)} \frac{q^{-c(\lambda^+_{(r)}\setminus\lambda)}}{[d(\mu,\lambda,\lambda^+_{(r)})][d(\mu',\lambda,\lambda^+_{(r)})]}\cdot\frac{h[\lambda]}{h[\lambda^+_{(r)}]}\\ &= \frac{1}{[c(\lambda\setminus\mu) - c(\lambda\setminus\mu')]}\sum_{r=1}^{b(\lambda)}\left(\frac{q^{-c(\lambda\setminus\mu)}}{[d(\mu,\lambda,\lambda^+_{(r)})]} - \frac{q^{-c(\lambda\setminus\mu')}}{[d(\mu',\lambda,\lambda^+_{(r)})]}\right)\cdot\frac{h[\lambda]}{h[\lambda^+_{(r)}]}\\ &= \frac{q^{-c(\lambda\setminus\mu)}}{[c(\lambda\setminus\mu) - c(\lambda\setminus\mu')]}\sum_{r=1}^{b(\lambda)}\frac{1}{[d(\mu,\lambda,\lambda^+_{(r)})]}\cdot\frac{h[\lambda]}{h[\lambda^+_{(r)}]}\\ &\quad - \frac{q^{-c(\lambda\setminus\mu')}}{[c(\lambda\setminus\mu) - c(\lambda\setminus\mu')]}\sum_{r=1}^{b(\lambda)}\frac{1}{[d(\mu',\lambda,\lambda^+_{(r)})]}\cdot\frac{h[\lambda]}{h[\lambda^+_{(r)}]}\\ &= 0.\end{aligned}$$

This proves (7). □

PROOF OF (8). In the same way as in the proof of (7), we have the following partial fraction:

$$\begin{aligned}&\frac{q^{c(\mu\setminus\mu^-_{(s)})}}{[d(\mu^-_{(s)},\mu,\lambda)][d(\mu^-_{(s)},\mu,\lambda')]}\\ &\quad= \frac{1}{[c(\lambda'\setminus\mu) - c(\lambda\setminus\mu)]}\left(\frac{q^{c(\lambda\setminus\mu)}}{[d(\mu^-_{(s)},\mu,\lambda)]} - \frac{q^{c(\lambda'\setminus\mu)}}{[d(\mu^-_{(s)},\mu,\lambda')]}\right).\end{aligned}$$

Hence using the result of (5), we have the following:

$$
\begin{aligned}
&\sum_{s=1}^{b'(\mu)} \frac{q^{c(\mu\setminus\mu^-_{(s)})}}{[d(\mu^-_{(s)},\mu,\lambda)][d(\mu^-_{(s)},\mu,\lambda')]}\cdot\frac{h[\mu]}{h[\mu^-_{(s)}]}\\
&=\frac{q^{c(\lambda\setminus\mu)}}{[c(\lambda'\setminus\mu)-c(\lambda\setminus\mu)]}\sum_{s=1}^{b'(\mu)}\frac{1}{[d(\mu^-_{(s)},\mu,\lambda)]}\cdot\frac{h[\mu]}{h[\mu^-_{(s)}]}\\
&\quad-\frac{q^{c(\lambda'\setminus\mu)}}{[c(\lambda'\setminus\mu)-c(\lambda\setminus\mu)]}\sum_{s=1}^{b'(\mu)}\frac{1}{[d(\mu^-_{(s)},\mu,\lambda')]}\cdot\frac{h[\mu]}{h[\mu^-_{(s)}]}\\
&=\frac{q^{c(\lambda\setminus\mu)}[c(\lambda\setminus\mu)]-q^{c(\lambda'\setminus\mu)}[c(\lambda'\setminus\mu)]}{[c(\lambda'\setminus\mu)-c(\lambda\setminus\mu)]}\\
&=-q^{(c(\lambda\setminus\mu)+c(\lambda'\setminus\mu))}\,.
\end{aligned}
$$

This completes the proof of (8). □

4. Proof of Theorem 0.2

Finally, we prove Theorem 0.2. By the equation (1.3) our goal is to show the following:

$$
\left(\prod_{d_s\in D}[d_s]^2\right)\sum_{s=1}^{b'(\mu)}\frac{1}{[d_s]^3}\cdot\frac{h[\mu]}{h[\mu^-_{(s)}]}=\left(\prod_{e_r\in E}[e_r]^2\right)\sum_{r=1}^{b(\lambda)}\frac{1}{[e_r]^3}\cdot\frac{h[\lambda]}{h[\lambda^+_{(r)}]}\,. \tag{4.1}
$$

In the proof of the equation (6) in the previous section, we have calculated that

$$
\frac{1}{[d_s]^2}\cdot\frac{h[\mu]}{h[\mu^-_{(s)}]}=-q^{-c(\lambda\setminus\mu)}\cdot\frac{q^{-d_s}}{q^{2d_s}-1}\cdot\frac{\prod_{e_r\in\hat{E}}(q^{2d_s}-q^{-2e_r})}{\prod_{d_i\in D\setminus\{d_s\}}(q^{2d_s}-q^{2d_i})}\,.
$$

Using this, we can calculate the left hand side of the equation (4.1) as follows:

$$
\begin{aligned}
&\left(\prod_{d_s\in D}[d_s]^2\right)\sum_{s=1}^{b'(\mu)}\frac{1}{[d_s]^3}\cdot\frac{h[\mu]}{h[\mu^-_{(s)}]}\\
&=-q^{-c(\lambda\setminus\mu)}\left(\prod_{d_s\in D}[d_s]^2\right)\sum_{d_s\in D}\frac{q-q^{-1}}{q^{d_s}-q^{-d_s}}\cdot\frac{q^{-d_s}}{q^{2d_s}-1}\cdot\frac{\prod_{e_r\in\hat{E}}(q^{2d_s}-q^{-2e_r})}{\prod_{d_i\in D\setminus\{d_s\}}(q^{2d_s}-q^{2d_i})}\\
&=-(q-q^{-1})q^{-c(\lambda\setminus\mu)}\left(\prod_{d_s\in D}[d_s]^2\right)\sum_{d_s\in D}\frac{\prod_{e_r\in\hat{E}}(q^{2d_s}-q^{-2e_r})}{(q^{2d_s}-1)^2\prod_{d_i\in D\setminus\{d_s\}}(q^{2d_s}-q^{2d_i})}\\
&=-\frac{q^{-c(\lambda\setminus\mu)}\prod_{d_s\in D}(q^{d_s}-q^{-d_s})^2}{(q-q^{-1})^{2|D|-1}}
\end{aligned}
$$

$$\times \sum_{d_s \in D} \frac{(q^{2d_s}-1)^{-2} \prod_{e_r \in \hat{E}}((q^{2d_s}-1)+(1-q^{-2e_r}))}{\prod_{d_i \in D\setminus\{d_s\}}((q^{2d_s}-1)-(q^{2d_i}-1))}$$

($|\hat{E}| = |D|$ by Remark 1.1 and apply the equation (2.1))

$$= -\frac{q^{-c(\lambda\setminus\mu)} \prod_{d_s \in D}(q^{d_s}-q^{-d_s})^2}{(q-q^{-1})^{2|D|-1}}$$

$$\times \sum_{d_s \in D}\left(\frac{(q^{2d_s}-1)^{-1}\sum_{e_r \in \hat{E}}(\prod_{e_i \in \hat{E}\setminus\{e_r\}}(1-q^{-2e_i}))}{\prod_{d_i \in D\setminus\{d_s\}}((q^{2d_s}-1)-(q^{2d_i}-1))}\right.$$

$$\left.+\frac{(q^{2d_s}-1)^{-2}\prod_{e_r \in \hat{E}}(1-q^{-2e_r})}{\prod_{d_i \in D\setminus\{d_s\}}((q^{2d_s}-1)-(q^{2d_i}-1))}\right)$$

(apply the equations (2.4),(2.5))

$$= -\frac{q^{-c(\lambda\setminus\mu)} q^{-\sum_{d_s \in D} 2d_s} \prod_{d_s \in D}(q^{2d_s}-1)^2}{(q-q^{-1})^{2|D|-1}}$$

$$\times\left((-1)^{|D|-1}\cdot\frac{\sum_{e_r \in \hat{E}}(\prod_{e_i \in \hat{E}\setminus\{e_r\}}(1-q^{-2e_i}))}{\prod_{d_s \in D}(q^{2d_s}-1)}\right.$$

$$\left.+(-1)^{|D|-1}\cdot\frac{(\prod_{e_r \in \hat{E}}(1-q^{-2e_r}))\sum_{d_s \in D}(\prod_{d_i \in D\setminus\{d_s\}}(q^{2d_i}-1))}{\prod_{d_s \in D}(q^{2d_s}-1)^2}\right)$$

$$= -\frac{q^{-c(\lambda\setminus\mu)} q^{-\sum_{d_s \in D} 2d_s}(-1)^{|D|-1}}{(q-q^{-1})^{2|D|-1}}$$

$$\times\left(\left(\prod_{d_s \in D}(q^{2d_s}-1)\right)\sum_{e_r \in \hat{E}}\left(\prod_{e_i \in \hat{E}\setminus\{e_r\}}(1-q^{-2e_i})\right)\right.$$

$$\left.+\left(\prod_{e_r \in \hat{E}}(1-q^{-2e_r})\right)\sum_{d_s \in D}\left(\prod_{d_i \in D\setminus\{d_s\}}(q^{2d_i}-1)\right)\right)$$

($|D| = |\hat{E}|$ by Remark 1.1)

$$= -\frac{q^{-c(\lambda\setminus\mu)} q^{-\sum_{d_s \in D} d_s} q^{-\sum_{e_r \in \hat{E}} e_r}(-1)^{|D|-1}}{(q-q^{-1})^{|D|+|\hat{E}|-1}}$$

$$\times\left(\left(\prod_{d_s \in D}(q^{d_s}-q^{-d_s})\right)\sum_{e_r \in \hat{E}}\left(q^{e_r}\prod_{e_i \in \hat{E}\setminus\{e_r\}}(q^{e_i}-q^{-e_i})\right)\right.$$

$$\left.+\left(\prod_{e_r \in \hat{E}}(q^{e_r}-q^{-e_r})\right)\sum_{d_s \in D}\left(q^{-d_s}\prod_{d_i \in D\setminus\{d_s\}}(q^{d_i}-q^{-d_i})\right)\right)$$

(Lemma 1.3)

$$
\begin{aligned}
&= (-1)^{|D|}\Bigg(\Big(\prod_{d_s\in D}[d_s]\Big)\sum_{e_r\in\hat{E}}\Big(q^{e_r}\prod_{e_i\in\hat{E}\setminus\{e_r\}}[e_i]\Big)\\
&\quad+\Big(\prod_{e_r\in\hat{E}}[e_r]\Big)\sum_{d_s\in D}\Big(q^{-d_s}\prod_{d_i\in D\setminus\{d_s\}}[d_i]\Big)\Bigg)\\
&= (-1)^{|D|}\Big(\prod_{f\in D\sqcup\hat{E}}[f]\Big)\Big(\sum_{e_r\in\hat{E}}\frac{q^{e_r}}{[e_r]}+\sum_{d_s\in D}\frac{q^{-d_s}}{[d_s]}\Big).
\end{aligned}
$$

On the other hand, using the equation (1.7) we can calculate the right hand side of the equation (4.1) as follows:

$$
\begin{aligned}
&\Big(\prod_{e_r\in E}[e_r]^2\Big)\sum_{r=1}^{b(\lambda)}\frac{1}{[e_r]^3}\cdot\frac{h[\lambda]}{h[\lambda^+_{(r)}]}\\
&=\Big(\prod_{e_r\in E}[e_r]^2\Big)\sum_{r=1}^{b(\lambda)}\frac{(q-q^{-1})^3}{(q^{e_r}-q^{-e_r})^3}\cdot\frac{q^{-c(\lambda^+_{(r)}\setminus\lambda)}\prod_{d_s\in\hat{D}_0}(q^{2e_r}-q^{-2d_s})}{\prod_{e_i\in E\setminus\{e_r\}}(q^{2e_r}-q^{2e_i})}\\
&=\Big(\prod_{e_r\in E}[e_r]^2\Big)\sum_{e_r\in E}\frac{(q-q^{-1})^3}{q^{-3e_r}(q^{2e_r}-1)^3}\cdot\frac{q^{-e_r-c(\lambda\setminus\mu)}\prod_{d_s\in\hat{D}_0}(q^{2e_r}-q^{-2d_s})}{\prod_{e_i\in E\setminus\{e_r\}}(q^{2e_r}-q^{2e_i})}\\
&=(q-q^{-1})^3q^{-c(\lambda\setminus\mu)}\Big(\prod_{e_r\in E}[e_r]^2\Big)\sum_{e_r\in E}\frac{q^{2e_r}}{(q^{2e_r}-1)^2}\frac{\prod_{d_s\in\hat{D}}(q^{2e_r}-q^{-2d_s})}{\prod_{e_i\in E\setminus\{e_r\}}(q^{2e_r}-q^{2e_i})}\\
&\qquad\Big(-c(\lambda\setminus\mu)=\textstyle\sum_{d_s\in\hat{D}}d_s+\sum_{e_r\in E}e_r\ \text{by Lemma 1.3}\Big)\\
&=(q-q^{-1})^3\cdot q^{\sum_{d_s\in\hat{D}}d_s+\sum_{e_r\in E}e_r}\cdot\frac{\prod_{e_r\in E}(q^{e_r}-q^{-e_r})^2}{(q-q^{-1})^{2|E|}}\\
&\quad\times\Big(\sum_{e_r\in E}\frac{1}{q^{2e_r}-1}\cdot\frac{\prod_{d_s\in\hat{D}}(q^{2e_r}-q^{-2d_s})}{\prod_{e_i\in E\setminus\{e_r\}}(q^{2e_r}-q^{2e_i})}\\
&\quad+\sum_{e_r\in E}\frac{1}{(q^{2e_r}-1)^2}\cdot\frac{\prod_{d_s\in\hat{D}}(q^{2e_r}-q^{-2d_s})}{\prod_{e_i\in E\setminus\{e_r\}}(q^{2e_r}-q^{2e_i})}\Big)\\
&=(q-q^{-1})^3\cdot q^{\sum_{d_s\in\hat{D}}d_s}\cdot q^{-\sum_{e_r\in E}e_r}\cdot\frac{\prod_{e_r\in E}(q^{2e_r}-1)^2}{(q-q^{-1})^{2|E|}}\\
&\quad\times\Big(\sum_{e_r\in E}\frac{(q^{2e_r}-1)^{-1}\prod_{d_s\in\hat{D}}((q^{2e_r}-1)+(1-q^{-2d_s}))}{\prod_{e_i\in E\setminus\{e_r\}}((q^{2e_r}-1)-(q^{2e_i}-1))}
\end{aligned}
$$

$$+\sum_{e_r\in E}\frac{(q^{2e_r}-1)^{-2}\prod_{d_s\in\hat{D}}((q^{2e_r}-1)+(1-q^{-2d_s}))}{\prod_{e_i\in E\setminus\{e_r\}}((q^{2e_r}-1)-(q^{2e_i}-1))}\Bigg)$$

($|\hat{D}|=|E|-2$ by Remark 1.2 and apply the equations (2.1),(2.4),(2.5))

$$=\frac{q^{\sum_{d_s\in\hat{D}}d_s-\sum_{e_r\in E}e_r}\prod_{e_r\in E}(q^{2e_r}-1)^2}{(q-q^{-1})^{2|E|-3}}$$

$$\times\Bigg((-1)^{|E|-1}\cdot\frac{\prod_{d_s\in\hat{D}}(1-q^{-2d_s})}{\prod_{e_i\in E}(q^{2e_i}-1)}$$

$$+(-1)^{|E|-1}\cdot\frac{\sum_{d_s\in\hat{D}}(\prod_{d_i\in\hat{D}\setminus\{d_s\}}(1-q^{-2d_s}))}{\prod_{e_i\in E}(q^{2e_i}-1)}$$

$$+(-1)^{|E|-1}\cdot\frac{(\prod_{d_s\in\hat{D}}(1-q^{-2d_s}))\sum_{e_r\in E}(\prod_{e_i\in E\setminus\{e_r\}}(q^{2e_i}-1))}{\prod_{e_i\in E}(q^{2e_i}-1)^2}\Bigg)$$

$$=(-1)^{|E|-1}\cdot\frac{q^{\sum_{d_s\in\hat{D}}d_s-\sum_{e_r\in E}e_r}}{(q-q^{-1})^{2|E|-3}}$$

$$\times\Bigg(\Bigg(\prod_{e_r\in E}(q^{2e_r}-1)\Bigg)\Bigg(\prod_{d_s\in\hat{D}}(1-q^{-2d_s})\Bigg)$$

$$+\Bigg(\prod_{e_r\in E}(q^{2e_r}-1)\Bigg)\sum_{d_s\in\hat{D}}\Bigg(\prod_{d_i\in\hat{D}\setminus d_s}(1-q^{-2d_i})\Bigg)$$

$$+\Bigg(\prod_{d_s\in\hat{D}}(1-q^{-2d_s})\Bigg)\sum_{e_r\in E}\Bigg(\prod_{e_i\in E\setminus\{e_r\}}(q^{2e_i}-1)\Bigg)\Bigg)$$

($2|E|-3=|E|+|\hat{D}|-1$ by Remark 1.2)

$$=(-1)^{|E|-1}\cdot\frac{q-q^{-1}}{(q-q^{-1})^{|E|+|\hat{D}|}}$$

$$\times\Bigg(\Bigg(\prod_{e_r\in E}(q^{e_r}-q^{-e_r})\Bigg)\Bigg(\prod_{d_s\in\hat{D}}(q^{d_s}-q^{-d_s})\Bigg)$$

$$+\Bigg(\prod_{e_r\in E}(q^{e_r}-q^{-e_r})\Bigg)\sum_{d_s\in\hat{D}}\Bigg(q^{d_s}\prod_{d_i\in\hat{D}\setminus\{d_s\}}(q^{d_i}-q^{-d_i})\Bigg)$$

$$+\Bigg(\prod_{d_s\in\hat{D}}(q^{d_s}-q^{-d_s})\Bigg)\sum_{e_r\in E}\Bigg(q^{-e_r}\prod_{e_i\in E\setminus\{e_r\}}(q^{e_i}-q^{-e_i})\Bigg)\Bigg)$$

$$=(-1)^{|E|-1}\Bigg((q-q^{-1})\Bigg(\prod_{e_r\in E}[e_r]\Bigg)\Bigg(\prod_{d_s\in\hat{D}}[d_s]\Bigg)$$

$$+\Big(\prod_{e_r\in E}[e_r]\Big)\sum_{d_s\in\hat{D}}\Big(q^{d_s}\prod_{d_i\in\hat{D}\setminus d_s}[d_s]\Big)$$

$$+\Big(\prod_{d_s\in\hat{D}}[d_s]\Big)\sum_{e_r\in E}\Big(q^{-e_r}\prod_{e_i\in E\setminus\{e_r\}}[e_i]\Big)\Big)$$

$$=(-1)^{|E|-1}\Big(\prod_{f\in\hat{D}\sqcup E}[f]\Big)\Big((q-q^{-1})+\sum_{d_i\in\hat{D}}\frac{q^{d_i}}{[d_i]}+\sum_{e_i\in E}\frac{q^{-e_i}}{[e_i]}\Big).$$

Hence it is suffice to prove the following:

$$(-1)^{|D|}\prod_{f\in D\sqcup\hat{E}}[f]=(-1)^{|E|-1}\prod_{f\in\hat{D}\sqcup E}[f],\tag{4.2}$$

$$\sum_{d_s\in D}\frac{q^{-d_s}}{[d_s]}+\sum_{e_r\in\hat{E}}\frac{q^{e_r}}{[e_r]}=q-q^{-1}+\sum_{d_s\in\hat{D}}\frac{q^{d_s}}{[d_s]}+\sum_{e_r\in E}\frac{q^{-e_r}}{[e_r]}.\tag{4.3}$$

First we verify the equation (4.2). Since the right hand side of the equation (4.2) contains no zero factor, we have the following:

$$\frac{(-1)^{|E|-1}\prod_{f\in\hat{D}\sqcup E}[f]}{(-1)^{|D|}\prod_{f\in D\sqcup\hat{E}}[f]}=\frac{(-1)^{|E|-1}\prod_{f\in\hat{D}\sqcup\hat{E}}[f]\prod_{f\in E\setminus\hat{E}}[f]}{(-1)^{|D|}\prod_{f\in\hat{D}\sqcup\hat{E}}[f]\prod_{f\in D\setminus\hat{D}}[f]}$$

$$=(-1)^{|E|-|D|-1}\frac{\prod_{f\in D\setminus\hat{D}}[f]}{\prod_{f\in E\setminus\hat{E}}[f]}$$

$$=(-1)^{|E\setminus\hat{E}|-1}\frac{\prod_{f\in D\setminus\hat{D}}[f]}{\prod_{f\in E\setminus\hat{E}}[f]}.$$

Here we used the fact that $|D|=|\hat{E}|$ and that E contains $\hat{E}$. Although $D\setminus\hat{D}$ (resp. $E\setminus\hat{E}$) varies depending on the position of $\lambda\setminus\mu$ (See Figure 7 and Table 1), it is easily checked that the last term of the equation above is identically equal to 1. Thus the equation (4.2) is verified.

Finally we find that the equation (4.3) holds as follows:

$$q-q^{-1}+\sum_{d_s\in\hat{D}}\frac{q^{d_s}}{[d_s]}+\sum_{e_r\in E}\frac{q^{-e_r}}{[e_r]}-\Big(\sum_{d_s\in D}\frac{q^{-d_s}}{[d_s]}+\sum_{e_r\in\hat{E}}\frac{q^{e_r}}{[e_r]}\Big)$$

$$=q-q^{-1}+\sum_{d_s\in\hat{D}}\frac{q^{d_s}-q^{-d_s}}{[d_s]}-\sum_{e_r\in\hat{E}}\frac{q^{e_r}-q^{-e_r}}{[e_r]}$$

$$-\sum_{d_s\in D\setminus\hat{D}}\frac{q^{-d_s}}{[d_s]}+\sum_{e_r\in E\setminus\hat{E}}\frac{q^{-e_r}}{[e_r]}$$

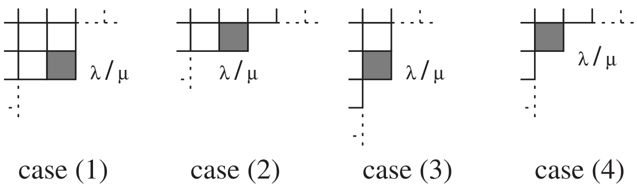

FIGURE 7. Possible positions of $\lambda \setminus \mu$.

TABLE 1. $D \setminus \hat{D}$ and $E \setminus \hat{E}$

	$D \setminus \hat{D}$	$E \setminus \hat{E}$
case (1)	$\{+1, -1\}$	$\emptyset$
case (2)	$\{+1\}$	$\{+1\}$
case (3)	$\{-1\}$	$\{-1\}$
case (4)	$\emptyset$	$\{+1, -1\}$

$$
\begin{aligned}
&= q - q^{-1} + \sum_{d_s \in \hat{D}} (q - q^{-1}) - \sum_{e_r \in \hat{E}} (q - q^{-1}) \\
&\quad - \sum_{d_s \in D \setminus \hat{D}} \frac{q^{-d_s}}{[d_s]} + \sum_{e_r \in E \setminus \hat{E}} \frac{q^{-e_r}}{[e_r]} \\
&= (q - q^{-1})(1 + |\hat{D}| - |\hat{E}|) - \sum_{d_s \in D \setminus \hat{D}} \frac{q^{-d_s}}{[d_s]} + \sum_{e_r \in E \setminus \hat{E}} \frac{q^{-e_r}}{[e_r]} \\
&\qquad (\ |\hat{D}| = |E| - 2 \text{ by Remark 1.2 and } E \supset \hat{E}\ ) \\
&= (q - q^{-1})(|E \setminus \hat{E}| - 1) - \sum_{d_s \in D \setminus \hat{D}} \frac{q^{-d_s}}{[d_s]} + \sum_{e_r \in E \setminus \hat{E}} \frac{q^{-e_r}}{[e_r]} \\
&= \begin{cases} (q - q^{-1})\cdot(-1) - (q^{-1} - q) + 0 & \text{(case (1))} \\ (q - q^{-1})\cdot 0 - q^{-1} + q^{-1} & \text{(case (2))} \\ (q - q^{-1})\cdot 0 - (-q) + (-q) & \text{(case (3))} \\ (q - q^{-1})\cdot 1 - 0 + (q^{-1} - q) & \text{(case (4))} \end{cases} \\
&= 0\,.
\end{aligned}
$$

Thus we have completed the proof of Theorem 0.2.

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Present Address:
DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF THE RYUKYUS,
SENBARU, NISHIHARA-CHO, OKINAWA, 903–0213 JAPAN.
e-mail: kosuda@math.u-ryukyu.ac.jp